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SOME REMARKS ON QUADRATIC PROGRAMMING WITH 0-1 VARIABLES

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Abstract. — *The aim of this paper is to show that* (1) *every bivalent* (0,1) *quadratic programming problem is equivalent to one having a positive (negative) semi-definite matrix in the objective function ;* (2) *to establish conditions for different classes of local optimality* ; (3) *to show that any problem of bivalent* (0,1) *programming is equivalent* (a) *to the problem of minimizing a real valued function, partly in* (0,1), *and partly in non-negative variables,* (b) *to the problem of finding the minimax of a real valued function in bivalent* (0,1) *variables.*

INTRODUCTION

Numerous problems in various fields of operations research (investment problems, graphs, etc.) lead naturally to problems of quadratic programming with variables which can take on only the values 0 and 1.

The available methods for solving mathematical programs in 0-1 variables, are either dealing only with the linear case (and hence unapplicable for our problems), or dealing with the most general cases (and hence not taking into account the particularities of a quadratic program). Specific methods for the solution of quadratic bivalent programs have been studied by H. P. Kunzi and W. Oettli [4], V. Ginsburgh and A. Van Peeterssen [2] and the present authors [5].

Our aim in this paper is to study some general properties of quadratic 0-1 programs. We shall deal here with :

a) The relationship between a quadratic 0-1 program and the associated continuous program ;

b) Conditions for different types of local optima ;

c) Possibilities of reducing a quadratic program to

c.1) an unconstrained quadratic minimization problem,

c.2) an unconstrained quadratic minimax problem.

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A *Boolean variable* x_i is a variable which takes its values from the two element Boolean algebra $B_2 = \{0, 1\}$.

A vector X with n Boolean components will be called a *Boolean vector*. The set of these vectors will be denoted by B_2^n.

A mapping $f(X)$ from B_2^n into the field R of reals will be called a *pseudo-Boolean function.*

We define the distance $d(X, Y)$ of two vectors X and Y belonging to B_2^n by putting :

$$d(X, Y) = \sum_{i=1}^{n} (x_i - y_i)^2 \; ; \tag{0.1}$$

$d(X, Y)$ represents the number of different components of X and Y.

We define the k-neighbourhood $W_k(X)$ of X in B_2^n as the set of those vectors Y belonging to B_2^n which are at distance k from X :

$$W_k(X) = \{ Y \in B_2^n, d(X, Y) = k \} \tag{0.2}$$

$f(X^*)$ is a (*globally*) *minimizing point* of the pseudo-Boolean function $f(X)$ if :

$$f(X^*) \leqslant f(X) \text{ for any } X \in B_2^n. \tag{0.3}$$

X^* is a *locally minimizing* point of f if :

$$f(X^*) \leqslant f(X) \text{ for any } X \in W_1(X^*), \tag{0.4}$$

and more generally X^* will be a *k-minimizing point* of f if :

$$f(X^*) \leqslant f(X) \text{ for any } X \in W_k(X^*). \tag{0.5}$$

Given a real valued n by n matrix $\tilde{Q} = (\tilde{q}_{ij})$ and a real valued n vector p we define the *pseudo Boolean quadratic function* $f(X)$ as :

$$f(X) = X'\tilde{Q}X + p'X \tag{0.6}$$

Remarking that $x_i^2 = x_i$ for every i, $i = 1, ..., n$ we add the component p_i of the vector p to the i-th diagonal element of the matrix $\tilde{Q}$. Let us denote by $Q = (q_{ij})$ the new matrix defined by :

$$q_{ij} = \begin{cases} \tilde{q}_{ij} & i \neq j \\ \tilde{q}_{ij} + p_i & i = j \end{cases}$$

From now on we will represent a pseudo-Boolean quadratic function simply by :

$$f(X) = X'QX. \tag{0.7}$$

The matrix Q will always be assumed to be symmetric. otherwise as $X'QX = \frac{1}{2} X'(Q + Q')$ the matrix $\frac{1}{2}(Q + Q')$ is symmetric, showing that our assumption is not restrictive.

1. THE ASSOCIATED CONTINUOUS PROGRAM

By a problem of quadratic pseudo-Boolean programming under linear constraints we shall mean the problem of minimizing

$$X'QX$$

subject to

$$AX \leqslant b \tag{1.1}$$

and to

$$X \in B_2^n, \tag{1.2}$$

where A is a given $m \times n$ matrix, Q is a given symmetric $n \times n$ matrix, b is a given m-vector, and X is an n-vector to be determined.

