Factor tables 1657–1817,
with notes on the birth of Number Theory

Maarten Bullynck

REVUE D'HISTOIRE DES MATHÉMATIQUES

FACTOR TABLES 1657–1817,
WITH NOTES ON THE BIRTH OF NUMBER THEORY

Maarten Bullynck

———————————

Abstract. — The history of the construction, organisation and publication of factor tables from 1657 to 1817, in itself a fascinating story, also touches upon many topics of general interest for the history of mathematics. The considerable labour involved in constructing and correcting these tables has pushed mathematicians and calculators to organise themselves in networks. Around 1660 J. Pell was the first to motivate others to calculate a large factor table, for which he saw many applications, from Diophantine analysis and arithmetic to philosophy. About a century later (1770), J.H. Lambert launched a table project that was to engage many (human) computers and mathematicians in the (re)production and extension of Pell's table. Lambert also pointed out

———————————

that a theory of numbers, of divisors and factoring methods was still lacking. Lambert's ideas were taken up by his colleagues at the Berlin Academy, and indirectly by L. Euler in St. Petersburg. Finally, the many number-theoretical essays that were written in the context of Lambert's table project contributed significantly to the birth of higher arithmetic around 1800, soon to be marked by the works of A.-M. Legendre and C.F. Gauss.

RÉSUMÉ (Tables des diviseurs 1657–1817, avec des notes sur la naissance de la théorie des nombres)

La fabrication, l'organisation et la publication des tables de diviseurs, de 1657 à 1817, constitue non seulement une histoire fascinante en soi, mais soulève en même temps des enjeux plus généraux de l'histoire des mathématiques. Le travail considérable que demande la fabrication et la correction de ces tables a poussé les mathématiciens et calculateurs à s'organiser en réseau scientifique. John Pell a été ait le premier qui incita, autour de 1660, d'autres mathématiciens à produire une grande table de diviseurs. Il en vantait l'utilité, non seulement pour l'analyse diophantienne, mais aussi pour l'arithmétique et même pour la philosophie. Un siècle plus tard, en 1770, Jean Henri Lambert lança un vaste projet, engageant beaucoup de calculateurs et mathématiciens, pour (re) construire et étendre la table de Pell. Ce faisant Lambert insistait sur le fait qu'une théorie des nombres, des diviseurs et des méthodes de factorisation faisait toujours défaut. Ces idées étaient reprises par les collègues de Lambert à l'Académie de Berlin et indirectment par Leonhard Euler à St. Petersbourg. Les nombreux textes sur la théorie des nombres écrits dans le cadre de ce projet de Lambert contribuaient de manière importante à la naissance de l'arithmétique supérieure aux alentours de 1800 dans les travaux de A.-M. Legendre et C.F. Gauss.

1. INTRODUCTION

The aim of the present paper is a historiographical appraisal of the construction of factor tables as a proper part of the scientific and social history of mathematics and as an important chapter or tradition within the history of number theory. Of course, like other scientific experts, the makers of tables which list prime numbers or factors of positive integers have, at least since the 18th–19th century, duly cultivated the memory of the work of their forerunners. In fact, in the case of factor tables, there was also a more specific reason to document and consult older tables. Since the entries in a factor table can neither be approximated nor interpolated with the help of the surrounding values, comparison with already existing tables was and is essential to test a new table. The most complete list of prime and/or factor tables compiled before the twentieth century can be found

in Lehmer [1909, i–vii] and Dickson [1919–1927, I, pp. 347–356]; the most extensive commentary on and analysis of factor and prime tables was provided by Glaisher [1878], in a companion essay to his *Factor Table for the Fourth Million.*

A topic closely connected with the construction of factor tables is the development of primality tests and factoring algorithms. Especially since the advent of the digital computer and still more since the invention of RSA-encryption, primality tests and factoring algorithms are considered as an important research field for mathematics and its applications. Before 1945, however, the topic figured mainly in research on and construction of tables in number theory. The history of primality tests and factoring algorithms has already been well documented by Dickson [1919–1927, I, pp. 357–374] and more recently been reappraised by Williams & Shallit [1994] and Mollin [2002].

This paper wants to go beyond the mere chronological list of tables and factoring methods and embed them into their proper historical context, scientifically, socially and philosophically. We will show that the circumstances of production of the earliest prime and factor tables provide insight into the way in which mathematicians and calculators organised themselves in communities or networks in the 17th and 18th centuries. Further, it will be demonstrated that the use and production of factor tables brought up specific problems, questions and viewpoints. Conditioned by the peculiarities of factor and prime tables, a particular frame of reference with its own concepts and partial theories came, in time, to be articulated, and was eventually to have rather an important impact on the emergence of number theory. Indeed, it is claimed here that the theories and methods that provide the theoretical background for factor tables constitute one of the contexts that should complement the classical story told about the formation of number theory; viz. the Greek heritage of Pythagoras and Diophantus, its transformation in the hands of Bachet, Fermat and Euler, finally culminating in A.-M. Legendre's *Essai* (1798) and C.F. Gauss' *Disquisitiones Arithmeticae* (1801). [1]

---

[1] On the role of the Pythagorean, Diophantine and Fermatian problems for the formation of number theory, see e.g. [Weil 1984, Chap. IV] and [Shanks 1993, Chap. 1].

The paper consists of three main parts. The first and second part are devoted to the big table projects of the 17th and 18th centuries respectively. In the first part, the project instigated by John Pell around 1660 and its impact is analysed. The second part is the story of Johann Heinrich Lambert's table project around 1770. Finally, the third part shows how the impetus of Lambert's project added coherence to the seemingly unorganised mass of mathematical publications on numbers and their properties that appeared in the last third of the 18th century, as well as how these, in their turn, provided the immediate background for Legendre's and Gauss' books.

2. JOHN PELL'S TABLE PROJECT

2.1. *The state of the art in prime and factor tables anno 1660*

Some smaller prime and/or factor tables were calculated and printed before the mid-17th century, before John Pell would advocate that numerical tables in general and the table of divisors in particular are important means and tools for doing mathematics and solving Diophantine problems, transcending their mere role as a collection of data. These smaller lists and tables served mainly as auxiliaries for factoring larger numbers that appear either in the determination of perfect or amicable numbers, or as the constant and last coefficient of an equation. [2]

The oldest extant prime and factor tables, within the Western tradition at least, were actually rather lists than tables, i.e., rather an enumeration one after the other (a linear order) than a formatted display of numbers that uses two dimensions for ordering its items. In the explanation of division and of numbers that measure only themselves (i.e. what we now call primes) Leonardo Pisano gave a list of the primes up to 97 in his *Liber Abaci* [Pisano 1202, p. 57]. Some four centuries later, Pietro Antonio Cataldi (1548–1626) gave a list of all primes up to 741, and a list displaying all divisors (except 1, but including composite divisors e.g. 2, 3, 4 and 6 for 12) of

---

On the shaping of this discipline after the publications of Gauss and Legendre refer to the rewarding collection of essays in [Goldstein et al. 2007].

[2]   The factorisation of the constant of an equation helps, of course, to determine the roots of the equation: $(x - a_1)(x - a_2) \cdots (x - a_n) = x^n + \cdots \pm a_1 a_2 \ldots a_n$.

the numbers up to 750 [Cataldi 1603, pp. 28–40]. As the title of Cataldi's treatise, *Trattato de Numeri Perfetti,* indicates the list was used in connection with perfect numbers, i.e., numbers that are equal to the sum of their divisors (including 1 but excluding the number itself). By dividing the tested number by the successive primes of his list, Cataldi proved that $2^{17} - 1$ and $2^{19} - 1$ (known as Mersenne numbers) were prime, hence $2^{16}(2^{17} - 1)$ and $2^{18}(2^{19} - 1)$ were perfect numbers. Cataldi's claim to the effect that the exponents 23, 29, 31, 37 also generate perfect numbers, could not be checked against his small list and was later proven false, except for 31, by Fermat and Euler [Dickson 1919–1927, I, pp. 10–19].

The tables published by Paul Guldin around 1640, and the lists published by Frans van Schooten in 1657 were the immediate context of Pell's project, rather than the earlier lists. Both appeared at a time when the *logistica speciosa,* symbolic algebra, was starting to spread within Europe. In fact, all factor tables of the 17th century appear in the company of combinatorial tables and problems. Tables giving all combinations of *n* letters, thus exploring the possibilities of the new algebra, are in a sense the specious equivalent of the factor tables. Using both kind of tables, one could try to identify a number *n* of the general algebraic form *aab* appearing in an equation with the help of a factor table, checking on all numbers that factor as a square of primes multiplied by another prime. Thus, factor tables could provide a way to pass from a general problem in the *logistica speciosa* to a specific problem (one with concrete numbers) in the *logistica numerosa.* This link between arithmetic and combinatorics, between factor tables and combinatorial tables, is specific to the 17th century and disappeared in the 18th century.[3]

The first genuine factor table may have been published by the Jesuit mathematician Paul Guldin (1577–1643).[4] Guldin was no amateur when it came to tables. In the first book of his series *De Centro Gravitatis,* Guldin [1635, *post* p. 228] added a hundred pages tabulating all squares and cubes of the numbers from 1 up to 10000. Similarly, at the end of the fourth and

---

3 Although C. F. Hindenburg "rediscovered" the link, passing from factor tables to combinatorial analysis, cf. p. 180.

4 Guldin's table is missing in [Dickson 1919–1927] and seems so far to have escaped scholarly scrutiny.

last book of the series (and after tables of combinations of letters, of π, of segments of spirals, ellipses and circles) Guldin [1641, pp. 383–401], included one more table, the *Tabula ultima*, giving all prime factors (except 5) for all odd numbers up to 9999. As Guldin recounts in the second book of his series:

> to know whether a given number is one that is prime and incomposite, we exhibit a catalogue of them, from 1 to 10000, later on. We once constructed this catalogue for our private use and we call it the *Ultimate Table*, because it has the last position amidst the tables we display at the end of these books, so that you can find it almost at a glance. [5]

At the end of the table, Guldin added, as a kind of conclusion, that, between 1 and 9999, there were 1226 primes and 8773 composites, or about 7 times more composites than primes [Guldin 1641, pp. 401, cf. Figure 1]. [6]

Some fifteen years after Guldin, the Dutch mathematician Frans van Schooten (1615–1660), best known as friend, translator and editor of René Descartes, published a list of all primes up to 9973 [Schooten 1657, pp. 393–403]. [7] The list appeared in Van Schooten's *Exercitationes Mathematicae*, which consisted of five books. The first four books contained geometrical problems, problems from Euclid's *Elements*, from Apollonius' works, and showed how Descartes' *calculus geometricus* could be applied to these. The fifth and last book contained "Miscellaneous Problems" and may be situated in the then newly emerging tradition of books on

---

5   Original: "Ut vero cognoscas an propositus numerus sit unus ex Primis & Incompositis, exhibemus infra eorundem Catalogum, ab unitate usq;ad 10000. quem olim ad nostrum priuatum usum construximus, & hic *Tabulam Ultimam* vocamus, eo quod locum ultimum inter Tabulas, in fine huius Operis positas obtineat, ex qua unico quasi intuitu, tuum assequeris propositum." [Guldin 1640, p. 15]

6   Guldin is three off, there are 1229 primes under 10000.

7   The Dutch version [Schooten 1659] of [Schooten 1657] was published two years later (although van Schooten first wrote in Dutch and then translated it into Latin). The *syllabus numerorum primorum* was reprinted there as *Tafel der eerste getallen*, with a correction. In the table of the Latin version, 809 was dropped by accident in the process of printing; this error is corrected in the Dutch version. Van Schooten has 1229 primes under 10000, the correct count. Later, Jacques Ozanam [1697, pp. 30–32] put van Schooten's table in another format and reprinted it (with the 809 error) in his *Récreations mathématiques et physiques*. In discussing van Schooten's *Exercitationes mathematicae* we will refer to the Dutch version.

TABULA ULTIMA. 401

9791		9841	13.757	9891	3.7.157	9941		9991	97.103
9793	7	9843	3.17.59.193	9893	13.761	9943	61.163	9993	3
9795	3.653	9845	11.179	9895		9945	3.13.17	9995	
9797	97.101	9847	43.229	9897	3	9947	7.29	9997	13.769
9799	41.239	9849	3.7.67	9899	19.521	9949		9999	3.11.101
9801	3.11	9851		9901		9951	3.31.107		
9803		9853	167	9903	3	9953	37.269		
9805	37.53	9855	3.73	9905	7.283	9955	11.181		
9807	3.7.467	9857		9907		9957	3		
9809	17.577	9859		9909	3.367	9959	23.433		
9811		9861	3.19.173	9911	11.17.53	9961	7.59		
9813	3	9863	7	9913	23.431	9963	3.41		
9815	13.151	9865		9915	3.661	9965			
9817		9867	3.11.13.23	9917	47.211	9967			
9819	3	9869	71.139	9919	7.13.109	9969	3		
9821	7.23.61	9871		9921	3	9971	13		
9823	11.19.47	9873	3	9923		9973			
9825	3.131	9875	79	9925	397	9975	3.7.19		
9827	31.317	9877	7.17.83	9927	3	9977	11.907		
9829		9879	3.49.89	9929		9979	17.587		
9831	3.29.113	9881	41.241	9931		9981	3		
9833		9883		9933	3.7.11.43	9983	67		
9835	7.281	9885	3.659	9935		9985			
9837	3	9887		9937	19.523	9987	3		
9839		9889	11.29.31	9939		9989	7		

Sunt ergo ab unitate inclusive, usque ad 10,000 exclusive, Numeri Primi & Incompositi, in universum, numero 1,226.

Compositi ergo manent numero 8,773, qui superant septuplum Primorum, hoc numero 191, itidem Primo.

F I N I S.

OMNIA AD MAIOREM DEI GLO.
RIAM, BEATISSIMÆQVE VIRGINIS
VENERATIONEM.

FIGURE 1. The last page of Guldin's *Tabula Ultima*

recreational mathematics. Quite a lot of these miscellaneous problems were problems in combinatorics and arithmetic. They were mainly derived from Michael Stifel's edition of Christoph Rudolff's *Coss* [1553] and

Claude-Gaspard Bachet de Méziriac's edition of Diophantus' arithmetic books [1621]. Although the problems were seemingly disconnected, van Schooten often inserted programmatic remarks on the excellence of algebra and especially Descartes' analysis for solving these problems. [8] The *Exercitationes Mathematicae* may therefore be seen as an early attempt to develop some general method(s) to attack problems with numbers.

In this Cartesian context we find the list of all primes up to 10000. The list follows immediately after a table that solves the (combinatorial) problem of finding the forms of all numbers with the same number of divisors [Schooten 1659, pp. 365–367]. [9] Van Schooten explains that the prime list is useful for solving problems of parts and divisors, for avoiding fractions, for finding the roots of equations, for calculating logarithms, and in fact "helpful for nearly all sorts of calculations." [10]

2.2. *John Pell's Table of Incomposits*

As his biography and correspondence [Malcolm & Stedall 2005] amply show, the English scholar John Pell (1611–1685) was intimately acquainted with the scientific thought and research of his times, especially in matters mathematical. Pell travelled and lived for many years in the Protestant parts of Europe, spending a decade in the Low Countries (1643–1652), where he taught mathematics at the universities of Amsterdam and Breda. He also served from 1654 to 1658 as Cromwell's delegate at Zürich. Following his return to England after 1658, Pell became one of the founding members of the Royal Society in London. Rather secretive about his discoveries, and inclined to start many projects without finishing them, Pell

---

[8]   Van Schooten remarked for instance that, contrary to Stifel's opinion, algebra is useful for finding amicable numbers, i.e., pairs of numbers one of which equals the sum of the divisors of the other, and vice versa [Schooten 1659, pp. 390–391]. Similarly, van Schooten showed how a single algebraic trick could solve a rather disconnected collection of 15 problems [Schooten 1659, pp. 459–462]. Cf. also with [Bullynck 2009a, p. 68]

[9]   E.g. a^3b, abc and a^7 all have 10 divisors.

[10]   Original full quote: "Hier by komt, dat dese getallen mede niet weynig tottet minderen der gebroocke getallen, in 't delen der Æquatien of Vergelijckingen, en in hare wortelen te soecken, gelijck oock in het vinden der Logarithmi of Reden-tallen, en eyndelijck by-na in alle reeckeningen behulpsaem zijn." [Schooten 1659, p. 365]

attained a reputation among his contemporaries as a serious mathematician.

John Pell entertained particular ideas on mathematics and on the organisation and transmission of knowledge. As his pamphlet *Idea on Mathematicks* (1638) shows, Pell supported the kind of reorganisation of knowledge professed in Samuel Hartlib's circle, a reform inspired by (or at least close to) Comenius' ideas.[11] In the case of mathematical knowledge, this reform entailed an altered presentation and a compression, where in Pell's view due place should be given to "the usefullest Tables and the Precepts for their use, in solving all Problems" [Pell 1638/1650, p. 40]. Among these was proposed a table of sines/logarithms to solve higher equations that Pell often mentioned, but never published. Hartlib had characterised Pell as a man who "vrges mainly a perfect Enumeration of all things" (1639)[12] and the emphasis on complete enumeration is indeed a recurring theme in Pell's work. In his *Idea of Mathematics*, Pell had announced that a mathematical-philosophical analogue of the book catalogue[13] might be constructed:

> And it may be som would like the Method of that work [the catalogue] so vvell, as to extend it farther, and applie it to other studies; in *speculation* imitating this my wariness, that *no falsehood bee admitted, and no truth omitted*; and *for practice* ensuring themselves, *anie subject* being propounded, to determine the number of all the Problemes that can bee conceived concerning it, and *anie Probleme* beeing propounded, demonstratively to shew *either all the means* of it's solution, *or the impossibilitie* of it: and if *so*, then whether it bee *not yet*, or *not at all* possible. [Pell 1638/1650, p. 45]

In this context of a table exhausting all possible true statements, Pell happened upon the idea, a powerful metaphor, that knowledge could be organised through combining "prime truths" [Malcolm & Stedall 2005, pp. 263–5]. True statements would thus be certain combination of prime

---

11   See [Malcolm & Stedall 2005, Parts I and II]. Yates [1972, p. 233] also suspects a strong influence of John Dee, and of course, the Hartlib project was close in time and spirit to Bacon's proposal for the advancement of science, and the foundation of the Royal Society.

12   Quoted after [Malcolm & Stedall 2005, p. 63].

13   Pell writes of pandects, a kind of catalogue listing all books on a certain topic with "orderly, rational and uniform completeness."

truths. This kind of connection between tables, and especially prime and factor tables, and the (re)organisation of knowledge would prove to be a powerful idea and persist all through the 18th century, although altering its modes.

While Pell was reticent regarding his projects in mathematics, and only seldom finished a project or published a result, his teaching in continental Europe (Amsterdam and Zürich) spurred some enthusiasm and indirectly revealed some of Pell's mathematical occupations and ideas. For example, Pell had been interested in Diophantine problems since the 1640s. He had corresponded with Father Mersenne in 1639–40 ; had given lectures on Diophantine problems in Amsterdam from 1644 to 1646, with Vossius in the audience; and later in Zürich to J.H. Rahn (1622–1676) in the years 1654 to 1658. [14] Pell frequently announced or promised an edition of Diophantus (in his own tri-column style) but nothing was ever published [Malcolm & Stedall 2005, p. 289–290 et passim], although Vossius announced such a publication in his historical work *De scientiis mathematicis* of 1650. [15]

More on Pell's mathematics was intimated by Rahn, who published a *Teutsche Algebra* in 1659. The provenance of the ideas and notations in this book have been a matter of debate, but it has now been ascertained that Rahn, regarding method, notation and presentation, heavily relied on Pell's lectures to write his book. [16] It is in this book that one can find the first factor table that took more effort and time than a day well spent on calculation [Rahn 1659, pp. 37–48]. It gives only the smallest factors of the numbers less than 24,000 which are not divisible by 2 and 5. [17]

---

[14] See [Malcolm 2000, p. 276] on Mersenne plus the references there; [Vossius 1650, p. 37–38]; and [Malcolm 2004, p. 253] on Rahn, see also Malcolm & Stedall [2005], keyword Diophantus.

[15] Vossius [1650, p. 37–38] writes, regarding an expected edition of Diophantus: "Longe tamen majora jamdiu expextamus a vir variae erudtionis, Mathematico acutissimo, Ioanne Pellio […] in Amstelodamensi nostro, collega carissimo: ubi Diophantum publice cum summa admiratione omnium enarrantem, ac intima & difficillima quaeque penetrantem, saepius audivimus."

[16] Scriba [1974] and Malcolm [2000; 2004], on the basis of new unpublished correspondence, have removed nearly any doubt that Rahn's original text owes much, and certainly its innovations, to Pell's teachings.

[17] According to [Malcolm & Stedall 2005, p. 200] it was a student of Pell, Balthasar Keller, who calculated the table; Rahn only copied and published it.

Through the initiative of John Collins, a member of the then newly-founded Royal Society, Rahn's book was translated into English by Thomas Brancker (1633–1676) during the years 1665 to 1668. [18] Through Collins' mediation, Pell was involved in reading, correcting and supplementing the translation; in the end he replaced almost half of Rahn's text with his own [Malcolm 2004, pp. 250–252]. [19] For this translation, Brancker calculated the factor table afresh up to 100,000, following Pell's directions. After the publication of the English translation in 1668, the book would be generally known as Pell's *Algebra*, and the *Table of Incomposits* as Pell's *Table*, although Keller and Brancker, independently, had calculated the table, and Rahn wrote the original work. But indeed, both the *Algebra* and the *Table* are products directly inspired by Pell's particular mindset and the details of their execution depended on Pell's general philosophy. Also, as we will show, Pell's additions to the translation should be interpreted as a kind of indirect dialogue with van Schooten's *Exercitationes*. Pell's factor table and its importance to him has to be set within this philosophical context and this particular dialogue.

Some general characteristics of Pell's *Algebra* match up with Pell's philosophy of knowledge, where the "rationally organised library" aspect of indexing, tabulating and enumeration is of paramount importance. Pell had developed a "method" to present his algebra which an anonymous reviewer in the *Philosophical Transactions* [20] characterised as follows:

> the Method is such, that most of the Book, if not all, may be understood by those not vers'd in the English tongue, that are vers'd in Specious Algebra, most of the Questions being propounded in Symbols, and the progress of the work so described by the Marginal quotations, that for those exercised in Algebra, that would transcribe a Problem in this Method, it were sufficient, only to take the Margent, omitting the work it self, till farther leisure is afforded to perform it. [21]
> [Phil. Trans. 3 (1668), 689]

---

18   As H.M. Pycior [1997] showed, this English translation was one of the attempts following the initiative of John Collins to provide Britain with an up-to-date algebra book in English.

19   Pell added pages 79–82 and 100–192.

20   Actually Collins, according to Malcolm & Stedall [2005, p. 205].

21   Compare this description to Malcom's: "a sequence of marginal annotations summarizing the working out of the problem in the text" [Malcolm 2000, p. 287].

