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LAGRANGE'S ESSAY "RECHERCHES SUR LA MANIÈRE DE FORMER DES TABLES DES PLANÈTES D'APRÈS LES SEULES OBSERVATIONS"

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ABSTRACT. — The memoir presented by Lagrange, which this paper examines, is usually considered as an elegant, but scarcely practicable, contribution to numerical analysis. The purpose of this study is to show the significance of the novel mathematical ideas it contains, and in particular to look at this essay from the perspective of generating function theory, for which the theoretical foundations would be laid some little time later by Laplace. This excursus of Lagrange's does indeed proffer an abundance of procedures that were to become standard in this latter theory.

Further, Lagrange's memoir introduces some quite extraordinary elements, e.g. an algorithm for the approximation of an integral series by means of rational fractions — quite analogous in some cases to the determination of Padé approximants; or the introduction of polynomials formally akin to Chebyshev polynomials, to cater for tasks that would only devolve to the latter in the 20th century.

RÉSUMÉ. — L'ESSAI DE LAGRANGE «RECHERCHES SUR LA MANIÈRE DE FORMER DES TABLES DES PLANÈTES D'APRÈS LES SEULES OBSERVATIONS». Le mémoire de Lagrange étudié dans cet article est généralement considéré comme une contribution à l'analyse numérique, élégante mais difficilement utilisable. Mon propos est de montrer l'importance des idées mathématiques novatrices qu'il recèle et notamment de replacer cet essai dans la perspective de la théorie des fonctions génératrices fondée un peu plus tard par Laplace. Ce texte de Lagrange s'avère en effet riche en procédures qui deviendront ensuite standard dans cette théorie.

On y trouve de plus des éléments assez étonnants comme un algorithme d'approximation d'une série entière au moyen de fractions rationnelles, très semblable dans certains cas au calcul des approximants de Padé, ou l'introduction de polynômes qui, formellement, correspondent aux polynômes de Chebyshev, dans un rôle qui leur sera dévolu seulement au XX^{e} siècle.

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1. INTRODUCTION

The aim of this paper is to provide a thorough analysis of this essay of Lagrange. Contrary to the opinion sometimes expressed by other historians or mathematicians, I believe it to be a very important piece of mathematics. Since the structure of Lagrange's essay is rather complex, and many topics are discussed, I will list here four points of major interest I shall be considering.

Lagrange's essay contains many results which might be defined as belonging to generating function theory. I shall be dealing mainly with this subject in my paper, and consequently I shall set out in this introduction a brief outline history of the origin of that theory.

A well-defined philosophical and methodological point of view concerning the best way of dealing with experimental data is expressed by Lagrange. I will also point out this aspect in my introduction.

Lagrange's essay further has the practical purpose of producing astronomical tables. It has not proved very successful in this respect. I shall refer in this introduction to some historical opinions on the matter, as a topic to be discussed more specifically in the development of my paper.

Some mathematical results in this essay are "ahead of their time". I am well aware of how sensitive a theme the matter of forerunners is. But we shall encounter an algorithm which coincides with the attempt to calculate the Padé approximants of a given series; and a class of polynomials which (formally) coincides with Chebyshev polynomials. Be that as it may, we may not preclude an investigation of these results, for fear of speaking of "forerunners".

1.1. Generating functions

Generating function theory is nowadays a well-established part of discrete mathematics.[1] This is a particularly fascinating field, since, in spite of its richness in terms of deep mathematical results, such is the power and simplicity of its fundamental idea that its development follows a clear and intelligible pattern. We have no need to face technical complications, which sometimes make a mathematical theory hard reading.

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[1] First originating in strict connection with probability theory, this may now be considered to be closer to combinatorics, and bound to probability theory only inasmuch as combinatorics and probability theory overlap.

This is the main idea: suppose we have a sequence of numbers

$$d_0, d_1, d_2, \dots \tag{1}$$

and we want to know something about it.

For example, we may ask about the possibility of obtaining a simple formula, dependent on n, to express every d_n. Or we wonder whether there exists some relation which connects a generical d_n with the ones preceding it in the sequence.

Now, to use Wilf's words: "*Although giving a simple formula for the members of the sequence may be out of the question, we might be able to give a simple formula for* the sum of a power series, whose coefficients are the sequence that we're looking for" [Wilf 1990/1994, p. 1].

That is, the power series

$$\sum_{n=0}^{\infty} d_n x^n \tag{2}$$

may be a simpler object to deal with than the sequence itself.

In some cases it is possible to construct a simple closed form for the function expressed by series (2) even though sequence (1) may look rather complicated. In what follows we shall encounter plenty of examples of such a situation.

A function $f(x)$ such that $f(x) = \sum_{n=0}^{\infty} d_n x^n$ is said to be the *generating function* of the sequence.[2]

The "birth" of the theory of generating functions is usually (and rightly) attributed to Laplace, on account of his celebrated "Mémoire sur les suites" [Laplace 1782]. As might be expected, Laplace did not start the theory from scratch. On the contrary (and without any wish to deny the great originality of Laplace's work), a great merit of Laplace's "Mémoire" is the very fact that it organises and collects many previously-obtained results, as well as his own personal achievements, into a theoretical unity.

All mathematicians knew about geometrical series, at least from the mid-seventeenth century. But such is the triviality of the sequence

[2] Within generating function theory one looks at a generating function as an algorithm that produces, by the rules of differentiation, the terms of a given series, with no concern about convergence.

$1, 1, 1, \dots$ that no-one could be interested in looking at the correspondence $(1, 1, 1, \dots) \mapsto (1-x)^{-1}$. This was simply interpreted as an identity $1 + x + x^2 + \cdots = (1-x)^{-1}$.

It is the theory of recurrent series,[3] which goes back to de Moivre, that yielded the first fundamental results. A *recurrent series* is, by definition, a sequence such as (1), such that every term is a finite linear combination, with constant coefficients, of the same number of terms immediately preceding it.

Therefore a recurrent series is given by a sequence $d_0, d_1, d_2, \dots$, such that we may express a generical d_i as

$$d_i = \lambda_1 d_{i-1} + \lambda_2 d_{i-2} + \cdots + \lambda_n d_{i-n}, \tag{3}$$

where $\lambda_1, \dots, \lambda_n$ are constants (where $i \geq n$ and the first n terms are given). De Moivre called these constants *scale of relation* (or *index*) [1718/1756, p. 221], a terminology subsequently adopted by all authors in this field.

The remarkable discovery made by de Moivre (see, for example [1718/1756, pp. 220–222]) is that recurrent series precisely correspond to rational functions.

Let a rational function $P(x)/Q(x)$ be given, such that $Q(0) \neq 0$, and that the degree of P is less[4] than the degree of Q. Consider its Taylor development $d_0 + d_1 x + d_2 x^2 + \cdots = P(x)/Q(x)$, and let

$$Q(x) = c_0 + c_1 x + \cdots + c_n x^n.$$

On account of $P(x) = (d_0 + d_1 x + d_2 x^2 + \cdots)(c_0 + c_1 x + \cdots + c_n x^n)$, for $i > n$, we have

$$0 = \sum_{k=0}^{n} c_k d_{i-k}. \tag{4}$$

Since $c_0 \neq 0$ we may solve (4) with respect to d_i. We deduce

$$d_i = -\frac{c_1}{c_0} d_{i-1} - \frac{c_2}{c_0} d_{i-2} - \cdots - \frac{c_n}{c_0} d_{i-n}\,,$$

[3] *Recurring series*, in de Moivre's original terminology [1718/1756, p. 220].

[4] Let $f = P/Q$. If the degree of P is higher than the degree of Q we may effect the division, to yield $f = q + R/Q$. It is clear that the series development of f coincides with that of R/Q after a finite number of terms have been modified by the polynomial q. We may restrict ourselves to the hypothesis given without any loss of generality.

and we have proved that the $d_0, d_1, d_2, \ldots$ constitute a recurrent series. By the very structure of the proof it is clear that, conversely, if we have a sequence whose generical term verifies equation (3), we can form the polynomial $Q(x) = 1 - \lambda_1 x - \cdots - \lambda_n x^n$. The choice of an arbitrary polynomial $P(x)$ of degree lower than n, corresponds exactly to the choice of the original arbitrary initial values $d_0, d_1, \ldots, d_{n-1}$.