This problem will be called Problem I. To Problem I we associate the following Problem II :

Minimize

$$X'QX$$

subject to

$$AX \leqslant b \tag{1.3}$$

and to

$$0 \leqslant x_j \leqslant 1 \qquad (j = 1, ..., n). \tag{1.4}$$

Numerous procedures are available for solving Problem II *when Q is a positive semidefinite matrix* (see, e.g. [1], [6], etc.).

Obviously, by a rounding procedure we can obtain from the optimal solution of Problem II a certain « approximation » of the optimal solution of Problem I.

In order to make use of this remark, we have to solve Problem II ; the simplest way seems to utilize the following :

Theorem 1. *Given a symmetric $n \times n$ matrix Q, there exists a positive definite $n \times n$ matrix R and an n-vector d such that if*

$$f(X) = X'QX$$

$$g(X) = X'RX + dX,$$

then,

$$f(X) = g(X) \text{ for every } X \in B_2^n.$$

Proof. Let γ be an arbitrary real number, and let

$$g_\gamma(X) = X'QX + \gamma \sum_{i=1}^{n} (x_i^2 - x_i),$$

or

$$g_\gamma(X) = X'(Q + \gamma I)X - \gamma \sum_{i=1}^{n} x_i.$$

From the fact that $x_i^2 = x_i$ for any $x_i \varepsilon B_2$, it follows that $f(X) = g_\gamma(X)$ for any $X \in B_2^n$.

Q being a symmetric matrix, its eigen-values are reals. Let λ denote the smallest of these eigen-values. The smallest eigen-value of $Q + \gamma I$ will hence be $\lambda + \gamma$. Choosing γ such that $\lambda + \gamma$ should be positive, we assure the positive definiteness of $Q + \gamma I$, thus proving the theorem.

In order to make a reasonably good choice of the γ, let us remark the followings. If γ_1 and γ_2 are two reals ($\gamma_1 < \gamma_2$) satisfying the conditions $\lambda + \gamma_h > 0$ ($h = 1, 2$), and if $P - h$ denotes the problem of minimizing g_{γ_h} under constraints (1.3) and (1.4) (assumed to be consistent), then let us denote by X_h the optimal solution of $P - h$. Let us further denote the center of the hypercube B_2^n by $C = \left(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right)$, and let the *distance* $d(V_1, V_2)$ between two real vectors $V_1 = (V_{11}, \ldots, V_{1n})$ and $V_2 = (V_{21}, \ldots, V_{2n})$ be

$$d(V_1, V_2) = \sum_{i=1}^{n} (V_{1i} - V_{2i})^2.$$

The following result holds :

Theorem 2. *If* $\gamma_1 < \gamma_2$ *then* $d(X_1, C) > d(X_2, C)$.

Proof. From the fact that X_1 is an optimal solution of $P - 1$, and X_2 is a feasible solution of the same problem, it follows that

$$g_1(X_1) \leqslant g_1(X_2).$$

Analogously,

$$g_2(X_2) \leqslant g_2(X_1).$$

These relations can be rewritten as

$$X_1'QX_1 + \gamma_1 \sum_{i=1}^{n} (x_{1i}^2 - x_{1i}) \leqslant X_2'QX_2 + \gamma_1 \sum_{i=1}^{n} (x_{2i}^2 - x_{2i}) \quad (1.5)$$

$$X_2'QX_2 + \gamma_2 \sum_{i=1}^{n} (x_{2i}^2 - x_{2i}) \leqslant X_1'QX_1 + \gamma_2 \sum_{i=1}^{n} (x_{1i}^2 - x_{1i}). \quad (1.6)$$

Adding (1.5) and (1.6) we get

$$(\gamma_1 - \gamma_2) \sum_{i=1}^{n} (x_{1i}^2 - x_{1i}) \leqslant (\gamma_1 - \gamma_2) \sum_{i=1}^{n} (x_{2i}^2 - x_{2i}).$$

As $\gamma_1 < \gamma_2$, it follows that

$$\sum_{i=1}^{n} (x_{1i}^2 - x_{1i}) \geqslant \sum_{i=1}^{n} (x_{2i}^2 - x_{2i}). \quad (1.7)$$

Hence

$$\sum_{i=1}^{n} \left(x_{1i} - \frac{1}{2}\right)^2 \geqslant \sum_{i=1}^{n} \left(x_{2i} - \frac{1}{2}\right)^2,$$

or

$$d(X_1, C) \geqslant d(X_2, C), \tag{1.8}$$

proving the theorem.