This is the essence of Pell's tri-column method. From right to left one had three columns per line: 1) the text and its algebraic transliteration; 2) a numbering of the line; 3) a summary of the operations that led to that line using the numberings, e.g., 3*7 means: line 3 times line 7.[22] The complete and consecutive numbering of all lines in the *Algebra* impressed upon the whole book a near tabular arrangement. This arrangement made the book accessible to a variety of readers, such as those not fluent in English, who could follow the argument by its numbers only. It had the additional pedagogical advantage that one could choose one's own style of reading, choosing to follow primarily the algebra or the sequence of numbers, according to preference.

If we now take a closer look at the specific material that Pell added to Brancker's translation (pp. 79–82 and 100–192), we see that Pell supplied an additional layer of the book that had to do with indeterminate or Diophantine analysis. It seems that Pell took advantage of the English translation of Rahn's book to make up, at least in part, for the edition or commentary of Diophantus he never got around to producing. Instead of attempting a general treatment of these topics, however, Pell expressed thoughts that revolved around a particular set of problems. Thus problems XV and XVI, which deal with Pythagorean triangles, allowed Pell to give a short exposition on what exactly constitutes an indeterminate problem. All further problems (XXVII–XXXI) were problems which

> Bachet […] left obscure; and […] the celebrated DesCartes and Van Schooten have left doubtful, as not being by them throughly understood. [Phil. Trans. 3 (1668), 689][23]

Indeed, Pell's problems XXVII–XXVIII correspond to Diophantus V.19 and van Schooten's Problem XIII; Pell's XXIX–XXXI to van Schooten's

---

[22]   More on the tri-column method may be found in [Stedall 2002, pp. 137–138] and in general see her discussion of Pell's mathematical style, characterized as being close to "a mechanisation of unit steps", [Malcolm & Stedall 2005, pp. 235–319].

[23]   Fermat's work is not mentioned at all. However, the correspondence between Pierre de Fermat and the English mathematicians Brouncker and Wallis during the years 1657–1658 had been published by Wallis [1658]. Pell must have known about the correspondence but does not seem to have engaged with it. Instead, his dialogue is only with members of his "generation" like van Schooten.

XII. [24] Likewise, the *Table of Incomposits* corresponds to van Schooten's V (the *syllabus numerorum primorum*) and pp. 194–195 ("XXIX different examples of a Composit") to van Schooten's III and IV.

From this list it seems that Pell was engaged in a discussion with van Schooten's *Exercitationes mathematicae* and its (Cartesian) style of attack on Diophantine problems. Pell's main criticism of the solutions presented in van Schooten (solutions that were due to Bachet, van Ceulen and Descartes) was that these mathematicians had failed to appreciate the indeterminate nature of these problems. Namely, van Schooten's sources had given only one (or two) answer(s) to the problems, whereas they are "capable of innumerable answers" [Rahn 1668, pp. 80; 116; 138]. Pell adapted the methods so as to produce "innumerable" answers and then added a "review" of the list of solutions. [25] This "review" is actually an analysis of the order of presenting the solutions. For Problem XXVII this "review" states:

> this Pattern shews you a disorderly mixture of Answers in Great Numbers amongst Smaller Numbers. [...] So that here is need of another Rule for the orderly selecting of values of b and c, apt to lead us, in order, to Answers falling under any prescribed limit [as for example 100,000] that so we may not be cumbted with huge Numbers, when there are many smaller ones fit to answer the Question. [Rahn 1668, p. 142]

This disorder displays "inverted repetitions" (i.e. (a, b, c) and (c, b, a)) and "confused Anticipations" (smaller hypotenuses before larger ones). Near the end of the "review" Pell can say that

> In the two preceding Pages you have some Solutions of Probl. XXIX proposed p. 131, which was declared capable of innumerable Answers. And therefore I prescribed a Limit [No side greater than 100,000] Pag. 152, I required

---

[24] Stedall's essay on Pell's mathematics [Malcolm & Stedall 2005, pp. 247–328] is rather brief on Pell's work on Diophantine problems. In the case of problem XXIX the comment states only: "A further problem [...] fills most of the remaining sixty pages" (p. 311). The sources and solutions to problem XXIX are discussed in [Costabel 1950; Hofmann & Costabel 1952], but an analysis of Pell's treatment is unfortunately lacking, as Costabel [1986, p. 324] later acknowledged.

[25] This "review" can be found on pp. 121–128 for Problem XXVII, pp. 142–174 for Problem XXIX. The solutions are actually integer triangles and thus consist of three numbers (a, b, c), Pell's "review" focuses mainly on the hypotenuse of the triangle.

that the Enumeration of them should be orderly, pag. 159. I declared that I would have that Enumeration Complete, giving All the answers that do not exceed 100,000 in their greatest side. [Rahn 1668, p. 168]

This conclusion sums up the feature that is particular to Pell's take on Diophantine problems: he sought methods to generate an orderly and complete enumeration of answers. [26] Finally, after the "review", Pell gives another method of solution where the disentanglement of the disorder in the answers is easier. [27]

The crucial tools in the "review" of van Schooten's methods and the list of solutions were tables. [28] For both Problems XXVII and XXIX Pell used a table of squares [29] and the *Table of Incomposits*. [30] More specifically, the *Table of Incomposits* was pivotal in establishing Pell's orderly enumeration. It was used to find the greatest common divisor of three numbers (this being easier with the help of a table than with a repeated Euclidean algorithm). Now, to avoid the "inverted repetition" it was mandatory to divide by the greatest common divisor at a certain point in the solution process (p. 147). In order to repair the "confused anticipation" one then had to reorder the list of solutions according to their greatest common divisors (p. 152 ff.).

---

[26]   Two additional remarks may be added here. First, one can wonder if Pell's "orderly and complete Enumeration" is connected in any way to Regula VII of Descartes' *Regulae ad directionem ingenii* where a "sufficienti et ordinata enumeratione complecti" [Descartes 1701, p. 18] is demanded. Although this requirement is absent in Descartes' *Discours de la Méthode* (1637) and the *Regulae* were only published posthumously, Pell may have had knowledge of Descartes' manuscript during his years in the Low Countries. Second, in the analysis of the sources of Problem XXIX Pierre Costabel and J. E. Hofmann both remarked that in the 17th century, with the notable exception of Pierre de Fermat's work, there is "une impuissance [...] à pousser les questions de théorie des nombres jusqu'aux considérations exhaustives et au point de vue existentiel" [Hofmann & Costabel 1952, p. 326]. One should add Pell as an exception.

[27]   These are Problems XXVIII for XXVII and XXX for XXIX (pp. 131 and 174–188).

[28]   It is informative to compare Pell's approach to number problems with Frenicle de Bessy's methods as analysed in [Goldstein 2001]. Frenicle used tables as a heuristic tool, while Pell used them as a systematic instrument for generating and ordering solutions.

[29]   See p. 130; 148. In 1668 Pell used Paul Guldin's table of squares [Guldin 1635, post p. 228], later he calculated a square table of his own [Pell 1672].

[30]   See pp. 129; 147; 152–153.

It seems that Diophantine problems and specifically the generation of solutions in the right order stimulated Pell to calculate auxiliary tables. Both Brancker's *Table of Incomposits* and Pell's own *Table of Squares* are instances of such auxiliary tools, directly linked with specific problems. Both tables also have the same upper limit (100,000) and fulfill the condition that all numbers in the table should be in natural order without gaps. This substratum clarifies the philosophical importance Pell ascribed to a table of "prime truths". Such a table would not only combinatorially generate all composed truths, but would also be able to generate them without gaps, in the right order. Using such a table, one can prove or disprove a composed statement. If the statement is in the table of calculated solutions, it is true, if not, it is false; a "disorderly mixture of answers" does not allow for such a verification of truth or falsehood. As a matter of fact, the table of incomposites seems to have constituted, for Pell, the equivalent of the book catalogue, as suggested in his *Idea of Mathematics* (cf. quote p. 141 above). [31] Of course, as van Schooten had already pointed out, the *Table of Incomposits* has many possible applications. The announcement of Pell's *Algebra* in the *Philosophical Transactions* listed some of these:

> Thirdly, as to the Table of Incomposits, no Book but this extends it to above Ten thousands, some of the uses whereof are declared in the Title [i.e. "factors or coefficients"], others in the Book; and even in Common Arithmetick, it is of excellent Use for the Abbreviation of Fractions, and for giving of all the aliquot parts of a Number proposed, useful for the Depression and Resolution of Æquations, as is taught by Albert Gerard [sic], and van Schooten. [Phil. Trans. 3 (1668), p. 689]

The reduction of a fraction to its least denominator was indeed mentioned [p. 34] and taken up again in the explanation of the Table.

Pell's *Table* was arranged in 21 columns and 40 rows, for the hundreds and units respectively. These dimensions are a consequence of the fact that numbers divisible by 2 and 5 (though not 3) are excluded and that only the smallest factor is given. The result is:

---

[31]   This follows from a letter of Henry Oldenburg to Leibniz where Oldenburg relates Pell's discoveries following Pell's own words [Gerhardt 1849–1863, I, p. 98]. In this letter, the table is called "cribrum Erasthosthenis", Erathosthenes' sieve.

a complete and orderly enumeration of all incomposits between 0 and 100000. [Rahn 1668, p. 193]

Little is said of the actual calculation of this *Table of Incomposits*. Brancker wrote that Pell had taught him methods of calculating and extending the table, but no theoretical details are given. [32]

> He [Pell] shewed me the way of making the *Table of Incomposits*, of examining it, and of continuing as far as I would. He encouraged me to extend it to 100 thousand. [Rahn 1668, non-pag. preface]

Moreover, Brancker himself nearly despaired at the state of correctness of the table:

> I was very sensible of the bad effects of perfunctoriness in Supputating, Transcribing or Printing of it [the Table]. My care therefore was not small, yet pag. 198 is almost filled with Errata, and I dare not warrant that non have escaped unseen. [Rahn 1668, non-pag. preface]

Page 198 contains a list of 96 errors, some of them printing errors. During the publication process of the *Algebra* (Jan.–Feb. 1667), John Collins had written to John Wallis (1616–1703) regarding the translation. In a letter to Collins that arrived after publication, Wallis gave a list of 145 errata in the Table, at least 10 of which Brancker had not spotted. Collins communicated the list to Pell, who in his turn communicated it to Brancker, who in the meanwhile had found 19 additional errors. [33] This slow and inaccurate process of control led Brancker to conclude: "I yet doubt its exactness".

Wallis published his "Catalogue of Errors" some years later in *A Discourse on Combinations, Alternations and Aliquot Parts* [Wallis 1685a, pp. 135–136] appended to his *Treatise of Algebra* [Wallis 1685b]. [34] Wallis claimed to have examined the whole table "in the same method and with the same pains as if I were to Compute it anew", and listed 30 additional errors or misprints. More convinced than Brancker, Wallis added:

---

[32]   See, however, [Malcolm & Stedall 2005, pp. 256–7] for earlier work of Pell on factoring, by way of summing up the digits.

[33]   All letters in July 1668, [Beeley & Scriba 2005, II, pp. 469–470; 525–528; 533–535].

[34]   This text, including the error list, is also reprinted in the Latin translation of the *Treatise* in [Wallis 1693, p. 483ss.].

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
01	1.P	P	3	7	P	3	P	P	3	17	7	3	P	P	3	19	P	3	P	P
03	P	P	7	3	13	P	3	19	11	3	17	P	3	P	23	3	7	13	3	11
07	P	P	3	P	11	3	P	7	3	P	19	3	17	P	3	11	P	3	13	P
09	3	P	11	3	P	P	3	P	P	3	P	P	3	7	P	3	P	P	3	23
11	P	3	P	P	3	7	13	3	P	P	3	11	7	3	17	P	3	29	P	3
13	P	P	3	P	7	3	P	23	3	11	P	3	P	13	3	17	P	3	7	P
17	P	3	7	P	3	11	P	3	19	7	3	P	P	3	13	37	3	17	23	3
19	P	7	3	11	P	3	P	P	3	P	P	3	23	P	3	7	P	3	17	19
21	3	11	13	3	P	P	3	7	P	3	P	19	3	P	7	3	P	P	3	17
23	P	3	P	17	3	P	7	3	P	13	3	P	P	3	P	P	3	P	P	3
27	3	P	P	3	7	17	3	P	P	3	13	7	3	P	P	3	P	11	3	41
29	P	3	P	7	3	23	17	3	P	P	3	P	P	3	P	11	3	7	31	3
31	P	P	3	P	P	3	P	17	3	7	P	3	P	11	3	P	7	3	P	P
33	3	7	P	3	P	13	3	P	7	3	P	11	3	31	P	3	23	P	3	P
37	P	P	3	P	19	3	7	11	3	P	17	3	P	7	3	29	P	3	11	13
39	3	P	P	3	P	7	3	P	P	3	P	17	3	13	P	3	11	37	3	7
41	P	3	P	11	3	P	P	3	29	P	3	7	17	3	11	23	3	P	7	3
43	P	11	3	7	P	3	P	P	3	23	7	3	11	17	3	P	31	3	19	29
47	P	3	13	P	3	P	P	3	7	P	3	31	29	3	P	7	3	P	P	3
49	7	P	3	P	P	3	11	7	3	13	P	3	P	19	3	P	17	3	43	P
51	3	P	P	3	11	19	3	P	23	3	P	P	3	7	P	3	13	17	3	P
53	P	3	11	P	3	7	P	3	P	P	3	P	7	3	P	P	3	P	17	3
57	3	P	P	3	P	P	3	P	P	3	7	13	3	23	31	3	P	7		19
59	P	3	7	P	3	13	P	3	7	3	19	P	3	P	3	P	P	3	11	3
61	P	7	3	19	P	3	P	P	3	31	P	3	13	P	3	7	11	3	P	37
63	3	P	P	3	P	P	3	7	P	3	P	P	3	29	7	3	P	41	3	13
67	P	P	3	P	P	3	23	13	3	P	11	3	P	7	3	P	P	3	P	7
69	3	13	P	3	7	P	3	P	11	3	P	7	3	37	13	3	P	29	3	11
71	P	5	P	7	3	P	11	3	13	P	3	31	3	P	P	3	7	P	3
73	P	P	3	P	11	3	P	P	3	7	29	3	19	P	3	11	7	3	P	P
77	7	3	P	13	3	P	P	3	P	3	7	19	3	11	3	7	19	3	P	3
79	P	P	3	P	P	3	7	19	3	11	13	3	P	7	3	P	23	3	P	P
81	3	P	P	3	13	7	3	11	P	3	23	P	3	P	P	3	41	13	3	7
83	P	3	P	P	3	11	P	3	P	3	7	P	3	P	3	P	7	3	P
87	3	11	7	3	P	P	3	P	P	3	P	P	3	19	P	3	7	3	P
89	P	3	17	P	3	19	13	3	7	23	3	29	P	3	P	7	3	P	P	3
91	7	P	3	17	P	3	P	7	3	P	P	3	P	13	3	37	19	3	31	11
93	3	P	P	3	17	P	3	13	19	3	P	3	7	P	3	P	11	3	P
97	P	P	3	P	7	3	17	P	3	P	P	3	P	11	3	P	P	3	7	P
99	3	P	13	3	P	P	3	17	29	3	7	11	3	P	P	3	P	7	3	P

FIGURE 2. A specimen of Brancker's *Table of Incomposits*

the Table will then be very accurate; and (I think) without any Error. [Wallis 1685a, p. 135]

Wallis mentioned the table in connection with the problem of decomposing compound numbers into their aliquot parts, presenting the problem both in a numerical (1123) and a coefficient/literal (*aabc*) notation.

This followed the approach of van Schooten's Problems III and IV. Historically, Wallis' *Treatise* and *Discourse* both helped to strengthen an "arithmetic approach to algebra" [Pycior 1997, p. 125]. This approach had already been announced by Oughtred and Pell [35], but Wallis explictly advocated the use of the more powerful and universal methods of algebra and arithmetic in demonstrations over the geometric approach [Pycior 1997, pp. 118–134].

The *Table of Incomposits* was often recycled. It was first reprinted by John Harris in his *Lexicon Technicum*, following the entry *Incomposite Numbers* [Harris 1707 & 1710, II, Incomposite]. Harris' entry was almost a word for word repetition of Brancker's description of the Table (i.e., [Rahn 1668, pp. 193 and 196]) and gave the table, correcting the errors given by Brancker but not those given by Wallis, exactly as printed in the *Algebra*. [36]

Pell's Table was also reprinted as an appendix to volume XIII (1765) of the famous *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*. [37] The table is described under the rubric *premiers, nombres*.

> A l'occasion des nombres premiers, nous insérons, à la fin de ce volume, une table qui nous paroît assez bien étendue, & qui est tirée d'un livre anglois d'algebre assez ancien & assez peu connu [Diderot & D'Alembert 1751–1765, XIII, p. 289]

As in Harris' *Lexicon*, the table was reprinted with Brancker's corrections, but not Wallis'. [38]

---

[35]   On Pell's influence on Wallis' work see [Stedall 2002, pp. 141–153] and [Malcolm & Stedall 2005, pp. 313–20].

[36]   The second edition of Harris' *Lexicon* dropped the table but kept the description.

[37]   Dickson [1919–1927, I, p. 349] consulted a later edition of the *Encyclopédie* (1780, vol. II) and failed to note that this is a reprint of Pell's table.

[38]   For example, a superficial inspection revealed that the entry for 99443 is changed to 77 in Harris' table according to Brancker's instructions. In the *Encyclopédie*, it has been correctly changed to 277. However, it seems unlikely that a recalculation was done, since the errors Wallis indicated went unnoticed and in 1770 Lambert found 60 more errors (see 3.1). It is more likely that Rallier des Ourmes contributed the entry to the *Encyclopédie*; he published a small article on factor tables later that year [Rallier des Ourmes 1768] and had perhaps corrected a few entries while adapting the *Lexicon*-reprint of Pell's table for inclusion in the *Encyclopédie*.

2.3. *German Reception in the early 18th Century*

The endeavours of Brancker would remain unsurpassed for more a century, but his table to 100,000 was also hard to get hold of. While, for instance, Wallis' *Treatise on Algebra* and its Latin translation were quite accessible in continental Europe, the book by Rahn, Brancker and Pell was not.[39] Only the inclusion of the table in the *Encyclopédie* in 1765 made it accessible to a larger public. However, many knew of Pell's table, mainly through the mediation of Wallis. The most ironic part then of the history of factor tables in the 18th century is that many interested in mathematics knew that Pell's table existed, but were unable to obtain a copy, and thus had to calculate the table again. This was the case of Poetius and Lambert.

The reception of 17th century English algebra on the continent, and especially in the German-speaking, protestant countries, was a complex and multi-faceted process in which many personalities and media figured, but that process has so far not been adequately described or studied. Our treatment will therefore be rather short and will focus mainly on some developments that are important for the history of factor tables. It is, however, important to keep in mind that this history has to be situated in the more general context of the transmission of ideas between England and protestant Germany.

G.W. Leibniz (1646–1717) was one person who was rather well informed of the doings of the British algebraists through his correspondence with the secretary of the Royal Society, Henry Oldenburg, who was of German origin. Oldenburg (and indirectly Collins, who composed drafts of Oldenburg's letters) had started the correspondence with Leibniz in 1670, a correspondence that lasted until 1679 [Gerhardt 1849–1863, I, pp. 11–168]. In 1673 Leibniz also visited London and met with Hooke

---

[39] E.g., Poetius, Lambert, Kästner and Lagrange knew the book by hearsay, but none of them was ever able to inspect a copy. Rahn's German version seems to have been even rarer. All four, however, knew Wallis' work, Poetius and Kästner often quoted from it.

and Boyle, but also Pell. [40] When Wallis' *Treatise on Algebra* appeared in 1685, Leibniz reviewed it for the *Acta Eruditorum* [Leibniz 1686a]. [41] In 1695 Leibniz started a correspondence with Wallis that lasted until 1700 [Gerhardt 1849–1863, IV, pp. 1–82].

Through Leibniz's mediation there was a considerable reception of English algebra in the German states. When Augustinus Vagetius sollicited Leibniz's opinion on how to write a textbook on algebra in 1696, Leibniz [1923–2006, III, 6, pp. 780-81] replied that algebra and arithmetic (letters and numbers) should best be explained at the same time, an idea rather close to Wallis'. Following Leibniz's recommendations, Johann Michael Poetius was one of the writers who drew largely on Wallis in writing a textbook, and Christian August Hausen (1693–1743), mathematics professor in Leipzig, was one of the first to introduce the English style of algebra in Germany, using Newton's *Universal Arithmetick* in his courses [ADB 1875-1912, 15, pp. 440–441]. One of Hausen's students was Abraham Gotthelf Kästner (1719–1800), who would later become professor of mathematics at Göttingen and write influential textbooks that stressed the need to base arithmetic on the concept of number [Kästner 1758, non-pag. Vorrede]. Thus, although Wallis' arithmetic approach to mathematics was more or less forgotten in Britain, or rather, was superseded by the mathematics of Isaac Newton, who dominated the British mathematical scene throughout the 18th century, the idea got 'absorbed' and transformed in 18th century Germany.

---

[40]   In this context, consider Leibniz's letter to Abbé Galloys from December 1678. Leibniz claimed to have a method to resolve all Diophantine problems, giving all solutions in proper order or showing its impossibility, after a discourse on the use of tables in literal algebra [Gerhardt 1849–1863, I, p. 185]. The words of this letter match up very closely with Pell's own statements, though Leibniz's letter does not mention Pell.

[41]   Later, Leibniz also reviewed I. Newton's *Arithmetica Universalis* [Leibniz 1686b].

As mentioned, one of Wallis' readers was J.M. Poetius [42], who followed Wallis' work in writing his *Anleitung zu[r] arithmetischen Wissenschaft, vermittelst einer parallelen Algebra* (1728). [43] Poetius wrote this arithmetic and algebra textbook following a suggestion by Leibniz. [44]

> I will in this book follow Mr. Leibniz's proposition, and conjoin common arithmetic as much as possible with literal calculus [i.e., algebra], but I will set the latter apart in a smaller print, so that beginners (or also those who shy away from Algebra) may skip it at first reading according to their preference. In this way, one will have on the one hand the main rules and examples of the operations, on the other hand one will comprehend the reasons from which these rules spring, and consequently learn to understand the method of proceeding through demonstrations. [Poetius 1728 & 1738, p. 54] [45]

This explanation of the different levels of reading is somewhat reminiscent of Pell, but, in fact, Poetius here subscribes to a method and programme Christian Wolff (1679–1754) had started. In the introduction to *Auszug aus den Anfangs=Gründen aller mathematischen Wissenschaften,* Wolff [1717] had argued for the introduction of arithmetic into general education, on the one hand because of its practical use, on the other because of its logical order and structure, both aspects working together for the enlightenment of common sense. [46] Methodologically, this implied two

---

[42]   Unfortunately, we have been unable to find any biographical information on Poetius so far.

[43]   The dependence is clear from the content, but also shows through the numerous references. Poetius does, however, quote also many other authors such as Oughtred, French textbooks and, of course, many German writers.

[44]   The reference is to Leibniz's letter to Vagetius.