In the first book of Euler's *Introductio*, recurrent series take up a lot of space. Chapter XIII is expressly devoted to them. In chapter XVII we find a brilliant use of recurrent series (originally expounded by Daniel Bernoulli, as Euler notes) to approximate the roots of equations.

But what is of special importance is that, by using the results about the resolution of rational functions into partial fractions described in chapter II, Euler is able to express the general term of a recurrent series, whose generating function is $P(x)/Q(x)$, by the help of the roots of the polynomial $Q(x)$.

As may be expected, Lagrange, who throughout his work made extensive use of power series — to such effect that, at the close of his career, he could consider the whole of calculus to be a theory of power series —, made some outstanding contributions to the building up of the theory of generating functions.

Two results at least must be mentioned. In a classic paper [Lagrange 1759] he arrived at a famous result, by explaining the relation between recurrent series and finite difference equations, considered in parallel with linear differential equations having constant coefficients.

Another remarkable result of Lagrange's, though it did not appear at the outset in the context of generating functions proper, is his celebrated inversion formula. Nowadays it is a fundamental tool of the theory[5] even though, initially, the formula appeared in the more restricted context of equations with literal coefficients [Lagrange 1770b, p. 25].[6] Since the inclusion of the Lagrange inversion formula into the theory of generating functions only took place subsequently, we shall dispense with dealing with it here.

[5] One need only refer to what is said by Wilf [1990/1994, chap. 5].

[6] The topic is considered more generally in the *Théorie des fonctions analytiques* (see [Lagrange 1797/1847, chap. XV]).

1.2. Astronomical tables and laws

The production of astronomical tables usually corresponds to the following mathematical situation: we have a sequence of data such as the one we abstractly considered previously $d_0, d_1, d_2, \ldots$ and we know that it comes from the evaluation of a function, having a prescribed form, at equally spaced values of a variable t, $t_0, t_1, t_2, \ldots$ For the sake of simplicity we may assume that these values are $0, 1, 2, \ldots$ We have some general information about the form of that function: it must be a (general) trigonometric polynomial such as

$$p(t) = \sum_{j=1}^{N} A_j \sin(a_j + \alpha_j t), \tag{5}$$

and consequently, what we are actually dealing with is a sequence, every term of which comes, so to speak, with information about its genesis[7]

$$d_n = p(n) = \sum_{j=1}^{N} A_j \sin(a_j + \alpha_j n). \tag{6}$$

What we want is to reconstruct function $p(t)$ from (6).

Since we do not even know *a priori* how many terms have to be added up to give $p(t)$, the task looks awesome. It is to overcome that difficulty that Lagrange introduces a "revolutionary" idea: we do *not* need to search directly for function $p(t)$. Instead we require to construct the *generating function* of the sequence (6).

From a modern standpoint this looks like substituting for the search for a function of a given type another investigation of the same nature. Which may be smart, one might say, but anything but astounding. This, however, was far from true in Lagrange's time.

Just to make one point, we have to observe that a "generating function" was not deemed a function at all in Lagrange's time: this is simply a procedure to establish a correspondence of a given number for every positive integer. But we have neither an analytical description (in general cases), nor have we a meaning for the evaluation at an arbitrary "real"

[7] Lagrange explains, at the beginning of his essay, how the inequalities in the movements of the planets are represented by functions of this kind [Lagrange 1775, pp. 507–511].

number. The very concept of a function mapping from $\mathbb{N}$ into $\mathbb{R}$ is lacking in the eighteenth century. Formula (6) indeed does apparently look like that, but what it means is simply that originally we have a function, as given by (5), but we restrict its evaluation to particular points.

The restriction to the search for a generating function of sequence (1), leaving in the background the problem of finding the function $p(t)$, in effect brings about a radical change in the point of view. Lagrange, as a rule so modest, this time emphasises the significance of what he is doing.

This is how he explains the general purpose of his research:

"*On s'occupe depuis longtemps à rechercher* a priori *les inégalités des mouvements des planètes d'après les principes de la gravitation universelle; mais personne, que je sache, n'a encore entrepris de donner des méthodes directes et générales pour trouver ces mêmes inégalités* a posteriori*, c'est-à-dire, d'après les observations seules. C'est à remplir ce dernier objet dans toute son étendue qu'est destiné le Mémoire que j'ai l'honneur de présenter à l'Académie Royale des Sciences*" [Lagrange 1775, p. 507].

The story of the publication of this paper of Lagrange's has been masterly told by Bru and Crépel in [Condorcet, *Arith. pol.*]. Such was the enthusiasm of d'Alembert and Condorcet, when Lagrange's paper reached Condorcet at the end of 1773, that it was decided to publish it immediately in the 1772 volume of the *Mémoires de l'Academie des Sciences*, which was late.[8]

Bru and Crépel have also reproduced the *Compte rendu* Condorcet wrote about Lagrange's essay. In it, we find descriptions of the effect that Lagrange's work would have such as "*une carrière nouvelle et immense qu'il ouvroit aux mathématiciens*", or conclusions like: "*Il y a peu d'ouvrages d'analyse plus utiles aux progrès de la Physique, & plus propres à produire une révolution dans l'étude de cette Science*" [Concorcet, *Arith. pol.*, pp. 108–109].

It is possible, Bru and Crépel observe, that some kind of "politics" was at stake. But it is undeniable that the "proto-positivist" spirit of Lagrange's essay had really impressed the two encyclopedists.

Lagrange's essay is highly important for its philosophical attitude, but as it didn't achieve its practical aim, this very philosophical significance has been forgotten. I do not know of any modern author that considers it

[8] See also [Taton 1974].

from this point of view.

1.3. Historical judgements

As we recalled previously, Lagrange's essay had, after all, the practical purpose of improving the technique of producing astronomical tables. In this respect it cannot be considered a success. Here are some opinions on Lagrange's work.

Delambre, in his obituary of Lagrange, remembers it by these words:

"*Parmi ces jeux de son génie qui cherchait des difficultés pour mieux montrer sa force, se rangerait encore le Mémoire où il indique les moyens de construire les Tables astronomiques, d'après une suite d'observations, et sans connaître la loi des mouvements célestes. C'est le problème que résolvaient de tout temps les Astronomes, par les voies les plus élémentaires. Les moyens de Lagrange sont plus analytiques et plus savants; mais dans l'exemple même qu'il a choisi, et qui est des plus simples, il est permis de douter que les moyens qu'il emploie soient les plus sûrs et les plus faciles*" [Delambre 1867, p. xxx].

Burkhardt, who considers Lagrange's paper as a contribution to trigonometric (generalised) interpolation, in his brief (but extremely interesting) excursus on the history of trigonometric interpolation expresses the same appreciation:

"*The rapid progress of perturbation theory in these years is the reason why Lagrange's method has found no use. In a subsequent place* (Berl. astr. Jahrb. 1783 [80], Nr. 12; Œuvres 7, p. 547) *Lagrange observes that his method has never been used because the word continued fraction is unusual for the astronomers, who prefer to use linear equations*".[9]

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[9] "*Dass Lagrange's Verfahren keine praktische Anwendung fand, liegt wohl an den raschen Fortschritten der deduktiven Störungstheorie in jenen Jahren. An späterer Stelle ... stellt Lagrange sein Verfahren noch einmal dar, indem er das Wort Kettenbruch als den Astronomen ungewohnt vermeidet und nur von der Auflösung linearer Gleichungen redet*" [Burkhardt 1904, p. 675, footnote 127].

In brackets Burkhardt quotes both the first German translation of a paper read by Lagrange at the Berlin Academy of sciences on 3rd September 1778, and published in 1780, and the original French text reproduced in the *Œuvres* from Lagrange's papers in the Bibliothèque de l'Institut de France. He has erroneously indicated the page number in the *Œuvres* as 548. I have corrected it in the quotation. In what follows I will refer to this text as [Lagrange 1778]. Be that as it may, these are the words of Lagrange to which he implicitly refers: "*La méthode que j'ai donnée pour cet objet ... est peut-être ce qu'il y a de plus direct pour cette recherche; mais, comme cette méthode est*

A similar judgment is echoed in [Pearson 1978]. Pearson also interprets the subject of Lagrange's paper as a matter of generalised interpolation. By considering a series written in the form given by (6) he observes that "*Clearly Lagrange is supposing the series to consist of N Fourier's series ... What he wants to show is that there is a linear relation between successive terms, which may therefore be determined one from another*" [Pearson 1978, p. 619]. We will analyse in detail in the following paragraph what here is only hinted.