It follows from Theorem 2, that in order to get a good starting solution of *PI* from the rounded optimal solution of *PII* it is advisable to choose γ as small as possible.

2. CONDITIONS FOR k-MINIMALITY

A vector X is a k-minimizing point for the function f if :

$$f(X) \leqslant f(Y) \qquad \text{for any} \qquad Y \in W_k(X). \tag{2.1}$$

Let us denote by J the set of the indices of the k differing components of X and Y :

$$x_i = y_i \qquad i \notin J \tag{2.2}$$

$$x_i = 1 - y_i \qquad i \in J \tag{2.3}$$

$$J \subset \{1, \ldots, n\} \qquad \text{and } |J| = k.$$

Condition (2.1) is expressed by

$$f(X) - f(Y) = \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} x_i x_j - \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} y_i y_j \leqslant 0. \tag{2.4}$$

Using (2.2) and (2.3) we get

$$\begin{aligned} f(X) - f(Y) = & \sum_{i \notin J} \sum_{j \notin J} q_{ij} x_i x_j - \sum_{i \notin J} \sum_{j \notin J} q_{ij} y_i y_j \\ & + \sum_{i \in J} x_i \sum_{j \notin J} 2 q_{ij} x_j + \sum_{i \in J} \sum_{j \in J} q_{ij} x_i x_j \\ & - \sum_{i \in J} y_i \sum_{j \notin J} 2 q_{ij} y_j - \sum_{i \in J} \sum_{j \in J} q_{ij} y_i y_j, \end{aligned} \tag{2.5}$$

or,

$$f(X) - f(Y) = \sum_{i \in J} (2x_i - 1) \left[\sum_{j \notin J} 2 q_{ij} x_j + \sum_{j \in J} q_{ij} \right], \tag{2.6}$$

Hence, X is a k-minimizing point for the function f iff for every set of indices J, such that $|J| = k$, the following relation holds :

$$\sum_{i \in J} (2x_i - 1) \left[\sum_{j \notin J} 2 q_{ij} x_j + \sum_{j \in J} q_{ij} \right] \leqslant 0 \tag{2.7}$$

In particular

for $J = \{1, 2, \ldots, n\}$, (2.7) simplifies to

$$\sum_{i=1}^{n} (2x_i - 1) \left(\sum_{j=1}^{n} q_{ij} \right) \leqslant 0 \, ; \tag{2.8}$$

for $J = \{l\}$, (2.7) simplifies to

$$(2x_l - 1)\left(\sum_{j \neq l} 2q_{lj}x_j + q_{ll}\right) \leqslant 0 \, ; \tag{2.9}$$

for $J = \{1, 2, \ldots, n\}, - \{l\}$ (2.7) simplifies to

$$\sum_{j \neq l} (2x_j - 1)(2q_{lj}x_l + \sum_{j \neq l} q_{lj}) \leqslant 0 \tag{2.10}$$

Remark. We point out that a 1-minimizing and 2-minimizing point X^* is not necessarily a globally minimizing point. Consider for this, the following example in B_2^3 :

Let

$$Q = \begin{pmatrix} 5 & 0 & -3 \\ 0 & 5 & -3 \\ -3 & -3 & 5 \end{pmatrix}$$

The point (1, 1, 1) is both 1-minimizing and 2-minimizing, but it is not globally minimizing ; the globally minimizing point is (0, 0, 0), as it can be seen from Table 1.

Table. 1

x_1	x_2	x_3	$X'QX$
0	0	0	0
1	0	0	5
0	1	0	5
0	0	1	5
1	1	0	10
1	0	1	4
0	1	1	4
1	1	1	3

3. MINIMIZATION UNDER CONSTRAINTS

The problem (III) we shall consider in his section is the following :

Minimize : $f(X) = X'QX$

under the following constraints :

$$\begin{aligned} \varphi_j(X) &\leqslant 0 \qquad j = 1, ..., m \\ \varphi_j(X) &= 0 \qquad j = m+1, ..., l \\ X &\in B_2^n ; \end{aligned} \tag{3.1}$$

here $\varphi_j(X)$ are pseudo-Boolean functions of X. We shall assume that these functions $\varphi_j(X)$ are integer valued. As X has to satisfy the set of constrains (3.1), we have to define the concept of locally minimizing points of a pseudo-Boolean function under pseudo-Boolean constraints.