[45]   Original: "Ich werde hierinne dem Vorschlag des Herrn von Leibniz folgen, und die Buchstaben=Rechnung mit der gemeinen Rechnung so viel möglich conjungiren, jedoch jene bey dieser vermittelst des kleinern Drucks a parte setzen, damit die Anfänger, oder auch diejenige, so sich vor der Algebra scheuen, sie nach Gefallen im ersten Durchlesen auch übergehen können. denn also wird man dennoch eines Theils die Haupt=Reguln und Exempel zu denen Operationen haben, andern Theils aber wird man die Gründe, woraus dieselben Reguln entsprungen, verstehen, und einfolglich den Modum procedendi per Demonstrationes begreiffen lernen."

[46]   We should add that Erhard Weigel (1625–1699), professor in Jena, was the first to launch these ideas and that he had a direct impact on both Leibniz and Wolff, see also [Vleeschauwer 1932]. See also [Bullynck 2008, pp. 565–567].

changes to the standard style of arithmetic book then prevalent in Germany. First, a multi-layered text was given, appealing to both beginning and advanced students, often even distinguishing this in print, with a big font for the main concepts and rules and a small font for the advanced comments and proofs. Second, all rules were proven and all words or concepts neatly defined to emphasize the order and structure. Poetius implemented both requirements. However, contrary to Wolff, he did not rely on Euclidean proofs with definitions and logical deductions, but endorsed the development started by the British authors of algebra texts, using algebraic demonstrations. This is the meaning of arithmetic in parallel with algebra: Poetius gives the demonstration of an arithmetic rule by transliterating the rule into algebra. For example, to show that 5-3 gives 2 because 3 plus 2 makes 5, Poetius would use $c - b = a$ and convert it into $c = b + a$.

The construction and structure of signs and numbers was an important topic in Poetius' book, in line not only with the English tradition, but also with Leibniz's and Wolff's philosophy of *cognitia symbolica*, knowing through signs. Knowledge of how (numerical) signs are structured enhances the use of these signs to know and investigate the world.[47] In an introductory chapter Poetius stressed the Indian provenance of the Hindu-Arabic numerals[48], and devoted four pages to the explication of non-decimal positional systems.[49] The conclusion of this introductory chapter was:

---

[47] For a good introduction into 18th century semiotics see [Meier-Oeser 1998]; more specifically for semiotics within mathematics see [Knobloch 1998].

[48] This was a matter of dispute between Wallis and Vossius. See the *Treatise* by Wallis [1685b, p. 8] and *De scientiis mathematicis* by Johan Gerard Vossius [1650].

[49] We list the references mentioned by Poetius to show the popularity of the topic in Germany at the time. The Tetractys of Erhard Weigel [1672] was a positional number system to base four. After Weigel, G.W. Leibniz published his idea of a dyadic (i.e. binary) number system in 1703. Leibniz [1703/1720, p. 226] thought that the binary system might provide a way to find a law behind the progression of prime numbers, see [Mahnke 1912/13; Zacher 1973]. Leibniz's idea was followed and expanded upon by Dangicoure in the *Misc. Berol.* (1710), Pelecanus (1712) and Wiedeburg (1718). These last two give an introduction to calculating with binary numbers (including division). Both Weigel and Leibniz regarded their invention of a non-decimal numeration system (with the four basic arithmetic operations included) as a philosophical tool, reflecting and representing some intrinsic quality or structure of reality in terms

In this way, one needs but few signs and names to designate expressions of both the largest and smallest numbers. [Poetius 1728 & 1738, p. 13] [50]

This remark, pertaining to the semiotics of signs (to use a contemporary way of describing this), is typical for the context in which the first German factor table appeared. Using another analogy from medicine, Poetius would call his factor table an *Anatomia Numerorum, Oder Zergliederung der Zahlen Von 1 bis 10000*. It lists all factors of all numbers to 1,000, and of all odd numbers not divisible by 5 and 3 between 1,000 and 10,000.

In the introduction, Poetius referred to Pell's table to 100,000, mentioned in Wallis. Since Poetius had been unable to find a copy, he had calculated a (smaller) table by himself up to 10,000 [Poetius 1728 & 1738, pp. 39–40]. This anatomy of numbers was followed by a *Practicam,* a section on the advantages and uses of the table, such as the manipulation of fractions, arithmetic and geometric series. Diophantine problems are not mentioned. As far as the construction of the table is concerned, Poetius referred back to the main text on arithmetic, where he had explained the "Kenn-Zeichen" of the prime numbers, or the "Symbolum primi geneticum" [Poetius 1728 & 1738, p. 141]. A first class of such characteristics was given under the heading "Division", where the rules are given to determine whether a number is divisible by 2, 3, 4, 5, 7, 9 and 11. These rules were standard in most textbooks on arithmetic, and were generally called the "Kenn-Zeichen" of the numbers. The classic *Demonstrative Rechenkunst* by Clausberg [1732], for instance, devoted some 60 pages to them. For Poetius, these characteristics were tools for the construction of factor tables. A second class of factoring auxiliaries appears in the section on powers:

---

of the number two or four. It may be remarked that the development of number systems by Weigel and Leibniz seems to be independent of Caramuel's invention of non-decimal number bases in 1670. Caramuel does not give the rules of calculating with the numbers (except for base 60), while Weigel and Leibniz do. Also, whereas for Weigel and Leibniz their number systems constituted a possible perfect notation for both science and philosophiy, Caramuel seems to propose a pluralistic philosophy of numeration systems, cf. [Høyrup 2008, pp. 15–17].

[50] Original: "Auf solche Art hat man zu den Expressionen so wohl der größten, als auch der kleinsten Zahlen nicht viele Zeichen und Nahmen von nöthen." Authorities quoted at this point are Wolff [1713] and Christian August Hausen [1715].

To find whether a number with large aliquot parts has factors, or if it is a
prime number?
We mentioned this problem in §397, and this problem also belongs to the use
of square tables.
One subtracts the given number (if it is not a square) from the next greater
square, until the remainder is a perfect square, then the root of the greater
square + the root of the smaller square gives the larger factor, and the root of
the greater square − the root of the smaller square yields the smaller factor.
[Poetius 1728 & 1738, p. 299] [51]

With a table of squares at hand, this elementary method can indeed help
to determine the factors. Poetius appears to have been the first factor table
maker who explicitly indicated the methods he used in the construction.
Curiously enough, only methods working with one number at a time are
mentioned, whereas Eratosthenes' sieve does not occur at all [52].

Poetius' factor table was later reprinted in the *Vollständiges mathemati-
sches Lexicon* [Wolff & Richter 1734/1742, II, pp. 530ss.], originally edited
by Christian Wolff (1716), later reworked and extended against Wolff's will
by G.F. Richter, who inserted the table in Volume II (1742). [53] Some years
later, a Nürnberg military man, Peter Jäger, calculated a list of primes to
the full 100,000 (actually to 100,999) and offered his complete table for
sale at the steep price of 2,000 Thalers. Halle's professor of medecine, J.G.
Krüger [1746], however, published the table without paying. Apparently
without knowing of Pell or Poetius, H. Anjema of Franeker (Netherlands)
also undertook the calculation of all factors of numbers under 100,000, but

---

51   Original: "Zu finden, ob eine ungrade schwehrtheilige Zahl einige Factores hat,
oder ob sie eine Prim-Zahl ist?
Hiervon ist schon oben im 397 § gedacht worden, und gehört dieses Problema auch
mit zur Nutzung der Quadrat-Taffeln.
Man subtrahiere die vorgegebene Zahl, (so sie nicht selbst eine Quadratzahl ist,) von
denen nechstfolgenden grössern Quadraten, so lange biß der Rest ein vollkommen
Quadrat, so giebt des grösseren Wurtzel + des kleinern Wurtzel den grössern Fac-
torem, und d es grössern Wurtzel – des kleinern wurtzel, den kleinern Factorem."
52   As Verdonk [1966, p. 167] has shown, the original descriptions of the *cribrum
Erasthostenis* in Nichomachus and Boethius seem to have been little known in the 16th
century. This seems to have persisted in the 17th century. Only in the 18th century
did Rallier des Ourmes [1768] publish a description, and Horsley [1772] edited both
text excerpts.
53   See [Kästner 1786, pp. 556-57] for the publication history.

died when he had reached 10,000. [54] After his death, the editors Sam. and Joh. Luchtmanns published the extant part. Anjema's table consisted of 302 pages, because he not only indicated all factors for all numbers, but also included 1 and the number itself in the list [Kästner 1786, pp. 558-59].

3. JOHANN HEINRICH LAMBERT'S TABLE PROJECT

> *"It seems at any rate that calculation and the construction of tables have become Mr Lambert's second nature, as it seems to cost him no more time and effort than plain writing."*[55]

This was the state of the art regarding factor tables in the year 1770, when J.H. Lambert published several appeals for tables in general, and factor tables in particular. This event would dramatically change the history of factor tables and leave a mark far into the 19th century. Johann Heinrich Lambert (1728–1777) was born in Mulhouse (Alsace) to poor parents, but through private study and determination he became a philosopher, physicist, linguist and mathematician of importance. In 1748, Lambert became tutor of the children of the Swiss confederation president von Salis in Chur. In this way, Lambert not only became acquainted with many works in the rich library of the Salis family, but in 1756–58 he also visited Göttingen, Hannover, Utrecht, Leiden, Turin with his pupils. This trip brought him into contact with scholars in various European centers of learning. Among the mathematical books which Lambert read quite early on were Wolff's *Anfangsgründe* and Poetius' *Anleitung*.

As he recorded in his scientific diary, the *Monatsbuch*, Lambert started thinking about the divisors of integers in June 1756. An essay by G.W. Krafft (1701–1754) in the St. Petersburg *Novi Commentarii* seems to have

---

54 Some biographical detail can be found in [*Mathematical Tables and other Aids to Computation* 3 (24), (Oct. 1948), pp. 331–332].

55 Original: "Wenigstens muß das Rechnen und das Tabellenmachen dem H. Lambert schon so zur andern Natur worden seyn, daß es ihm nicht mehr Zeit und Mühe kostet, als gemeine Schrift." (review of Lambert's *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, Band II (1770), in: *Allgemeine Deutsche Bibliothek* 14 (2), p. 322, 1771.)

triggered Lambert's interest [Bopp 1916, p. 17, 40]. A physicist, astronomer and mathematician, Krafft was Leonhard Euler's colleague in St. Petersburg and had already published on perfect and amicable numbers. His [Krafft 1751/1753] was a survey of known facts and techniques to factor large integers. Krafft referred to van Schooten's list of primes and Poetius' factor table and indicated some errors in the latter. Then he went on to discuss the "prime-formula" $6n\pm1$ (due to Jacob Bernoulli and Leibniz [1678]) [56] and explained how versions of Fermat's Little Theorem (as it is called today; an odd prime p divides $a^p - a$ for all a) may be used for factoring, referring to Euler [1732/1738]. [57]

3.1. *Simple Ideas, Prime Numbers and Tables*

From 1760 to 1765 Lambert worked on his major philosophical works [Bopp 1916, p. 47] in which he merged Wolff's *cognitia symbolica* with John Locke's anatomy of concepts [Locke 1690]. Lambert wanted a reform of philosophy where the basic principles would not be fairly arbitrary definitions (as with Wolff), but would be acquired through an anatomy of the concepts available (following Locke). This reform was to enable philosophy to make progress by accumulating data, theorems and theories, just as

---

[56] I.e., all prime numbers are necessarily of the form $6n\pm1$, because $6n\pm2$ or $6n\pm3$ are divisible by 2 or 3 respectively.

[57] Euler's paper only contained a statement of Fermat's Little Theorem, not a proof. Euler proved the theorem in 1736 (published 1741) and showed in 1747 (published 1750) how to use it for factoring. Krafft did not refer to those later papers, but seems to have known their content as he gave a proof and application of Fermat's little theorem.

science did. The scientific method is explained in the *Neues Organon* [Lambert 1764] [58] The anatomy of basic concepts is performed in *Anlage zur Architectonic* [Lambert 1771]. [59]

To carry out this anatomy of concepts and to re-combine simple concepts ("das Chaos auseinander lesen"), tables were of great value to Lambert. Lambert often used a topical table as a heuristic tool in his investigations, whether philosophical or scientific:

> A topical system, which would be the abstraction of what can be thought, determined, researched for any object […] an inventory, a form of all things […] that one can use when one wants to know something, both the thing in itself and in its relationships to other things. [Lambert *Briefe*, I, pp. 284–85] [60]

Lambert saw the anatomy of a number system as an instance of such a topical table:

> The architecture of numbers is the abstraction of all those things where one calculates with numbers or discrete quantities. It is a general type, a form, and the relationships and transformations of numbers have arithmetic as their own theory. [Ibid.] [61]

---

[58]   The title of this work, of course, refers to both Aristotle's *Organon* and Francis Bacon's *Novum Organum* (1620). Lambert claims that his *Organon* is more complete than those of his predecessors, because it is comprised of four parts: The Dianoiology (laws of thought), the Alethiology (the doctrine of truth), the Semiotics (the doctrine of signs), and the Phenomenology (the doctrine of appearances). Among the many methods proposed by Lambert are an algebraic calculus for logical deduction (in the Dianoiology); an algebraic view of language (in the Semiotics) and an algebraic calculus to determine the degree of verisimilitude of historical accounts (in the Phenomenology).

[59]   This work was published in 1771, but had been already finished in 1765.

[60]   Original: "Ein topisches System, welches ein Abstractum wäre, von allem was sich bey einem jeden Objecte gedenken, betrachten, bestimmen, untersuchen läßt […] ein Inventarium, ein Formular etc. von allem […], was bey jeder Sache, wenn sie an sich und nach ihren Verhältnissen erschöpft werden sollte, zu suchen ist." (Lambert an Holland, 15.8.1768) Lambert's topical table was published in *Nova Acta Eruditorum* [Lambert 1768].

[61]   Original: "Nun ist das Zahlengebäude gleichsam das Abstractum alles dessen, wo man mit Zahlen rechnet oder aller Discreten-Quantitäten. Es ist ein allgemeiner Typus, ein Formular davon, und die Verhältnisse und Verwandlungen der Zahlen haben die Arithmetik als ihre eigene Theorie."

In this analogy, it is hardly surprising that a table of factors becomes the metaphor for a philosophy having a table of simple concepts at its disposition. [62]

> [In this anatomy of concepts] one takes a concept and looks up its inner determinations, which are more or less like its factors and prime numbers. [Lambert *Briefe*, I, p. 24] [63]

The anatomy of concepts is paralleled by an anatomy of numbers, *anatomia numerorum*, a peculiar combination of words that one encounters not only in the title of Poetius' table, but also in the title of [Lambert 1769].

Reforming philosophy meant for Lambert also reforming the organisation of knowledge. In 1765, Lambert became a member of the Berlin Academy, a position he kept until his death in 1777. During this Berlin period he devoted himself to the dissemination and advancement of sciences in general, and to the editing of tables in particular. From 1770 onwards, two extensive table projects absorbed nearly all of Lambert's time. The first one, supported by the Academy and in collaboration with his colleagues Bernoulli, Schulze, Lagrange and Bode, was a collection of astronomical tables [Bernoulli et al. 1776]. The aim was to bring together all useful and necessary astronomical tables in one collection so that they would be accessible to the individual astronomer and would eliminate printing and calculation errors through comparison and recalculation. In this way, Lambert claimed:

> if all the best astronomical tables […] were to be lost, they could be reconstructed from our collection [Lambert *Briefe*, V, p. 154] [64]

In 1770, Lambert started up a similar kind of compilation of mathematical tables.

---

[62]   As far as we know, Lambert hit upon this metaphor independently of John Pell (see 2.2 above).

[63]   Original: "[Bey dieser Anatomie der Begriffe] hält man sich schlechthin an den Begriff selbst, und sucht seine inneren Bestimmungen auf, welche gleichsam seine Factores und numeri primi sind." (Lambert to Holland, 21.04.1765)

[64]   Original: "Wenn die besten astronomischen Tafeln […] sollten verlohren gehen, so würden sie aus unserer Sammlung wieder hergestellt werden können."

There are numbers, proportions, formulae and calculations that deserve to be done and written down once and for all, because they occur very often, so as to avoid the trouble of finding or calculating them over and over again. This is the reason why in all parts of mathematics one has tried to put everything in tables that can possibly be put into tables. [Lambert 1770, p. 1] [65]

The publication of the *Zusätze zu den logarithmischen und trigonometrischen Tafeln* [Lambert 1770] was a personal project of Lambert's (without the support of the Academy). Therefore, at the same time, Lambert launched an urgent appeal to the general public to help him extend this collection of tables.

Prominently featured in Lambert's collection of tables is a factor and a prime table, filling pages 2 to 117 of the 210 pages of the book. As Lambert recounted in the introduction to the *Zusätze*, Poetius' table to 10,000 was the first factor table he saw and although Poetius referred to Pell's table, Lambert had been unable to find a copy. At first, Lambert wanted to use Poetius' table for his collection:

I satisfied myself with the table calculated by Poetius, and just brought it into a more flexible order. [66] I occasionally showed my table, before its printing, to Mr de la Grange. He did not know of any other tables similar to it, and thus he wished to have copies of the table once it was printed, to send them to his correspondents. As the printing was delayed, Mr de la Grange looked to see whether he could find more of these tables. He did not search in vain. Pell's table, that in fact goes to 100,000 and thus goes 10 times further than Poetius or Anjema, could be found in the *Dictionnaire encyclopédique* and in Harris' Lexicon of Arts and Sciences. As I thereupon looked into Wallis' Opera, I found the 30 printing errors Wallis had indicated in Pell's tables and Pell himself had missed, all just as Poetius mentions. [Lambert 1770, pp. 4–5] [67]

---

[65]   Original: "[es gibt] Zahlen, Verhältnisse, Formeln und Rechnungen, die eben daher, daß sie öfters vorkommen, ein für allemahl gemacht und aufgezeichnet zu werden verdienen, damit man der Mühe, sie immer von neuen zu finden oder zu berechnen, überhoben seyn könne. Dieses ist der Grund, warum man in allen Theilen der Mathematick, was sich in Tabellen bringen liesse, in Tabellen zu bringen gesucht hat."

[66]   This specimen, a factor table to 10,200 was printed in [Lambert 1765-1772, II, pp. 52–53].

[67]   Original: "[ich ließ] es bey der von Poetius berechneten Tafel bewenden, und begnügte mich sie in eine geschmeidigere Ordnung zu bringen. Ich zeigte hierauf meine Tafel, ehe sie abgedrückt wurde, gelegentlich dem Herrn de la Grange. Es

NUMERORUM.   69

	992	995	998	1001	1004	1007	1010	1013	1016	1019
3	13	19	11	—	—	—	7	17	—	181
9	11	151	—	—	31	7	—	13	17	101
11	7	191	151	11	—	13	83	7	—	223
17	47	11	—	53	—	23	7	71	307	—
21	313	23	173	7	137	47	—	11	13	—
23	—	—	—	59	233	-7	13	—	151	227
27	67	—	7	223	29	11	—	19	—	7
29	13	—	—	—	7	263	31	107	11	—
33	—	7	—	11	67	—	71	—	7	13
39	7	11	—	13	47	131	23	7	37	—
41	—	13	7	239	11	—	79	—	—	7
47	61	7	11	17	—	—	37	—	7	97
51	—	—	31	—	13	7	—	43	11	269
53	7	113	13	—	17	53	139	7	—	43
57	—	29	61	47	7	19	11	79	59	—
59	—	—	—	37	—	17	7	—	277	11
63	•17	—	37	7	11	13	—	—	—	—
69	53	17	7	—	—	—	211	167	19	7
71	37	—	—	109	7	11	53	17	293	107
77	—	—	—	7	13	179	61	—	17	—
81	7	—	—	17	89	31	—	7	—	11
83	101	11	7	—	—	97	271	—	23	7
87	43	53	59	19	17	—	7	11	61	—
89	—	7	23	—	317	13	—	53	7	79
93	31	13	191	—	—	7	43	41	—	29
99	109	137	283	11	7	—	17	—	13	—

	991	994	997	1000	1003	1006	1009	1012	1015	1018
51	13	11	23	7	17	251	157	19	173	179
57	229	71	7	—	—	17	—	13	41	7
61	17	79	—	13	—	11	7	109	—	37
63	53	7	67	47	—	43	17	131	7	—
67	131	17	—	11	167	7	31	—	47	23
69	7	—	19	—	29	—	11	7	13	—
73	—	11	17	19	7	—	37	—	—	—
79	41	31	113	7	—	83	241	—	157	—
81	—	53	11	41	37	7	—	—	—	13
87	11	—	—	13	7	107	—	—	29	139
91	—	7	73	101	—	17	11	199	7	—
93	281	37	—	7	—	—	23	—	19	11
97	7	—	23	199	11	101	13	7	283	19
99	19	29	7	31	13	—	—	11	—	7

E 4   102000.

101279 est num. prim.

FIGURE 3. A specimen of Lambert's *Table of Divisors*, with C.F. Gauss' handwritten correction

Lambert used Poetius' table, Pell's table in the two reprinted versions and Wallis' corrections to check the table before publication. Following Pell's table, Lambert excluded numbers divisible by 2 and 5 and only noted the smallest factor, but he also excluded the numbers divisible by 3 and likewise changed the arrangement of the table. As Lambert had explained in [Lambert 1765-1772, II, pp. 42–53], the regularities of the decimal system could be used for checking the table, and he advised the following arrangement: the hundreds still figure over the columns, but are now arranged in three separate progressions ($3n$, $3n + 1$, $3n + 2$); the units are still in the rows but those divisible by 2, 3, and 5 are excluded. In this way, there are 3000 numbers per page, and certain regularities are easier to observe. Lambert noted more than 60 errors in addition to those given by Wallis [Lambert 1770, p. 5].

3.2. *Lambert's Contributions to a Theory of Composite and Incomposite Numbers*

Lambert did more than deliver the factor table. He also addressed the absence of any coherent theory of prime numbers and divisors. Filling such a lacuna could be important for the discovery of new and more primality criteria and factoring tests. For Lambert the absence of such a theory was also an occasion to apply the principles laid out in his philosophical work. A fragmentary theory, or one with gaps, needed philosophical and mathematical efforts to mature.

> To this aim [prime recognition] and others I have looked into the theory
> of prime numbers, but only found certain isolated pieces, which did not seem
> possible to make easily into a connected and well formed system. Euclid has

---

war ihm ebenfalls weiter nichts davon bekannt, und so bezeugte er ein Verlangen, die Tafel, wenn sie einmal abgedrückt wäre, zu haben, und selbst an seine Correspondenten Exemplarien davon zu verschicken. Da es sich inzwischen mit dem Abdrucke verzögerte, so suchte der Herr de la Grange, ob er nichts weiteres von solchen Tafeln finden könne. Er suchte auch nicht vergebens. Pells Tafel, die in der That biß auf 100000, und demnach 10mal weiter als die von Poetius und Anjema geht, findet sich sowohl in dem Dictionaire encyclopňĕedique als in des Harris Lexicon der Künste und Wissenschaften. Und da ich daraufhin noch in den operibus Wallisi nachsuchte, so fand ich darinn auch die von Poetius erwähnte 30 Druckfehler, die Wallis in Pells Tafel angemerkt hat, und die sich unter den von Pell selbst angemerkten nicht fanden."

few, Fermat some mostly unproven theorems, Euler individual fragments, that anyway are farther away from the first beginnings, and leave gaps between them and the beginnings. [Lambert 1770, p. 20] [68]

This lack of theory was much regretted by Lambert. In a comprehensive discussion on the structure and acquirement of scientific knowledge in his *Neues Organon* [Lambert 1764, I, pp. 386–450], Lambert had given many tools and strategies for finding and repairing the gaps in a theory, for building a theory and making it more complete. The theory of numbers, in Lambert's opinion, needed such a treatment badly.