E.S. Pearson (the son of K. Pearson and editor of [Pearson 1978]) notes, in a comment on the same page, that Lagrange's results "*are used in his later works*". He does not give a precise reference, but it is clear that he is refering to [Lagrange 1778], the contents of which are partly analised at p. 622 of [Pearson 1978]. E.S. Pearson does not reproduce all the arguments of his father. He contents himself with adding a short note where he sums up his opinions: "*K.P. now discusses two methods proposed by Lagrange but concludes that while the theory is suggestive it is doubtful whether it will stand the touchstone of numerical practice*".

Goldstine also, in his book about the history of numerical analysis [Goldstine 1977] considers jointly the papers of 1775 with the previously-mentioned memoir of 1778 on interpolation.[10] As he considers the text of 1778 to be somewhat of "*a refinement and improvement on the first one*" [Goldstine 1977, p. 171] he pays, initially, little attention to the paper that is the subject of my essay. But a few pages further on he reconsiders his strategy, and observes: "*Up to this point, we have more or less avoided discussing recurring or recurrent series or sequences. Since they played such a large role in the work of Stirling, de Moivre, Euler, etc., and since they are inherently important, perhaps we should say more on the topic*" [Goldstine 1977, p. 176].

What follows, in Goldstine's book, after that somewhat underwhelming declaration, is a brief, clear analysis of Lagrange's 1775 essay, with no great deal of attention expended on some details that are, in the plan of

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fondée sur la théorie des fractions continues, qui n'est peut-être pas assez familière aux astronomes, nous allons en proposer une autre qui a l'avantage de ne demander que des opérations élémentaires" [Lagrange 1778, p. 547].

[10] In his bibliography, at the end of the book, Goldstine quotes also another German translation of [Lagrange 1778], that is Lagrange, *Mathematische Elementarvorlesungen*, Leipzig 1880.

my exposition, very important.

Goldstine's opinion, insofar as it does in part coincide with mine, is that the significance of Lagrange's essay lies more in the manipulations of power series he achieved than in the providing of new tools for astronomy. Where my opinion differs from Goldstine's is in the fact that I believe that Lagrange did not restrict himself to dealing with recurrent series but went far beyond in the direction of generating function theory.

Wilson in his book-length paper about perturbations and solar tables explicitly contrasts Laplace's to Lagrange's attitude: "*The basic orientation* [*of Laplace*] *is ever toward the practical result. Lagrange, by contrast, remains above all the lover of beautiful forms, algebraic or geometrical: there is sometimes impracticality in his elegant and elaborate solutions to the problems*". In a note he adds, after having recalled Delambre's remarks quoted a few lines before:

"*One thinks, for instance, of Lagrange's 'Recherches sur la manière de former des tables des planètes d'après les seules observations',…, of which no use, so far as the present writer knows, was ever made: the course of astronomical conquest lay in the opposite direction*" [Wilson 1980, p. 219].

Lagrange's paper is something like a melting pot of eighteenth-century techniques, some of which, like continued fractions,[11] are still in use in modern mathematics more or less in the same form they took on at that time. Finding a formula that literally coincides with a modern one (possibly after giving it a face-lift by use of modern notations) may well be a discovery of something that was unknown but may equally be an arbitrary rereading. That is quite clear; but it is also completely obvious that there are no recipes to avoid this danger. We can but trust in our experience and in our own esthetic judgment.

[11] The main aspects of the theory were set out for the first time by Euler [1744], and subsequently a whole chapter of [Euler 1748] was devoted to them. Lagrange had made extensive use of continued fractions in the sixties. Just to quote two important papers near in time to the essay we will be considering, we may mention [Lagrange 1769, § III], and the "Additions" to the French translation of Euler's *Algebra* [Lagrange 1773].

2. THE FUNDAMENTAL THEOREM

Let us have a sequence $d_0, d_1, d_2, \ldots$ and suppose we know *a priori* that

$$d_n = \sum_{j=1}^{N} A_j \sin(a_j + \alpha_j n). \tag{7}$$

Lagrange begins by proving that such a sequence may be generated not only by a rational function but also *by a very special rational function.*

This is how Lagrange states the matter:

"*Toute série dont un terme quelconque est représenté par la formule*

$$A \sin(a + m\alpha) + B \sin(b + m\beta) + C \sin(c + m\gamma) + \cdots,$$

m étant le nombre des termes précédents, est une série récurrente dont l'échelle de relation dépend uniquement des angles $\alpha, \beta, \gamma, \ldots$" [Lagrange 1775, p. 511].

Therefore a sequence $d_0, d_1, d_2, \ldots$ for which (7) holds, is a particular recurrent series corresponding to a rational function whose denominator depends only on $\alpha_1, \alpha_2, \alpha_3, \ldots$ Other peculiarities will become clear as the proof unfolds.

The construction of the generating function may be easily obtained by using only the results contained in Euler's *Introductio* (§§ 217–218). Lagrange prefers a more direct application of the complex variable to solve the problem.[12] What follows is an outline, in modern terms, of his proof.[13]

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[12] Euler too had considered the possibility of using geometric series, but he noted that it was feasible to give another method "*if we wish to avoid complex expressions*" [1748, § 218]. Euler's method consists in a direct proof of (9) by induction. An interesting analysis of this part of Euler's *Introductio* is in [Panza 1992, vol. 2, pp. 500–515]. Euler gives a more general result about sums of this type in §§ 258–260.

[13] Taton [1974, pp. 4–5] gives a long list of faults present in the *Œuvres* of Lagrange as edited by Serret and Darboux. Taton comes down particularly harshly against the modernisation of notations. So, before introducing an even stronger modernisation, a few remarks are called for. Firstly it should be noticed that in the particular case of the edition of this essay, the original text in the *Mémoires de l'Académie royale des sciences* is quite similar to the one reproduced in the *Œuvres*. The main difference is that in the *Mémoires* we find $\sin \cdot x$, while in the *Œuvres* there is simply $\sin x$. Consequently I will use the text of the *Œuvres* and not the original as the first is more accessible to a modern reader. As regards my own modernisations, I will use $a_0, a_1, a_2, \ldots, a_m, \ldots$, to denote a

In the expression for d_n, as given by (7), we use Euler's formulas to substitute $\sin\varphi$ for $(\mathrm{e}^{i\varphi} - \mathrm{e}^{-i\varphi})/2i$. By use of the usual properties of geometrical series Lagrange arrives at

$$\sum_{n=0}^{\infty} d_n x^n = \sum_{j=1}^{N} A_j \frac{\sin a_j - \sin(a_j - \alpha_j)\cdot x}{1 - 2\cos\alpha_j \cdot x + x^2}. \tag{8}$$

Observe that, by writing (7) in the form

$$d_n = \sum_{j=1}^{N} A_j \sin(a_j + \alpha_j n) = \sum_{j=1}^{N} A_j \cos a_j \cdot \sin(n\alpha_j) + A_j \sin a_j \cdot \cos(n\alpha_j),$$

and by using the identities

$$\begin{cases} \sum_{n=0}^{\infty} \sin(n\alpha)\cdot x^n = \dfrac{\sin\alpha \cdot x}{1 - 2\cos\alpha \cdot x + x^2}, \\ \sum_{n=0}^{\infty} \cos(n\alpha)\cdot x^n = \dfrac{1 - \cos\alpha \cdot x}{1 - 2\cos\alpha \cdot x + x^2}, \end{cases} \tag{9}$$

given by Euler in § 218 of his *Introductio*, we equally[14] arrive at (8). Lagrange will require and prove those very identities a few pages further on, but at this juncture he has preferred, as we have noted, a more direct use of the complex variable.[15]

In any case, (8) makes it apparent that sum $\sum_{n=0}^{\infty} d_n x^n$ may be expressed as a rational function.

sequence; but Lagrange uses the perfectly equivalent notation $T, T', T'', \dots, T^{(m)}, \dots$. I will use the same notation, without the final dots, to indicate the coefficients of a polynomial of degree m while Lagrange has $(0), (1), (2), \dots, (m)$ or $[0], [1], \dots, [m]$, or even $[(0)], \dots, [(m)]$. Obviously Lagrange does not use the symbol Σ in the modern sense, but he does use expressions like "*a function of the form ... gives a series whose general term is ...*" or similar ones which (in my opinion) are perfectly equivalent to what the modern notation expresses. About the problem of the edition of Lagrange's works see also [Pepe 1986].

