Recherche Coopérative sur Programme n° 25

S. Hosono

A. Klemm

S. Theisen

An Extended Lecture on Mirror Symmetry

Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1995, tome 47
« Conférences de M. Audin, D. Bernard, A. Bilal, B. Enriquez, E. Frenkel, F. Golse, M. Katz, R. Lawrence, O. Mathieu, P. Von Moerbeke, V. Ovsienko, N. Reshetikhin, S. Theisen », , exp. n° 2, p. 33-76

<http://www.numdam.org/item?id=RCP25_1995__47__33_0>

© Université Louis Pasteur (Strasbourg), 1995, tous droits réservés.

L'accès aux archives de la série « Recherche Coopérative sur Programme n° 25 » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

Numdam

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

An Extended Lecture on Mirror Symmetry

S. Hosono†, A. Klemm* and S. Theisen◇

†Department of Mathemathics, Toyama University
Toyama 930, Japan

*Department of Mathematics, Harvard University
Cambridge, MA 02138, U.S.A.

◇Sektion Physik der Universität München
Theresienstraße 37, D - 80333 München, FRG

We give an introduction to mirror symmetry of strings on Calabi-Yau manifolds with an emphasis on its applications e.g. for the computation of Yukawa couplings. We introduce all necessary concepts and tools such as the basics of toric geometry, resolution of singularities, construction of mirror pairs, Picard-Fuchs equations, etc. and illustrate all of this on a non-trivial example.

1. Introduction

It is the purpose of these notes to give a pedagogical introduction to mirror symmetry and its applications. We start with a review of some general concepts of string theory in view of mirror symmetry. General introductions to string theory can be found in [1].

Some basic properties of closed string theory are best discussed in the geometrical approach, i.e. by looking at the classical σ-model action. It is defined by a map Φ from a compact Riemann surface Σ_g of genus g (the world-sheet with metric $h_{\alpha\beta}$) to the target space X (the space-time) $\Phi : \Sigma_g \to X$ and an action $S(\Phi, G, B)$, which may be viewed as the action of a two dimensional field theory. The latter depends on the dynamical field Φ, whereas the metric G of X and an antisymmetric tensorfield B on X are treated as background fields. As the simplest example one may take a bosonic action, which reads

$$S = \frac{1}{2\pi\alpha'}\int_{\Sigma_g} d^2\sigma\sqrt{h}\Big(h^{\alpha\beta}G_{ij}(\phi)\partial_\alpha\phi^i\partial_\beta\phi^j + \epsilon^{\alpha\beta}B_{ij}(\phi)\partial_\alpha\phi^i\partial_\beta\phi^j + \dots\Big) \qquad (1.1)$$

Delivered by S. Theisen at the 58ieme Rencontre entre Physiciens Theoriciens et Mathematiciens

where ϕ^i ($i = 1, \dots, \dim(X)$) and σ^α ($\alpha = 1,2$) are local coordinates on Σ_g and X respectively. The dots indicate further terms, describing the coupling to other background fields such as the dilaton and gauge fields. The first quantized string theory can then be perturbatively defined in terms of a path integral as[1]

$$\mathcal{S}(X) = \sum_g \int_{\mathcal{M}_g} \int D\Phi e^{iS(\Phi,G,B,\dots)} \tag{1.2}$$

For a particular background to provide a classical string vacuum, the sigma model based on it has to be conformally invariant [3]. This means that the energy-momentum tensor, including corrections from σ-model loops, must be traceless, or, equivalently, that the β-functions must vanish. Vanishing of the dilaton β-function demands that we quantize in the critical dimension, whereas the β-functions associated to the metric and the anti-symmetric tensor impose dynamical equations for the background, in particular that is has (to lowest order in α') to be Ricci flat, i.e. that the metric satisfy the vacuum Einstein equations[2].

Since we are dealing with strings, it is not the classical geometry (or even topology) of X which is relevant. In fact, path integrals such as (1.2) are related to the loop space of X. Much of the attraction of string theory relies on the hope that the modification of the concept of classical geometry to string geometry at very small scales will lead to interesting effects and eventually to an understanding of physics in this range. At scales large compared to the scale of the loops (which is related to α') a description in terms of point particles should be valid and one should recover classical geometry. The limit in which the classical description is valid is referred to as the large radius limit.

One particular property of strings as compared to ordinary point particles is that there might be more than one manifold X which leads to identical theories; i.e. $\mathcal{S}(X_1) = \mathcal{S}(X_2)$[3]. The case where two manifolds have just different *geometry* is usually referred to as duality symmetry [5]. Mirror symmetry [6] [7] [8] [9]. relates, in the generic case, identical string theories on *topologically* different manifolds. These symmetries are characteristic features

[1] We always assume that we quantize in the critical dimension. Integration over the world-sheet metric can then be converted to an integration over the moduli space $\mathcal{M}_g$ with suitable measure which, for general correlation functions, depends on the number of operator insertions [2]. By $\mathcal{S}(X)$ we mean the generating function of all correlators of the string theory on X.

[2] In fact, this point is subtle, as for X a Calabi-Yau manifold (cf. section two), higher (≥ 4) σ-model loop effects modify the equations of motion for the background. It can however be shown [4] that σ-models for Calabi-Yau compactifications are conformally invariant and that by means of a (non-local) redefinition of the metric one can always obtain $R_{i\bar{j}} = 0$ as the equations of motion for the background.

[3] This resembles the situation of quantized point particles on so called isospectral manifolds. However in string theory the invariance is more fundamental, as no experiment can be performed to distinguish between X_1 and X_2.

of string geometry. For the case of mirror symmetry, which is the central topic of these lectures, this will become evident as we go along.

So far the analysis of (parts of) $\mathcal{S}(X)$ can be explicitly performed only for very simple target spaces X, such as the torus and orbifolds. Much of our understanding about the relation between classical and string geometry is derived from these examples. As a simple example we want to discuss compactification of the bosonic string on a two dimensional torus [10]; for review, see also [11]. This also allows us to introduce some concepts which will appear in greater generality later on.

The torus $T^2 = \mathbb{R}^2/\Gamma$ is defined by a two-dimensional lattice Γ which is generated by two basis vectors e_1 and e_2. The metric, defined by $G_{ij} = e_i \cdot e_j$, has three independent components and the antisymmetric tensor $B_{ij} = b\epsilon_{ij}$ has one component. For any values of the altogether four real components does one get a consistent string compactification. We thus have four real moduli for strings compactified on T^2. We can combine them into two complex moduli as follows: $\sigma = \frac{|e_1|}{|e_2|}e^{i\phi}$ and $\tau = 2(b+iA)$ where ϕ is the angle between the two basis vectors which we can, without loss of generality, choose to be $0 \leq \phi \leq \pi$ and $A = \sqrt{|\det G|} > 0$ is the area of the unit cell of Γ. σ parameterizes different complex structures on the torus and is usually called the Teichmüller parameter. The imaginary part of τ parameterizes the Kähler structure of the torus. We have used the antisymmetric tensor field to complexify the Kähler modulus. The role the two moduli play is easily recognized if one considers $ds^2 = G_{ij}dx^i dx^j \equiv \tau_2 dz d\bar{z}$ where the relation between the real coordinates x_i and the complex coordinates $z, \bar{z}$ only involves σ but not τ.

If one now considers the spectrum of the theory, one finds various symmetries. They restrict the moduli space of the compactification which is naively just two copies of the upper half complex plane. With the definition of the left and right momenta

$$p_L^2 = \frac{1}{\sigma_2\tau_2}|(m_1 - \sigma m_2) - \tau(n_2 + \sigma n_1)|^2 , \quad p_R^2 = \frac{1}{\sigma_2\tau_2}|(m_1 - \sigma m_2) - \bar{\tau}(n_2 + \sigma n_1)|^2$$

the spectrum of the energy and conformal spin eigenvalues can be written as

$$m^2 = p_L^2 + p_R^2 + N_L + N_R - 2 , \quad s = p_L^2 - p_R^2 + N_L - N_R \tag{1.3}$$

where n_i and m_i are winding and momentum quantum numbers, respectively, $N_{L,R}$ are integer oscillator contributions and the last term in m^2 is from the zero point energy. The symmetries of the theory are due to invariance of (1.3) under the transformation $(\sigma, \tau) \mapsto (\tau, \sigma)$, $(\sigma, \tau) \mapsto (-\bar{\sigma}, -\bar{\tau})$, $(\sigma, \tau) \mapsto (\sigma + 1, \tau)$ and $(\sigma, \tau) \mapsto (-\frac{1}{\sigma}, \tau)$ accompanied by a relabeling and/or interchange of the winding and momentum quantum numbers. The transformation which interchanges the two types of moduli generates in fact what we call mirror symmetry. The torus example is however too simple to exhibit a change of topology as it is its own mirror. The transformations which reflect the string property are those which require an interchange of momentum and winding modes. The last three of the generators given above are not of this type and they are also present for the point particle moving on a torus (then $n_1 = n_2 = 0$). It is the addition of the mirror symmetry generator which introduces the stringy behavior. Interchange of the two moduli must be accompanied by $n_2 \leftrightarrow m_2$. Composing the mirror transformation with some of the other generators given above, always involves interchanges of winding and momentum quantum numbers. E.g. for the transformation $\tau \to -\frac{1}{\tau}$ we have to redefine $m_1 \to n_2, m_2 \to -n_1, n_2 \to -m_1, n_1 \to m_2$ and if we set $b = 0$ then this transformation identifies compactification on a torus of size A to compactification on a torus of size $1/A$. Integer shifts of τ are discrete Peccei-Quinn symmetries. One can show that the interactions (correlation functions) are also invariant under these symmetries.