X^* is a *locally minimizing point* for the function $f(X)$ *under the set of constraints* (3.1.) if

1) X^* fulfills the set of constraints (3.1.)

2) for every $Y \in W_1(X^*)$, either $f(X^*) \leqslant f(Y)$ or Y violates at least one of the constraints (3.1) ($j = 1, ..., l$).

A. *Introducing slack variables.*

We introduce the slack variables $u_j (j = 1, ..., m)$, and reformulate (IV) the program (III) :

Minimize $f(X) = X'QX$

so that

$$\begin{aligned} &\varphi_j(X) + u_j = 0, && j = 1, ..., m \\ &\varphi_j(X) = 0, && j = m+1, ..., l \\ &X \in B_2^n ; u_j \geqslant 0 && j = 1, ..., m \end{aligned} \tag{3.2}$$

We can use « Lagrangean multipliers » (as defined in [3]) and formulate the program as one without constraints. For this sake, let us denote by B^+ and B^-, an upper and a lower bound of $f(x)$ in B_2^n (for example the sum of all its positive and all its negative coefficients). We have :

Theorem 3. (See [3]).

(α) *If $X^* = (x_1^* ..., x_n^*)$, is an optimal solution of problem (III), then there exists a vector $U^* = (u_1^*, ..., u_m^*)$, such that (X^*, U^*) is an optimal solution of the following problem* (V) :

Minimize

$$F(x_1, ..., x_n, u_1, ..., u_m) = f(x_1, ..., x_n)$$

$$+ (B^+ - B^- + 1)\left(\sum_{j=1}^{m} (\varphi_j(X) + u_j)^2 + \sum_{j=m+1}^{l} \varphi_j^2(X)\right) ; \tag{3.6}$$

$$x_i \in \{0, 1\} ; \quad i = 1, ..., n ; \quad u_j \geqslant 0, \quad j = 1, ..., m.$$

(β) *If* (X^*, U^*) *is an optimal solution of problem (V) and* $F(x_1^*, \ldots, x_n^*, u_1^*, \ldots, u_m^*) \leqslant B^+$, *then the constraints* (3.1.) *are consistent and* X^* *is an optimal solution of problem (III).*

(γ) *If* (X^*, U^*) *is an optimal solution of problem (V), and* $F(x_1^*, \ldots, x_n^*, u_1^*, \ldots, u_m^*) > B^+$, *then the constraints* (3.1) *are inconsistent.*

Proof. Let us first notice that

$$B^- \leqslant f(X) \leqslant B^+ \tag{3.7}$$

(α) Given an opimal solution X^* of problem (III), we have :

$$\varphi_j(X^*) = 0 \qquad j = m+1, \ldots, l$$
$$\varphi_j(X^*) \leqslant 0 \qquad j = 1, \ldots, m$$

We define the vector U^* by

$$u_j^* = -\varphi_j(X^*) \geqslant 0 \qquad j = 1, \ldots, m$$

Let us suppose that there exists a vector (Y^*, V^*), $(Y^* \in B_2^n ; V^* \geqslant 0)$ such that

$$F(Y^*, V^*) < F(Y^*, U^*). \tag{3.8}$$

It follows that Y^* fulfils the system (3.1). Indeed, if not, then there exists an index j_0 such that either

$$j_0 \in (1, \ldots, m) \qquad \text{and} \qquad \varphi_{j0}(Y^*) \geqslant 1 \tag{3.9}$$

or

$$j_0 \in (m+1, \ldots, l) \qquad \text{and} \qquad \varphi_{j0}(Y^*) \neq 0 \tag{3.10}$$

In the first case, $\varphi_{j0}(Y^*) \geqslant 1$ and $v_{j0}^* \geqslant 0$ imply

$$(\varphi_{j0}(Y^*) + v_{j0}^*)^2 \geqslant 1.$$

In the second case, we see that

$$\varphi_{j0}^2(Y^*) \geqslant 1. \tag{3.11}$$

In both cases we deduce that

$$\sum_{j=1}^{m} (\varphi_j(Y^*) + v_j^*)^2 + \sum_{j=m+1}^{l} \varphi_j^2(Y^*) \geqslant 1 \tag{3.12}$$

and

$$F(Y^*, V^*) \geqslant f(Y^*) + B^+ - B^- + 1 \geqslant B^+ + 1 \tag{3.13}$$

On the other hand

$$F(X^*, U^*) = f(X^*) \leqslant B^+ \tag{3.14}$$

From (3.13) and (3.14) we get

$$F(X^*, U^*) < F(Y^*, V^*) \tag{3.15}$$

which contradicts (3.8.) Hence Y^* fulfils the constraints (3.1).