[14] Euler freely uses complex variables in § 219 to expand fractions of type

$$\frac{1}{2}\,\frac{f - ig}{[1 - r(\cos x + i\sin x)]^k}$$

into powers of $z = r(\cos x + i\sin x)$. See also [Burkhardt 1914, pp. 825–826]. Euler's attitude need not be considered contradictory. It merely implies that sometimes complex variables are convenient and sometimes not.

[15] Poisson, at the beginning of his 1823 paper gave anew the result of Euler and Lagrange, paying some attention to the problem of convergence, which we do not need

Closer inspection of the form of this rational fraction shows that the denominator, $Q(x)$, is a polynomial of degree exactly $2N$, as it is the product of N monic polynomials of degree 2. Besides, every polynomial whose product gives $Q(x)$ is a *reciprocal polynomial of even degree*,[16] that is the coefficients of the terms equidistant from the middle term are equal. It is clear that a product of two reciprocal polynomials is a reciprocal polynomial, and consequently the same property holds for $Q(x)$.[17]

Also, the roots of the denominator $Q(x)$ are of a special kind. In fact, every term of the product which composes $Q(x)$ when equated to zero yields the two complex-conjugate roots $\cos\alpha_j \pm i\sin\alpha_j$, whose modulus is 1.

$P(x)$ is simply a polynomial of degree no higher than $(2N-1)$ and it has no apparent special properties.

Considering the preceding result, given a sequence of numbers such as (1), we are to search for a rational function $P(x)/Q(x)$, where $Q(x)$ is a reciprocal polynomial of degree $2N$, having roots of suitable form, such that

$$d_0 + d_1 x + d_2 x^2 + \cdots = \frac{P(x)}{Q(x)} \cdot \tag{10}$$

The problem now is to set out an algorithm to find this function.

The structure of the proof of the fundamental theorem is such that no construction of the required function may be obtained from it. We

to consider here, since Lagrange is working with formal power series. It is noteworthy that this result, which Euler actually proved once for all, did not acquire the status of a common theorem. Perhaps the explanation lies in the fact that it is an easy consequence of the result about the sum of a geometric series, not so trivial however that every justification might be left out.

Burkhardt more generally observes that "*Euler's method for finding the sum of trigonometric series has beeen rediscovered and exposed again many times*" ("Eulers Verfahren zur Auffindung der Summen von trigonometrischen Reihen ist dann noch öfter von neuem gefunden und dargestellt worden" [Burkhardt 1914, p. 828]). A long list of contributions follows, that includes the work of Poisson, but does not include the work of Lagrange we are considering. Burkhardt studied and described it as a work on trigonometric interpolation rather than a work on trigonometric series (see note 9). Burkhardt did no proffer any explanation for the continued repetition of results which had been, for the most part, firmly established in Euler's *Introductio*.

[16] Lagrange deals with the properties of reciprocal and opposite polynomials in an appropriate *Remarque* [Lagrange 1775, pp. 552–553].

[17] It is also easy to prove that the rational fraction $P(x)/Q(x)$ cannot be reducible, at least in non-trivial cases.

do not know how many terms sum (7) has. Whatever that number may be, we may understand which is the final form of $P(x)/Q(x)$, since that form remains the same for all addenda, and the process of summation preserves it.

To construct $P(x)/Q(x)$ we may firstly assume that (7) consists of just one term. If that be the case we are home and dry. Failing that we must assume that (7) consists of two terms ... and so on. This broad notion of what we have to do is within our grasp. But to convert all that into an algorithm is quite a different kettle of fish. Let us see what Lagrange does.

3. THE ALGORITHM

As often happens, it is a good idea to generalise. Lagrange disregards some of the peculiarities of the series given and simply assumes that it comes from a rational function of type[18] $(n-1,n)$. He constructs a general algorithm simply designed to approximate the given series

$$d_0 + d_1 x + d_2 x^2 + \cdots \tag{11}$$

up to the terms of order 2, 4, 6, etc. by successive functions of type $(0,1)$, $(1,2)$, $(2,3)$, ... until the Taylor development of the last rational function found coincides with the given series.

As is widely known, given an arbitrary series such as (11), the problem of finding a rational function $P(x)/Q(x)$ of type (m,n) such that

$$d_0 + d_1 x + d_2 x^2 + \cdots - \frac{P(x)}{Q(x)} = \mathrm{O}[x^{m+n+1}],$$

the so-called "Hermite problem", does not always admit a solution.[19]

None of which is of consequence when we are to investigate a series originating in astronomical measurements, since we are assured of a solution with a prescribed form. But when considering the problem in its generality the difficulty may not be disregarded.

[18] In general, by the symbol (h,k) I denote an irreducible rational function with a numerator of degree h and denominator of degree k.

[19] Elementary counter-examples are given in [Lorentzen, Waadeland 1992, pp. 376–379].

Lagrange was well aware of this, and even though he did not give a complete analysis, he observed that sometimes his algorithm did not succeed in yielding a function of the expected type. He also tried to calculate at least the degree of the denominator of the approximating function in this case, but his result is not general, as we will see.

At any rate, the algorithm (which actually constituted one of the first attempts to construct "Padé approximants" of a given type) is really beautiful.[20] A brief exposition of the "analysis" of the problem (Lagrange restricts himself to giving the "synthesis") may be helpful.

Assume the problem has been solved and that we have a rational function $P(x)/Q(x)$ whose Taylor development is (11). We have the equality:

$$d_0 + d_1 x + d_2 x^2 + \cdots = \frac{P(x)}{Q(x)} \cdot$$

We now proceed to write $P(x)/Q(x)$ as a particular continued fraction. We write $P(x)/Q(x) = 1/(Q(x)/P(x))\cdot$ By taking the Taylor development arrested at the first order (or a simple kind of division, introduced by Newton in *De analysi* [*MP* 2, pp. 212–215]), we have[21]

$$\frac{Q(x)}{P(x)} = p_1 + q_1 x + x^{2+\lambda_1} \frac{R_1(x)}{P(x)} \cdot$$

If $R_1(x)$ is identical to 0 our research is at an end. Otherwise, we keep on dividing. We have

$$\frac{P(x)}{Q(x)} = \frac{1}{\dfrac{Q(x)}{P(x)}} = \frac{1}{p_1 + q_1 x + x^{2+\lambda_1} \dfrac{R_1(x)}{P(x)}}$$

$$= \frac{1}{p_1 + q_1 x + \dfrac{x^{2+\lambda_1}}{p_2 + q_2 x + x^{2+\lambda_2} \dfrac{R_2(x)}{R_1(x)}}} \cdot$$

Once again, we can test to see whether the remainder is 0. In this case we stop: otherwise, we continue. But since $P(x)/Q(x)$ is to be a rational

[20] An analysis of Lagrange's algorithm is also given in [Brezinski 1991, pp. 119–120].

[21] Lagrange at first considers the case in which every λ_i is zero. Only afterwards does he describe the general algorithm. I presume that the modern reader will find easier it to go directly through the general case.

function, a simple test upon the degree of the remainders shows that we must stop after a finite number of steps. Eventually we arrive at this particular continued fraction expansion

$$\frac{P(x)}{Q(x)} = \cfrac{1}{p_1 + q_1 x + \cfrac{x^{2+\lambda_1}}{p_2 + q_2 x + \cfrac{x^{2+\lambda_2}}{\ddots + \cfrac{x^{2+\lambda_n}}{p_n + q_n x}}}}. \tag{12}$$

If $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$ (this being the general case) $P(x)/Q(x)$ is a function of type $(n-1, n)$. Otherwise the degree of the denominator (at least) must be modified. Lagrange maintains that in that case it should be $n + \lambda_1 + \lambda_2 + \cdots + \lambda_n$ [1775, p. 534], but this result is not always true.