In an essay on periodic decimal fractions, Lambert had already constructed some primality tests and had found a new criterion for primality. [69]

In this way, there are given, for any prime a, progressions $1, m, m^2, m^3, m^4$, etc. for which a period of $a-1$ members is produced. This clearly never happens in the case of composite numbers and thus affords a criterion for prime numbers. [Lambert 1769, pp. 127–128] [70]

This criterion comes down to the (modern) statement that every prime number has a primitive root (using the word Euler would later introduce), or alternately, that there exists a positional number system with base m in which $\frac{1}{p}$, written down in this number system, will display a period of length $p - 1$. This criterion, however, is hardly efficient in practice, since its worst case amounts to writing $\frac{1}{p}$ in $(p-1)/2$ different positional systems (i.e. with base 2, 3, …, $(p-1)/2$).

In 1770, Lambert presented two sketches of what would be needed for something like a theory of numbers. The first dealt mainly with factoring methods [Lambert 1765-1772, II, pp. 1–41], while the second gave a

---

[68] Original: "Ich habe mich zu diesem Ende [der Primzahlerkennung] so wie auch zu andern Absichten um die Theorie der Primzahlen näher umgesehen, und da fand ich freylich nur einzelne abgebrochne Stücke, ohne sonderlich Anschein, daß dieselbe so bald sollten zusammengehängt und zum förmlichen System gemacht werden können. Euclid hat wenig, Fermat einzelne meistens unbewiesene Sätze, Euler einzelne Fragmente, die ohnehin von den ersten Anfängen weiter entfernet sind und zwischen sich und den Anfängen Lücken lassen."

[69] On decimal fractions and Lambert's essay, see [Bullynck 2009b].

[70] Original: "Sic et pro quovis numero primo a dantur progressiones $1, m, m^2, m^3, m^4$, etc. quae periodum producant $a - 1$ membrorum, quod cum de numeris compositis nunquam locum habeat, patet, et hincpeti posse numerorum primorum criterium."

more axiomatic treatment [Lambert 1770, pp. 20–48]. In the first essay, Lambert explained how, for composite number with small factors, Eratosthenes' sieve could be used and optimised. For larger factors, Lambert explained that approximation from above, starting by division by numbers that are close to the square root of the tested number p, was more advantageous. [71] For both methods, Lambert advised the use of tables. [72] The second essay had more theoretical bearings. Lambert rephrased Euclid's theorems for use in factoring, included the greatest common divisor algorithm, and put the idea of relatively prime numbers to good use. He also noted that binary notation, because of the frequent symmetries, could be helpful. Finally, Lambert also recognized Fermat's little theorem as a good, though not infallible criterion for primality, "but the negative example is very rare" [Lambert 1770, p. 43]. [73]

Through Lambert's efforts, the topic of factoring came to be discussed in the Berlin Academy. J.-L. Lagrange (1736–1813) showed clear interest in factor tables [74], and Johann III Bernoulli (1744–1807), pursuing Lambert's method of factoring with decimal periods, also regretted the absence of a theory [Bernoulli 1771/1773, p. 318]. In this atmosphere, Academy member Nikolaus von Beguelin (1716–1789), the former tutor of the Prussian crown prince, wrote some essays on factoring. To this end, Beguelin devised a new number system that combined the advantages of algebraic and numerical notation.

> Although the science of numbers is necessarily geometrical, founded on the
> principle of contradiction, we know that the number signs & and the methods

---

[71] One should bear in mind that Lambert's range of tested numbers is between 1 and a few million, say, up to 10-digit numbers. Nowadays we test much larger numbers, using methods that fall into 3 categories: small, medium and large.

[72] With Eratosthenes' sieve, a table of primes can help in selecting the next sieve number or recognizing primes. When trying to find a large factor, a table that holds the last two or three digits of squares and a table of repeated division can help in recognizing squares or could reduce some divisions to copying down the right numbers.

[73] The exceptions are composite numbers, say c, for which $a^{c-1} \equiv 1 \pmod{c}$ for all a relatively prime to c. Nowadays these exceptions are called Carmichael numbers.

[74] See the introduction to Lambert's *Zusätze* [Lambert 1770, p. 4], Lambert's correspondence [Lambert *Briefe*, V, pp. 51–52; 120–121; 194] and Lagrange's correspondence [Lagrange *Œuvres*, XIII, p. 193].

of expressing the various combinations are not of absolute necessity. It is a question of choice or convention. [...] It is evident that the greater the number of primitive elements, the more the arithmetical operations will be simplified, & the sooner also we can hope to see the nature of numbers & their mutual relationships in their expressions. [Beguelin 1772b, p. 296] [75]

Though the 18th century had proven Leibniz wrong in expecting that the binary notation would reveal the mysteries of prime numbers, other notations could be helpful for finding factors, as Lambert had pointed out, providing examples of binary numbers which can be factorised 'at sight'. Beguelin applied his principle of sufficient reason ("raison suffisante") to this problem of finding an optimal notation. The binary system would be the best, if it did not posess two drawbacks that Beguelin pointed out: the notation of the numbers becomes too long, and one cannot transpose the digits at will as one can in literal algebra, that is, in algebra ab equals ba, but in positional notations 12 does not equal 21. His solution was a mixture of systems: he wrote down the exponents of two that occur in the binary expression of a number. For example, the number 19 is 10101 in binary notation (or more explicitly: $1 \cdot 2^4 + 1 \cdot 2^2 + 1 \cdot 2^0$), and can be written in Beguelin's notation as 0.2.4. (or 4.0.2. etc.) [Beguelin 1772b, p. 297].

Using the advantages of his notation, Beguelin constructed formulae for numbers with 2, 3, and 4 factors that show the general framework of the notation. In this way, Beguelin had an approach that mechanised Lambert's remarks on binary numbers. Instead of manipulating the binary number to find symmetries, Beguelin could look them up in a table of formulae. Unfortunately, Beguelin soon found out that the problem was thus reduced to a form of the combinatorial problem of partition, that is, writing a number as a sum of certain other numbers having a specific form. For most forms, Beguelin's approach reduced factoring to a problem in additive number theory which was at least as difficult. For numbers that

---

[75]   Original: "Quoique la science des nombres soit de nécessité géométrique, fondée sur le principe de contradiction, on fait que les signes des nombres, & les méthodes d'en exprimer les diverses combinaisions ne sont pas d'une nécessité absolue. C'est une affaire de choix, ou de convention. [...] Il est évident que plus le nombre des élémens primitifs, plus les operations arithmétiques seront simplifiées, & plutôt aussi on pourra se promettre d'appercevoir la nature des nombres & leurs rapports mutuels dans leur expression."

in Beguelin's notation had no gaps (i.e., contained all exponents starting from 0), the so-called Mersenne numbers $2^n - 1$, or have many gaps, such as $2^n + 1$, Beguelin [1777] tediously deduced formulae. Beguelin's essays again display the connection between a rational philosophy of *cognitia symbolica* and the research done on number systems.

Finally, it should be remarked that in 1773 and 1775, Lagrange published his *Recherches d'arithmétique* in the *Mémoires* of the Berlin Academy, merging Euler's and his own results on quadratic forms, presenting the first general treatment of reduction and equivalence of binary quadratic forms (in later terminology). In his preface, he explicitly referred to the use of this theory for factoring numbers:

> These studies are devoted to numbers that can be represented by the formula $Bt^2 + Ctu + Du^2$ [...] First I will give a way to find the different forms the divisors of these numbers can have; then I will give a method to reduce these forms to the smallest possible number; I will show how to set up tables for practical use, and I will show how to use these tables in the search for divisors of numbers.[Lagrange 1773 and 1775, p. 265] [76]

Lagrange had tabulated the linear divisors of a certain family of quadratic forms. These were useful as a tool in finding particular forms of prime numbers, or forms that exclude them. Lagrange closed his paper with an application, showing how to factor 10,001, 10,003 and 100,003 with the help of these tables, and with a list of divisibility criteria. [77]

3.3. *Lambert's Call for the Production of Factor Tables*

Lambert's first public appeal for the production of mathematical tables was expressed in the 2nd part of his *Beyträge zum Gebrauch der Mathematik*

---

[76]   Original: "Ces Recherches ont pour objet les nombres qui peuvent être représentés par la formule $Bt^2 + Ctu + Du^2$ [...] Je donnerai d'abord la manière de trouver toutes les différentes formes dont les diviseurs de ces sortes de nombres sont susceptibles; je donnerai ensuite une méthode pour réduire ces formes au plus petit nombre possible; je montrerai comment on en peut dresser des Tables pour la pratique, et je ferai voir l'usage de ces Tables dans la recherche des diviseurs des nombres."

[77]   The tables are in pp. 311–312 (1773); pp. 329–330; 332–333 (1775), or also in [Lagrange *Œuvres*, III, p. 695] on pages 757–758, 766–767 & 769–770. The theorems on divisibility on pp. 345–349 (1775) resp. pp. 783–788. The divisibility criteria had been announced two years before by Johann III Bernoulli [1771/1773, p. 321].

(1770). In the description of his first specimen of a factor table (to 10,200), Lambert had encouraged other mathematical practitioners to extend the table. To all calculators who wanted to join his project, Lambert had promised honour, if not scientific immortality, of the sort that Napier, Briggs, and others had earned:

> I will remark that I publish this [factor] table principally so that its flexible arrangement would motivate someone to add 9 more [tables of the same length], or if he wants real immortal fame, 99 more [i.e. to 1,020,000] [Lambert 1765-1772, II, p. 49] [78]

In the introduction to the table collection, the *Zusätze* (1770), Lambert elaborated on his argument. He deplored that the factor table had been computed at least four times already (Pell/Brancker, Poetius, Anjema, Krüger/Jäger), but that those duplicate efforts had not advanced the table in any way.

> I said that in the future these tables will be calculated from scratch again. […] Because it is a tedious labour, to calculate the table of all divisors of the numbers from 1 to 102000 again, I have a plea for the journalists and other writers who will see this work. Namely, they will act out of humanity and do good service to the mathematical sciences if they contribute as much as possible to the advertisement of this work. Because if someone in the future feels like calculating such tables, he will better spend his time […] extending the table, instead of recomputing it once more.[Lambert 1765-1772, II, pp. 8–9] [79]

---

[78] Original: "Vielmehr werde ich anmerken, daß ich die Tabelle vorzüglich deswegen durch den Druck bekannt mache, daß etwann jemand durch die so geschmeidige Einrichtung derselben sich bewegen lasse, noch 9 andere, oder wenn er sich einen recht unsterblichen Namen machen will, noch 99 andere beyzufügen. [d.h. bis 1020000]"

[79] Original: "Ich sagte erst, daß solche [Faktoren- und Prim-]Tafeln auch künftig noch von neuem werden berechnet werden. […] Da es indessen eine langwierige Arbeit ist, die Tafeln der Theiler der Zahlen von 1 biß 102000 von neuem zu berechnen, so werde ich an die Herren Journalisten und an jede andere Schriftsteller, denen dieses Werckchen vorkommen wird, eine Bitte thun. Sie werden nemlich aus Menschenliebe handeln, und den mathematischen Wissenschaften einen guten Dienst thun, wenn sie zur Bekanntmachung dieses Werkchens so viel möglich beytragen. Denn wer nur auch künftig Lust hat, solche Tafeln zu berechnen, der wird dann immer besser seine Zeit darauf verwenden […] [die Tabelle] weiter [zu führen], als das bereits berechnete nochmals [zu berechnen]"

Here again, we see Lambert's vision of science at work: science should progress, and its results should be known and accessible to everyone.

In 1771 and 1772 two updates on the table project were published. A few people had joined Lambert's table project, and made some progress which had to be made public to avoid duplication. Lambert inserted a note in the widely read review journal *Allgemeine Deutsche Bibliothek* (1771, vol. 14, nr. 1, pp. 305–306), communicating that one person (Oberreit) had calculated the factor table to 150,000 and planned to proceed to 200,000, and that another (Wolfram) had calculated hyperbolic logarithms. He also repeated that everyone was invited to complete the system of tables. Lambert concluded thus:

> My address is:
> A Monsieur Monsieur Lambert, Professeur Royal, Membre de l'Acad. R. de Berlin et diverses Academies et Sociétés des Sciences, à Berlin.

Parallel to this public appeal, Lambert had also added a standard appeal to all the private letters he sent off during the years 1770–1771:

> Every now and then there are lovers of mathematics who like to calculate. And I have reason to hope that my invitation […] will not be in vain. If, dear Sir, you should find someone in your vicinity who would like to undertake such calculations, it would be very agreeable to me. [Lambert *Briefe*, I, pp. 367–368] [80]

The last publicly printed update of Lambert's table project appeared in the third and last volume of Lambert's *Beyträge* (1772). It included a four page list of errors in the factor table which the officer Wolfram had found and a list of tables that Lambert had already received or calculated himself, amongst them a factor table to 339,000 [Lambert 1765-1772, III, non-pag. *Vorrede*]. The table project had been mostly organized through letters,

---

[80]   Original: "Es gibt hin und wieder Liebhaber der Mathematik, die gerne rechnen. Und ich habe Ursache zu hoffen, daß die Einladung […] nicht ohne Frucht seyn werde. Sollten Sie, mein Herr, in dortigen Gegenden jemand finden, der zu solchen Berechnungen Lust hätte, so würde es mir sehr angenehm seyn." (Lambert to Kant). Cf. Bernoulli's footnote on this page. In this context, it is also interesting to note that Lambert proposed to Röhl (Sept. 1771) that the calculation of tables might provide convenient topics for doctoral students [Lambert *Briefe*, II, pp. 391–392].

everything else can be easily adjusted in a written correspondence [*A.D.B.* **14** (1), p. 305]

This exchange with about a dozen correspondents comprises most of parts IV and V in Lambert's *Deutscher Gelehrter Briefwechsel*, edited by Johann III Bernoulli from 1782 to 1787, after Lambert's death.

3.4. *Calculators in Correspondence*

3.4.1. *Early Answers to Lambert's Appeal*

One year after the publication of the *Zusätze* the first contributions arrived in Lambert's hands. All in all, between 1770 and 1777, the year of his death, Lambert entertained a correspondence with 11 persons on tabular topics, 6 of whom were working on factor tables.[81] Restricting ourselves to factor tables, the contributions occurred in two quite different phases: A first one 1770–1776 (Wolfram, Oberreit, von Stamford, Rosenthal) and a second one 1776–1777 (Felkel, Hindenburg). It is remarkable how differentiated the social provenance of Lambert's collaborators was: Wolfram was an artillery officer, Oberreit an accountant, von Stamford an engineer, Rosenthal a baker, Felkel a teacher and Hindenburg a student and later professor. This list already gives a hint why the first phase is different from the second; Felkel and Hindenburg were near professionals, thinking in terms of career, whereas the others were amateurs.

As already mentioned, Isaac Wolfram had contributed a list of errors to the factor table in Lambert's *Zusätze*, his other main contributions to the table project were logarithms.[82] Oberreit had devoted himself exclusively to factor tables. He had computed up to 150,000 in 1770, to 260,000 in the Summer of 1771, and to 339,000 in 1772 [Lambert 1765-1772, II, non-pag. *Vorrede*]. In 1774–5 Lambert had received Oberreit's factor table up to 500,000, but also the announcement that professional problems

---

[81]   These are Wolfram, Oberreit, von Stamford, Rosenthal, Felkel, Hindenburg on factor tables; Schönberg and Röhl on square and cube tables; Baum on sine tables; Schulze and Eißenhardt on issues of publication. The correspondences are in [Lambert *Briefe*, II, IV & V].

[82]   Some biographic details and a thorough survey of Wolfram's contributions in [Archibald 1950].

made further calculations quite impossible for him. Acting on this problem, Lambert found von Stamford willing to calculate the part 504,000 to 1 million, Seeing that the work did not advance quickly enough, however, he divided the work between von Stamford (504,000 to 750,000) and Rosenthal (750,000 to 1 million). [83] In the end, two tables were finished, the one by Oberreit to 504,000 and the one by Rosenthal from 750,000 to 1 million. Unfortunately neither of them were published. J.K. Schulze, who was designated to continue Lambert's table project after 1777, had Lambert's table-related *Nachlass* at hand. [84] He published Wolfram's logarithms and Röhl's squares and cubes in [Schulze 1778], but he never got around to publishing Oberreit's table. [85] After Lambert's death, Rosenthal sent his table to J.G. Kästner, professor at Göttingen University, but the remaining gap between 500,000 and 750,000 prevented publication [Kästner 1786, pp. 564–565]. [86]

Lambert had had a clear plan for publishing the tables. He repeatedly promised his correspondents they would get their "paper and ink" paid back when it came to a publication, and that he would take care of finding a publisher [Lambert 1781–1787, V, p. 61]. But publishers tended to be rather unwilling to invest in volumes of tables which were hard to print and hard to sell. Knowing very well the mechanisms of the printing trade of his day, Lambert devised strategies to find a publisher:

> I have to find various ways to gradually publish [these tables]. I thought the *Leipziger Buchmesse* would offer the best opportunity to find publishers, in particular those who cannot find enough manuscripts because they live in remote places. From time to time one can easily convince a publisher who has had a bad experience with a fashionable book to count more on the durability of sales

---

83   The respective correspondence with Oberreit, von Stamford and Rosenthal is in [Lambert *Briefe*, II, pp. 366–382 & V, pp. 10–23 & V, 24–33], a summary of the content of these letters is given by Glaisher [1878, pp. 111–113].

84   The rest, letters and unpublished essays, was with Johann III Bernoulli.

85   See Bernoulli's footnote [Lambert *Briefe*, I, p. 368].

86   In a letter to Z. Dase [1856, pp. 76–77] (dated 1850), C.F. Gauss wrongly described this table as a factor table for numbers between 500,000 and 750,000. This was probably due to a slightly confused recollection of Kästner's lectures and books which Gauss had read some 50 years prior to this letter. Cf. [Glaisher 1878, p. 113].

than on rapidity, which is usually very iffy. That has indeed always been my best argument. [Lambert 1781–1787, V, p. 313] [87]

A more unfortunate result of Lambert's appeal was the case of J. Neumann, who, upon reading the 2nd part of the *Beyträge*, had decided to extend and correct Lambert's small table (to 10,200) to 100,100. Working on his own without contact with Lambert, Neumann [1785] finished and published his table in Dessau. His table lists all factors, not only the smallest one. At the time of publication, his table had unfortunately already been superseded by Lambert's own table to 102,000 in the *Zusätze*, although, as Kästner [1786, pp. 562–3] remarked, it could be used for checking. Incidentally, Vega [1797, I, pp. 1–86] used exactly this factor table in his collection of mathematical tables.

3.4.2. *Mechanising the Production of Factor Tables*

In January 1776, the Vienna-based teacher Anton Felkel (1740–1800?) announced to Lambert that he had found an apparatus, consisting of rods that could mechanically find the divisors of all integers. He had come up with this device by reading Lambert's *Zusätze*, especially the remarks on the best arrangement of a table. Felkel's device was a mechanisation of factor table making, based on Eratosthenes' sieve. In his letter, Felkel had inserted a short announcement of his method, of a soon-to-be-published table to 144,000, and of a promise to extend it to 1,000,000. Felkel's idea was that Lambert would publish the announcement in a journal to help him find financial support (and to fend off competitors) [Lambert 1781–1787, V, pp. 41–44]. Lambert inserted Felkel's circular in the *Leipziger Neue Zeitungen von gelehrten Sachen* (L.G.Z., nr. 63, 5 August 1776, pp. 507–510), but advised Felkel to collaborate with the other calculators and start on the second million. At the end of March 1776, however, Felkel wrote back

---

[87] Original: "[ich] muß auf verschiedene Mittel bedacht seyn, sie nach und nach herauszugeben. Ich dächte inzwischen, daß sich auf der Leipziger Messe die beste Gelegenheit anbieten sollte, Verleger zu finden, zumal solche die, weil sie an abgelegenen Orten wohnen, in ihrer Gegend nicht immer genug Manuscripte aufbringen können. Zuweilen läßt sich ein Buchhändler, dem ein Mode=Buch fehlgeschlagen, leicht bereden, mehr auf die Dauerhaftigkeit als auf die meistens sehr mißliche Schnelligkeit des Verkaufs zu setzen. Dieses war auch in der That immer mein bester Beweggrund." (Lambert to von Schönberg)
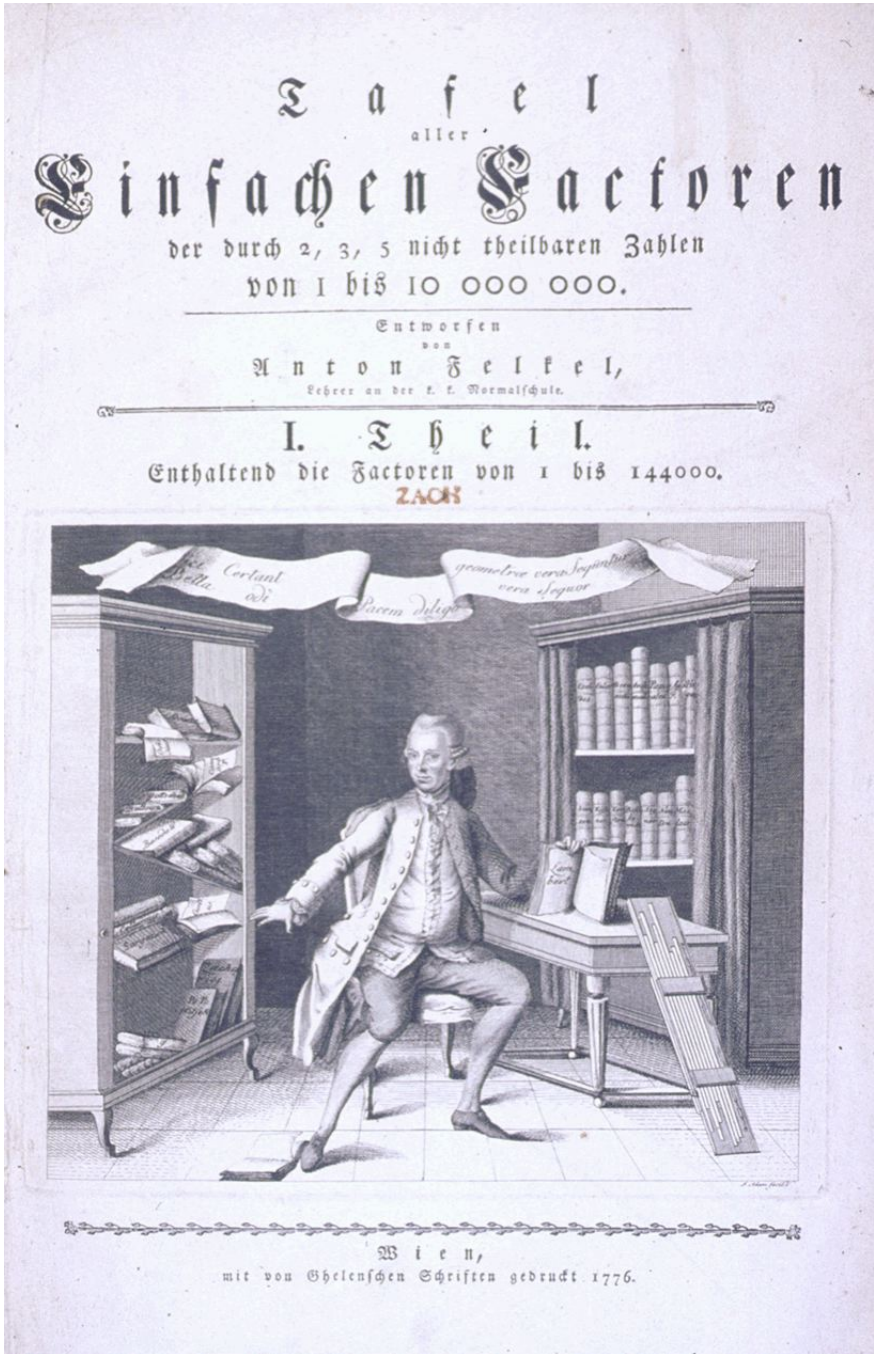
FIGURE 4. The frontispice of Felkel's 1776 Factor Table, depicting him with a book by Lambert in his hand, his machine at his table

to Lambert that he had changed his plan. Constrained by the design of his machine, processing 240 numbers per hour, on the one hand, and encouraged by his patrons in Vienna on the other, Felkel had decided to play *cavalier seul.* He now announced a table from 1 to 2,016,000, the numbers arranged in a way differing from Lambert's set-up [Lambert 1781–1787, V, pp. 62–70]. He held back with details on his method and machine, to avoid the danger of "seeing my own work thwarted" [Lambert *Briefe*, V, p. 67]. Felkel's ensuing letters of June and July kept insisting that Lambert ought to convince the other calculators to discontinue their work [Lambert *Briefe*, V, pp. 70–80].