This lecture deals with mirror symmetry of strings compactified on Calabi-Yau spaces. In section two we will review some of the main features of Calabi-Yau compactifications, in particular the correspondence of complex structure and Kähler moduli with elements of the cohomology groups $H^{2,1}$ and $H^{1,1}$, respectively. For X and X^* to be a mirror pair of Calabi-Yau manifolds (we will use this notation throughout) one needs that $h^{p,q}(X) = h^{3-p,q}(X^*)$ (for three dimensional Calabi-Yau manifolds this is only non-trivial for $p, q = 1, 2$). The *mirror hypothesis* is however much more powerful since it states that the string theory on X and X^* are identical, i.e. $\mathcal{S}(X) = \mathcal{S}(X^*)$. In particular it implies that one type of couplings on X can be interpreted as another type of couplings on X^* after exchanging the role of the complex structure and the Kähler moduli.

In section three we give a description of Calabi-Yau compactification in terms of symmetric $(2,2)$ superconformal field theory. The moduli of the Calabi-Yau space correspond to exactly marginal deformations of the conformal field theory. They come in two classes. Mirror symmetry appears as a trivial statement, namely as the change of relative sign, which is pure convention, of two $U(1)$ charges [6][7]. By this change the two classes of marginal perturbations get interchanged. This does not change the conformal field theory and thus leads to the same string vacuum. In the geometrical interpretation this is however non-trivial, as it entails the mirror hypothesis which implies the existence of pairs of topologically different manifolds with identical string propagation.

We will apply mirror symmetry to the computation of Yukawas couplings of charged matter fields. They come in two types, one easy to compute and the other hard to compute. On the mirror manifold, these two couplings change role. What one then does is to compute the easy ones on either manifold and then map them to one and the same manifold via the so-called mirror map. In this way one obtains both types of couplings. This will be explained in detail in section six. Before getting there we will show how to construct mirror pairs and how to compute the easy Yukawa couplings. This will be done in sections four and five. In section seven we present an example in detail, where the concepts introduced before will be applied. In the final section we draw some conclusions.

Before continuing to section two, let us give a brief guide to the literature. The first application of mirror symmetry was given in the paper by Candelas, de la Ossa, Green and Parkes [12] where the simplest Calabi-Yau manifold was treated, the quintic in $\mathbb{P}^4$ which has only one Kähler modulus. Other one-moduli examples were covered in [13] (for hypersurfaces) and in [14] (for complete intersections). Models with several moduli were examined in refs. [15] [16] (two and three moduli models) and [17]. Other references, especially to the mathematical literature, will be given as we go along. A collection of papers devoted to various aspects of mirror symmetry is [9]. Some of the topics and results to be discussed here are also contained in [15][18] [19] [20]. These notes draw however most heavily from our own papers on the subject.

2. Strings on Calabi-Yau Manifolds

One of the basic facts of string theory is the existence of a critical dimension, which for the heterotic string, is ten. To reconciliate this with the observed four-dimensionality of space-time, one makes the compactification Ansatz that the ten-dimensional space-time through which the string moves has the direct product form $X_{10} = X_4 \times X_6$ where X_6 is a six-dimensional compact internal manifold, which is supposed to be small, and X_4 is four-dimensional Minkowski space. If one imposes the 'phenomenologically' motivated condition that the theory has $N = 1$ supersymmetry in the four uncompactified dimensions, it was shown in [21] that X_6 has to be a so-called Calabi-Yau manifold [22] [23].

Def.[4]*: A Calabi-Yau manifold is a compact Kähler manifold with trivial first Chern class.*

The condition of trivial first Chern class on a compact Kähler manifold is, by Yau's theorem, equivalent to the statement that they admit a Ricci flat Kähler metric. The necessity is easy to see, since the first Chern class $c_1(X)$ is represented by the 2-form $\frac{1}{2\pi}\rho$ where ρ is the Ricci two form, which is the 2-form associated to the Ricci tensor of the Kähler metric: $\rho = R_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$. Locally, it is given by $\rho = -i\partial\bar{\partial} \log \det((g_{i\bar{j}}))$. One of the basic properties of Chern classes is their independence of the choice of Kähler metric; i.e. $\rho(g') = \rho(g) + d\alpha$. If now $\rho(g) = 0$, $c_1(X)$ has to be trivial. That this is also sufficient was conjectured by Calabi and proved by Yau [24].

Ricci flatness also implies that the holonomy group is contained in $SU(3)$ (rather than $U(3)$; the $U(1)$ part is generated by the Ricci tensor $R_{i\bar{j}} = R_{i\bar{j}k}{}^{k}$). If the holonomy is $SU(3)$ one has precisely $N = 1$ space-time supersymmetry. This is what we will assume in the following. (This condition e.g. excludes the six-dimensional torus T^6, or $K_3 \times T^2$, which would lead to extended space-time supersymmetries.)

Another consequence of the CY conditions is the existence of a unique nowhere vanishing covariantly constant holomorphic three form, which we will denote by $\Omega = \Omega_{ijk} dz^i \wedge dz^j \wedge dz^k$ $(i,j,k = 1,2,3)$, where z^i are local complex coordinates of the CY space. Since Ω is a section of the canonical bundle[5], vanishing of the first Chern class is equivalent to the triviality of the canonical bundle.

A choice of complex coordinates defines a complex structure. The transition functions on overlaps of coordinate patches are holomorphic functions. There are in general families of possible complex structures on a given CY manifold. They are parameterized by the so-called complex structure moduli. Using Kodaira-Spencer defomation theory [25], it was shown in [26] that for Calabi-Yau manifolds this parameter space is locally isomorphic to an open set in $H^1(X, T_X)$. For algebraic varieties the deformation along elements of $H^1(X, T_X)$ can often be described by deformations of the defining polynomials (cf. section four).

In addition to the complex structure moduli there are also the Kähler moduli. They parameterize the possible Kähler forms. A Kähler form is a real closed $(1,1)$ form $J =$

[4] Here and below we restrict ourselves to the three complex-dimensional case.

[5] The canonical bundle is the highest (degree $\dim_{\mathbb{C}}(X)$) exterior power of the holomorphic cotangent bundle T^*_X.

$\omega_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}$ (with the associated Kähler metric $g_{i\bar{j}} = i\omega_{i\bar{j}}$) which satisfies the positivity conditions

$$\int_C J > 0\,, \quad \int_S J^2 > 0\,, \quad \int_X J^3 > 0 \tag{2.1}$$

for all curves C and surfaces S on the CY manifold X. Since $\frac{1}{3!}J^3$ is the volume form on X, one concludes that the Kähler form cannot be exact and consequently one has $\dim(H^{1,1}(X)) \equiv h^{1,1} \geq 1$ for X Kähler. If there are more than one harmonic $(1,1)$ forms on X, i.e. if $h^{1,1} > 1$, then $\sum_{a=1}^{h^{1,1}} t'_a h_a$, $t' \in \mathbb{R}$, with $h_a \in H^{1,1}_{\bar{\partial}}(X)$ will define a Kähler class, provided the Kähler moduli lie within the so-called Kähler cone, i.e. (2.1) is satisfied.

From the local expression of the Ricci form it follows that it depends on the complex structure and on the volume form of the Kähler metric. The question now arises whether by changing the Kähler form and the complex structure Ricci flatness is preserved. This means that the moduli of the CY manifold must be associated with deformations of the *Ricci flat* Kähler metric: $\delta g_{i\bar{j}}$ with Kähler deformations and δg_{ij} and $\delta g_{\bar{i}\bar{j}}$ with deformations of the complex structure. If one examines the condition $\rho(g + \delta g) = 0$ one finds that $\delta g_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}}$ is harmonic, i.e. we can expand it as $\delta g_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}} = \sum_{a=1}^{h^{1,1}} \delta t'_a h_a$. Likewise, $\Omega_{ij}{}^{\bar{l}}\delta g_{\bar{l}\bar{k}}dz^i \wedge dz^j \wedge d\bar{z}^{\bar{k}} = \sum_{\alpha=1}^{h^{2,1}} \delta\lambda_\alpha b_\alpha$, with $b_a \in H^{2,1}(X)$. Here we have employed the unique Ω.

One can show that the only independent non trivial Hodge numbers of CY manifolds are $h^{0,0} = h^{3,0} = 1$ and $h^{1,1}$ and $h^{2,1}$ depending on the particular manifold. In addition we have $h^{p,q} = h^{q,p}$ (complex conjugation), $h^{p,q} = h^{3-p,3-q}$ (Poincaré duality) and $h^{0,p} = h^{0,3-p}$ (isomorphism via Ω). For Kähler manifolds the Betti numbers are $b_r = \sum_{p+q=r} h^{p,q}$ and the Euler number is thus $\chi(X) = \sum_{r=0}^{\dim(X)} b_r$.

If we were geometers we would only be interested in the deformations of the *metric* and the number of (real) moduli would be $h^{1,1} + 2h^{2,1}$. However, in string theory compactified on CY manifolds we have additional massless scalar degrees of freedom from the non-gauge sector, namely those coming from the (internal components of the) antisymmetric tensor field $B_{i\bar{j}}$. As a result of the equations of motion it is a harmonic $(1,1)$ form and its changes can thus be parameterized as $\delta B_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}} = \sum_{a=1}^{h^{1,1}} \delta t''_a h_a$ where t''_a are real parameters. One combines $(i\delta g_{i\bar{j}} + \delta B_{i\bar{j}})dz^i \wedge dz^{\bar{j}} = \sum_{a=1}^{h^{1,1}} \delta t_a h_a$ where now the $t_a = t''_a + it'_a$ are complex parameters. This is referred to as the complexification of the Kähler cone.