As above we can also deduce that

$$v_j^* = -\varphi_j(Y^*), \qquad j = 1, ..., m$$

hence :

$$F(Y^*, V^*) = f(Y^*) \tag{3.16}$$

From (3.16) and (3.8) we deduce that

$$F(Y^*, V^*) = f(Y^*) < F(X^*, U^*) = f(X^*) \tag{3.17}$$

or

$$f(Y^*) < f(X^*),$$

contradicting the fact that X^* is an optimal solution of problem (III).

(β) Conversely, let (X^*, U^*) be an optimal solution of problem (V). It follows then, that X^* satisfies the constraints (3.1) and

$$u_j^* = -\varphi_j(X^*), \qquad j = 1, ..., m \tag{3.19}$$

because if not, we could reason as above deducing

$$F(X^*, U^*) \geqslant f(X^*) + B^+ - B^- + 1 > B^+ \tag{3.20}$$

Now it can be easily seen that X^* is an opimal solution of problem (III).

(γ) If the constraints (3.1) are consistent, let Y^* be a vector satisfying them and let us put

$$(v_j = -\varphi_j(Y^*) \qquad j = 1, ..., m \tag{3.21}$$

Hence,

$$F(Y^*, V^*) = f(Y^*) \leqslant B^+ \tag{3.22}$$

which contradicts the assumption (γ).

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B. *Minimax formulation*

Let us consider the following problem (VI) :

Find the minimum over all $X \in B_2^n$ of the maximum over all $V \in B_2^m$, of $F(X, V)$, where

$$F(X, V) = f(X) + (B^+ - B^- + 1)\left(\sum_{j=1}^{m} v_j\varphi_j(X) + \sum_{j=m+1}^{l} \varphi_j^2(X)\right) \tag{3.23}$$

and where

$$X = (x_1, ..., x_n) \in B_2^n \tag{3.24}$$

$$V = (v_1, ..., v_m) \in B_2^m. \tag{3.25}$$

X^*, V^*) wil l be called a *minimaxing point* of problem (VI) if :

$$\begin{aligned} F(X^*, V^*) &\geqslant F(X^*, V), && \text{for any} \quad V \in B_2^m \\ F(X^*, V^*) &\leqslant \underset{V \in B^m{}_2}{\text{ax}}\ F(X, V), && \text{for any} \quad X \in B_2^n \end{aligned} \tag{3.26}$$

and (X^*, V^*) will be called a *locally minimaxing point* of problem (VI) if :

$$F(X^*, V^*) \geqslant F(X^*, V), \qquad \text{for any} \qquad V \in B_2^m$$
$$F(X^*, V^*) \leqslant \operatorname*{Max}_{V \in B_2^m} F(X, V), \qquad \text{for any} \qquad X \in W_1(X^*) \tag{3.27}$$

Theorem 4. *Every pseudo-Boolean program under linear constraints is equivalent to a minimax problem without constraints. The relations between the optimizing points are the following :*

(α) *If $X^* = (x_1^*, \ldots, x_n^*)$ is a globally minimizing point of problem (III), then there exists a $V^* \in B_2^m$ such that (X^*, V^*) is a minimaxing point of problem (VI).*

(β) *If (X^*, V^*) is a minimaxing point of problem (VI) and $F(X^*, V^*) \leqslant B^+$ then X^* is a globally minimizing point of problem (III).*

(γ) *If (X^*, V^*) is a minimaxing point of problem (VI), and $F(X^*, V^*) > B^+$ then the constraints* (3.1.) *are inconsistent.*

(δ) *If X^* is a locally minimizing point of problem (III) then there exists $V^* \in B_2^m$, such that (X^*, V^*) is a locally minimaxing point of problem (VI).*

(ε) *If (X^*, V^*) is a locally minimaxing point of problem (VI) and*

$$F(X^*, V^*) \leqslant B^+$$

then X^* is a locally minimizing point of problem (III).

Proof.