Lambert's reaction to Felkel was irritation and silence; he expressed his disappointment to Rosenthal [Lambert 1781–1787, V, p. 30]. In the meantime, triggered by Felkel's announcement in the *L.G.Z.*, the publisher S.L. Crusius had a note inserted in the same *Zeitung* (L.G.Z., nr. 64, 8 August 1776, pp. 515–522), stating that the Magister Carl Friedrich Hindenburg (1741–1808) from Leipzig had also found a mechanism for producing factor and other tables in 1774, some time ago, and now planned to publish a description of the mechanism and a factor table to the fifth million [Hindenburg 1776b]. Hindenburg's mechanism was a continuation of Lambert's ideas:

> The advantage mostly results from a careful study of the structure of the decimal number system, and is in itself so considerable that it surpasses everything one could hope and wish for in that it changes the tedious looking up of divisors into a nearly immediate finding, and it produces the prime numbers in their natural order without searching and without loss of time. The method is, as would be suspected, totally mechanical and so reliable that it becomes impossible to make errors which would not immediately be betrayed by a contradiction. This circumstance takes away the danger of the usual miscalculations which are inevitable with such huge quantities of numbers. [Hindenburg 1776b, pp. 144–45] [88]

---

[88]   Original: "Dieser Vortheil ist größtentheils das Resultat einer sehr sorgfältigen Untersuchung des Decimalzahlengebäudes, und ist in seiner Art so beträchtlich, daß es alles übertrifft, was man nur wünschen und hoffen konnte, indem er das mühsame Aufsuchen der Theiler, in ein fast augenblickliches Finden verwandelt, und selbst die Primzahlen, in ihrer natürlichen Ordnung nach einander, ohne sie zu suchen, und also ohne allen Zeitverlust, giebt. Das Verfahren hierbey ist, wie man leicht

Upon the publication of the announcement, Hindenburg opened his correspondence with Lambert, forwarding him the text from the *L.G.Z.* (August 1776). Trying to save the collaborative spirit, Lambert immediately informed all his factor table correspondents on August 13 of Felkel's and Hindenburg's plans, adding complaints when writing to his correspondents of longer standing, and adding an insistent request in the letters to Felkel and Hindenburg to divide amongst them the 2nd and 3rd million [Lambert *Briefe*, V, pp. 81–82 & 151–154]. To Rosenthal, Lambert wrote:

> These Gentlemen apparently want to best each other, but it would be clearly better instead for one to start where the other stops. The first one praises his machine, the second his method. Time must tell what there is in both. [Lambert 1781–1787, V, p. 30] [89]

Instead of the scientific collaboration that Lambert so vividly promoted, a series of disputes, (unfulfilled) promises and discussions ensued. [90]

Felkel's reaction to Hindenburg's announcement was immediate. In September he published a more extensive, though not more informative, announcement of his plans, promising now a table to 10 million, to assert his priority in mechanising the production of factor tables [Felkel 1776a]. As Crusius had antedated Hindenburg's announcement to May 1776, Felkel did the same, antedating his to June 1776. Hindenburg meanwhile had prepared a manuscript for Lambert and Kästner that described his method, and although nearly all presses were busy during the *Leipziger Messe*, he succeeded in publishing his *Beschreibung* at the end of 1776 [Hindenburg 1776a]. By that time, Felkel, in his turn, had printed a first specimen of his table to 144,000 [Felkel 1776b]. Both sent their work to Lambert, with Felkel included an error list for his table [Lambert *Briefe*, V, pp. 112–113].

---

vermuthen wird, ganz mechanisch, und so zuverläßig, daß es unmöglich wird einen Fehler zu begehen, der sich nicht sogleich auf der Stelle durch einen Widerspruch verrathen sollte: ein Umstand, der das gewöhnliche, bey einer so großen Menge Zahlen, ganz unvermeidliche Verrechnen nicht befürchten läßt."

[89] Original: "Diese Herren wollen, wie es scheint, einander zuvorkommen, anstatt dass unstreitig besser wäre, wenn der eine da anfienge, wo der andere aufhöret. Der eine rühmt seine Maschine, der andere seine Methode. Die Zeit muss lehren, was an beiden ist."

[90] See also [Glaisher 1878, pp. 113–118] for an account of the priority discussion.

Following this series of announcements and publications, Lambert presented the plans of Felkel and Hindenburg to the Berlin Academy. In this matter, he communicated closely with Lagrange, who acted as a kind of second referee for Felkel's and Hindenburg's pretentions and productions. [91] The judgment was made in favour of Hindenburg, because his procedure was, so to speak, 'open source', whereas Felkel spent 2 long pages describing his mechanism without being all too clear. Lambert made up for this obscure description and explained it himself in a posthumously published review of both Felkel's and Hindenburg's books [Lambert 1778]. In this review, Lambert also mentioned a serious drawback of Felkel's table: It grouped numbers not in groups of $3n + 0, 1, 2$ as Lambert had done, but in groups of $30n + \ldots$ [92] and it used letters as abbreviations for the factors, which made any connection with previous (and later) work quite difficult.

After Lambert's death in 1777, the dispute continued. Felkel travelled to Leipzig in the Autumn of 1783 to discuss the matter with Hindenburg [Lambert *Briefe*, V, pp. 487-88]. The conflict was, however, not resolved, and each issued a new announcement, Felkel a circular in Halle [Felkel 1784], Hindenburg an advertisement for his table to the million in the *Leipziger Messkatalog*. The tables to the n-th million they promised were never printed. However, Felkel had calculated a table to the second million during the years 1775–1776. The first part to 144,000 was printed (see [Felkel 1776b]) and two additions up to 408,000 were issued. These would later be used by Vega [1797, I, pp. 87–128] for his list of primes. [93] Most copies of these additions were not sold and were unfortunately destroyed and/or lost during the Austrian-Turkish War (1787–1791); the paper of Felkel's table was recycled for gunpowder cartridges. From 1793 to 1794,

---

[91] The correspondence with Lagrange over Felkel's first circular, Felkel's table and Hindenburg's *Beschreibung* can be found in [Lambert 1781–1787, V, pp. 51–52 & 120–121 & 194].

[92] This ordering is similar to Euler's plan, see 4.1.

[93] A description of Felkel's rare 1776 tables to 408,000 and their arrangement is given by Glaisher [1878, pp. 106–108]. In [Glaisher 1873/1874] there remains some confusion between the 1770 original of Lambert's *Zusätze* and the Latin 1798 edition. Glaisher seems to have seen only the 1798 edition, and thus gets the data slightly confused sometimes, thinking that Felkel has priority over Lambert.

Felkel occupied himself with finding methods to reduce the bulk of the tables to 5 or 10 millions. He finally took refuge in non-decimal place value systems, a topic he had pursued while studying periodic decimal fractions [Felkel 1785]. Felkel considered 15 bases for the tables to 24 million, while 65 were needed for a table to 100 million. In 1798, Felkel produced a Latin translation of Lambert's *Zusätze,* commissioned and published by the Academy of Sciences in Lisbon [Felkel 1798]. The introduction recounted his story, and the book contained a table to 102,000, not in Lambert's but in Felkel's arrangement [Glaisher 1878, pp. 119–122]. Hindenburg, as J. Bernoulli confirmed [Lambert *Briefe*, I, pp. 386–87 & V, 242], had prepared the manuscript for the first two millions, but as had been the case with his his *Primtariffe*[94], they never appeared in print.

3.4.3. *A Short Description of the Sieve Mechanisms*

Both Felkel's and Hindenburg's inventions [95], as well as a third one proposed by Lambert himself, were mechanisations of multipication, based on the simple idea that every multiple of n in the natural order of a positional number system is n units removed from the preceding and from the preceding multiple. The most time-expensive step in Eratosthenes' sieve, checking the multiples of already known prime numbers, is simplified by these devices. They all use Lambert's remark in the *Beyträge* that a well chosen arrangement of a table displays certains patterns. For example, the multiples of 7 can be discerned by the eye in Lambert's arrangement. Such patterns allow to check certain multiples as well as the consistency of the factor table.

Felkel's mechanism, depicted on the frontispiece of his *Tafel* (Figure 4), is a variation on multiplication rods. Because Felkel based his device on a step-30 procedure, there are 8 of these rods corresponding respectively to $30n + 1, 7, 11, 13, 17, 19, 23, 29$ (the numbers of the form $30n + a$ which are not divisible by 2, 3 or 5). On each rod, all integers $30n + a$, with $n = 0$ to 99, are inscribed but the digits of the thousands are dropped. If one now

---

94   See [Bullynck 2009b].
95   Bischoff [1804/1990, pp. 73–83] contains detailed drawings and descriptions of Felkel's and Hindenburg's devices.

wants to find the multiples of 47, say, one first has to calculate the multiples of 47 below 1000 by hand, and then look these numbers up on the 8 rods. Then, one has to align the rods so that the multiples are all on a horizontal line. Now, one has to calculate the first multiple of 47 exceeding 1000, drop the 1 of the thousands (but take it down on a sheet of paper), and look up the remaining digits on the first rod. The numbers (on the other rods) that are horizontally aligned with that number are the further multiples of 47. This procedure (with some slight improvements) can be repeated to find further multiples [Lambert 1778, pp. 494–95].

As Lambert remarked, this mechanism has some disadvantages. First, one has to take down the thousands by hand, which is a possible source of errors. Second, the mechanism is limited by the limits of the rods, though one can produce extra rods [Lambert 1778, p. 494] [Lambert *Briefe*, V, pp. 120–121]. Hindenburg noticed one more inconvenience in a letter to Lambert, namely, that the numbers that are found are on the rods and have to be copied down (by dictation or by sight) on paper, so that "machine and journal are separated" [Lambert 1781–1787, V, p. 204]. Lambert also noticed in his review that this same mechanism can be transformed into a cylinder with circular discs on it. On one disc the numbers 0 to 99 are inscribed, on the other a segment is indicated, corresponding to a number p whose multiples one wants to know. By turning this second disc on the first one, one can find the last 2 digits of the desired multiples in natural order; by counting how many complete circles have already been turned, one can manually take down the other digits [Lambert 1778, p. 495]. This system has, of course, the same disadvantages as Felkel's system.

Hindenburg's solution avoided the first and third disadvantages, and adapted the procedure to printing practice, since it was an ink-and-paper implementation of Erastosthenes' sieve procedure. As is explicit in the title of his work: *Beschreibung einer ganz neuen Art, nach einem bekannten Gesetze fortgehende Zahlen, durch Abzählen oder Abmessen bequem und sicher zu finden,* Hindenburg relied (as Felkel and Lambert did) on the property of the positional system that every multiple of n taken in order is n removed from the next one. To exploit this property, he designed perforated cartridges that fit on a folio page with 10 times 30 cells on which were imprinted all
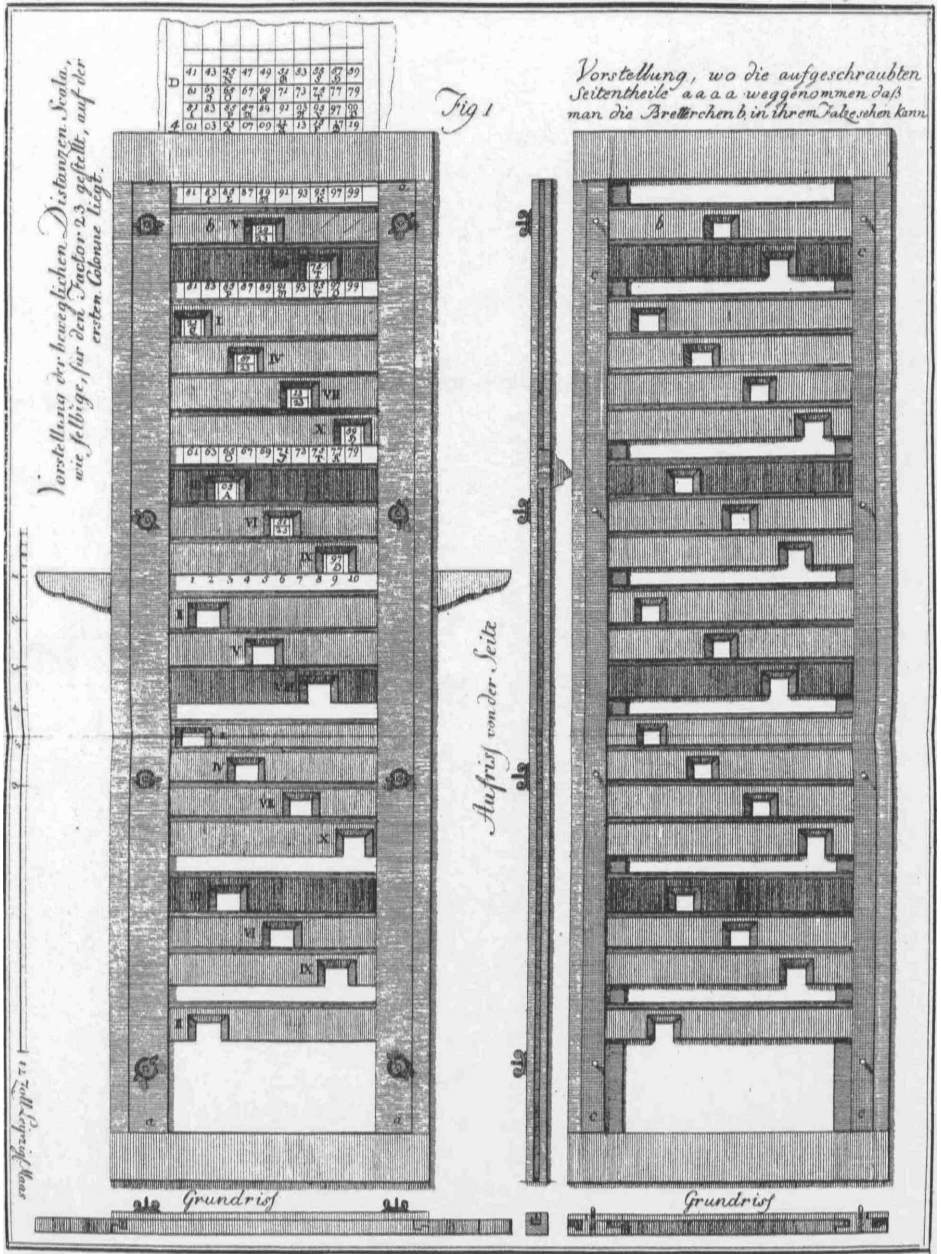
FIGURE 5. C.F. Hindenburg's sizeable paper-and-ink prime sieve
set up for the number 23

odd numbers from left to right, and were continued left to right in the next row. The perforations were, relative to the grid on the folio, at a distance n from each other. This cartridge method worked for 7, 11, 13, 17 and 19 [96]; for larger numbers, Hindenburg conceived of a cartridge of adaptable size (see Figure 5). This cartridge is a frame of wood with various sliders in it that can be adjusted and/or changed. The sliders cover exactly one row of 10 cells on the folio, except for one cell at position n $(1 < n < 10)$. Thus, the number 23 can be sieved using two sliders without holes and one with a hole at position 3, followed by one slider without a hole and one with a hole at position 6, etc. It is clear that arranging the sliders takes quite some time, but the pattern repeats itself after 23 folio pages (in our example), and it is also possible to re-use one portion of the work with a slight reshuffling after one or two folios.

Using the cartridges for all prime numbers in their natural order, Hindenburg wrote the factor m and its "distance" n (which he called "Ordnungszahl", index) in the measured cell, $m.n$ making up the number of the cell. The end product was many folios that only differed in one respect from the to-be-printed tables, namely, that they still contained the multiples of 3 and 5. Hindenburg insisted from the beginning on the fact that this procedure could be used for constructing other tables. In his *Beschreibung*, he explained how to use this device for making tables of squares, of triangular numbers, of remainders after division, and even how to use it for solving linear Diophantine equations [Hindenburg 1776a, pp. 39–92 and 106–116]. [97] The foundations of the later Combinatorial Analysis, as laid down in [Hindenburg 1781], sprang from this study of the positional number system [Hindenburg 1776a, pp. 92–104], as Hindenburg

---

96   3 and 5 can be discerned by the eye because of the arrangement of the folio, viz., all 5-tuples were contained in two horizontal columns, all triples could be found be a diagonal.

97   This last issue was investigated theoretically in [Hindenburg 1786a]. Hindenburg later realised that his folios could be used directly for printing the factor tables if he deleted the multiples of 3 and 5 in his standard folios [Lambert 1781–1787, V, p. 178 note]. Although this had the advantage of minimising copying errors, it had the disadvantage that the folios could then no longer be used for other tables such as tables of squares, of remainders after division, etc.

[1795b, p. 247] explained. [98]   Hindenburg insisted again on the idea of how remarkable it was that so many numbers could be represented with such a small set of signs. By showing that it does not matter if one uses hindu-arabic ciphers or letters, Hindenburg [1776a, pp. 96–100] rediscovered the combinatorial nature of positional number systems with respect to an arbitrary base. [99] He also pointed out that a positional number system essentially implies the possibility of mechanisation. Referring to Leupold's encyclopedic work on arithmetical instruments [Leupold 1727], Hindenburg indicated how his system was related to the abacus, to multiplication rods, and to many other computational devices [Hindenburg 1776a, pp. 101–104].

4. FACTOR TABLES AND THE BIRTH OF NUMBER THEORY

> *"Most of the important classical theorems in number theory were discovered as a by-product of the production and inspection of tables."* [100]

Lambert had predicted that his factor table project "may in the future be an important part of the history of mathematics" [Lambert *Briefe*, II, p. 30]. In line with this thought, Johann III Bernoulli carefully documented the whole history while editing the scientific correspondence of Lambert between 1781 and 1787. He asked the human computers involved in the project for their part of the correspondence, and also for notes and additions to the exchange of letters. Also A.G. Kästner [1786, pp. 549–564], professor of mathematics in Göttingen, wrote extensively on the topic. Later on, however, partly through external circumstances, in particular the Napoleonic Wars, and partly with the advent of a new generation of mathematicians after 1800, the history of Lambert's project

---

[98]   Cf. Bernoulli's remark in the *Vorrede* of [Lambert *Briefe*, V]. Note also that one issue (vol. 2, nr. 2) of the *Leipziger Magazin für reine und angewandte Mathematik* (1787), of which Hindenburg was the editor, was devoted in part to factoring. None of the essays in the *Magazin*, however, went beyond what Hindenburg himself had described much earlier (as he did not fail to point out in his editorial comments).

[99]   Hindenburg (p. 100) referred in this context to [Beguelin 1772b].

[100]   [Lehmer 1969, p. 118].

was largely forgotten. [101] Indirectly, however, Lambert's project stimulated some of the most important mathematicians around 1800, and as such contributed substantially to the birth of number theory as an independent discipline.

4.1. *Euler, Factor Tables, and the Science of Numbers*

As long as Lambert participated and communicated actively, from 1770 through 1777, the table project caused a considerable stir in the research community, which, however, was quite small at the time. Tables and factoring were a much discussed topic at the Berlin Academy (see p. 165 above). Lagrange also made sure that a specimen of Lambert's Tables was sent to his correspondent d'Alembert [Lagrange *Œuvres*, XIII, pp. 202–203]. Apart from the project of astronomical tables and the factor tables, a third collaborative project was under way at the Academy around the same time. Following a wish of d'Alembert, Johann III Bernoulli began a French translation of Leonhard Euler's *Algebra* which had been published in 1770 in St. Petersburg. He was assisted by J.L. Lagrange, who wrote the well-known supplement on indeterminate analysis which complemented Euler's second volume. When the French *Élémens d'Algèbre* appeared in 1774, Bernoulli wrote in the foreword:

> … Nor will I say anything about the notes I added to the first part […] but they can throw light on various points in the history of mathematics & make known a great number of little-known subsidiary tables. [Euler 1774, p. xvj] [102]

Indeed, in the notes to Bernoulli's translation we find all the tables listed that Lambert had mentioned in the preface to his *Zusätze*, and we also see praise of Lambert's own work on tables [Euler 1774, pp. 26–28].