Let us come back to Lagrange's essay. So far we have assumed that the function $P(x)/Q(x)$ is available to us. However we only have its series development $d_0 + d_1 x + d_2 x^2 + \cdots$. Hence the problem is to find the development into a continued fraction by sole use of the given series.

Let us start by writing

$$\frac{1}{\cfrac{Q(x)}{P(x)}} = \frac{1}{\cfrac{1}{d_0 + d_1 x + d_2 x^2 + \cdots}}.$$

The type of division previously mentioned may be extended without any difficulty into a power series [Lagrange 1775, pp. 525–529].

Let us suppose we have two series developments

$$a_0 + a_1 x + a_2 x^2 + \cdots, \quad b_0 + b_1 x + b_2 x^2 + \cdots, \quad \text{where } b_0 \neq 0.$$

We may determine the polynomial of degree 1, $a_0/b_0 + (a_1 b_0 - a_0 b_1)x/b_0^2$, which clearly is such that

$$a_0 + a_1 x + \cdots - \Big(\frac{a_0}{b_0} + \frac{a_1 b_0 - a_0 b_1}{b_0^2} x\Big)(b_0 + b_1 x + \cdots) = x^{2+\lambda} R(x),$$

where $R(x) \equiv 0$ or $\lambda \geq 0$ and $R(0) \neq 0$.

We apply this division to the power series $1 = 1 + 0x + 0x^2 + \cdots$ and $d_0 + d_1 x + d_2 x^2 + \cdots$. This gives for the first step of the algorithm the quotient $p_1 + q_1 x = 1/d_0 - d_1 x / d_0^2$.

The remainder $R_1(x)$ is obviously given by the difference

$$1 - (p_1 + q_1 x)(d_0 + d_1 x + d_2 x^2 + \cdots) = x^{2+\lambda_1}(\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots)$$
$$= x^{2+\lambda_1} R_1(x).$$

If $R_1(x) \neq 0$, we take as dividend $d_0 + d_1 x + d_2 x^2 + \cdots$, and as divisor $R_1(x)$ and we proceed to the second step.

We carry on subsequent divisions until we arrive at the final result as given by formula (12). The algorithm described is a general method for approximating (in general situations) a given power series by rational functions of prescribed type.[22] It has clearly an independent value, and Lagrange underlines its relevance:

"*La solution du problème précédent n'est ... qu'une simple application de la théorie des fractions continues; mais, quoique cette théorie ait déjà été traitée par plusieurs grands géomètres, il paraît que l'application dont il s'agit peut néanmoins être regardée comme neuve à plusieurs égards, et surtout relativement au point de vue sous lequel nous venons de l'envisager*" [Lagrange 1775, p. 542].

An example may be helpful to clarify the details of Lagrange's algorithm.

Example 1.

This is actually the second example given by Lagrange [1775, p. 534].[23]

Consider the sequence

$$1, 1, 1, 2, 4, 6, 7, 7, 7, 8, 10, 12, 13, 13, 13, 14, 16, \ldots$$

[22] Lagrange does not stop to note that his algorithm, when applied to a finite number of arbitrary data (in general situations) produces, in the worst case, a rational function with a denominator of degree equal at most to half of the number of data. This is of no great interest if we are looking at the possibility of giving an "economical" description of the series of data, but it may be read as a remarkable result about trigonometrical interpolation. I think that Condorcet had grasped this point, when he remarked in his *Compte rendu*: "*il ne s'agira plus que de chercher la série récurrente la plus simple, dont cette suite de nombres représente les coëfficiens, & l'expression générale de cette série, qui, comme il est aisé de le voir, ne peut monter à un degré plus élevé que la moitié du nombre des observations*" [*Arith. Pol.*, p. 108].

[23] The nature of the sequence chosen makes it evident that Lagrange is interested in discovering some kind of structure in the sequence considered in its own right. Which is exactly the mindset of generating function theory.

We convert it into the series

$$s = 1 + x + x^2 + 2x^3 + 4x^4 + 6x^5 + 7x^6 + 7x^7 + 7x^8 + 8x^9 + \cdots.$$

Now the quotient of 1 and s is $(1-x)$, while the remainder is

$$-x^3 - 2x^4 - 2x^5 - x^6 - x^9 - 2x^{10} - 2x^{11} - x^{12} - x^{15} \cdots.$$

We divide the remainder by $-x^3$, and take as dividend s and as divisor the remainder. We make another division and we obtain as quotient $(1-x)$. We keep on dividing and taking the remainder, and so on, until we arrive at

$$\cfrac{1}{1 - x + \cfrac{-x^3}{1 - x + \cfrac{x^2}{1-x}}}.$$

This continued fraction expansion may be converted in the usual form, which yields

$$\begin{aligned} s &= 1 + x + x^2 + 2x^3 + 4x^4 + 6x^5 + 7x^6 + 7x^7 + 7x^8 + 8x^9 + \cdots \\ &= \frac{1 - 2x + 2x^2}{1 - 3x + 4x^2 - 3x^3 + x^4}. \end{aligned}$$

4. THE ROOTS OF THE DENOMINATOR AND CHEBYSHEV POLYNOMIALS

To return to the analysis of the paper, suppose we have a sequence of numbers $d_0, d_1, d_2, \ldots$. Let us consider the series $d_0 + d_1 x + d_2 x^2 + \cdots$. With the help of the algorithm described, we can construct a rational function $P(x)/Q(x)$. But we now have to check that the degree of Q is an even number $2N$ and that Q is a reciprocal polynomial having its roots in the requisite form.

That is, we still have to verify that Q may be factored in the form

$$(x^2 - 2\cos\alpha_1 \cdot x + 1)(x^2 - 2\cos\alpha_2 \cdot x + 1) \cdots (x^2 - 2\cos\alpha_N \cdot x + 1). \tag{13}$$

To facilitate this task Lagrange introduces a general technique to transform a reciprocal polynomial of degree $2N$ into a polynomial of degree N whose (real) roots may be used in a simple way to calculate the roots of the original polynomial.[24]

By the substitution of variables $z = x + x^{-1}$, a reciprocal polynomial $Q(x)$, having degree $2N$, may be transformed into $x^N \widetilde{Q}(z)$, where $\widetilde{Q}(z)$ is a polynomial of degree N in z. This is self-evident if N is a small number. But if N is large, we have to face the problem of writing $x^n + x^{-n}$ as a polynomial in $z = x + x^{-1}$. Lagrange shows how this happens for $n = 2, 3$ and immediately writes

$$(14)\quad x^n + \frac{1}{x^n} = z^n - nz^{n-2} + \frac{n(n-3)}{2} z^{n-4} - \frac{n(n-4)(n-5)}{2 \cdot 3} z^{n-6} + \cdots$$

leaving it to the reader to verify (14).

For the sake of completeness, I shall give a proof in modern terms, noting, however, that equation (14) has, for Lagrange, a complete transparency not requiring any proof.

Let us write (14) in modern form as

$$(15)\qquad x^n + \frac{1}{x^n} = \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} z^{n-2k}.$$

Now consider the identity

$$(16)\qquad x^{n+1} + \frac{1}{x^{n+1}} = \left(x + \frac{1}{x}\right)\left(x^n + \frac{1}{x^n}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right).$$

Identity (16) means that, if we denote by $L_n(z)$ the polynomial in z which expresses $x^n + x^{-n}$ as a polynomial in $z = x + x^{-1}$, we have the recursive definition

$$(17)\qquad L_{n+1}(z) = zL_n(z) - L_{n-1}(z).$$

By dint of (17) it is elementary to prove that the coefficients have the form given in (15). It is enough to remark that

$$\frac{n-1}{n-k-1}\binom{n-k-1}{k} + \frac{n-2}{n-k-1}\binom{n-k-1}{k-1} = \frac{n}{n-k}\binom{n-k}{k}.$$

[24] Lagrange describes this technique at an ealier point in his text. I prefer to introduce it here. See [Lagrange 1775, pp. 515–517].