(α) If X^* is a globally minimizing point of problem (III), then

$$\varphi_j(X^*) \leqslant 0 \qquad j = 1, \ldots, m \tag{3.29}$$

$$\varphi_j(X^*) = 0 \qquad j = m+1, \ldots, l \tag{3.30}$$

Let us take V^* such that

$$v_j^* = 0, \qquad j = 1, \ldots, m\ ;$$

then,

$$F(X^*, V^*) = \operatorname*{Max}_{V \in B^m{}_2} F(X^*, V)$$

and

$$F(X^*, V^*) = f(X^*). \tag{3.31}$$

Let suppose that there exists a vector $Y \in B_2^n$ such that

$$\operatorname*{Max}_{V} F(Y, V) < f(X^*) \tag{3.32}$$

It follows from (3.32) that Y fulfils the set of constraints (3.1). Indeed, if not, then at least one constraint is violated. There exists either an index $j_0 \in (1, \ldots, m)$ such that $\varphi_{j0}(Y) \geqslant 1$ implying $v_{j0} = 1$, and

$$v_{j0}\varphi_{j0}(Y) \geqslant 1, \tag{3.33}$$

or an index $j_0 \in (m+1, \ldots, l)$ such that $\varphi_{j0}(Y) \neq 0$, implying

$$\varphi_{j0}^2(Y) \geqslant 1 \tag{3.34}$$

Every term of the sum

$$\sum_{j=1}^{m} v_j\varphi_j(Y) \tag{3.35)}$$

will be non-negative following the choice of V :

$$\begin{array}{lll} \varphi_j(Y) < 0 & \text{implies} & v_j = 0 \\ \varphi_j(Y) > 0 & \text{implies} & v_j = 1 \\ \varphi_j(Y) = 0 & & v_j \text{ free} \end{array}$$

From (3.33), (3.34) and (3.35) we get

$$\operatorname*{Max}_V F(X, V) \geqslant f(Y) + B^+ - B^- + 1 \geqslant B^+ + 1. \tag{3.36}$$

From (3.31) and (3.36) we obtain

$$F(X^*, V^*) < \operatorname*{Max}_V F(Y, V) \tag{3.37}$$

which contradicts (3.32). Hence Y fulfils the set of constraints (3.1).

As above, we deduce that

$$\operatorname*{Max}_V F(Y, V) = f(Y), \tag{3.38}$$

so that relation (3.32) becomes

$$f(Y) < f(X^*) \tag{3.39}$$

contradicting the fact that X^* is a minimizing point for problem (III).

(β) Conversely, let (X^*, V^*) be a minimaxing point of problem (VI) ; then, X^* satisfies the constraints (3.1). If not, we could reason as above deducing

$$F(X^*, V^*) = \operatorname*{ax}_V F(X^*, V) \geqslant f(X^*) + B^+ - B^- + 1 > B^+.$$

Now it is obvious that X^* is also an optimal solution of problem (III).

(γ) If the constraints (3.1) are consistent, let us denote by Y a vector satisfying them. We get

$$\operatorname*{Max}_V F(Y, V) = f(Y) \leqslant B^+ \tag{3.40}$$

which contradicts the assumption in (γ).

$(\delta) X^*$ satisfies the constraints (3.1). Let us take V^* such that $v_j^* = 0$ $(j = 1, \ldots, m)$; then,

$$F(X^*, V^*) = \underset{V \in B^n_2}{\text{Max}} F(X^*, V)$$

$$F(X^*, V^*) = f(X^*) \leqslant B^+ \tag{3.41}$$

For every $X \in W_1(X^*)$ one of the following alternatives holds :

(1) X satisfies the contraints and

$$f(X^*) \leqslant f(X) \tag{3.42}$$

It follows then, that

$$\underset{V}{\text{Max}}\, F(X, V) = f(X) \tag{3.43}$$

From (3.41), (3.42) and (3.43) we get

$$F(X^*, V^*) \leqslant \underset{V}{\text{ax}}\, F(X, V)$$

(2) X does not satisfy the constraints and hence

$$\underset{V}{\text{Max}}\, F(X, V) > B^+ \tag{3.44}$$

It follows then that

$$F(X^*, V^*) < \underset{V}{\text{Max}}\, F(X, V) \tag{3.45}$$

and X^* is a locally minimizing point of problem (VI).

(ε) From she assumption we deduce that X^* satisfies the contrainst. Then

$$F(X^*, V^*) = f(X^*). \tag{3.46}$$

For every feasible point $X \in W_1(X^*)$ we have

$$F(X^*, V^*) \leqslant \underset{V}{\text{Max}}\, F(X, V) = f(X) \tag{3.47}$$

From (3.46) and (3.47) we deduce

$$f(X^*) \leqslant f(X) \tag{3.48}$$

and hence X^* is a locally minimizing point of problem (III).

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