Recall (and see below) that for strings on CY manifolds, the massless sector of the theory consists of a universal sector, containing the graviton, an antisymmetric tensor field (by duality related to the axion) and a dilaton, and a matter sector with $h^{1,1}$ 27-plets and $h^{2,1}$ $\overline{27}$-plets of E_6 and a certain number of E_6 singlets. E_6 invariance restricts the possible Yukawa couplings to the following four types: $\langle 27^3 \rangle$, $\langle \overline{27}^3 \rangle$, $\langle 27 \cdot \overline{27} \cdot 1 \rangle$ and $\langle 1^3 \rangle$. In the following we will only treat the former two couplings [27]. Not much is known about the remaining two[6].

---

[6] except for cases in which the corresponding conformal field theory can be treated exactly, e.g. for Calabi-Yau spaces with an toroidal orbifold limit they can be calculated for the untwisted sector at the orbifold point [28] and for Calabi-Yau spaces with Gepner [29] model interpretion at

One considers the coupling of two fermionic and one bosonic field (Yukawa coupling) in the ten-dimensional field theory. All these fields are in the fundamental (248) representation of E_8, the gauge group of the uncompactified heterotic string[7]. (The bosonic field is the E_8 gauge field.) One then expands the fields in harmonics on the internal CY manifold and arrives at couplings which factorize into two terms: one is a cubic coupling of three fields on the four-dimensional Minkowski space and the other an overlap integral over the internal manifold of three zero-modes (we are interested in massless fields) of the Dirac and the wave operators, respectively. The second factor is the effective Yukawa coupling of the four-dimensional field theory. Under $E_8 \supset E_6 \times SU(3)$ we have the decomposition $248 = (27,3) \oplus (\overline{27},\bar{3}) \oplus (1,8) \oplus (78,1)$. The $(78,1)$ gives the E_6 gauge fields and the $(1,8)$ is the spin connection which has been identified with the $SU(3)$ part of the E_8 gauge connection [21].

The four-dimensional matter fields transform as 27 and $\overline{27}$ of E_6 and the zero modes in the internal CY manifold carry a $SU(3)$ index in the 3 and $\bar{3}$ representations, respectively. Group theory then tells us that there are two different kinds of Yukawa couplings among the charged matter fields: $\langle 27^3\rangle$ and $\langle \overline{27}^3\rangle$. The zero modes on the internal manifold can be related to cohomology elements of the CY space and one finds [27] that the two types of Yukawa couplings are of the form:

$$\begin{aligned}
\kappa^{0(27)}_{abc}(X) \equiv \kappa^0_{abc}(X) &= \int_X h_a \wedge h_b \wedge h_c, && a,b,c = 1,\ldots,h^{1,1}\\
\kappa^{(\overline{27})}_{\alpha\beta\gamma}(X) \equiv \bar{\kappa}_{\alpha\beta\gamma}(X) &= \int_X \Omega \wedge b^i_\alpha \wedge b^j_\beta \wedge b^k_\gamma\, \Omega_{ijk}, && \alpha,\beta,\gamma = 1,\ldots,h^{2,1}
\end{aligned} \tag{2.2}$$

where h_a are the harmonic $(1,1)$ forms and $b^i_\alpha = (b_\alpha)^i_{\bar{j}} d\bar{z}^{\bar{j}}$ are elements of $H^1(X,T_X)$ which are related to the harmonic $(2,1)$ forms via the unique element of $H^3(X)$: $(b_\alpha)^i_{\bar{j}} = \frac{1}{2\|\Omega\|^2}\bar{\Omega}^{ikl}(b_\alpha)_{kl\bar{j}}$, $\|\Omega\|^2 = \frac{1}{3!}\bar{\Omega}^{ijk}\Omega_{ijk}$. Note that while the former couplings are purely topological the latter do depend on the complex structure (through Ω). Both types of cubic couplings are totally symmetric. Note also that by the discussion above there is a one-to-one correspondence between charged matter fields and moduli: $27 \leftrightarrow (1,1)$ moduli and $\overline{27} \leftrightarrow (2,1)$ moduli[8].

These results for the couplings have been derived in the (classical) field theory limit and do not yet incorporate the extended nature of strings. This issue will be taken up next.

In general, to compute the string Yukawa couplings, one has to take into account sigma model and string perturbative and non-perturbative effects. One can show that both types of Yukawa couplings do not receive corrections from sigma model loops and string loops. The couplings $\bar{\kappa}$ are, in fact, also unmodified by non-perturbative effects on

---

one point (the Gepner point) in moduli space.

[7] We do not consider the second E_8 factor here. It belongs to the so-called hidden sector.

[8] This identification is a matter of convention. Here we have identified the $\bar{3}$ of $SU(3)$ with a holomorphic tangent vector index. The 3 of $SU(3)$ is a holomorphic cotangent vector index and one uses (Dolbeault theorem) $H^1(X,T^*_X) \simeq H^{1,1}(X)$.

the world-sheet, which, due to absence of string loop corrections, is just the sphere [30]. The couplings κ^0 do however receive corrections from world-sheet instantons [31]. These are non-trivial holomorphic embeddings of the world-sheet $\Sigma_0 \simeq \mathbb{P}^1$ in the CY manifold. In algebraic geometry they are known as rational curves C on X. Then there are still non-pertubative string-effects, i.e. possible contributions from infinite genus world-sheets. We do not know anything about them and will ignore them here. The possibility to incorporate them in the low-energy effective action has been discussed in [32].

The couplings in eq.(2.2) are computable in classical algebraic geometry, and, were they the whole truth to the Yukawa couplings of strings on CY manifolds, they would blatantly contradict the mirror hypothesis: κ^0 is independent of moduli whereas $\bar{\kappa}$ depends on the complex structure moduli. In fact, the mirror hypothesis states that the full $\langle 27^3 \rangle$ couplings on the manifold X depend on the Kähler moduli in such a way that they are related to the $\langle \overline{27}^3 \rangle$ couplings on the mirror manifold X^* via the mirror map. The main topic of these lectures is to explain what this means and to provide the tools to carry it through. The dependence on the Kähler moduli is the manifestation of string geometry and is solely due to the extended nature of strings.

When computing the Yukawa-couplings in conformal field theory as correlation functions of the appropriate vertex operators, inclusion of the non-perturbative σ-model effects means that in the path-integral one has to sum over all holomorphic embeddings of the sphere in X. This is in general not feasible since it requires complete knowledge of all possible instantons and their moduli spaces. In fact, this is where mirror symmetry comes to help.

We have thus seen that, modulo the remark on non-perturbative string effects, the Yukawa couplings are $\bar{\kappa}(X)$ and for the instanton corrected couplings $\kappa(X)$ one expects an expansion of the form ($\imath : C \hookrightarrow X$)

$$\kappa_{abc} = \kappa^0_{abc} + \sum_C \int_C \imath^*(h_a) \int_C \imath^*(h_b) \int_C \imath^*(h_c) \frac{e^{2\pi i \int_C \imath^*(J(X))}}{1 - e^{2\pi i \int_C \imath^*(J(X))}}, \tag{2.3}$$

which generalizes the Ansatz made in [12], which led to a successfull prediction for the numbers of instantons on the quintic hypersurface in $\mathbb{P}^4$, to the multi-moduli case. This ansatz was justified in ref. [33] in the framework of topological sigma models [34]. The sum is over all instantons C of the sigma model based on X and the denominator takes care of their multiple covers. J is the Kähler form on X.

One sees from (2.3) that as we go 'far out' in the Kähler cone to the 'large radius limit', the instanton corrections get exponentially supressed and one recovers the classical result.

3. Superconformal Field Theory and CY Compactification

Let us now turn to an alternative view of string compactification. We recall that the existence of a critical dimension is due to the requirement that the total central charge (matter plus ghost) of the Virasoro algebra vanishes. The critical dimensions for the bosonic and fermionic strings then follow from the central charges of the Virasoro algebras generated by the reparametrization and local $n = 1$ world-sheet supersymmetry ghost systems, which are $c = -26$ and $\hat{c} = \frac{2}{3}c = -10$, respectively. If we want a four-dimensional Minkowski space-time, we need four left-moving free bosonic fields and four right-moving free chiral superfields, contributing $(\bar{c}, c) = (4, 6)$ to the central charge. (Barred quantities refer to the left-moving sector.) Compactification might then be considered as an internal conformal field theory with central charge $(\bar{c}_{\text{int}}, c_{\text{int}}) = (22, 9)$. There are however internal conformal field theories which satisfy all consistency requirements (e.g. modular invariance, absence of dilaton tadpoles, etc.) which do not allow for a geometric interpretation as compactification. In the case of CY compactification, the internal conformal field theory is of a special type. The left moving central charge splits into a sum of two contributions, $\bar{c}_{\text{int}} = 22 = 13 + 9$, where the first part is due to a $E_8 \times SO(10)$ gauge sector (at level one; $E_8 \times SO(10)$ is simply laced of rank 13). The remaining contribution combines with the right-moving part to a symmetric (i.e. the same for both left and right movers) $(\bar{n}, n) = (2, 2)$ superconformal field theory with central charges $(9, 9)$. A right moving (global) $n = 2$ extended algebra is necessary for space-time supersymmetry [35], whereas the symmetry between left and right movers are additional inputs which allow for the geometrical interpretation as CY compactification with the spin connection embedded in the gauge connection [21].