Formula (17), as I have set it out, will look rather familiar to the modern reader. It immediately reminds us of the usual definition of Chebyshev polynomials, which is

$$T_{n+1}(z) = 2z\,T_n(z) - T_{n-1}(z). \tag{18}$$

Comparison between (17) and (18) shows[25] that $L_n(2z) = 2\,T_n(z)$.

In other words, these polynomials $L_n(z)$ are the "Chebyshev polynomials" of degree n on the interval $[-2, 2]$. These polynomials were introduced by Lanczos (see [Lanczos 1952], where he called them $C_n(z)$), who judged that these polynomials had simpler numerical properties than the ordinary Chebyshev polynomials.[26]

That the polynomials considered by Lagrange are Chebyshev polynomials only in a formal sense, is quite obvious. The fact that they may be defined in a very similar way might appear as sheer historical contingency. But what is remarkable, is the fact that they were introduced by Lagrange precisely to effect one of the tasks that nowadays are usually assigned to Chebyshev polynomials (see chapter 5, especially section 5.3 in [Rivlin 1974/1990]).

Chebyshev polynomials were introduced to solve a well-defined problem in approximation theory. But nearly a century later it was discovered that they also have nice arithmetical properties [Schur 1973].

Can we say that Lagrange was in some sense a "forerunner"? I believe this question to be pointless. What is astounding is the nature of mathematical objects: even when they seem aimed to solve a strictly defined problem they may have completely unforeseen applications.

If we substitute $\mathrm{e}^{i\vartheta}$ for x in (16), we obtain

$$\cos(n+1)\vartheta = 2\cos\vartheta \cdot \cos n\vartheta - \cos(n-1)\vartheta,$$

a common identity used to define Chebyshev polynomials.

In Euler's *Introductio* § 243, this identity is used to obtain the further equality

$$2\cos n\vartheta = 2^n \cos^n\vartheta - n\cdot 2^{n-2}\cos^{n-2}\vartheta + \cdots,$$

[25] We have $T_0(z) = 1$, $T_1(z) = z$, and consequently $L_0(z) = 2$, $L_1(z) = z$.

[26] I gave this information to prof. Rivlin, who not only pointed out Lanczos' paper to me, but has also had the kindness of sending me a photocopy of this paper which is very difficult to find. Prof. Rivlin was also a fund of useful information about the history of Chebyshev polynomials.

which rapidly leads to (14), but Lagrange does not mention Euler and it may be that he had another idea about the meaning of equality (14).

Be that as it may, to return to our purpose, let us assume we have a reciprocal polynomial of even degree $2N$ such as

$$Q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N + \cdots + a_2 x^{2N-2} + a_1 x^{2N-1} + a_0 x^{2N}.$$

We may write

$$\begin{aligned} Q(x) &= x^N \left[a_0 \left(x^N + \frac{1}{x^N} \right) + a_1 \left(x^{N-1} + \frac{1}{x^{N-1}} \right) + \cdots + a_N \right] \\ &= x^N \left[a_0 L_N(z) + a_1 L_{N-1}(z) + \cdots + \tfrac{1}{2} a_N L_0(z) \right]. \end{aligned}$$

Now let us make the substitution $z = x + x^{-1}$ directly in (13). We obtain

$$x^N (z - 2\cos\alpha_1)(z - 2\cos\alpha_2) \cdots (z - 2\cos\alpha_N).$$

Thus we may consider the particular features that we may expect for $Q(x)$ in another way: it must be a reciprocal polynomial of even degree $2N$, such that after the substitution $z = x + x^{-1}$, it assumes the form $x^N \widetilde{Q}(z)$, where $\widetilde{Q}(z)$ has exactly N real roots in the interval $[-2, 2]$ [Lagrange 1775, p. 574].

5. LAGRANGE'S PROCEDURE

It will be useful to summarise the procedure Lagrange suggests.[27]

Let us consider the sequence $d_0, d_1, d_2, \ldots$, considering it as the power series $d_0 + d_1 x + d_2 x^2 + \cdots$.

We apply the algorithm described to construct a rational fraction $P(x)/Q(x)$ such that $d_0 + d_1 x + d_2 x^2 + \cdots = P(x)/Q(x)$; by hypothesis such a function does exist, hence the continued fraction development must stop after a finite number of steps.

The denominator $Q(x)$ is to have the requisite form, and in it we effect the substitution $z = x + x^{-1}$, in order to have $\widetilde{Q}(z)$.

[27] Here I shall describe only the first solution given by Lagrange. I leave out the explanation of the other two solutions to avoid technical complexities.

We need to calculate the real roots[28] of $\widetilde{Q}(z)$ and we thus write

$$Q(x) = (x^2 - 2\cos\alpha_1 \cdot x + 1)(x^2 - 2\cos\alpha_2 \cdot x + 1) \cdots (x^2 - 2\cos\alpha_N \cdot x + 1).$$

Now we must decompose fraction $P(x)/Q(x)$ into partial fractions, to obtain

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{N} \frac{M_i + N_i x}{1 - 2\cos\alpha_i \cdot x + x^2}.$$

By availing ourselves of the identities[29]

$$\sum_{n=0}^{\infty} \sin n\alpha \cdot x^n = \frac{\sin\alpha \cdot x}{1 - 2\cos\alpha \cdot x + x^2},$$
$$\sum_{n=0}^{\infty} \cos n\alpha \cdot x^n = \frac{1 - \cos\alpha \cdot x}{1 - 2\cos\alpha \cdot x + x^2},$$

it is a straightforward matter to obtain for every addendum

$$\frac{M_i + N_i x}{1 - 2\cos\alpha_i \cdot x + x^2} = \sum_{n=0}^{\infty} A_i \sin(a_i + n\alpha_i) \cdot x^n,$$

with

$$\tan a_i = \frac{M_i \sin\alpha_i}{M_i \cos\alpha_i + N_i}, \quad A_i = \pm \frac{\sqrt{M_i^2 + 2M_i N_i \cos\alpha_i + N_i^2}}{\sin\alpha_i}, \tag{19}$$

where the sign in the second formula has to be chosen in suitable fashion.

Hence, we have

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{N} \frac{M_i + N_i x}{1 - 2\cos\alpha_i \cdot x + x^2} = \sum_{n=0}^{\infty} \sum_{i=1}^{N} A_i \sin(a_i + n\alpha_i) \cdot x^n.$$

The analysis is complete. We have found

$$d_n = \sum_{i=0}^{N} A_i \sin(a_i + n\alpha_i).$$

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[28] Lagrange himself refers here to [Lagrange 1769] and [Lagrange 1770a,b].

[29] Which Lagrange introduces at this point [Lagrange 1775, p. 575].

Example 2

Lagrange will examine the case of this series (taken from Mayer's *Tabulae solares*, published in 1750) only after introducing many improvements to his method (see *infra* p. 229).[30] I prefer to look at it directly, to clarify some numerical difficulties. Suppose we consider the sequence

$$456, -168, 274, -933, 220, 631, -232, 349, -823, -72, 772, \\ -237, 358, -657, 360, 860, -181, 305, -457, -616, \ldots$$

We convert it into the power series

$$456 - 168x + 274x^2 - 933x^3 + 220x^4 + 631x^5 + \cdots$$

and we search for a rational approximant of type $(2n-1, 2n)$ that has all the requisite particular features. The most suitable such approximant is given by

$$\frac{-474.52x^3 + 561.72x^2 + 219x + 456}{0.99988x^4 + 0.84251x^3 + 0.94414x^2 + 0.85005x + 1}.$$

The denominator may be judged sufficiently close to a reciprocal polynomial. We approximate it by

$$x^4 + 0.84630x^3 + 0.94414x^2 + 0.84630x + 1.$$

The roots of this polynomial are

$$0.34405 + 0.93895i, \quad 0.34405 - 0.93895i, \\ -0.76721 + 0.64140i, \quad -0.76721 - 0.64140i.$$

From this we deduce the values of two angles (in radians) $\alpha = 1.2196$ and $\beta = 2.4453$. We can now use formulas (19) to calculate the other constants to obtain function

$$f(n) = 457.49 \sin(1.4384 + n \cdot 1.2196) + 587.92 \sin(-0.0042476 + n \cdot 2.4453).$$

The first values of this function are

$$455.99, \quad -166.26, \quad 271.91, \quad -933.57, \quad 218.20, \\ 632.54, \quad -223.36, \quad 340.41, \quad -830.95, \quad \ldots$$

and give an approximation of the given sequence.