Leonhard Euler (1707–1783) had left the Berlin Academy for the St. Petersburg Academy in 1766, after a conflict with Frederick of Prussia, and

---

101   It was left to Glaisher [1878] to rediscover it, for his account of the production of factor tables.
102   Original: "Je ne dirai rien non plus des notes que j'ai ajoutées à la premiere Partie; […] elles peuvent d'ailleurs répandre du jour sur différens points d'histoire des Mathématiques, & faire connoitre un grand nombre de tables subsidiaires peu connues."

possibly also after a conflict with Lambert on running the Academy [Biermann 1985]. Euler plays only a marginal role in the history of tables, but a curious one. Even though he did not correspond on the topic either with Lambert or with Lagrange [103], he did so with Johann III Bernoulli and Nikolaus von Beguelin. Many of Euler's contributions concerning factoring methods were actually made in letters, presentations or papers that were initially conceived as a response to their work, with the notable exception of one essay. This essay, published in 1774 in the *Novi Commentarii* of the Petersburg Academy, discussed how to arrange a factor table to the first million in the best possible way [Euler 1774/1775]. Without any reference to Lambert's work, Euler argued for the importance of factor tables and described a method of arrangement that was based in the grouping into $30n + 1, 7, 11, 13, 17, 19, 23, 29$. Thinking of quarto pages (not the larger folio format, as most others did), Euler constructed subsidiary tables to facilitate the application of Eratosthenes' sieve over successive pages. At the end of his paper, Euler gave some (fairly error-ridden) samples from such a factor table. [104]

Euler's plan was known (i.e., its abstract in the *Journal Encyclopédique* of 1776 was read), but its influence was not tremendous. As Hindenburg remarked in a letter to Lambert, Euler's arrangement was essentially the same as Felkel's, except for the different format; but Felkel claimed he had not been aware of Euler's paper [Lambert *Briefe*, V, pp. 204 & 501]. After 1800, Euler was often mentioned in the introductions to factor tables, but instead of Euler's arrangement, Lambert's tended to be used throughout. The only exception that we are aware of is an unpublished table of primes by the Swede Schenmark, which was presented to Lexell in Lund. Regarding this table, Nikolas Fuss, Euler's assistant, wrote to Joh. III Bernoulli in December 1781:

---

103 Euler's last letter to Lagrange in 1775, that ends with a reference to [Euler 1774/1775], seems to be the only exception to this [Euler *Opera*, IV, 5, pp. 244-45]. A letter from Lambert to Euler with a specimen of the *Zusätze* from October 1771 received only a polite and short response from Euler's secretary [Juskevic et al. 1975, OO1419].

104 The editors of [Euler 1907, X–XIII] give a short overview of the history of factor tables from the perspective of Euler's paper, barely mentioning Lambert's earlier work in a footnote.

Mr. Lexell […] has brought from Lund a table of prime numbers in manus-
cipt, computed to 1 million after Mr. Euler's plan […] by Mr. Schenmark and a
few other computers under his direction. We first discussed publishing it at the
expense of the Academy, but then we hesitated since I communicated to Mr. Eu-
ler what you have remarked to me concerning Mr. Hindenburg, who must have
promised two million by Easter. [Bernoulli 1781/1783, p. 31] [105]

Bernoulli consequently advised against publishing Schenmark's tables,
protecting his friend Hindenburg's work. Schenmark's work was, however,
conserved, and later used by Burckhardt for checking his tables.

Euler's reactions to the factoring methods developed by Bernoulli and
Beguelin were more influential for the history of mathematics than his pro-
posals for the layout of a factor table; they announced important methods
that would soon be bolstered by C.F. Gauss' theoretical analysis. [106] In a
letter to Bernoulli regarding his publication on factoring numbers of the
form $10^n \pm 1$, Euler presented a simple, necessary and sufficient criterion to
decide whether p divides either $10^{2p+1} - 1$ or $10^{2p+1} - 1$ (p a prime), replac-
ing the piecemeal collection of rules that Bernoulli had assembled from
Euler's earlier papers [Euler 1772/1774b]. The result rested on Fermat's
theorem and on a theorem Euler did not state, but described as unproven:
the quadratic reciprocity law (as Legendre would later call it). Simulta-
neously to this letter, Euler read a paper at the St. Petersburg Academy
concerning the theory on which Bernoulli's paper was built, namely, on
residues of a series of powers a^n after division by a prime p (for $n = 0$ to
$p - 1$), and introduced the term "primitive root" [Euler 1772b]. Later in
the same year, Euler returned to the topic of residues of powers, focussing
in the second half of this new paper [Euler 1772/1774a] on the residues of
the even powers, i.e., on quadratic residues of squares. Euler gave criteria
for deciding whether ± 1 is or is not a quadratic residue of a prime p.

---

[105] Original: "M. Lexell […] a apporté de Lund, en manuscript, une Table des
nombres premiers, éxecutée jusqu'à un million d'après le plan que M. Euler a donné
[…] par M. Schenmark & quelques autres Calculateurs sous sa direction. On avoit
parlé d'abord de la faire imprimer aux dépens de l'Académie; mais on hésite depuis
que j'ai communiqué à M. Euler ce que Vous m'avez marqué touchant M. Hinden-
bourg, qui doit avoir promis 2 millions pour Pâques."
[106] See also [Bullynck 2009c, p. 158;166–175] for a analysis of Euler's work on arith-
metic after 1770.

Berlin Academy	Euler in St. Petersburg	Topic
J.H. Lambert, *Zusätze zu den logarithmischen und trigonometrischen Tafeln* (1770) Lambert to Euler, 18 October 1771	De tabula numerorum primorum usque ad millionem et ultra continuanda (E467, 1774/1775)	manufacture of factor tables
Jean III Bernoulli: Sur les fractions décimales périodiques. Suivi de: Recherches sur les diviseurs de quelques nombres très grands (1771/1773)	Extrait de la correspondance de M. Bernoulli (E461, 1772/1774) Demonstrationes circa residua ex divisione potestatum (E449, 1772/1774) Observationes circa divisionem quadratorum; Disquitio accuratior circa residua etc. (E552 & E554, 1772/1783) Earlier work: E54, E134, E271 (Fermat's little theorem); E242, E262, (quadratic residues) Later work: E557, E792	periodic fractions, quadratic residues, instances of quadratic reciprocity
Nikolaus von Beguelin: Solution particulière du Problème sur les nombres premiers (1775)	Extrait d'une lettre de M. Euler le père à M. Beguelin en mai 1778 (E498) Extrait d'une lettre de M. Fuss à M. Beguelin écrite de Pétersbourg le 19C20 juin 1778 Earlier work: E29, E164, E228, E241, E255, E256, E272, E283, E369 Later work: E699, E708, E715, E718, E719, E725	factorisation using quadratic forms with idoneal numbers
J.L. de Lagrange: Recherches arithmétiques (1773 and 1775)	De insigni promotione scientiae numerorum (E598, 1775/1785) Earlier work: E29, E279, E323, E452, E454, E559 Later work: E610, E683, E744	theory of quadratic forms

TABLE 1. A tabular overview of the interactions between research at the Berlin Academy and Euler's research. Indicated are Euler's previous work on the same topics and Euler's later (often posthumously published) work, using the numbers of his papers in the Eneström-index.

A similar pattern of interactions ensued some years later (1778) when Euler reacted to an essay by Beguelin [1775]. This essay had sprung from Beguelin's investigations on his new number system and proposed a new method for factoring numbers. The idea was to isolate a particular sequence of numbers, for instance triangular or square numbers, and gradually exclude elements until one or more were left.

> … it is only in determining the composite elements that one finds the primitive elements [107], through the gaps that result of this determination, in the manner of Eratosthenes. [Beguelin 1775, p. 301] [108]

Beguelin obtained his exclusions using a result by Euler, namely that only composite numbers can be decomposed in more than one way as a sum of squares $a^2 + b^2$. Using this property, Beguelin sieved the composites out of his series $ppxx + 1$ (p prime and fixed, x a variable integer). Beguelin slightly modified Euler's method: through a certain arrangement of lists, only one series of possible squares had to be checked instead of two [Beguelin 1775, p. 308].

The form $a^2 + b^2$ is not the only form with the property that a prime can only be expressed in exactly one way by the form, whereas composites either cannot be expressed by the form or have two or more expressions. The same holds true for $a^2 + 2b^2$, $a^2 + 3b^2$, … (but for instance not for $a^2 + 11b^2$ or $a^2 + 14b^2$). Now, Euler had found by induction that there was only a finite number of such forms. In a letter to Beguelin, Euler gave a list of all 65 n's for which $a^2 + n.b^2$ had the property of representing a prime in only one way [Euler 1776/1779]. These numbers were later coined idoneal numbers by Euler, but the letter to Beguelin contains the first mention (without the name) of these numbers. [109] Since

---

[107] "Primitive elements" here are p's that are prime and may be used to generate the elements of the sequence, for example in a sequence of the form $ppxx + 1$, x a variable integer.

[108] Original: "ce n'est qu'en déterminant les élémens composés qu'on trouve les primitifs, par les lacunes qui résultent de cette détermination, á la manière d'Erastosthene."

[109] Euler's idoneal numbers relate to what later would become the concept of the genus of a quadratic form and the the principal genus theorem. More on this can be found in [Steinig 1966] and [Lemmermeyer 2007, pp. 531–533].

Beguelin and Lagrange desired to know more details of this discovery and the associated novel method of factoring, Euler's assistant N. Fuss compiled a resumé which was published somewhat later in the *Mémoires* of the Berlin Academy [Fuss 1776/1779]. As in the case of the earlier paper by Bernoulli [1771/1773], Euler seized the opportunity to write some papers, this time about the topic of factoring large numbers using the ideoneal numbers. They were published only long after Euler's death, between 1801 and 1806. [110].

Finally, Euler also responded to Lagrange's *Recherches d'arithmétique* [Lagrange 1773], not in a letter but with a presentation to the St. Petersburg Academy October 26, 1775 (published posthumously in the *Opuscula Analytica*, 1785). Euler's objective in that presentation was to make Lagrange's survey more elegant and more complete (thus partly "anticipating" [Lagrange 1775]). Euler used the opportunity to point out that he had found many of the results much earlier by induction, but that only the joint efforts of Lagrange and himself had advanced this part of mathematics. According to Euler, Lagrange "ha[d] brought light into the science of numbers, that ha[d] hitherto been shrouded in darkness" [Euler 1775/1785, p. 163]. [111] Although Euler had occasionally used the term "science of numbers" before [112], this paper explicitly addressed this new science in its title: *De insigni promotione scientiae numerorum.* In this way, Euler indirectly connected Lagrange's and his own endeavours with Lambert's programme.

Euler's contributions dealt mainly with theoretical aspects of producing factor tables. Even though these contributions to factoring often re-used (partial, often unproven) results that Euler had obtained earlier [113], they

---

110 These are E708, E715, E718, E719, E725 [Euler 1907, II, pp. 249–260 & 198–214 & 215–219 & 220–242 & 261–262].

111 Original: "Eximia omnino sunt, quae La Grange […] demonstravit, et maximam lucem in scientia numerorum, quae etiamnunc tantis tenebris est involuta, accendunt.".

112 For instance in [Euler, I, 2, p. 611].

113 Some were mentioned in his correspondence with Christian Goldbach, some were published in the St. Petersburg *Commentarii*, see [Dickson 1919–1927, I, pp. 360–1] and Table I.

now appeared in a different context, as reactions to publications on factoring, and these publications in turn had been instigated by Lambert's project. Although Euler never referred to this context, the chronology of his papers is telling enough, and so is Euler's adaptation of results from Diophantine problems to the problem of factoring (large numbers), a problem that almost never appeared in his work before 1770. [114]

Although Euler elaborated further on these results in St. Petersburg, he did not embed them into a structured series of theorems and demonstrations, in the way that Lambert would have liked to see the gap bridged between the beginnings and the advanced parts of a theory of integers and factoring. This job remained to be done. However, the papers written at the Berlin and St. Petersburg Academies were stepping stones towards turning the tables, results, and partial theory of number-related problems into a coherent body of results, concepts, theorems and demonstrations, into something that could be called a "theory of numbers" proper. Finishing the job, in some sense, was the work of A.M. Legendre in his *Essai sur la théorie des nombres* (1798) and C.F. Gauss in his *Disquisitiones Arithmeticae* (1801).

4.2. *Tables and the Distribution of Primes*

As an effect of Lambert's project, it became possible to make more accurate empirical observations and conjectures about the distribution of primes using the various tables of factors and primes that had been printed. The question of the distribution of primes before 1770 had only been touched upon by Euler. In a letter from 1752 [Fuss 1843, I, p. 587], Euler had remarked to Goldbach, without proof, that the number of prime numbers relative to the number of integral numbers ($= x$) converges to $\ln x$. Euler repeated this observation in [Euler 1762/1763, p. 101]. In another letter from the same year 1752 [Fuss 1843, I, p. 595], Goldbach had claimed he could prove that no closed algebraic formula could generate only primes; this was later published and proven by Euler

---

[114]   Though the topic of finding large prime numbers did appear, factoring itself was not a major topic.

[Euler 1762/1763, p. 102]. In 1776, Hindenburg had somewhat light-heartedly dismissed the question of the progression or distribution of the primes:

> I have neither time nor motivation, to tediously and rigorously prove or disprove a theorem which is completely useless for my purpose, and which appears to develop an untimely curiosity rather than real use. [Hindenburg 1776a, p. 15] [115]

A more elaborately argued dismissal in the same spirit was given by Karl Christian Friedrich Krause (1781–1832) in 1804. Krause, now better known as a philosopher, had written a dissertation at the university of Jena on finding prime numbers, *De inventione numerorum primorum* (1801), probably under the supervision of the Hindenburg-influenced professor D.M.C. Stahl. [116] In 1804, Krause published a factor table to 100,000 and returned to the question of the distribution of primes, earlier dismissed by Hindenburg. Considering the successive application of the sieve proce-dure, eventually eliminating all composites, Krause arrived at a description of the "Primzahlgesetz".

> We have found the law [that governs the distribution of primes]. It is a law that is continuously changing with every series of prime numbers. It is an in-finitely multisided law which continuously acquires new specifications as one progresses. That is why we will not trouble ourselves any further to find a finite algebraic law, where an infinite one rules. [Krause 1804b, p. 12] [117]

Krause considered the distribution of primes as an infinite process of eliminating the multiples of 2, 3, 5 and so on, and since the series of prime

---

[115]   Original:"Ich habe weder Zeit noch Lust, einen für meine Absicht völlig un-brauchbaren Satz [wie sich die Primzahlen unter den Zahlen verteilen], der eher eine unzeitige Neugierde, als einen reellen Nutzen zu befördern scheint, durch einen weitläufigen strengen Beweis *a priori* zu unterstützen oder zu verwerfen."

[116]   According to [Riedel 1941], the manuscript seems to be lost now.

[117]   Original: "Wir haben das Gesetz gefunden, es ist nehmlich ein beständig durch jede Reihe der Primzahlen gesetzmäßig verändertes; es ist ein unendlich vielseit-iges, bei immer weiterem Fortschreiten weiter bestimmtes. Daher werden wir uns nicht weiter bemühen, ein endliches, algebraisches Gesetz aufzufinden, wo ein un-endliches waltet."

numbers was infinite (according to Euclid), their inverses formed an in-
finite decreasing series; but since the rate of decrease was neither regu-
lar nor rapidly decreasing itself, a finite algebraic formula was impossible
[Krause 1804b, p. 11]. [118]

4.2.1. *The empirical law, first version.*

The materials available in the late eighteenth century thanks to Lam-
bert's project did lead to more concrete results than Krause's rather vague
speculations. On being received by the Duke of Braunschweig in 1791,
the young Carl Friedrich Gauss (1777–1856) had been given Schulze's
Sammlung, and somewhat later (in 1793), Lambert's *Zusätze* and Hinden-
burg's *Beschreibung*. According to a letter to Encke (1849), Gauss had
begun counting in Lambert's and Schulze's tables as early as 1792–3,
"even before I had occupied myself with subtler investigations in higher
arithmetic" [Gauss *Werke*, II, p. 444]. In the margin of his copy of Hin-
denburg's work, Gauss entered his objections to Hindenburg's opinion
that it was uninteresting to formulate or prove a law of prime distribution.
On the contrary, Gauss was among the first to make counts and draw up
a formula, though he never published it. In his scientific diary, he had
noted:

> Comparationes infinitorum in numeris primis et factoribus cont[entorum]
> [1796] 31. Mai G[ottingae]
> Leges distributionis [1796] 19. Iun. G[ottingae] [Gauss 1863–1929, X/I,
> pp. 493 & 495]

The formula in question was that the number of primes up to a con-
verges to

$$\frac{a}{\ln a}$$

---

[118] Most famous in connection with finite algebraic formulae for primes is Euler's
$41 - x + xx$, where the first 40 terms (for $x = 0$ to 39) are all prime numbers. This
formula was first mentioned in his letter to Bernoulli [Euler 1772/1774b, p. 36]. Of
course, Euler had already proven some 40 years earlier Krause's claim that no finite
algebraic formula exists.

[Gauss *Werke*, X/1, pp. 11–16], a result already conjectured by Euler. Using Vega's list of primes (taken from Felkel's tables), Adrien-Marie Legendre (1753–1833) came up with a similar, though slightly more accurate formula in the second edition of his *Essai sur la Théorie des nombres* (1808):

$$\frac{x}{\log .x - 1.08566},$$

where $\log .x$ denotes the hyperbolic logarithm ln.

4.2.2. *New Tables.*

In 1811, Ladislas Chernac [1811], a professor of mathematics in Deventer of Hungarian origin, published the *Cribrum Arithmeticum* at his own expense. His table, of more than 1000 pages, gave all factors for the numbers not divisible by 2, 3, nor 5. His introduction contained a very complete list of all factor tables until 1811, even mentioning a table by Adolph Marci (Amsterdam, 1772) that was calculated in response to Lambert's appeal but seems to have been lost [Chernac 1811, pp. V–X; IX]. Chernac did not explain how he constructed his table, except for a reference to Nicomachus' *Arithmetic* and the title *Cribrum Arithmeticum*, which seems to denote both the method and the content of the table. Chernac's introduction closed with applications and examples: logarithms, divisions and several tricks where the factor table can be put to use, but no reference at all to more abstract number problems.

Three years later, Johann Karl Burckhardt (1773–1825) published the second million, and in 1816, the third million (he actually went to 3,036,000). In 1817, he republished the first million, corrected and in the same format as the other two. For this last edition, Burckhardt had compared Chernac's table with Schenmark's manuscript, had found some but few errors in Chernac's table, many in Schenmark's, and had checked each inconsistency by re-calculation [Burckhardt 1814, p. i]. In his own words, Burckhardt undertook these calculations while "one was occupied with comparing my moon tables with those of Mr. Burg, a circumstance which

FIGURE 6. Gauss' Table counting the primes in Lambert's *Zusätze* and a specimen of C.F. Gauss' specially printed paper slips for counting primes and composites for the first chiliad in Chernac's Table

prevented me from taking up other astronomical studies." [Burckhardt 1814, p. vij] [119]

Burckhardt had a remarkable career. Born in Leipzig, he had acquired an extensive knowledge of mathematics and astronomy as a 15-year-old through private study before enrolling at Leipzig University in 1792. There he studied with C.F. Hindenburg and became a *Magister* in 1794 with a dissertation on the expression of continued fractions through combinatorial signs. [120] Thanks to a scholarship and with Hindenburg's recommendation, Burckhardt went to the observatory in Gotha to work and study under the astronomer Franz-Xaver von Zach, a central figure in the internationalisation of astronomy at the time [Brosche 2001]. Upon Zach's recommendation, Burckhardt then went to Paris in 1798, where the astronomer Lalande hired him as assistant in the *Bureau des Longitudes.* In 1800–1802, Burckhardt and Gauss competed for the best calculations of the orbits of Ceres and Pallas. Burckhardt also translated Laplace's *Mécanique céleste* into German (1800–1802, with lengthy comments and additional examples), and he was known as a skilled and reliable computer. After Lalande's death in 1807, Burckhardt became the director of the observatory of the *Ecole Militaire.* He died in Paris in 1825. [121]

Burckhardt's very accurate factor tables were computed with the stencil method which would become standard for the production of factor tables in the 19th and early 20th centuries. Burckhardt described it as an improvement on Hindenburg's method in the *Beschreibung.* [122] First, Burckhardt eliminated all numbers divisible by 3 and 5 beforehand. [123] To this

---

119   Original: "j'ai entrepris et fort avancé ce travail dans le tems qu'on s'occupait de comparer mes Tables de la lune à celles de M. Burg, circonstance qui m'empêchait de commencer d'autres recherches astronomiques."

120   Hindenburg [1795a, pp. 174–178] described Burckhardt's procedure.

121   Cf. the biography in [ADB 1875-1912, 3, pp. 571–72], which consistently misspells "Hindenburg" as "Hindenberg."

122   As mentioned earlier, Hindenburg had made some of these improvements earlier, [Lambert 1781–1787, V, p. 178 note].

123   "[L]a moitié de l'ouvrage [i.e. the work with Hindenburg's sieve] [est faite] en pure perte; car dans les Tables imprimées on rejette les nombres divisibles par 3 ou par 5 [i.e. one does not print them], ce qui oblige de copier au net la partie de l'ouvrage qu'on conserve. J'ai évité ces deux inconvéniens et j'ai obtenu en même tems que dans mes Tableaux imprimés les facteurs" [Burckhardt 1814, p. v].

aim, Burckhardt had let a copper plate be engraved in 81 horizontal lines and 78 vertical ones, obtaining 80 times 77 little squares. Next to the first horizontal line were engraved all numbers under 300 not divisible by 2, 3 and 5, thus repeating the arrangement Lambert had advised and Hindenburg had followed. Due to the 77 columns, the multiples of 7 and 11 could immediately be engraved on the plate.

With this plate, the individual sheets were printed, immediately reducing the work to checking divisors above 11. For these larger divisors, e.g. 13, Burckhardt took an empty, squared sheet, started cutting out the squares that were multiples of 13 and stopped after the 13th column, since "this factor will return in the same order […] because of the distance" [Burckhardt 1814, p. v]. [124] By putting together two sheets, three sheets etc. this procedure could be expanded for larger divisors.

> for divisors over 500, I have preferred to find the multiples by successive additions […] I have checked the last multiple by a direct multiplication [Burckhardt 1814, p. vi] [125]

This procedure of cutting out squares in a sheet of squared paper is essentially the stencil method, not very much different from Hindenburg's sieve, but more practical. [126]

---

[124]   Original: "ce facteur retournera dans le même ordre, puisque la distance d'une colonne à l'autre est toujours de 300."

[125]   Original: "quant aux facteurs qui surpassent 500, j'ai préféré de trouver les multiples par des additions successives. […] le dernier multiple […] a été vérifié par une multiplication directe."

[126]   C.F. Gauss reviewed both Chernac's and Burckhardt's tables [Gauss 1863–1929, II, pp. 181–186], referring in the review of Chernac to Lambert's project and describing at length Burckhardt's procedure (without reference to Hindenburg). Also, Gauss summarized the history of factor tables around 1800, most probably using Kästner's account [Kästner 1786, pp. 549–564] in the Chernac review. Gauss repeated this same summary some 30 years later in a letter to Zacharias Dase [Dase 1856], stimulating Dase to undertake the calculation of the missing millions. The 6th, 7th, and 9th million were eventually calculated by Dase, the 4th and 5th were computed (with lots of errors) by Leopold Crelle; they are preserved at the Berlin Academy, cf. [Crelle 1853].

4.2.3. *The empirical law, second version.*

Using these new tables, Legendre had checked his empirical formula again in 1830, for the third edition of his book that was now simply entitled *Théorie des Nombres.* His comparison between tables and formula was quite favorable evidence for the formula [Legendre 1830, II, p. 65]. Gauss had pursued his counting as well, at times together with Goldschmidt. Gauss had even caused custom-made paper slips to be produced for counting the chiliads in Chernac's and Burckhardt's tables (Figure 6). In a letter to Encke, Gauss spelled out and corrected his formula for the distribution of primes. Gauss saw it in inverse proportion to the integral

$$\int \frac{da}{\ln a}.$$

Gauss compared this integral with Legendre's formula and found that his integral approximated the distribution of primes better [Johnson 1884].