The fact that we have a left as well as a right moving extended superconformal symmetry will be crucial for mirror symmetry. Before explaining this, let us briefly mention the relevant features of the $n = 2$ superconformal algebra [36]. It has four generators, the bosonic spin two energy momentum tensor T, two fermionic spin 3/2 super-currents $T_F^\pm$ and a bosonic spin one current J which generates a $U(1)$ Kac-Moody algebra. If we expand the fields in modes as $T(z) = \sum L_n z^{-n-2}$, $T_F^\pm(z) = \sum G_r^\pm z^{-r-3/2}$ and $J(z) = \sum J_n z^{-n-1}$ the algebra takes the form ($G_r^+ = (G_{-r}^-)^\dagger$)

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \\
\{G_r^\pm, G_s^\mp\} &= 2L_{r+s} \pm (r-s)J_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}\,, \quad \{G_r^\pm, G_s^\pm\} = 0 \\
[L_n, G_r^\pm] &= (\frac{n}{2} - r)G_{n+r}^\pm\,, \quad [L_n, J_m] = -mJ_{n+m} \\
[J_n, J_m] &= \frac{c}{3}n\delta_{m+n,0}\,, \quad [J_n, G_r^\pm] = \pm G_{n+r}^\pm
\end{aligned} \tag{3.1}$$

The moding of the fermionic generators is $r \in \mathbf{Z}$ in the Ramond (R) and $r \in \mathbf{Z} + \frac{1}{2}$ in the Neveu-Schwarz (NS) sector. The finite dimensional subalgebra in the NS sector, generated by $L_{0,\pm1}$, J_0 and $G_{\pm1/2}^\pm$ is $OSp(2|2)$. In a unitary theory we need

$$\langle\phi|\{G_r^\pm, G_{-r}^\mp\}|\phi\rangle = 2h \pm 2rq + \frac{c}{3}(r^2 - \frac{1}{4}) \geq 0 \tag{3.2}$$

for any state with $U(1)$ charge q (i.e. $J_0|\phi\rangle = q|\phi\rangle$) and conformal weight h (i.e. $L_0|\phi\rangle = h|\phi\rangle$). Setting $r = 0$ we thus find that in the R sector $h \geq \frac{c}{24}$ and in the NS sector (setting $r = \frac{1}{2}$) $h \geq \frac{|q|}{2}$.

There is a one-parameter isomorphism of the algebra, generated by $\mathcal{U}_\theta$, called spectral flow [37]

$$\begin{aligned}
\mathcal{U}_\theta L_n \mathcal{U}_\theta^{-1} &= L_n + \theta J_n + \frac{c}{6}\theta^2 \delta_{n,0} \quad \rightarrow \Delta h = \frac{3}{2c}\Delta(q^2) \\
\mathcal{U}_\theta J_n \mathcal{U}_\theta^{-1} &= J_n + \frac{c}{3}\theta \delta_{n,0} \quad \rightarrow \Delta q = +\frac{c}{3}\theta \\
\mathcal{U}_\theta G_r^\pm \mathcal{U}_\theta^{-1} &= G^\pm_{r\pm\theta}
\end{aligned} \tag{3.3}$$

States transform as $|\phi\rangle \rightarrow \mathcal{U}_\theta|\phi\rangle$. For $\theta \in \mathbf{Z}+\frac{1}{2}$ the spectral flow interpolates between the R and the NS sectors and for $\theta \in \mathbf{Z}$ it acts diagonally on the two sectors.

We have two commuting copies of the $n=2$ algebra. The left moving $U(1)$ current combines with the $SO(10)$ Kac-Moody algebra to form an E_6 algebra at level one. Hence the gauge group E_6 for CY compactifications. (The E_8 factor, which is also present, will play no role here.)

Let us first consider the R sector[9]. Ramond ground states $|i\rangle_R$ satisfy $G_0^\pm|i\rangle_R = 0$, i.e. $\{G_0^+, G_0^-\}|i\rangle_R = 0$. From (3.2) it follows that R ground states have conformal weight $h = \frac{c}{24}$. Under spectral flow by $\theta = \mp\frac{1}{2}$, the R ground states flow into *chiral/anti-chiral* primary states of the NS sector. They are primary states that satisfy the additional constraint $G^\pm_{-1/2}|i\rangle_{NS} = 0$. (Recall that primary states are annihilated by all positive modes of all generators of the algebra.) It follows from (3.3) that chiral/anti-chiral primary states satisfy $h = \pm\frac{1}{2}q$. The $OSp(2|2)$ invariant NS vacuum $|0\rangle$ is obviously chiral and anti-chiral primary. Under spectral flow by $\theta = \pm 1$ it flows to a unique chiral (anti-chiral) primary field $|\rho\rangle$ ($|\bar\rho\rangle$) with $h = \frac{c}{6}$ and $q = \pm\frac{c}{3}$. It follows from (3.2) that for chiral primary fields $h \leq \frac{c}{6}$.

We now look at the operator product of two chiral primary fields $\phi_i(z)\phi_j(w) \sim \sum_k (z-w)^{h_k - h_i - h_j}\psi_k(w)$ where the ψ_k are necessarily chiral but not necessarily primary. $U(1)$ charge conservation requires $q_k = q_i + q_j$ and due to the inequality $h \geq \frac{|q|}{2}$ with equality for primary fields we conclude that in the limit $z \rightarrow w$ only the chiral primaries survive. They can thus be multiplied pointwise and therefore form a ring $\mathcal{R}$ under multiplication: $\phi_i\phi_j = \sum_k c_{ij}{}^k\phi_k$ where the structure constants are functions of the moduli (cf. below). This ring is called chiral primary ring. The same obviously holds for anti-chiral fields forming the anti-chiral ring.

Note that spectral flow by θ is merely a shift of the $U(1)$ charge by $\frac{c}{3}\theta$ and the accompanying change in the conformal weight. Indeed, in terms of a canonically normalized boson ($\phi(z)\phi(w) = -\ln(z-w)+\ldots$) we can express the $U(1)$ current as $J(z) = \sqrt{\frac{c}{3}}\partial\phi(z)$. Any field[10] with $U(1)$ charge q can be written as $\phi_q = e^{i\sqrt{\frac{3}{c}}q\phi}\mathcal{O}$ where $\mathcal{O}$ is neutral under $U(1)$. The conformal weight of ϕ_q is $\frac{3}{2c}q^2 + h_{\mathcal{O}}$ and of $\phi_{q+\frac{c}{3}\theta}$ it is $\frac{3}{2c}(q+\frac{c}{3}\theta)^2 + h_{\mathcal{O}}$, in agreement with (3.3). We thus find that the spectral flow operator $\mathcal{U}_\theta$ can be written as $\mathcal{U}_\theta = e^{i\theta\sqrt{\frac{c}{3}}\phi}$ and also $\rho(z) = \mathcal{U}_1(z) = e^{i\sqrt{\frac{c}{3}}\phi(z)}$.

The foregoing discussion of course applies separately to the left and right moving sectors and we in fact have four rings: (c,c) and (a,c) and their conjugates (a,a) and

[9] The following paragraphs draw heavily from ref.[7].

[10] Recall the correspondence between states and fields: $|\phi\rangle = \lim_{z\rightarrow 0}\phi(z)|0\rangle$.

(c, a). Here c stands for chiral and a for anti-chiral.

We will now make the connection to our discussion of CY compactification and set $(\bar{c}, c) = (9, 9)$. We have already mentioned that $N = 1$ space-time supersymmetry for the heterotic string requires $n = 2$ superconformal symmetry for the right movers. This is however not sufficient. The additional requirement is that for states in the right-moving NS sector $q_R \in \mathbb{Z}$. The reason for this is the following [35]. The operator $\mathcal{U}_{\frac{1}{2}}(z)$ takes states in the right-moving NS sector to states in the right-moving R sector i.e. it transforms space-time bosons into space-time fermions (and vice versa). In fact, $\mathcal{U}_{\frac{1}{2}}(z) = e^{i\frac{\sqrt{3}}{2}\phi(z)}$ is the internal part of the gravitino vertex operator, which, when completed with its space-time and super-conformal ghost parts, must be local with respect to all the fields in the theory. When considering space-time bosons this leads to the requirement of integer $U(1)$ charges[11] (which have to be in the range $-3, \ldots, +3$)[12]. Since we are dealing with *symmetric* $(2, 2)$ superconformal theories both q_L and q_R must be integer for states in the (NS, NS) sector.

We now turn to the discussion of the moduli of the Calabi-Yau compactification. In an effective low energy field theory they are neutral (under the gauge group $E_6 \times E_8$) massless scalar fields with vanishing potential (perturbative and non-perturbative) whose vacuum expectation values determine the 'size' (Kähler moduli) and 'shape' (complex structure moduli) of the internal manifold. In the conformal field theory context they parameterize the perturbations of a given conformal field theory by exactly marginal operators. Exactly marginal operators can be added to the action without destroying (2,2) super-conformal invariance of the theory. Their multi-point correlation functions all vanish. In fact, one can show [31][6][38] that there is a one-to-one correspondence between moduli of the (2,2) superconformal field theory and chiral primary fields with conformal weight $(\bar{h}, h) = (\frac{1}{2}, \frac{1}{2})$. The chiral primary fields are (left and right) (anti)chiral superfields whose upper components (they survive the integral over chiral superspace) have conformal weight (1,1) and are thus marginal. The lower components with weights $(\frac{1}{2}, \frac{1}{2})$ provide the internal part of the charged matter fields (27 and $\overline{27}$ of E_6). The gauge part on the left moving side and the space-time and superconformal ghost parts on the right-moving side account for the remaining half units of conformal weight for the massless matter fields. We have thus a one-to-one correspondence between charged matter fields and moduli: extended world-sheet supersymmetry relates the 27's of E_6 to the marginal operators in the (c, c) ring and the $\overline{27}$'s of E_6 to the marginal operators in the (a, c) ring[13].

---

[11] To see this, take the operator product of the gravitino vertex operator (e.g. in the 1/2 ghost picture) $\psi_\alpha(z) = e^{-\varphi/2} S^\alpha e^{i\frac{\sqrt{3}}{2}\phi}(z)$ (S^α is a $SO(4)$ spin field) with a space-time boson (in the zero ghost picture) with vertex operator $V = (\text{s.t.}) e^{i\frac{q}{\sqrt{3}}\phi}$. The operator products of the spin field with the space-time parts (s.t) are either local, in which case we need $q = 2\mathbb{Z} + 1$, or have square root singularities, and we need $q = 2\mathbb{Z}$.