[30] About Mayer's tables see [Wilson 1980, pp. 262–264].

6. THE METHOD IMPROVED (PART I)

Is the approximation given in the example a good one or not? We do not have many tools available to evaluate this. When we sought to calculate the expansion of the given series into a continued fraction we stopped once we could assume that the numerical values of the coefficients of the remainder would be very small [Lagrange 1775, p. 593]. But we were given no criterion to evaluate this situation. We were left to our own judgement. Beside which, the polynomial we found at the denominator was only approximately reciprocal, and once again it was left to our intuition to choose a polynomial that would be right.

But what is worse is the fact that, as will be apparent from the very nature of the algorithm, the approximation we found is a *global* one. We must have something like a *distance* of two sequences if we are to conclude we have found a suitable approximation.

Thus, applying Lagrange's method to a real situation may lead to some real difficulties. These difficulties go far beyond the practical task of finding fraction $P(x)/Q(x)$ or of calculating the roots of $Q(x)$.

But I have stressed over and over again that the main contribution of Lagrange's paper (in my opinion) lies more in the direction of developing the theory of generating functions than in promoting numerical calculus. Whatever the case, to overcome some difficulties, Lagrange did propose an improvement to his method. Once again, this improvement turns out to be a remarkable achievement in generating function theory.

We have seen Lagrange's ability to use the correspondence

$$(d_0, d_1, d_2, \ldots) \longmapsto \sum_{n=0}^{\infty} d_n x^n = f(x),$$

to deduce relations among the elements of the sequence, by analysing the form of the function $f(x)$. But actually this correspondence is useful in both directions,[31] and what Lagrange proceeds with, in a very few, but extremely interesting, pages, is a careful examination of some kinds of

[31] A well-known example is provided by the functional equation $u(x) = x \cdot \mathrm{e}^{u(x)}$. The calculus of its Taylor coefficients for $x = 0$ may be obtained by help of differential calculus rules. But it is a simpler task to look at the series generated and to deduce, by use of Lagrange's inversion formula, that their values constitute the sequence n^{n-1} [Wilf 1990/1994, pp. 168–169].

relations between a given sequence and of the corresponding generating functions, in the case where said function is rational.[32] At the end of his analysis he gives a simpler method of finding a rational function which is generated by a sequence of the form (7). Once again, Lagrange was not propounding a general *theory* of generating functions as Laplace would, a few years later,[33] but his results are indeed very impressive.

A given sequence $(d_0, d_1, d_2, \ldots)$ was thought, initially, to be the sequence of values of a function of $\mathbb{N}$ into $\mathbb{R}$, but it is quite natural to give a precise meaning also, in the case of a recurrent series, to its extension to a function mapping from $\mathbb{Z}$ into $\mathbb{R}$. What we are looking for is a sequence $(\ldots, d_{-2}, d_{-1}, d_0, d_1, d_2, \ldots)$ such that the same law holds for the construction of all the elements.

Let us consider the case where the sequence is generated by a rational function $P(x)/Q(x)$ of type $(n-1, n)$. Then Lagrange notes that the function $-x^{-1}P(x^{-1})/Q(x^{-1})$ generates the required sequence, *i.e.* $(d_{-1}, d_{-2}, d_{-3}, \ldots)$ [Lagrange 1775, pp. 548–551].

Given a polynomial like $P(x) = \sum_{k=0}^{n} a_k x^k$, the *contrary polynomial* is, by definition $\widetilde{P}(x) = \sum_{k=0}^{n} a_k x^{n-k}$.

It is evident that $-x^{-1}P(x^{-1})/Q(x^{-1})$ is equal to $-\widetilde{P}(x)/\widetilde{Q}(x)$, where the polynomials $\widetilde{P}(x)$ and $\widetilde{Q}(x)$ are the contrary polynomials to $P(x)$ and $Q(x)$, respectively.

Suppose we have a sequence, which in practical applications will be given by a finite number of terms and will be considered as beginning at a given place. We can always consider it as being a sequence of the form

$$(\ldots, d_{-2}, d_{-1}, d_0, d_1, d_2, \ldots).$$

If we know *a priori* that the given sequence is determined by a rational function of type $(n-1, n)$ which has as its denominator a reciprocal polynomial, we must have $Q(x) = \widetilde{Q}(x)$ and both functions $P(x)/Q(x) \pm \widetilde{P}(x)/Q(x)$ will present special features.

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[32] In fact we find that Euler in his *Introductio* has also used this possibility. The development of a rational function in partial fractions, just to take the first example, may be viewed as an instrument to be used to analyse a given sequence.

[33] This is a reference to [Laplace 1782], previously quoted. An interesting analysis of this work of Laplace is given by Panza [1992, vol. II, pp. 615–650].

Indeed, it is a straightforward matter to observe that $P(x) \pm \widetilde{P}(x)$ will be divisible by $(1-x)$ and $(1+x)$ respectively. By division, they will yield reciprocal polynomials of degree $(n-2)$.

In other words, we construct the series

$$\frac{(d_0 + d_{-1}) + (d_1 + d_{-2})x + (d_2 + d_{-3})x^2 + \cdots}{1-x} = \big[(d_0 + d_{-1}) + (d_1 + d_{-2})x + \cdots\big](1 + x + x^2 + \cdots),$$

$$\frac{(d_0 - d_{-1}) + (d_1 - d_{-2})x + (d_2 - d_{-3})x^2 + \cdots}{1+x} = \big[(d_0 - d_{-1}) + (d_1 - d_{-2})x + \cdots\big](1 - x + x^2 - \cdots),$$

which are very easily obtained,[34] we know that their generating functions are of the type $(n-2, n)$ and that both numerators and both denominators are reciprocal polynomials.

7. THE METHOD IMPROVED (PART II)

What we have seen shows that we may restrict ourselves to applying the algorithm for development into continued fractions to a sequence of which we know *a priori* that its generating function is of the type $(2n-2, 2n)$ and such that both the numerator and the denominator are reciprocal polynomials.

Lagrange [1775, pp. 559–562] goes on to consider this special case. Let a rational function be given by

$$\frac{P_{2n-2}}{Q_{2n}} = \frac{b_0 + b_1 x + \cdots + b_{n-1}x^{n-1} + \cdots + b_2 x^{2n-4} + b_1 x^{2n-3} + b_0 x^{2n-2}}{a_0 + a_1 x + \cdots + a_n x^n + \cdots + a_2 x^{2n-2} + a_1 x^{2n-1} + a_0 x^{2n}}.$$

We may write, as before,

$$\frac{P_{2n-2}}{Q_{2n}} = \frac{1}{x}\,\frac{b_0(x^{n-1} + x^{-(n-1)}) + b_1(x^{n-2} + x^{-(n-2)}) + \cdots}{a_0(x^n + x^{-n}) + a_1(x^{n-1} + x^{-(n-1)}) + \cdots}.$$

If we set $y = x/(1+x^2)$, *i.e.* $y^{-1} = x + x^{-1}$, we find (remembering the previously-given definition of L_n)

$$\frac{P_{2n-2}}{Q_{2n}} = \frac{1}{x} \cdot \frac{b_0 L_{n-1}(y^{-1}) + b_1 L_{n-2}(y^{-1}) + \cdots}{a_0 L_n(y^{-1}) + a_1 L_{n-1}(y^{-1}) + \cdots}.$$

[34] We need not use multiplication to obtain these series. The only operations required are additions and subtractions.

Developing all the calculations, we arrive at

$$\frac{P_{2n-2}}{Q_{2n}} = \frac{y}{x} \cdot \frac{p_{n-1}}{q_n} = \frac{1}{1+x^2} \cdot \frac{p_{n-1}}{q_n},$$

i.e. $(1+x^2)P_{2n-2}/Q_{2n} = p_{n-1}/q_n$.