4.3. *Introducing Number Theory*

The birth of a new discipline is usually accompanied by legitimation practices, often including an account of its prehistory leading inevitably to the new science. The two works that were to found number theory as a discipline in its own right, Gauss' *Disquisitiones Arithmeticae* (1801) and Legendre's *Essai sur la Théorie des nombres* (1798), used their prefaces to do exactly this; they construct their own historical lineage. [127] The lineages constructed in the two treatises are very similar : starting from Books VII and VIII of Euclid's Elements, and passing through Diophantus and Fermat, they finally arrive at Euler and Lagrange.

The immediate precursor was Lagrange's *Recherches Arithmétiques.* Legendre explicitly referred to this text as the first general theory on indeterminate questions ("la Théorie de Lagrange", [Legendre 1798, p. X]) and Gauss may well have translated Lagrange's title into the title of his own *Disquisitiones Arithmeticae* [Weil 1984, pp. 319–320]. A second important

---

[127] To complement our observations on the birth of number theory, [Goldstein & Schappacher 2007a] is essential reading. In general, see [Goldstein et al. 2007] for the development of number theory after Gauss (and Legendre).

impetus, however, had come from Euler's *De insigni promotione scientiae numerorum*, posthumously published in 1785. Legendre's first publication on number theory, which remained an important part in his later book, was written precisely to prove some of the theorems Euler had put forward in that paper [Legendre 1785, pp. 523–524]. It is also one of the most frequently quoted papers in Gauss' work.

Legendre's legitimization of number theory was rather tentative; he placed himself in the tradition of Diophantus:

> I do not make a distinction between number theory and indeterminate analysis, and I regard these two parts as one and the same branch of algebraic analysis. [Legendre 1798, p. xj] [128]

Gauss, on the contrary, explicitly addressed a new discipline, probably taking up Lambert's suggestion to fill in the gaps between elementary and advanced arithmetic, and separated its content from mere Diophantine analysis.

> The inquiries which this volume will investigate pertain to that part of Mathematics which concerns itself with integers. […] The Analysis which is called indeterminate or Diophantine […] is not the discipline to which I refer but rather a special part of it, just as the art of reducing and solving equations (Algebra) is a special part of universal Analysis. [Gauss 1801, Preface]

For Gauss, arithmetic comprised "all investigations on the general properties and relations between numerical quantities", so that "integers are the sole object of arithmetic", containing both elementary arithmetic (reckoning) and higher arithmetic (now called number theory).

Whereas Legendre proposed to present and expand Lagrange's theory, focussing on quadratic Diophantine problems, Gauss' reference framework was broader from the start. Euclid's books VII and VIII, on composite and incomposite numbers, constituted the basis of this new discipline according to Gauss' preface. In this respect, Gauss seemed to continue John Wallis' programme, of re-writing Euclid in arithmetical

---

[128] Original: "Je ne sépare point la Théorie des Nombres de l'Analyse indéterminée, et je regarde ces deux parties comme ne faisant qu'une seule et même branche de l'Analyse algébrique."

terms, thus founding mathematics on numbers [Wallis 1657, pp. 14–20].[129] Most probably, C. S. Remer's *Demonstrativische Rechenkunst* (1739), which the 8-year-old Gauss received as a gift in 1785 and which he called his "liebes Büchlein" [Maennchen 1928, p. 17], acquainted Gauss early on with such ideas. Remer, who often cited Poetius as one of his sources, dealt extensively with the topics of divisors, odd and even numbers, prime and composite numbers and factoring methods, including Eratosthenes' sieve procedure [Remer 1739, pp. 232–321]. The book also contained a large section on "the properties of numbers in relation to each other", discussing properties of the greatest common divisor process [Remer 1739, pp. 324–356].[130] Add to this, that between 1791 and 1795 Gauss had also perused the tabular works edited by Schulze and Lambert, before going to the university of Göttingen, and one sees a clear interest in integers and factoring developing over the years.

When Gauss started to write his book on arithmetic that would ultimately become the *Disquisitiones Arithmeticae*, his original plan (dating from 1796–97) proposed to conclude the book with his construction of the 17-sided polygon and a treatise on the general solution of higher order congruences, the *caput octavum* which was only published posthumously [Bachmann 1911; Merzbach 1981, pp. 6–8]. During the later re-workings of his text, Gauss kept expanding the section V on quadratic forms, building on his student readings of Euler's, Lagrange's, and Legendre's works that he could now consult in the rich mathematical library of the Göttingen university (December 1795 through May 1796).[131] As a consequence, some of the original focus disappeared from the ultimate publication, which stops at section VII.

---

[129]   There was a copy of Wallis' work in the library of the Collegium Carolinum where Gauss studied [Küssner 1979, p. 37]. The idea of arithmetisation was discussed by e.g. Poetius, Lambert, and Kästner in protestant Germany. Of course, Lambert's tentative presentation of a theory of numbers in [Lambert 1770] is also along these lines and was read by Gauss at age 16. Gauss' Sections I and II may be called the more mature equivalent of Lambert's essay.

[130]   Dr. Christian Siebeneicher (Bielefeld) most kindly drew my attention to Remer's book and pointed out its relevance for this topic.

[131]   See his letters to Zimmermann, [Poser 1987, pp. 20 and 24].

However, if one leaves out for a moment section V in the structure of the *Disquitiones Arithmeticae* and then tries to focus on the older sections I, II, III and VI and considers their internal coherence, one notices their connection with the developments we have described from the period 1770–1800. Section I defines what a congruence is and shows how the concept of congruence can account easily for simple divisibility tests by 9, 11 or 7 (art. 12). Section II contains elementary theorems of number theory, such as unique factorisation, solution of linear congruences, theorems on Euler's ϕ-function, combinations etc. needed for later sections. Section II, short, is related to Lambert's attempt at providing the essentials for a theory of integers and factoring but goes beyond it in depth and scope. Section III (the theoretical part, including Fermat's little theorem) and part of section VI (the application, art. 312–318) treat power residues (using Euler's term "primitive root") and decimal periods, referring to Lambert's, Bernoulli's and Euler's work. [132] Finally, the end of Section VI (art. 329–334) is concerned with factoring methods, taking up Lagrange's linear divisors of quadratic forms and Euler's ideoneal numbers (for which the relevant theorems are proven in section V). Putting all this together, one can observe a sub-focus of the *Disquitiones* on factoring and other topics that gained prominence during the years 1770–1800; this only becomes apparent in the light of the prehistory we have investigated above. It is of great use when one tries to put Gauss' work into a historical context. The *Disquisitiones*, which are often described as a book of wonders falling from the sky, then becomes a phenomenon that can be accounted for. It turns out to be the brilliant culmination of half a century of research which started from Euler's various texts and was actively stimulated by Lambert's project on factor tables and his appeal for a coherent and complete theory of numbers in general, and in particular of factoring.

5. CONCLUSIONS

19th century mathematics was marked by a rapidly growing professionalisation, due to the growing importance (and sometimes introduction)

---

132   For more detail on this part, see [Bullynck 2009b].

of mathematics in the university curriculum and the foundation of mathematical seminars. [133] As a consequence, in harmony with the industrial idea of division of labour, table-making became a rather mechanical job, for reckoners and less talented mathematicians, and often got separated from academic mathematics. A prime example of this was de Prony's logarithmic and trigonometric table project in Paris, where the job was divided between Legendre, who set up the formulae, and jobless haircutters, who calculated additions and subtractions [Grattan-Guinness 1990b]. The factor tables of Burckhardt, calculated in his 'spare time', and those of the reckoning prodigy Zacharias Dase were also typical expressions of this tendency. The reports of the British Committee on Mathematical Tables [Cayley 1875/1876; Glaisher 1873/1874] constituted the eventual outcome of this evolution. They listed all existing tables so as to produce the missing tables as efficiently as possible. Renowned mathematicians such as Cayley and Stokes pointed out the most urgent tasks, and J.W.L. Glaisher executed and/or commissioned the missing tables. [134]

But before 1800, the picture was different. Important mathematicians like Pell, Wallis, Lambert, and Euler spent quite some time on factor tables, producing, correcting or promoting them. The main medium for the organization and promotion of these factor tables was private correspondence, although scientific societies (Collins at the Royal Society, Lambert at the Berlin Academy) acted as catalysts in this scientific communication, and literary periodicals were also called into service in this regard . Due the expansion and vulgarization of written communication in the 18th century, Lambert could also use popular journals and book publications for the dissemination of his appeal, and could hope for non-professional, but well-educated amateurs to join in on his plans.

Apart from its importance for the production and layout of factor tables, Lambert's appeal also had a considerable impact on the birth of number theory. His scientific essays on tables and numbers put factoring on the academic agenda and pointed out that a coherent theory of numbers, free from gaps, was a scientific desideratum. Through Lambert's active

---

[133]   See e.g. [Jahnke 1990] for Germany, [Grattan-Guinness 1990a] for France.
[134]   For the history of the Committee and its dissolution in the 1930s which announced the computer era, see [Thompson 1949].

propagation, these questions spread not only in academic circles, inspiring contributions by Bernoulli, Béguelin, Lagrange, and Euler, but acquired an even wider public in the German states. The popular professors Kästner (Göttingen), Karsten, Klügel (Halle) and Hindenburg (Leipzig) often referred to Lambert's project in their textbooks and lectures, introducing a generation of university students to the problem of tables, factoring and some kind of "theory of numbers". This scientific project was one of the roots of Gauss' *Disquisitiones Arithmeticae*, which would ultimately create that missing discipline, the theory of numbers.

REFERENCES

[ADB]   *Allgemeine deutsche Biographie*, Leipzig: Historische Commission bei der Kgl. Akademie der Wissenschaften, 1875-1912; 56 volumes.

ARCHIBALD (Raymond Clare)
[1950]   New Information Concerning Isaac Wolfram's Life and Calculations, *Mathematical Tabels and Other Aids to Computation*, 4 (1950), pp. 185–200.

BACHET DE MÉZIRIAC (Claude-Gaspar)
[1621]   *Diophanti Alexandrini Arithmeticorum Libri sex et de Numeris Multangulis Liber unus*, Paris: Drouart, 1621.

BACHMANN (Paul)
[1911]   *Über Gauss' Zahlentheoretische Arbeiten*, 1911; in: [Gauss 1863–1929, volume X/2 (Ergänzungsband)].

BEELEY (Paul) & SCRIBA (Christoph J.), eds.
[2005]   *The correspondence of John Wallis*, vol. II 1666-Sept. 1668, Oxford: Oxford Univ. Press, 2005.

BEGUELIN (Niklaus von)
[1772]   Essais sur un Algorithme déduit du principe de la raison suffisante, *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, 1772, pp. 296–352.
[1775]   Solution particulière du Problème sur les nombres premiers, *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, XXV (1775), pp. 300–322.

[1777] Application de l'Algorithme exponentiel à la recherche des facteurs des nombres de la forme $2^n + 1$, *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, XXVII (1777), pp. 239–264.

BERNOULLI (Jean III)

[1771/1773] Sur les fractions décimales périodiques. Suivi de: Recherches sur les diviseurs de quelques nombres très grands compris dans la somme de la progression géométrique $1 + 10^1 + 10^2 + 10^3 + \cdots + 10^T = S$, *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, III (1771/1773), pp. 273–317 & 318–337.

[1781/1783] Extrait de la correspondance de M. Bernoulli, *Histoire de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, XIII (1781/1783), pp. 31–35.

BERNOULLI (Johann III), BODE (Johann Elert), LAGRANGE (Joseph Louis) & LAMBERT (Johann Heinrich), eds.

[1776] *Sammlung astronomischer Tafeln*, Berlin: Decker, 1776; 3 volumes.

BIERMANN (Kurt-Rolf)

[1985] Wurde Leonhard Euler durch J.H. Lambert aus Berlin vertrieben?, in *Festakt und wissenschaftliche Konferenz aus Anlaß des 200. Todestages von Leonhard Euler*, Berlin: Akademie-Verlag, 1985, pp. 91–99.

BISCHOFF (Johann Paul)

[1804/1990] *Versuch einer Geschichte der Rechenmaschine*, München: Systhema, 1804/1990.

BOPP (Karl)

[1916] Johann Heinrich Lamberts Monatsbuch mit den zugehörigen Kommentaren, sowie mit einem Vorwort über den Stand der Lambertforschung, in *Abhandlungen der Königlich Bayerischen Akademie der Wissenschaften Mathematisch-physikalische Klasse*, 1916, pp. 1–84.

BROSCHE (Peter)

[2001] *Der Astronom der Herzogin - Leben und Werk von Franz Xaver von Zach*, Frankfurt-am-Main: H. Deutsch, 2001.

BULLYNCK (Maarten)

[2008]   The transmission of numeracy: integrating reckoning in Protestant
          North-German elementary education, 1770-1810, *Paedagogica Histor-
          ica*, 44(5) (2008), pp. 563–585.

[2009a]   Modular Arithmetics before C.F. Gauss, *Historia Mathematica*, 36(1)
          (2009), pp. 48–72.

[2009b]   Decimal Periods and their Tables: A German Research Topic (1765-
          1801), *Historia Mathematica*, 36(2) (2009), pp. 137–160.

[2009c]   Leonhard Eulers Wege zur Zahlentheorie, in *Mathesis & Graphe. Leon-
          hard Euler zum 300. Geburtstag (W. Velminski and H. Bredekamp, eds.)*,
          Berlin: Akademie-Verlag, 2009, pp. 157–175.

BURCKHARDT (Johann Karl)

[1814]   *Table des diviseurs pour tous les nombres du deuxième million, ou plus exacte-
          ment depuis 1020000 à 2028000, avec les nombres premiers qui s'y trouvent,*
          Paris: Courcier, 1814.

CATALDI (Antonio)

[1603]   *Trattato de' numeri perfetti*, Bologna, 1603.

CAYLEY (George)

[1875/1876] Report of the Committee, consisting of Profs. Cayley, Stokes,
          Thomson, Smith and M. Glaisher on Mathematical Tables, in *Report
          of the Meeting of the Britisch Association for the Advancement of Science*,
          1875/1876, pp. 305–336; continuation of [Glaisher 1873/1874].

CHERNAC (Louis)

[1811]   *Cribrum arithmeticum sive Tabula continens numeros primos a compositis seg-
          regatos, occurrentes in serie numerorum ab unitate progredientium usque ad
          decies centena millia, et ultra haec, ad viginti millia, 102000 : numeris com-
          positis, per 2, 3, 5 non dividuis, adscripti sunt divisores simplices, non minimi
          tantum, sed omnino omnes*, Deventer: De Lange, 1811.

CLAUSBERG (Christlieb von)

[1732]   *Demonstrative Rechenkunst, oder Wissenschaft, gründlich und kurz zu rech-
          nen; Worinnen nicht nur sowohl die gemeinen, als allerhand vortheilhafte
          Rechnungs=arten überhaupt, nebst sehr compendiösen Proben, sondern auch
          die Wechsel=, Arbitragen= und andere kaufmännische Rechnungen auf eine*

sonderbare, kurze Manier gründlich und deutlich gelehret, anbey eine Beschrei-bung der Europäischen Münzen, Wechsel=Arten und Usanzen, auch Ver-gleichung der Gewichte und Ellen=Maasse; nicht weniger die wahre Berech-nung des Interusurii, wie auch unterschiedene andere Mathematische und curiöse Rechnungen; imgleichen eine Probe einer bus auf 32 Ziffern verfer-tigten neuen Logarithmischen Tabelle, Leipzig: auf Kosten des Autoris gedruckt, 1732.

COSTABEL (Pierre)

[1950]   Sur un problème de Roberval et un cas particulier d'analyse dio-phantienne, *Revue d'histoire des sciences et de leurs applications*, 3 (1950), pp. 80–86.

[1986]   Le Registre académique 'Journaux d'Angleterre', in *Costabel, Pierre (ed.), Mariotte, savant et philosophe, +1784. Analyse d'une renommée*, Paris: Vrin, 1986, pp. 321–325.

CRELLE (August Leopold)

[1853]   Wie eine Tafel der untheilbaren Factoren der Zahlen bis zu beliebiger Höhe möglichst leicht und sicher aufzustellen sei, *Journal für die reine und angewandte Mathematik*, 51(1) (1853), pp. 61–99.

DASE (Zacharias)

[1856]   *Aufschlüsse und Proben seiner Leistungen als Rechenkünstler. Mitgetheilt von ihm selbst aus seinem Album*, Berlin: Selbstverlag, 1856.

DESCARTES (René)

[1701]   Regulae ad directionem ingenii, in *Opuscula posthuma, physica et math-ematica*, Amsterdam: Blaev, 1701, pp. 1–66 (Folio 2).

DICKSON (Leonard Eugene)

[1919–1927]   *History of the Theory of Numbers*, Washington: Carnegie Institute, 1919–1927; 3 volumes.

DIDEROT (Denis) & D'ALEMBERT (Jean le Rond), eds.

[1751–1765]   *Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers*, Paris & Neufchâtel: Briasson & Samuel Faulche, 1751–1765; 17 vol-umes.

EULER (Leonhard)

[Opera] *Leonhardi Euleri Opera Omnia*, Basel: Birkhäuser; 84 volumes in 4 series, 1911-... Series I Opera mathematica, 30 vol.; series II Opera mechanica et astronomica (32 vol.); series III Opera physica, Miscelanea (12 vol.); series IVA Commercium epistolicum (10 vol.).

[1732/1738] Observationes de theoremate quodam Fermatiano aliisque ad numeros primos spectantibus, *Commentarii academiae scientiarum Petropolitanae*, 6 (1732/1738), pp. 103–107; [Euler 1907, I, pp. 1–3].

[1762/1763] De numeris primis valde magnis, *Novi commentarii academiae scientiarum Petropolitanae*, 9 (1762/1763), pp. 99–153; [Euler 1907, II, pp. 1–45].

[1772] *Disquitio accuratior circa residua ex divisione quadratorum altiorumque potestatum per numeros primos relicta*, 1772; [Euler 1907, II, pp. 513–543].

[1772/1774a] Demonstrationes circa residua ex divisione potestatum per numeros primos resultantia, *Novi Commentarii academiae scientiarum Petropolitanae*, 18 (1772/1774), pp. 85–135; [Euler 1907, I, pp. 516–537].

[1772/1774b] Extrait d'une lettre de M. Euler le père à M. Bernoulli, *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, IV (1772/1774), pp. 35–36; [Euler 1907, II, pp. 335–337].

[1774] *Élemens d'algèbre*, Lyon: Bruyset, 1774; translated and rearranged by Jean Bernoulli, with additions from J.L. Lagrange, 2 volumes.

[1774/1775] De tabula numerorum primorum usque ad millionem et ultra continuanda, in qua simul omnium numerorum non primorum minimi divisores exprimantur, *Novi commentarii academiae scientiarum Petropolitanae*, 19 (1774/1775), pp. 132–183; [Euler 1907, II, pp. 359–404].

[1775/1785] De insigni promotione scientiae numerorum, in *Opuscula Analytica*, vol. 2, Petersburg: Kais. Ak. d. Wiss., 1775/1785, pp. 275–314; [Euler, IV, pp. 163–196].

[1776/1779] Extrait d'une lettre de M. Euler le père à M. Beguelin en mai 1778, *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, 1776/1779, pp. 337–339; [Euler 1907, II, pp. 418–420].

[1907] *Commentationes Arithmeticae*, Leipzig, Berlin: Teubner, 1907; 2 volumes. Later volumen II and III in [Euler, Series I].

FELKEL (Anton)

[1776a] *Nachricht von einer Tafel, welche alle Factoren aller Zahlen von 1 bis 1 Million, dann einer andern, welche alle Factoren, aller durch 2, 3, 5 nicht theilbaren Zahlen enthält, von 1 bis 10 Millionen, als ein Vorbericht zur ausführlichen Beschreibung der neuen Factorentafeln und ihrer Berechnungsart,* Wien, 1776; reprinted in: [Lambert 1781–1787, V, pp. 85–105].

[1776b] *Tafel aller Einfachen Factoren der durch 2, 3, 5 nicht theilbaren Zahlen von 1 bis 10 000 000. I. Theil: Enthaltend die Factoren von 1 bis 144000,* Ghelen: Wien, 1776.

[1784] *Nachricht von den wiederauflebenden Felkelschen Factorentafeln; den Abdruck der ersten Million u.f. und die Fortsetzung der Berechnung (von 2016001 an) bis 10 Millionen betreffend,* 1784; reprinted in [Lambert 1781–1787, V, pp. 123–131].

[1785] Verwandlung der Bruchperioden nach den Gesetzen verschiedener Zahlsysteme, *Abhandlungen der Böhmischen Gesellschaft der Wissenschaften,* 1 (1785), pp. 135–174.

[1798] *J.H. Lambert supplementa tabularum logarithmicarum et trigonometricarum auspiciis almae academiae regiae scientiarum Olisiponensis cum versione introductionis Germanicae in Latinum sermonem, secundum ultima auctoris consilia amplificata,* Lisbon: Typographie regiae academiae scientiarum, 1798.

FUSS (Nikolaus)

[1776/1779] Extrait d'une lettre de M. Fuss à M. Beguelin écrite de Pétersbourg le 19–20 juin 1778, *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin,* 1776/1779, pp. 340–346.

FUSS (Paul-Heinrich), ed.

[1843] *Correspondance mathématique et physique de quelques célèbres géomètres du XVIIIe siècle,* St. Petersburg: L'Académie impériale des sciences de Saint-Pétersbourg, 1843; 2 volumes.

GAUSS (Carl Friedrich)

[Werke] *Werke, 12 volumes,* Göttingen: Königliche Akademie der Wissenschaften, 1863–1929; Reprint Hildesheim, New York: Olms, 1973.

[1801] *Disquisitiones Arithmeticae,* Leipzig: Fleischer, 1801; Reprint Bruxelles: Culture et Civilisation (1968); [Gauss 1863–1929, I].

GERHARDT (Carl Immanuel), ed.

[1849–1863] *G.W. Leibniz: Mathematische Schriften*, Berlin and Halle: Schmidt, 1849–1863; 7 volumes. Reprint, Hildesheim: Olms, 1971.

GLAISHER (James Whitbread Lee)

[1873/1874] Report of the Committee, consisting of Profs. Cayley, Stokes, Thomson, Smith and M. Glaisher on Mathematical Tables, in *Report of the Meeting of the Britisch Association for the Advancement of Science*, 1873/1874, pp. 1–175; continued in: *Rep. Brit. Ass. for Adv. of Sc.* 1875/1876, 305–336.

[1878] On factor tables, with an account of the mode of formation of the factor table for the fourth million, *Proceedings of the Cambridge Philosophical Society*, 3 (1878), pp. 99–138.

GOLDSTEIN (Catherine)

[2001] L'expérience des nombres de Bernard Frenicle de Bessy, *Revue de synthèse*, 2-3-4 (2001), pp. 425–454.

GOLDSTEIN (Catherine) & SCHAPPACHER (Norbert)

[2007] *A Book in Search of a Discipline, 1801–1860*; in: [Goldstein et al. 2007, pp. 3–66].