[12] Space-time supersymmetry and the existence of the unique state $|\rho\rangle$ with $q = \frac{c}{3}$ thus requires that c be an integer multiple of 3.

[13] Note that this identification is again a matter of convention. The arbitrariness here is due to a trivial symmetry of the theory under the flip of the relative sign of the left and right $U(1)$ charges (cf. below).

We have encountered the four chiral rings. The fields in the four rings can all be obtained from the (R,R) ground states via spectral flow. The additive structure of the rings is therefore isomorphic, not however their multiplicative structures. They are in general very different. The (c,c) ring contains fields with $U(1)$ charges $(q_L, q_R) = (+1,+1)$ whereas the (a,c) contains fields with $(q_L, q_R) = (-1,+1)$, both with conformal weights $(h_L, h_R) = (\frac{1}{2}, \frac{1}{2})$. The latter are related, via spectral flow, to states in the (c,c) ring with charges $(2,1)$.

We now turn to a comparison between the chiral primary states and the cohomology of the CY manifold. We expect a close relationship since the (2,2) super-conformal field theories we are considering correspond to conformally invariant sigma models with CY target space. Let us first look at the (c,c) ring. In the conformal field theory there is a unique chiral primary state with $(q_L, q_R) = (3,0)$ whereas there is the unique holomorphic three form Ω in the cohomology of the CY space. By conjugation we have the state with charge $(0,3)$ and $\bar{\Omega} \in H^{0,3}(X)$. Also $(q_L, q_R) = (3,3) \leftrightarrow \Omega \wedge \bar{\Omega}$. The fields with $(q_L, q_R) = (1,1)$ are marginal and correspond to the complex structure moduli, whereas the fields with $(q_L, q_R) = (2,1)$ are related (via spectral flow by $(-1,0)$) to marginal states in the (a,c) ring with charges $(-1,1)$. They correspond to the complex structure deformations. In general, if we identify the left and right $U(1)$ charges with the holomorphic and anti-holomorphic form degrees, respectively, one is tempted to establish a one-to-one correspondence between elements of the (c,c) ring with charges (q_L, q_R) and elements of $H^{q_L, q_R}(X)$. This becomes even more suggestive if we formally identify the zero modes of the supercurrents with the holomorphic exterior differential and co-differential as $G_0^+ \sim \partial$, $\bar{G}_0^+ \sim \bar{\partial}$. Via the spectral flow (by $(\theta_L, \theta_R) = (-\frac{1}{2}, -\frac{1}{2})$) we can also identify $G^+_{-\frac{1}{2}} \sim \partial$ and $\bar{G}^+_{-\frac{1}{2}} \sim \bar{\partial}$. Furthermore, one can show [7] that each NS state has a chiral primary representative in the sense that there exists a unique decomposition $|\phi\rangle = |\phi_0\rangle + G^+_{-\frac{1}{2}}|\phi_1\rangle + G^-_{\frac{1}{2}}|\phi_2\rangle$ with $|\phi_0\rangle$ chiral primary. For $|\phi\rangle$ itself primary, $|\phi_2\rangle$ is zero. This parallels the Hodge decomposition of differential forms. In fact, the one-to-one correspondence between the cohomology of the target space of supersymmetric sigma models and the Ramond ground states has been established in [39]. Let us now compare the (c,c) ring with the cohomology ring, whose multiplicative structure is defined by taking wedge products of the harmonic forms. We know from (2.2) that in the large radius limit the Yukawa couplings are determined by the cohomology of the Calabi-Yau manifold X. However, once string effects are taken into account, the Yukawa couplings are no longer determined by the cohomology ring of X but rather by a deformed cohomology ring. This deformed cohomology ring coincides with the (c,c) ring of the corresponding super-conformal field theory. For discussions of the relation between chiral rings and cohomology rings in the context of topological σ-model we refer to Witten's contribution in [9].

On the conformal field theory level, mirror symmetry is now the following simple observation: the exchange of the relative sign of the two $U(1)$ currents is a trivial symmetry of the conformal field theory, by which the (c,c) and the (a,c) rings of chiral primary fields are exchanged. On the geometrical level this does however have highly non-trivial implications. It suggests the existence of two topologically very different geometric interpretations of a given $(2,2)$ internal superconformal field theory. The deformed cohomology rings are isomorphic to the (c,c) and (a,c) rings, respectively. Here we associate elements in the

(a, c) ring with charge (q_L, q_R) with elements of H^{3+q_L, q_R}.

If we denote the two manifolds by X and X^* then one simple relation is in terms of their Hodge numbers:

$$h^{p,q}(X) = h^{3-p,q}(X^*) \tag{3.4}$$

The two manifolds X and X^* are referred to as a mirror pair. The relation between the Hodge numbers alone is not very strong. A farther reaching consequence of the fact that string compactification on X and X^* are identical is the relation between the deformed cohomology rings. This in turn entails a relation between correlation functions, in particular between the two types of Yukawa couplings.

It is however not clear whether a given $(2,2)$ theory always allows for two topologically distinct geometric descriptions as compactifications [14]. In fact, if one considers so-called rigid manifolds, i.e. CY manifolds with $h^{2,1} = 0$, then it is clear that the mirror manifold cannot be a CY manifold, which is Kähler, i.e. has $h^{1,1} \geq 1$. The concept of mirror symmetry for these cases has been exemplified on the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold in [40] and was further discussed in [41].

To close this section we want to make some general comments about the structure of the moduli space of Calabi-Yau compactifications which will be useful later on (see [11] for a review of these issues). In this context it is useful to note that instead of compactifying the heterotic string on a given Calabi-Yau manifold, one could have just as well taken the type II string. This would result in $N = 2$ space-time supersymmetry. In the conformal field theory language this means that we take the same $(2,2)$ super-conformal field theory with central charge $(\bar{c}, c) = (9,9)$ but this time without the additional gauge sector that was required for the heterotic string. This results in one gravitino on the left and on the right moving side each. The fact that the identical internal conformal field theory might also be used to get a $N = 2$ space-time supersymmetric theory leads to additional insight into the structure of the moduli space which is the same for the heterotic as for the type II string and which has to satisfy additional constraints coming from the second space-time supersymmetry. This was used in [42] to show that locally the moduli manifold has the product structure

$$\mathcal{M} = \mathcal{M}_{h^{1,1}} \times \mathcal{M}_{h^{2,1}} \tag{3.5}$$

where $\mathcal{M}_{h^{1,1}}, \mathcal{M}_{h^{2,1}}$ are two Kähler manifolds with complex dimensions $h^{1,1}$ and $h^{2,1}$, respectively. The same result was later derived in refs. [43] [44] [45]. In ref.[45] it was shown to be a consequence of the $(2,2)$ super-conformal algebra. $N = 2$ space-time supersymmetry or super-conformal Ward identities can be used to show that each factor of $\mathcal{M}$ is a so-called special Kähler manifold. Special Kähler manifolds are characterized by a prepotential F from which the Kähler potential (and thus the Kähler metric) and also the Yukawa couplings can be computed. Locally on special Kähler manifolds there exist so-called special coordinates t_i which allow for simple expressions of the Kähler potential

[14] Mirror symmetric manifolds are excluded if $\chi \neq 0$.

K and the Yukawa couplings κ in terms of the prepotential as[15] [46]

$$
\begin{aligned}
K &= -\ln\Big[2(F-\bar{F}) - (t_i - \bar{t}_i)(F_i + \bar{F}_i)\Big]\,, \qquad \left(F_i = \frac{\partial F}{\partial t_i}\right) \\
\kappa_{ijk} &= F_{ijk}
\end{aligned}
\tag{3.6}
$$

with one set for each factor of $\mathcal{M}$[16]. To reproduce the classical Yukawa couplings (2.2) $F^{1,1}(X)$ is simply a cubic polynomial whereas $F^{2,1}$ is a complicated function of the complex structure moduli. Instanton corrections will modify $F^{1,1}$ such as to reproduce the couplings (2.3). Mirror symmetry then relates the pre-potentials on X and X^*. If λ_i are the complex structure moduli on X^* then one can find a local map (the mirror map) $\mathcal{M}_{2,1}(X^*) \to \mathcal{M}_{1,1}(X) : \lambda_i \mapsto t_i(\lambda)$ such that $F^{2,1}(\lambda)(X^*) = F^{1,1}(t)(X)$ and similarly for $F^{1,1}(X^*)$ and $F^{2,1}(X)$. We will find below that $\lambda_i(t)$ are transzendental functions containing exponentials. It is an interesting fact that $\lambda(q^i)$, with $q = e^{2\pi i t}$, is always an infinite series with integer coefficients.

4. Construction of Mirror Pairs

We have already alluded to the fact that a classification of consistent string vacua is still out of reach. In fact, even a classification of three-dimensional CY manifolds, providing just a subset of string vacua, is still lacking. (One only knows how to classify the homotopy types of the manifolds by virtue of Wall's theorem; [23], p. 173.) What has been achieved so far is to give complete lists of possible CY manifolds within a given construction. But even here one does not have criteria to decide which of these manifolds are diffeomorphic.

The constructions that have been completely searched for are hypersurfaces in four-dimensional weighted projective space [47] and complete intersections of k hypersurfaces in $3+k$ dimensional products of projective spaces [48]. In these notes we will limit ourselves to the discussion of hypersurfaces. Mirror symmetry for complete intersections has been discussed in [49] and [17].