Function p_{n-1}/q_n is only of type $(n-1, n)$ so that the continued fraction algorithm may be easier to apply to the series generated by this function. Once we have calculated p_{n-1}/q_n we need only substitute $x/(1+x^2)$ for its variable and divide the result by $1/(1+x^2)$ to find the original function we are seeking.

This leads on to the problem of analysing how all this operates at series level.

Let us consider the two series $t_0+t_1x+t_2x^2+\cdots$ and $\vartheta_0+\vartheta_1x+\vartheta_2x^2+\cdots$, and suppose that $f(x)$ and $g(x)$ are their generating functions. Suppose further that the generating functions are linked by the relation

$$g\Big(\frac{x}{1+x^2}\Big) = (1+x^2)f(x).$$

Lagrange [1775, p. 562] shows that, in this case

$$\begin{aligned} t_n &= \vartheta_n - (n-1)\vartheta_{n-2} + \frac{(n-2)(n-3)}{1\cdot 2}\vartheta_{n-4} - \cdots \\ &= \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} \vartheta_{n-2k} \end{aligned} \tag{20}$$

and

$$\begin{aligned} \vartheta_n &= t_n + (n-1)t_{n-2} + \frac{n(n-3)}{1\cdot 2}t_{n-4} + \cdots \\ &= \sum_{k=0}^{[n/2]} \frac{n-2k+1}{n-k+1}\binom{n}{n-k} t_{n-2k}. \end{aligned} \tag{21}$$

Let us stop a moment to examine these results. It is evident that they concern general series developments, with no requirement that we restrict ourselves to rational functions. In modern terms we may interpret these result as follows: we have a functional equation connecting two functions $f(x)$ and $g(x)$

$$g\Big(\frac{x}{1+x^2}\Big) = (1+x^2)f(x),$$

and supposing we know one of them from its series development, we can obtain the series development of the other by help of (20) or (21).

Suppose that $g(x) = \dfrac{1}{1-x} = 1+x+x^2+\cdots$. Then $f(x) = \dfrac{1}{1-x+x^2}$. Hence we have

$$t_n = \Big(\frac{3+i\sqrt{3}}{6}\Big)\Big(\frac{1-i\sqrt{3}}{2}\Big)^n + \Big(\frac{3-i\sqrt{3}}{6}\Big)\Big(\frac{1+i\sqrt{3}}{2}\Big)^n$$
$$= \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k}.$$

Or consider the particular case where $f(x) = 1/(1+x^2)$. It follows that $g(x) \equiv 1$. From (21) we get the identity

$$\sum_{k=0}^{m} \frac{2m-2k+1}{2m-k+1}\binom{2m}{2m-k}(-1)^k = 0, \quad m > 0. \tag{22}$$

These are but two of the innumerable identities concerning binomial coefficients, and, at first sight, this kind of deduction would not seem to be of great historical relevance. But it is precisely the very ease with which we move from Lagrange's mathematics to modern combinatorics that makes manifest (in my opinion) the legitimacy of reading his mathematics as I do. And to present this point of view is obviously to exert a historical judgement.[35]

A remark should be made. Every rational fraction p_{n-1}/q_n is such that the previously explained manipulations of variables lead to a fraction P_{2n-2}/Q_{2n} where both numerator and denominator are reciprocal. If we are able to propose a "good" series to calculate p_{n-1}/q_n, we have no cause to trouble ourselves about the problem of having a rational fraction whose denominator will be a reciprocal polynomial. But, once again, all these considerations involve a set of problems of approximation. Be that as it may, let us now turn to a further example.

[35] The interested reader who looks at [Gould 1990], just to take a nice example, will discover manipulations very similar to the ones of Lagrange described in my paper. The recent book [Graham, Knuth, Patashnik 1989] carries a dedication to Euler. Great emphasis is laid on the conviction that many instruments of eighteenth-century mathematics are still at work in modern discrete mathematics. I thank prof. B. Sury who has remarked a nice connection between identity (22) and Lagrange's interpolation formula.

Example 3

Take the example proposed by Lagrange [1775, pp. 588–589]. I avail myself of only part of the technical subtleties he brought to bear to improve calculations, since describing them in detail would be somehow tedious. Once again let us consider the sequence

$$456, -168, 274, -933, 220, 631, -232, 349, -823, -72, 772, -237,$$
$$358, -657, 360, 860, -181, 305, -457, -616, \ldots$$

But now let us consider it as being given by two sequences proceeding in opposite directions. The first begins with 772, while the second begins with the element immediately preceding

$$772, -237, 358, -657, 360, 860, -181, 305, -457, -616, \ldots$$
$$-72, -823, 349, -232, 631, 220, -933, 274, -168, 456, \ldots$$

We now form the sum and the difference of the two sequences

$$700, -1060, 707, -889, 271, 1080, -1114, 579, -625, -160, \ldots$$
$$844, 586, 9, -425, -991, 640, 752, 31, -289, -1072, \ldots$$

We convert these sequences into the power series

$$s_1 = 700 - 1060x + 707x^2 - 889x^3 + 271x^4 + 1080x^5 + \cdots$$

and

$$s_2 = 844 + 586x + 9x^2 - 425x^3 - 991x^4 + 640x^5 + \cdots$$

Now we construct the new series

$$\begin{aligned} S_1 = \frac{s_1}{1-x} &= (700 - 1060x + 707x^2 - 889x^3 + 271x^4 + 1080x^5 + \cdots) \\ &\qquad \times (1 + x + x^2 + x^3 + \cdots) \\ &= 700 - 360x + 347x^2 - 542x^3 - 271x^4 + 809x^5 + \cdots \end{aligned}$$

and

$$\begin{aligned} S_2 = \frac{s_2}{1+x} &= (844 + 586x + 9x^2 - 425x^3 - 991x^4 + 640x^5 + \cdots) \\ &\qquad \times (1 - x + x^2 - x^3 + \cdots) \\ &= 844 - 258x + 267x^2 - 629x^3 - 299x^4 + 939x^5 - 187x^6 + \cdots. \end{aligned}$$

Since we expect both series to be produced by rational functions of the type $(2n-2, 2n)$, we manipulate these series using procedure (21). The first yields

$$\vartheta_1 = 700 - 360y + 1047y^2 - 1262y^3 + 2170y^4 - 3159y^5 + \cdots,$$

while the second yields

$$\vartheta_2 = 844 - 258y + 1111y^2 - 1028y^3 + 2190y^4 - 3119y^5 + \cdots.$$

The first series may be approximated by a rational function of type $(1, 2)$

$$\frac{700 + 227.661y}{1 + 0.839y - 1.06y^2}.$$

Substitution of variables as per $y = x/(1+x^2)$ and subsequent division by $1 + x^2$ gives

$$\frac{700x^2 + 227.661x + 700}{x^4 + 0.839x^3 + 0.936x^2 + 0.839x + 1}.$$

In like manner, the second series gives

$$\frac{844x^2 + 452.096x + 844}{x^4 + 0.841x^3 + 0.940x^2 + 0.841x + 1}.$$

I leave out the final part of the calculation which would proceed in the very same manner we have already seen. It will be enough to observe that the angles calculated from the roots of the denominators are in both cases very close to one another.

At the end of the analysis of Lagrange's paper we are forced to conclude that, from a numerical point of view, it does not proffer many results that might be applied to concrete cases. The numerical examples we have considered explain the difficulty one may encounter in applying Lagrange's procedures to concrete cases, and the difficulties we met in them justify the perplexities expressed in § 1.3.

But the mastery Lagrange shows in dealing with the correspondence

$$(d_0, d_1, d_2, \ldots) \longmapsto f$$

is most impressive. Not only is he able to see how the usual operations reflect themselves in both sides of the correspondence, he also succeeds in predicting the effects of substituting variables and of other sophisticated manipulations. I think that, without being anachronistic, I may indeed conclude this analysis of Lagrange's essay by emphasising how powerful and modern his technique is when manipulating formal power series. Even if Lagrange did not give a theoretical foundation to the theory of generating functions, a task it would befall to Laplace to accomplish, he made manifest, in practice, the great interest that may attach to this aspect of mathematics.

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