GOLDSTEIN (Catherine), SCHAPPACHER (Norbert) & SCHWERMER (Joachim), eds.

[2007] *The Shaping of Arithmetic – After C.F. Gauss' Disquisitiones Arithmeticae*, Berlin, Heidelberg: Springer, 2007.

GRATTAN-GUINNESS (Ivor)

[1990a] *Convolutions in French mathematics 1800-1840: from the calculus and mechanics to mathematical analysis and mathematical physics, volume 1: The settings*, Basel: Birkhauser, 1990 (= Science networks historical studies 2).

[1990b] Work for the Hairdressers: The Production of de Prony's Logarithmic and Trigonometric Tables, *IEEE Annals for the History of Computing*, 12 (1990), pp. 177–185.

GULDIN (Paul)

[1635] *De Centri Gravitatis Inventione. Appendicis Loco, accesserunt Tabulae Numerorum Quadratorum & Cubicorum; Earundemq[ue] Fabrica & Usus*, Vienna: Gelbhaar, 1635 (= *Centrobaryca* Volume I).

[1640]   *De Usu Centri Gravitatis binarum specierum Quantitatis continuae; sive de Compositione et Resolutione Potestatum Rotundarum*, Vienna: Cosmoverius, 1640 (= *Centrobaryca* Volume II).

[1641]   *De Gloria, Ab Usu Centri Gravitatis binarum specierum Quantitatis continuae parta, sive Archimedes Illustratus*, Vienna: Cosmoverius, 1641 (= *Centrobaryca* Volume IV).

HARRIS (John)

[1707 & 1710] *Lexicon Technicum: Or, An Universal English Dictionary of Arts and Sciences*, London: Brown, Goodwin et al., 1707 & 1710; 2 volumes.

HAUSEN (Christian August)

[1715]   *De veteri et nova Geometria ad I. Maji A.C. MDCCXV. capienda Brevi Consideratione de Matheseos Semiotica*, Leipzig: Takkus, 1715.

HINDENBURG (Carl Friedrich)

[1776a]   *Beschreibung einer ganz neuen Art, nach einem bekannten Gesetze fortgehende Zahlen, durch Abzählen oder Abmessen bequem und sicher zu finden, nebst Anwendung der Methode auf verschiedene Zahlen, besonders auf eine darnach zu fertigende Factorentafel, mit eingestreuten, die Zahlenberechnung überhaupt betreffenden Anmerkungen*, Leipzig: Crusius, 1776.

[1776b]   *Nachricht von einer Tafel der kleinsten Theiler aller durch 2, 3, 5, nicht theilbaren, zwischen 1 und 5 Millionen fallenden zusammengesetzten Zahlen, nebst beygefügter Primtafel auf so weit (durch den Herausgeber Crusius vermittelt)*, 1776; reprinted in: [Lambert 1781–1787, V, 142–150].

[1781]   *Novi Systematis Permutationem Combinationum ac Variationum Primas Lineas et Logisticae Serierum Formulis Analytico-Combinatoriis per Tabulas Exhibendas Conspectum*, Leipzig: Breitkopf (short version) & Crusius (long version), 1781.

[1786]   Verbindungsgesetz cyklischer Perioden; Natur und Eigenschaften derselben; Ihr Gebrauch in der diophantischen oder unbestimmten Analytik, *Leipziger Magazin für reine und angewandte Mathematik*, 1(3) (1786), pp. 281–324.

[1795a]   Combinatorische Verfahren, zu Bestimmung der Werthe der continuirlichen Brüche, in und außer der Ordnung, *Archiv der reinen und angewandten Mathematik*, 1(1 & 2) (1795), pp. 47–68 & 154–195.

[1795b]   Historisch-kritisches Verzeichniß aller die combinatorische Analytik nahe oder entfernt angehenden Schriften, *Archiv der reinen und angewandten Mathematik*, 1(2) (1795), pp. 242–253 (continued in Heft 3).

HOFMANN (Joseph Ehrenfried) & COSTABEL (Pierre)

[1952]   A propos d'un problème de Roberval, *Revue d'histoire des sciences et de leurs applications*, 5 (1952), pp. 312–333.

HORSLEY (Samuel)

[1772]   Κόσκινον Ἐρατοσθένους or The Sieve of Eratosthenes. Being an account of his method of finding all the Prime Numbers, *Philosophical Transactions of the Royal Society*, 62 (1772), pp. 327–347.

HØYRUP (Jens)

[2008]   Baroque Mind-set and New Science a Dialectic of Seventeenth-Century High Culture, 2008; Max-Planck-Institut für Wissenschaftsgeschichte preprint 359.

JAHNKE (Hans-Niels)

[1990]   *Mathematik und Bildung in der Humboldtschen Bildungsreform*, Göttingen: Vandenhoeck & Ruprecht, 1990.

JOHNSON (William Woolsey)

[1884]   Mr. James Glaisher's Factor Tables and the Distribution of Primes, *Annals of Mathematics*, 1(1) (1884), pp. 15–23.

JUSKEVIC (Adolf P.), SMIRNOV (Vladimir I.) & HABICHT (Walter), eds.

[1975]   *Leonhardi Euleri Commercium Epistolicum. Descriptio commercii epistolici*, Basel: Birkhäuser, 1975; [Euler, Series 4, vol. I].

KÄSTNER (Abraham Gotthelf)

[1758]   *Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie und Perspectiv*, Göttingen: Vandenhoeck, 1758 (= *Anfangsgründe der Mathematik*, Theil I, Abth. 1).

[1786]   *Fortsetzung der Rechenkunst in Anwendungen auf mancherley Geschäffte*, Göttingen: Vandenhoek & Ruprecht, 1786 (= *Anfangsgründe der Mathematik*, Theil I, Abth. 2).

KNOBLOCH (Eberhard)

[1998]   *Zeichenkonzeptionen in der Mathematik von der Renaissance bis zum frühen 19. Jahrhundert*, 1998; in [Posner et al. 1998, pp. 1280–1292].

KRAFFT (Georg Wolfgang)

[1751/1753] De divisoribus numerorum indagandis, *Novi commentarii academiae scientiarum Petropolitanae*, 3 (1751/1753), pp. 109–124.

KRAUSE (Karl Christian Friedrich)

[1804] *Factoren- und Primzahlentafeln, von 1 bis 100000 neuberechnet und zweck-mässig eingerichtet, nebst einer Gebrauchsanleitung und Abhandlung der Lehre von Factoren und Primzahlen, worin diese Lehre nach einer neuen Methode abgehandelt, und die Frage über das Gesetz der Primzahlenreihe entschieden ist*, Jena and Leipzig: Gabler and Cnobloch, 1804.

KRÜGER (Johann Gottlob)

[1746] *Gedanken von der Algebra, nebst den Primzahlen von 1 bis 100000*, Halle: Lüderwald, 1746.

KÜSSNER (Martha)

[1979] *Carl Friedrich Gauß und seine Welt der Bücher*, Göttingen: Musterschmidt, 1979.

LAGRANGE (Joseph Louis)

[Œuvres] *Œuvres*, Paris: Gauthier-Villars; 14 volumes, edited by Serret, M.J.-A., 1867-1892. Reprint, Hildesheim, New York: Olms Verlag, 1973.

[1773 and 1775] Recherches arithmétiques, *Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin*, 1773 and 1775, pp. 265–312 (1773) and pp. 323–356 (1775); [Lagrange, III, 696–795].

LAMBERT (Johann Heinrich)

[1764] *Neues Organon oder Gedanken über die Erforschung und Bezeichnung des Wahren und dessen Unterscheidung vom Irrtum und Schein*, Leipzig: Johann Wendler, 1764.

[1765-1772] *Beyträge zum Gebrauche der Mathematik und deren Anwendung*, Berlin: Buchhandlung der Realschule, 1765-1772; 3 Teile, davon 2. Teil in 2 Abschnitten, 1765, 1770, 1772.

[1768] De Topicis Schediasma, *Nova Acta Eruditorum*, XVIII (1768), pp. 12–33.

[1769] Adnotata quaedam de numeris, eorumque anatomia, *Nova Acta Eruditorum*, LXIX (1769), pp. 107–128.

[1770] *Zusätze zu den Logarithmischen und Trigonometrischen Tabellen*, Berlin: Spener'sche Buchhandlung, 1770.

[1771] *Anlage zur Architectonic, oder Theorie des Ersten und des Einfachen in der philosophischen und mathematischen Erkenntniß*, Riga: Hartknoch, 1771.

[1778] Hindenburg: Beschreibung & Felkel: Tafel (review), *Allgemeine deutsche Bibliothek*, 33(2) (1778), pp. 494–497.

[Briefe]   *Deutscher gelehrter Briefwechsel*, edited by Johann III Bernoulli, Berlin: Selbstverlag, 1781–1787; 5 volumes.

LEGENDRE (Adrien-Marie)

[1785]   Recherches d'analyse indéterminée, *Mémoires de l'Academie des Sciences*, 88 (1785), pp. 465–559.

[1798]   *Essai sur la Théorie des nombres*, Paris: Courcier, 1798; 2nd revised edition 1808, 3rd revised edition, as [Legendre 1830].

[1830]   *Théorie des Nombres*, Paris, 1830 (= 3rd edition of [Legendre 1798]).

LEHMER (Derrick Norman)

[1909]   *Factor table for the first ten millions containing the smallest factor of every number not divisible by 2, 3, 5, or 7 between the limits 0 and 10017000*, Washington D.C.: Carnegie Institution of Washington, 1909.

LEHMER (Derrick Henry)

[1969]   Computer Technology applied to the Theory of Numbers, in *W. Le-Veque (ed.)*, Studies in Number Theory, Studies in Mathematics Mathematical Association of America, distributed by Prentice-Hall, 1969, pp. 117–151.

LEIBNIZ (Gottfried Wilhelm)

[1678]   Extrait d'une lettre écrite d'Hanovre par M. de Leibniz à l'auteur du Iournal, contenant une observation nouvelle de la maniere d'essayer si un nombre est primitif, *Journal des Scavans*, 1678, pp. 75–76.

[1686a]   John Wallis, A Treatise of Algebra, both historical and practical, Review, *Acta Eruditorum*, 1686, pp. 283–289.

[1686b]   Isaac Newton, Arithmetica universalis; sive De compositione et resolutione arithmetica liber, Review, *Acta Eruditorum*, 1686, pp. 519–526.

[1703/1720]   Explication de l'Arithmétique binaire, qui se sert des seuls caractères 0 et 1, avec des remarques sur son utilité, et sur ce qu'elle donne le sens des anciennes figures Chinoises de Fohy, *Mémoires de l'Academie des Sciences*, 1703/1720, pp. 85–90; [Gerhardt 1849–1863, VII, 223–228].

[1923–2006]   *Werke (Akademie-Ausgabe)*, Darmstadt & Berlin: Berlin-Brandenburgische Akademie der Wissenschaften und der Akademie der Wissenschaften in Göttingen, 1923–2006; 5 series and many volumes.

LEMMERMEYER (Franz)

[2007]   *The Development of the Principal Genus Theorem*, 2007; [Goldstein et al. 2007, pp. 529–562].

LEUPOLD (Johann)

[1727]   *Theatrum arithmetico-geometricum, das ist: Schau-Platz der Rechen- und Meßkunst*, Leipzig: Gleditsch, 1727.

LOCKE (John)

[1690]   *An Essay concerning humane understanding*, London: Th. Basset, 1690.

MAENNCHEN (Paul)

[1928]   *Methodik des mathematischen Unterrichts*, Frankfurt-am-Main: Diester-weg, 1928 (= Handbuch des Unterrichts an höheren Schulen zur Ein-führung und Weiterbildung in Einzeldarstellungen 13).

MAHNKE (Dieter)

[1912/13] Leibniz auf der Suche nach einer allgemeinen Primzahlgleichung, *Bibliotheca Mathematica (3. Folge)*, 13 (1912/13), pp. 29–63.

MALCOLM (Noel)

[2000]   The Publications of John Pell, F.R.S. (1611-1685): Some new light and some old confusions, *Notes and Records of the Royal Society of London*, 54 (3) (2000), pp. 275–292.

[2004]   An Unpublished Letter from Henry Oldenburg to Johann Heinrich Rahn, *Notes and Records of the Royal Society of London*, 58(3) (2004), pp. 249–266.

MALCOLM (Noel) & STEDALL (Jacqueline)

[2005]   *John Pell (1611–1685) and his correspondence with Sir Charles Cavendish: the mental world of an early modern mathematician*, Oxford: Oxford Univ. Press, 2005.

MEIER-OESER (Stephan)

[1998]   *Zeichenkonzeptionen in der Allgemeinen Philosophie von der Renaissance bis zum frühen 19. Jahrhundert*, 1998; in [Posner et al. 1998, pp. 1199–1232].

MERZBACH (Uta)

[1981]   An early version of Gauss's Disquisitiones Arithmeticae, in *Dauben, Joseph (ed.),* Mathematical Perspectives. Essays on Mathematics and its Historical Development, New York: Academic Press, 1981, pp. 167–178.

MOLLIN (Richard A.)

[2002]   A Brief History of Factoring and Primality Testing B.C., Before Computers, *Mathematics Magazine,* 75(1) (2002), pp. 18–29.

NEUMANN (Johann)

[1785]   *Tabellen der Prim-Zahlen und der Faktoren der Zahlen, welche unter 100100, und durch 2, 3, oder 5 nicht theilbar sind,* Dessau, 1785.

OZANAM (Jacques)

[1697]   *Récreations mathématiques et physiques, qui contiennent plusieurs problêmes d'arithmétique, de géometrie, d'optique, de gnomonique, de cosmographie, de mécanique, de pyrotechnie, & de physique. Avec un traité nouveau des horloges élementaires,* Jean Jombert: Paris, 1697.

PELL (John)

[1638/1650] *An Idea of Mathematicks,* London: Printed by William Du-Gard, 1638/1650 (first published 1638).

[1672]   *Tabula Numerorum Quadratorum Decies millium, una cum ipsorum Lateribus ab Unitate incipientibus ordine naturali usque as 10000 progredientibus,* London: Printed by Thomas Ratcliffe and Nath. Thompson, and are to be sold by Moses Pitt at the White Hart in Little Britain, 1672.

PISANO (Leonardo)

[1202]   *Liber Abaci,* 1202; translated by Sigler (L.E.) [2002], *Fibonacci's Liber Abaci. Leonardo Pisano's Book of Calculation,* New York, Berlin: Springer.

POETIUS (Johann Michael)

[1728 & 1738] *Anleitung zu der unter den Gelehrten jetzt üblichen arithmetischen Wissenschaft, vermittelst einer parallelen Algebra, worin zugleich vom Gebrauch ihrer Rechnungen in Facultäten und Disciplinen auch von unterschiedlichen arithmetischen Instrumenten und Maschinen gehandelt wird, samt einer Anatomia Numerorum von 1–10000,* Franckfurt & Leipzig (1728); Halle (1738): Fritz'sche Buchhandlung, 1728 & 1738.

POSER (Hans), ed.

[1987] *Briefwechsel zwischen Carl Friedrich Gauß und Eberhard August Wilhelm von Zimmermann*, Göttingen: Vandenhoeck & Ruprecht, 1987.

POSNER (Ronald), ROBERING (Klaus) & SEBEOK (Thomas A.), eds.

[1998] *Semiotik/Semiotics, Ein Handbuch zu den zeichentheoretischen Grundlagen von Natur und Kultur, A Handbook on the Sign-Theoretic Foundations of Nature and Culture*, Berlin - New York: de Gruyter, 1998; 2nd Teilband.

PYCIOR (Helena)

[1997] *Symbols, impossible numbers, and geometric entanglements: British algebra through the commentaries on Newton's* Universal arithmetick, Cambridge: Cambridge Univ. Press, 1997.

RAHN (Johann Heinrich)

[1659] *Teutsche Algebra oder Algebraische Rechenkunst zusamt ihrem Gebrauch*, Zürich: Johann Jakob Bodmer, 1659.

[1668] *An Introduction to Algebra Translated out of the High-Dutch into English by Thomas Brancker. Much Altered and Augmented by D. P[ell] Also a Table of Odd Numbers less than One Hundred Thousand, shewing those that are Incomposit and Resolving the rest into their Factors or Coefficients, &c.*, London: W.G. for Moses Pitt at the White-Hart in Little Britain, 1668.

RALLIER DES OURMES (Jean Joseph)

[1768] Méthode facile pour découvrir tous les nombres premiers contenus dans un cours illimité de la suite des impairs et tout d'un temps les diviseurs simples de ceux, qui ne le sont pas, *Mémoires de mathématique et de physique, présentés à l'Académie royale des sciences par divers sçavans*, 5 (1768), pp. 485–499.

REMER (Christian Stephan)

[1739] *Demonstrativische Anweisung zur Rechen=Kunst für diejenigen, in derselben den rechten Grund legen wollen, und welche im gemeinen Leben unentbehrlich*, Braunschweig: Schröder, 2nd edition, 1739.

RIEDEL (Ehrenfried)

[1941] *Das Gesetz der Primzahlenreihe. M. Hinweis auf Karl Chr. F. Krauses Dissertation Jena 1801: De inventione numerorum primorum … Nebst einem Feldpostbrief an den Verf. als Nachw. des Hrsg. und dem Anhang: Das Auftauchen*

der Primzahlfrage in einem pestalozzischen Rechenlehrgang der ersten beiden Grundschuljahre 1928-31, Dresden: Riedel, 1941.

SCHOOTEN (Frans van)

[1657]   *Exercitationes Mathematicae*, Leiden: Elsevier, 1657.

[1659]   *Mathematische oeffeningen, begrepen in vijf boecken*, Amsterdam: Gerrit van Goedesbergh, 1659.

SCHULZE (Johann Carl)

[1778]   *Neue und Erweiterte Sammlung logarithmischer, trigonometrischer und anderer zum Gebrauch der Mathematik unentbehrlicher Tafeln*, Berlin: Mylius, 1778; 2 volumes.

SCRIBA (Christoph J.)

[1974]   John Pell's English Edition of J.H. Rahn's *Teutsche Algebra*, in *Cohen, R., Stachel, J. and Wartofsky, M. (eds.),* Scientific, Historical and Political Essays in Honor of Dirk J. Struik, Dordrecht: D. Reidel, 1974, pp. 261–274 (= Boston Studies in the Philosophy of Science, 15).

SHANKS (Daniel)

[1993]   *Solved and unsolved problems in number theory*, New York: Chelsea, 1993; 1st ed., Spartan Books, Washington, DC, 1962.

STEDALL (Jacqueline)

[2002]   *A Discourse concerning algebra: English algebra to 1685*, Oxford: Oxford Univ. Press, 2002.

STEINIG (John)

[1966]   On Euler's ideoneal numbers, *Elemente der Mathematik*, 21 (1966), pp. 73–88.

STIFEL (Michael)

[1553]   *Die Coss Christoffe Rudolffs mit schönen Exempeln der Coss*, Königsberg: Alexandrus Lutomyslensis, 1553.

THOMPSON (Alexander John)

[1949]   The BAASMTC now RSTMC (containing: Final Report of Committee on Calculation of Mathematical Tables), *Mathematical Tables and Other Aids to Computation*, 3(25) (1949), pp. 333–340.

Vega (Georg von)

[1797] *Vega's logarithmisch-trigonometrische Tafeln, nebst andern zum Gebrauch der Mathematik eingerichteten Tafeln und Formeln*, Leipzig: Weidmann, 1797; 2 volumes.

Verdonk (J.J.)

[1966] *Petrus Ramus en de Wiskunde*, Assen: Van Gorcum, 1966.

Vleeschauwer (Herman Jan De)

[1932] La genèse de la méthode mathématique de Wolff. Contribution à l'histoire des idées au XVIIIe siècle, *Revue Belge de philosophie et d'histoire*, 11 (1932), pp. 651–677.

Vossius (Gerard Johan)

[1650] *De Quatuor Artibus Popularibus, de Philologi et Scientiis Mathematicis, cui operi subjuingitur, Chronologia Mathematicorum, Libri Tres*, Amsterdam: Blaeu, 1650.

Wallis (John)

[1657] *Mathesis Universalis, sive, Arithmeticum opus Integrum, tum Numerosam Arithmeticam tum Speciosam complectens*, London: Robinson, 1657.

Wallis (John), ed.

[1658] *Commercium epistolicum de quaestionibus quibusdam mathematicis nuper habitum*, Oxford: printed by Lichfeld for Robinson, 1658.

Wallis (John)

[1685a] *A Discourse of Combinations, Alternations and Aliquot Parts*, Oxford: printed by John Playford, for Richard Davis, 1685.

[1685b] *A Treatise of Algebra, both historical and practical. Shewing, the original, progress and advancement thereof, from time to time, and by what Steps it hath attained to the Heighth at which now it is*, London: printed by John Playford, for Richard Davis, 1685.

[1693] *De algebra tractatus; historicus & practicus, cum variis appendicibus*, Oxford: Sheldon, 1693.

Weigel (Erhard)

[1672] *Tetractys Summum tum Arithmeticae tum Philosophiae discursivae compendium, Artis magnae sciendi genuina radix*, Jena: Werther, 1672.

WEIL (André)
    [1984]   *Number Theory: An Approach through History from Hammurapi to Legendre*,
              Boston: Birkhäuser, 1984.

WILLIAMS (Hugh) & SHALLIT (Jeffrey)
    [1994]   Factoring integers before computers, in *Gautschi, Walter (ed.),* Mathe-
              matics of Computation 1943–1993: Half a Century of Computational
              Mathematics, Providence: AMS, 1994, pp. 481–531 (= Proceedings of
              the Symposium on Applied Mathematics, 48).

WOLFF (Christian)
    [1713]   *Elementa Matheseos Universae. Tomus I qui Commentatione de Methodo
              Mathematica, Arithmeticam, Geometriam, Trigonometriam planam et
              Analysin tam Finitorum, quam Infinitorum complectitur,* Halle: Renger,
              1713 (the edition consulted is of 1732).
    [1717]   *Auszug aus den Anfangs=Gründen aller mathematischen Wissenschaften,*
              Halle: Renger, 1717 (the edition consulted is the 2nd of 1724).

WOLFF (Christian) & RICHTER (Georg Friedrich)
    [1734/1742] *Vollständiges Mathematisches Lexicon, Darinnen alle Kunst=Wörter und
              Sachen, welche in der erwegenden und ausübenden Mathesi vor zu kommen
              pflegen… Überall aber zur Historie der Mathematischen Wissenschaften dien-
              liche Nachrichten eingestreuet,* Leipzig: Gleditsch, 1734/1742; 3 parts in 2
              volumes. 1st Volume, 1st edition by Wolff, 1716; 2nd and changed edi-
              tion: 1st Volume (1734) and 2nd Volume in 2 Parts (1742) by Richter.

YATES (Frances)
    [1972]   *The Rosicrucian Enlightment*, London, New York: Routledge, 1972;
              reprint 2004.

ZACHER (Hans-Joachim)
    [1973]   *Die Hauptschriften zur Dyadik von G. W. Leibniz: ein Beitrag zur Geschichte
              des binären Zahlensystems,* Frankfurt-am-Main: Klostermann, 1973 (=
              Veröffentlichungen des Leibniz-Archivs, 5).