Weighted n-dimensional complex projective space $\mathbb{P}^n[\vec{w}]$ is simply $\mathbb{C}^{n+1}\backslash\{0\}/\mathbb{C}^*$ where $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ acts as $\lambda\cdot(z_0,\ldots,z_n) = (\lambda^{w_0}z_0,\ldots,\lambda^{w_n}z_n)$. We will denote a point in $\mathbb{P}^n[\vec{w}]$ by $(z_0:\cdots:z_n)$. The coordinates z_i are called homogeneous coordinates of $\mathbb{P}^n[\vec{w}]$ [17] and $w_i \in \mathbb{Z}_+$ their weights. For $\vec{w} = (1,\ldots,1)$ one recovers ordinary projective space. Note however that $\mathbb{P}^n[k\vec{w}] \sim \mathbb{P}^n[\vec{w}]$. In fact, due to this and other isomorphisms (see Prop. 1.3.1 in [50]) one only needs to consider so-called well-formed weighted projective spaces. $\mathbb{P}^n[\vec{w}]$ is called well-formed if each set of n weights is co-prime. Weighted projective $\mathbb{P}^n[\vec{w}]$ can be covered with $n+1$ coordinate patches U_i with $z_i \neq 0$ in U_i. The transition functions

[15] From this it follows immediately that the $\langle\overline{27}^3\rangle$ couplings receive no instanton corrections since they would lead to a dependence on the Kähler moduli.

[16] We will denote the special coordinates on $\mathcal{M}_{h^{1,1}}$ by t and the ones on $\mathcal{M}_{h^{2,1}}$ by λ.

[17] We will sometimes denote them by $z_0,\ldots,z_n$ and othertimes by $z_1,\ldots,z_{n+1}$. We are confident that this will cause no confusion.

between different patches are then easily obtained. The characteristic feature to note about the transition functions of projective space is that in overlaps $U_i \cap U_j$ they are given by Laurent monomials. For example, consider $\mathbb{P}^2$ with homogeneous coordinates $(z_0 : z_1 : z_2)$ and the three patches U_i, $i = 0, 1, 2$ with inhomogeneous coordinates $\varphi_0(z_0 : z_1 : z_2) = (u_0, v_0) = (\frac{z_1}{z_0}, \frac{z_2}{z_0})$, $\varphi_1(z_0 : z_1 : z_2) = (u_1, v_1) = (\frac{z_0}{z_1}, \frac{z_2}{z_1})$ and $\varphi_2(z_0 : z_1 : z_2) = (u_2, v_2) = (\frac{z_0}{z_2}, \frac{z_1}{z_2})$. The transition functions on overlaps are then Laurent monomials; e.g. on $U_0 \cap U_1$ we have $(u_1, v_1) = (\frac{1}{u_0}, \frac{v_0}{u_0})$, and we have $\varphi_i(U_i \cap U_j) \simeq \mathbb{C} \times \mathbb{C}^*$, $\varphi_i(U_1 \cap U_2 \cap U_3) = (\mathbb{C}^*)^2$. The reason for dwelling on these well-known facts is that below we will treat this example also in the language of toric geometry and that it is the fact that transition functions between patches are Laurent monomials, that characterizes more general toric varieties.

Weighted projective spaces are generally singular. As an example, consider $\mathbb{P}^2[1,1,2]$, i.e. (z_0, z_1, z_2) and $(\lambda z_0, \lambda z_1, \lambda^2 z_2)$ denote the same point and for $\lambda = -1$ the point $(0:0:z_2) \equiv (0:0:1)$ is fixed but λ acts non-trivially on its neighborhood: we thus have a $\mathbb{Z}_2$ orbifold singularity at this point.

A hypersurface X in (weighted) projective space is defined as the vanishing locus of a (quasi)homogeneous polynomial, i.e. of a polynomial in the homogeneous coordinates that satisfies $p(\lambda^{w_0} z_0, \ldots, \lambda^{w_n} z_n) = \lambda^d p(z_0, \ldots, z_n)$ where $d \in \mathbb{Z}_+$ is called the degree of $p(z)$; i.e. we have

$$X_{\vec{w}} = \Big\{ (z_0 : \ldots : z_n) \in \mathbb{P}^n[\vec{w}] \Big| p(z) = 0 \Big\} \tag{4.1}$$

In order for the hypersurface to be a CY manifold one has to require that its first Chern class vanishes. This can be expressed in terms of the weights and the degree of the defining polynomial as[23]

$$c_1(X) = \Big(\sum_{i=0}^{n} w_i - d \Big) J \tag{4.2}$$

where J is the Kähler form of the projective space the manifold is embedded in. A necessary condition for a hypersurface in $\mathbb{P}^4[\vec{w}]$ to be a three-dimensional CY manifold is then that the degree of the defining polynomial equals the sum of the weights of the projective space. However one still has to demand that the embedding $X \hookrightarrow \mathbb{P}^n[\vec{w}]$ be smooth. This means that one has to require the transversality condition: $p(z) = 0$ and $dp(z) = 0$ have no simultaneous solutions other than $z_0 = \ldots = z_n = 0$ (which is not a point of $\mathbb{P}^n[\vec{w}]$).

There exists an easy to apply criterium [51] which allows one to decide whether a given polynomial satisfies the transversality condition. This criterium follows from Bertini's theorem (c.f. e.g. Griffiths and Harris in [22]) and goes as follows. For every index set $J = \{j_1, \ldots, j_{|J|}\} \subset \{0, \ldots, n\} = N$ denote by $z_J^{m^{(k)}}$ monomials $z_{j_1}^{m^{(k)}_{i_1}} \cdots z_{i_{|J|}}^{m^{(k)}_{i_{|J|}}}$ of degree d. Transversality is then equivalent to the condition that for every index set J there exists either (a) a monomial z_J^m, or (b) $|J|$-monomials $z_J^{m^{(k)}} z_k$ with $|J|$ distinct $k \in N \setminus J$.

Analysis of this condition shows that there are 7555 projective spaces $\mathbb{P}^4[\vec{w}]$ which admit transverse hypersurfaces. They were classified in [47]. In case that $q_i = d/w_i \in \mathbb{Z}$, $\forall i = 0, \ldots, k$ one gets a transverse polynomial of Fermat type: $p(z) = \sum_{i=0}^4 z_i^{q_i}$.

If the hypersurfaces X meet some of the singularities of the weighted projective space, they are themselves singular. Let us first see what kind of singular sets one can get. If the weights w_i for $i \in I$ have a common factor N_I, the singular locus S_I of the CY space is the intersection of the hyperplane $H_I = \{z \in \mathbb{P}^4[\vec{w}] | z_i = 0 \,\text{for}\, i \notin I\}$ with $X_{\vec{w}}$. We will see that the singular locus consists either of points or of curves.

As $\mathbb{P}^4[\vec{w}]$ is wellformed we have $|I| \leq 3$. Consider $|I| = 3$ and apply the transversality criterum to $J = I$. Obviously, only transversality condition (a) can hold, which implies that p will not vanish identically on H_I, hence $\dim(S_I) = 1$. It is important for the following to consider the $\mathbb{C}^*$-action on the normal bundle[18] to this curve. We write the $c_1(X) = 0$ condition as $\sum_{i \in I} w'_i + \sum_{j \notin I}(w_j/N_I) = (d/N_I)$, with $w'_i \in \mathbb{Z}_>$ and $(w_j/N_I) \notin \mathbb{Z}$. Because of (a) one has $d = \sum_{i \in I} m_i w_i = N_I \sum_{i \in I} m_i w'_i$, for $m_i \in \mathbb{Z}_{\geq}$, from which we can conclude $\sum_{j \notin I}(w_i/N_I) \in \mathbb{Z}$. Locally we can then choose (z_{k_1}, z_{k_2}) with $k_i \notin I$ as the coordinates normal to the curve. The $\mathbb{C}^*$-action which fixes S_I will therefore be generated by $(z_{k_1}, z_{k_2}) \mapsto (\lambda z_{k_1}, \lambda^{-1} z_{k_2})$, where we define $\lambda = e^{2\pi i/N_I}$. That is, locally the singularity in the normal bundle is of type $\mathbb{C}^2/\mathbb{Z}_{N_I}$.

Finally for $|I| \leq 2$ clearly $\dim(S_I) \leq 1$. From the analysis of the divisibility condition imposed by transversality and $c_1(X) = 0$ we can summarize that the $\mathbb{C}^*$ action in the normal bundle of singular curves or the neighborhood of singular points is in local coordinates always of the form $\mathbb{C}^2/\mathbb{Z}_{N_I}$ and $\mathbb{C}^3/\mathbb{Z}_{N_I}$ with

$$
\begin{aligned}
\lambda \cdot (z_1, z_2) &= (\lambda z_1, \lambda^{-1} z_2) \\
\lambda \cdot (z_1, z_2, z_3) &= (\lambda z_1, \lambda^a z_2, \lambda^b z_3) \quad \text{with} \quad 1 + a + b = N_I,\ a, b \in \mathbb{Z}
\end{aligned}
\tag{4.3}
$$

Note that for the case of fixed curves invariant monomials are $z_1^{N_I}, z_2^{N_I}$ and $z_1 z_2$, i.e. we can describe the singularity as $\{(u, v, w) \in \mathbb{C}^3 | uv = w^{N_I}\}$. This type of singularity is called a rational double point of type $A_{N_I - 1}$. The relation with the Lie-algebras from the A series and the discussion of the resolution of the singularities within toric geometry will be given below.

Let us mention that the types of singularities encountered here are the same as the ones in abelian toroidal orbifolds, discussed in [52] [53]. The condition on the exponents of the $\mathbb{C}^*$-action there is related to the fact that one considers only subgroups of $SU(3)$ as orbifold groups. This was identified as a necessary condition to project out of the spectrum three gravitinos in order to obtain exactly $N = 1$ spacetime supersymmety [54]. As we will briefly explain below, it also ensures $c_1(\hat{X}) = 0$, i.e. triviality of the canonical bundle of the resolved manifold $\hat{X}$. Desingularisations with this property are referred to as minimal desingularizations [55]. In ref.[47] the Hodge numbers of the minimal desingularizations of all 7555 transverse hypersurfaces were evaluated.

Among the singular points we have to distinguish between isolated points and exceptional points, the latter being singular points on singular curves or the points of intersection of singular curves. The order N_I of the isotropy group I of exceptional points exceeds that of the curve. In order to get a smooth CY manifold $\hat{X}$, these singularities have to be resolved by removing the singular sets and replacing them by smooth two complex dimensional manifolds which are then called exceptional divisors. Each exceptional divisor D provides, by Poincaré duality, a harmonic $(1,1)$ form h_D: $\int_D \alpha = \int_{\hat{X}} \alpha \wedge h_D$ for every

[18] The normal bundle on C is the quotient bundle $N_C = T_X|_C / T_C$.

closed $(2,2)$ form α and $h^{1,1}(\hat{X}) = \#$ exceptional divisors $+1$ where the last contribution counts the restriction of the Kähler form from to embedding space to $\hat{X}$ (to which one can also associate a divisor).

There are only few hypersurfaces, namely the quintic (i.e. $d=5$) in $\mathbb{P}^4$, the sextic in $\mathbb{P}^4[1,1,1,1,2]$, the octic in $\mathbb{P}^4[1,1,1,1,4]$ and finally the dectic in $\mathbb{P}^4[1,1,1,2,5]$, which do not require resolution of singularities and the Kähler form they inherit from the embedding space is in fact the only one, i.e. for these CY spaces $h^{1,1}=1$. One easily sees that these Fermat hypersurfaces do not meet the singular points of their respective embedding spaces. They were analyzed in view of mirror symmetry in [13].

If one considers the lists of models of ref.[47] one finds already on the level of Hodge numbers that there is no complete mirror symmetry within this construction. Also, if one includes abelian orbifolds of the hypersurfaces [56] the situation does not improve.

One can however get a mirror symmetric set of CY manifolds if one generalizes the construction to hypersurfaces in so-called toric varieties. Since they are not (yet) familiar to most physicists but relevant for describing the resolution of the above encountered singularities and for the discussion of mirror symmetry, we will give a brief description of toric varieties. For details and proofs we have to refer to the literature [57]. Toric methods were first used in the construction of Calabi-Yau manifolds in [55] and [52]. They have entered the discussion of mirror symmetry through the work of V. Batyrev [58] [59] [60] [49][61].

Toric varieties are defined in terms of a lattice $N \simeq \mathbb{Z}^n$ and a fan Σ. Before explaining what a fan is, we first have to define a (*strongly convex rational polyhedral*) cone (or simply cone) σ in the real vector space $N_{\mathbb{R}} \equiv N \otimes_{\mathbb{Z}} \mathbb{R}$:

$$\sigma = \Big\{\sum_{i=1}^{s} a_i n_i ; a_i \geq 0\Big\} \tag{4.4}$$

where n_i is a finite set of *lattice* vectors (hence rational), the generators of the cone. We often write simply $\langle n_1, \ldots, n_s\rangle$. Strong convexity means that $\sigma \cap (-\sigma) = \{0\}$, i.e. the cone does not contain lines through the origin. A cone σ is called *simplicial* if it is generated by linearly independent (over $\mathbb{R}$) lattice vectors. A cone generated by part of a *basis* of the lattice N is called a basic cone. If we normalize the unit cell of the lattice to have volume one, then a simplicial cone $\langle n_1, \ldots n_m\rangle$ is basic if $\det(n_1, \ldots, n_m) = 1$ (here n_i are the generators of minimal length). If a cone fails to be basic, not all lattice points within the cone can be reached as linear combinations of the generators with positive integer coefficients.

To every cone we can define the dual cone $\sigma^\vee$ as

$$\sigma^\vee = \Big\{x \in M_{\mathbb{R}} ; \langle x, y\rangle \geq 0 \text{ for all } y \in \sigma\Big\} \tag{4.5}$$

where M is the lattice dual to N and $\langle\,,\rangle : M \times N \to \mathbb{Z}$, which extends to the $\mathbb{R}$ bilinear pairing $M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$. $\sigma^\vee$ is rational with respect to M but strongly convex only if $\dim(\sigma) = \dim(M_{\mathbb{R}})$. (For instance the cones dual to one-dimensional cones in $\mathbb{R}^2$, are half-planes.). Also $(\sigma^\vee)^\vee = \sigma$. Given a cone we define

$$S_\sigma = \sigma^\vee \cap M \tag{4.6}$$

which is finitely generated by say $p \geq \dim M \equiv d_M$ lattice vectors n_i. In general (namely for $p \geq d_M$) there will be non-trivial linear relations between the generators of S_σ which can be written in the form

$$\sum \mu_i n_i = \sum \nu_i n_i \tag{4.7}$$

with μ_i and ν_i non-negative integers. A cone then defines an affine toric variety as

$$U_\sigma = \{(Z_1, \ldots, Z_p) \in \mathbb{C}^p | Z^\mu = Z^\nu\} \tag{4.8}$$

where we have used the short hand notation $Z^\mu = Z_1^{\mu_1} \cdots Z_p^{\mu_{d_M}}$. In the mathematics literature one often finds the notation $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma]) = \mathrm{Spec}(\mathbb{C}[Z_1, \ldots, Z_p]/I)$ where the ideal I is generated by all the relations $Z^\mu = Z^\nu$ between the gerators of S_σ.

To illustrate the construction, let us look at a simple example. Consider the cone σ generated by $(N+1)e_1 - Ne_2$ and e_2. Then S_σ is easily shown to be generated by $n_1 = e_1^*$, $n_2 = Ne_1^* + (N+1)e_2^*$ and $n_3 = e_1^* + e_2^*$ which satisfy $(N+1)n_3 = n_1 + n_2$. This leads to

$$U_\sigma = \{(Z_1, Z_2, Z_3) \in \mathbb{C}^3 | Z_1 Z_2 = Z_3^{N+1}\} \tag{4.9}$$

This is just the A_N rational double point discussed above.

A face τ of a cone σ is what one expects; it can be defined as $\sigma \cap u^\perp = \{v \in \sigma : \langle u, v\rangle = 0\}$ for some $u \in \sigma^\vee$. The constructive power of toric geometry relies in the possibility of gluing cones to fans.

A fan Σ is a family of cones σ satisfying:

(i) any face of a cone in Σ is itself a cone in Σ;

(ii) the intersection of any two cones in Σ is a face of each of them.

After having associated an affine toric variety U_σ with a cone σ, we can now construct a toric variety $\mathbb{P}_\Sigma$ associated to a fan Σ by glueing together the U_σ, $\sigma \in \Sigma$:

$$\mathbb{P}_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma \tag{4.10}$$

The U_σ are open subsets of $\mathbb{P}_\Sigma$. The glueing works because $U_{\sigma \cap \sigma'}$ is an open subset of both U_σ and U'_σ i.e. $U_{\sigma \cap \sigma'} = U_\sigma \cap U_{\sigma'}$. On the other hand $U_{\sigma^\vee \cap (\sigma')^\vee} \neq U_{\sigma^\vee} \cap U_{(\sigma')^\vee}$, hence gluing is natural for the cones σ. A last result we want to quote before demonstrating the above with an example is that a toric variety $\mathbb{P}_\Sigma$ is compact iff the union of all its cones is the whole space $\mathbb{R}^n$. Such a fan is called complete.

To see what is going on we now give a simple but representative example. Consider the fan Σ whose dimension one cones τ_i are generated by the vectors $n_1 = e_1$, $n_2 = e_2$ and $n_3 = -(e_1 + e_2)$. Σ also contains the dimension two cones $\sigma_1 : \langle n_1, n_2\rangle$, $\sigma_2 : \langle n_2, n_3\rangle$, $\sigma_3 : \langle n_3, n_1\rangle$ and of course the dimension zero cone $\{0\}$. We first note that Σ satisfies the compactness criterium. The two-dimensional dual cones are $\sigma_1^\vee : \langle e_1^*, e_2^*\rangle$, $\sigma_2^\vee : \langle -e_1^*, -e_1^* + e_2^*\rangle$, $\sigma_3^\vee : \langle -e_2^*, e_1^* - e_2^*\rangle$ and $U_{\sigma_1} = \mathrm{Spec}(\mathbb{C}[X, Y])$, $U_{\sigma_2} = \mathrm{Spec}(\mathbb{C}[X^{-1}, X^{-1}Y])$, $U_{\sigma_3} = \mathrm{Spec}(\mathbb{C}[Y^{-1}, XY^{-1}])$ each isomorphic to $\mathbb{C}^2$. These glue together to form $\mathbb{P}^2$. Indeed, if we define coordinates u_i, v_i via $U_{\sigma_i} = \mathrm{Spec}(\mathbb{C}[u_i, v_i])$ we get the transition functions for $\mathbb{P}^2$ between the three patches. Note also that $U_{\tau_i} = U_{\sigma_i \cap \sigma_j} \simeq \mathbb{C} \times \mathbb{C}^*$ and $U_{\sigma_1 \cap \sigma_2 \cap \sigma_3} = U_{\{0\}} = (\mathbb{C}^*)^2$. To get e.g. the weighted projective space $\mathbb{P}^2[1,2,3]$ one simply has to replace the generator n_3 by $n_3 = -(2e_1 + 3e_2)$. In fact, all projective spaces are toric varieties. This will become clear below.

We now turn to the discussion of singularities of toric varieties and their resolution. We have already given the toric description of the rational double point. The reason why the corresponding cone leads to a singular variety is because it is not basic, i.e. it can not be generated by a basis of the lattice N. This in turn results in the need for three generators for S_σ which satisfy one linear relation. The general statement is that U_σ is smooth if and only if σ is basic. Such a cone will also be called smooth. An n-dimensional toric variety X_Σ is smooth, i.e. a complex manifold, if and only if all dimension n cones in Σ are smooth. We do however have to require more than smoothness. We also want to end up with a CY manifold, i.e. a smooth manifold with $c_1 = 0$.

It can be shown that if Σ is a smooth fan (i.e. all its cones are smooth) X_Σ has trivial canonical bundle if and only if the endpoints of the minimal generators of all one-dimensional cones in Σ lie on a hyperplane (see the proof in the appendix to [52]). The intersection of the hyperplane with the fan is called the trace of the fan. An immediate consequence of this result is that a compact toric variety, i.e. corresponding to a complete fan, can never have $c_1 = 0$. One therefore has to consider hypersurfaces in compact toric varieties to obtain CY manifolds.

The singularities we are interested in are cyclic quotient singularities. A standard result in toric geometry states that X_Σ has only quotient singularities, i.e. is an orbifold, if Σ is a simplicial fan, i.e. if all cones in Σ are simplicial. Given a singular cone one resolves the singularities by subdividing the cone into a fan such that each cone in the fan is basic.

Let us demonstrate this on the rational A_N double point. The two-dimensional cone $\sigma^{(2)}$ was generated by $(N+1)e_1 - Ne_2$ and e_2 which is not a basis of the lattice $N = \mathbb{Z}^2$. We now add the one-dimensional cones $\sigma_m^{(1)}$ generated by $me_1 - (m-1)e_2$ for $m = 1, \ldots, N$. (In this notation σ_0 and σ_{N+1} are the original one-dimensional cones.) The two-dimensional cones $\sigma_m^{(2)} : \langle me_1 - (m-1)e_2, (m+1)e_1 - me_2 \rangle$, $m = 1, \ldots, N$ are then basic and furthermore we have a $c_1 = 0$ resolution.

The original singular manifold has thus been desingularized by gluing exceptional divisors D_m, $m = 1, \ldots, N$. To cover the nonsingular manifold one needs $N + 1$ patches. One generator of $(\sigma_i^{(2)})^\vee$ and one of $(\sigma_{i+1}^{(2)})^\vee$ are antiparallel in $\Sigma^\vee$. These generators therefore correspond to inhomogenous coordinates of one of N $\mathbb{P}^1$'s. The exceptional divisors are therefore $\mathbb{P}^1$'s. By inspection of the various patches one can see that the $\mathbb{P}^1$'s intersect pairwise transversely in one point to form a chain. The self-intersection number is obtained as the degree of their normal bundles, which is -2, so that the intersection matrix is the (negative of) the Cartan matrix of A_N; for details we refer to Fulton, ref.[57]. Such a collection of $\mathbb{P}^1$'s is called a Hirzebruch-Jung sphere-tree [62]. Recalling that the rational double points appeared in the discussion of *curve* singularities of CY manifolds we have seen that the resolved singular curves are locally the product of the curve C and a Hirzebruch-Jung sphere-tree.

Data of toric varieties depend only on linear relations and we may apply bijective linear transformations to choose a convenient shape. E.g. given the canonical basis e_1, e_2 in $\mathbb{R}^2$, we may use e_i as generators of the cone and $n_1 = \frac{1}{N+1}e_1 + \frac{N}{N+1}e_2$, $n_2 = e_2$, as generators of $N = \mathbb{R}^2$ to describe the A_N singularity. This form generalizes easily to higher dimensional cyclic singularities such as $\mathbb{C}^3/\mathbb{Z}_N$, the general form of the *point* singularities (4.3). They are described by e_i, $i = 1, 2, 3$ as generators of the cone and the lattice basis $n_1 = \frac{1}{N}(e_1 + ae_2 + be_3)$, $n_2 = e_2$, $n_3 = e_3$. The local desingularisation process consists again of adding further generators such as to obtain a smooth fan. One readily sees that as a consequence of (4.3) all endpoints of the vectors generating the nonsingular fan lie on the plane $\sum_{i=1}^{3} x_i = 1$, as it is necessary for having a trivial canonical bundle on the resolved manifold. The exceptional divisors are in 1-1 correspondence with lattice points inside the cone on this hyperplane (trace of the fan). Their location is given by

$$\mathcal{P} = \left\{ \sum_{i=1}^{3} \vec{e}_i \frac{a_i}{N} \,\middle|\, (a_1, a_2, a_3) \in \mathbb{Z}^3, \begin{pmatrix} e^{2\pi i \frac{a_1}{N}} & & \\ & e^{2\pi i \frac{a_2}{N}} & \\ & & e^{2\pi i \frac{a_3}{N}} \end{pmatrix} \in \mathbb{Z}_N, \sum_{i=1}^{3} a_i = N, a_i \geq 0 \right\} \tag{4.11}$$

The generator of the isotropy group $\mathbb{Z}_N$ on the coordinates of the normal bundle of the singular point is given in (4.3) and $\vec{e}_1, \vec{e}_2, \vec{e}_3$ span an equilateral triangle from its center. The corresponding divisors can all be described by compact toric surfaces which have been classified.

The toric diagrams for the resolution of singular points are thus equilateral triangles with interior points. If we have exceptional points then there will also be points on the edges, which represent the traces of the fans of the singular curves.

Whereas in the case of curve singularities, whose resolution was unique, this is not so for point singularities. Given the trace with lattice points in its interior, there are in general several ways to triangulate it. Each way corresponds to a different resolution. They all lead to topologically different smooth manifolds with the same Hodge numbers, differing however in their intersection numbers.

To obtain a Kähler manifold one also has to ensure that one can construct a positive Kählerform (2.1). This is guaranteed if one can construct an upper convex piecewise linear function on Σ (see Fulton, ref. [57]). The statement is then (see e.g. [60]) that the cone $K(\Sigma)$ consisting of the *classes* of all upper convex piecewise linear functions (i.e. modulo globally linear functions) on Σ is isomorphic to the Kähler cone of $\mathbb{P}_\Sigma$. This construction is reviewed in [16]. Below we will consider hypersurfaces in $\mathbb{P}_\Sigma$. The positive Kählerform on the hypersurface is then the one induced from the Kähler form on $\mathbb{P}_\Sigma$.

We will now give another description of toric varieties, namely in terms of convex integral polyhedra. The relation between the two constructions will become clear once we have demonstrated how to extract the cones and the fan from a given polyhedron. The reason for introducing them is that this will allow for a simple and appealing description of the mirror operation and of the construction of mirror pairs [59].

We start with a few definitions. We consider *rational* (with respect to a lattice N) *convex polyhedra* (or simply polyhedra) $\Delta \subset N_\mathbb{R}$ containing the origin $\nu_0 = (0, 0, 0, 0)$. Δ is called *reflexive* if its dual defined by

$$\Delta^* = \{ \, (x_1, \ldots, x_4) \mid \sum_{i=1}^{4} x_i y_i \geq -1 \text{ for all } (y_1, \ldots, y_4) \in \Delta \, \} \tag{4.12}$$

is again a rational polyhedron. Note that if Δ is reflexive, then Δ^* is also reflexive since

$(\Delta^*)^* = \Delta$. We associate to Δ a complete rational fan $\Sigma(\Delta)$ whose cones are the cones over the faces of Δ with apex at ν_0.

The l dimensional faces will be denoted by Θ_l. Completeness is ensured since Δ contains the origin. The toric variety $\mathbb{P}_\Delta$ is then the toric variety associated to the fan $\Sigma(\Delta^*)$, i.e. $\mathbb{P}_\Delta \equiv \mathbb{P}_{\Sigma(\Delta^*)}$.

Denote by ν_i, $i = 0, \ldots, s$ the integral points in Δ and consider the affine space $\mathbb{C}^{s+1}$ with coordinates a_i. We will consider the zero locus Z_f of the Laurent polynomial

$$f_\Delta(a, X) = a_0 - \sum_i a_i X^{\nu_i}, \quad f_\Delta(a, X) \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_4^{\pm 1}] \tag{4.13}$$

in the algebraic torus $(\mathbb{C}^*)^4 \subset \mathbb{P}_\Delta$, and its closure $\bar{Z}_f$ in $\mathbb{P}_\Delta$. Here we have used the convention $X^\mu \equiv X_1^{\mu_1} \ldots X_4^{\mu_4}$. Note that by rescaling the four coordinates X_i and adjusting an overall normalization we can set five of the parameters a_i to one.

$f \equiv f_\Delta$ and Z_f are called Δ-regular if for all $l = 1, \ldots, 4$ the f_{Θ_l} and $X_i \frac{\partial}{\partial X_i} f_{\Theta_l}$, $\forall i = 1, \ldots, n$ do not vanish simultaneously in $(\mathbb{C}^*)^4$. This is equivalent to the transversality condition for the quasi-homogeneous polynomials p. Varying the parameters a_i under the condition of Δ-regularity, we get a family of toric varieties.

In analogy with the situation of hypersurfaces in $\mathbb{P}^4[\vec{w}]$, the more general ambient spaces $\mathbb{P}_\Delta$ and so $\bar{Z}_f$ are in general singular. Δ-regularity ensures that the only singularities of $\bar{Z}_f$ are the ones inherited from the ambient space. $\bar{Z}_f$ can be resolved to a CY manifold $\hat{Z}_f$ iff $\mathbb{P}_\Delta$ has only so-called Gorenstein singularities, which is the case iff Δ is reflexive [59].

The families of the CY manifolds $\hat{Z}_f$ will be denoted by $\mathcal{F}(\Delta)$. The above definitions proceed in an exactly symmetric way for the dual polyhedron Δ^* with its integral points ν_i^* $(i = 0, \cdots, s^*)$, leading to families of CY manifolds $\mathcal{F}(\Delta^*)$.

In ref. [59] Batyrev observed that a pair of reflexive polyhedra (Δ, Δ^*) naturally provides a pair of mirror CY families $(\mathcal{F}(\Delta), \mathcal{F}(\Delta^*))$ as the following identities for the Hodge numbers hold

$$\begin{aligned} h^{1,1}(\hat{Z}_{f,\Delta^*}) &= h^{2,1}(\hat{Z}_{f,\Delta}) \\ &= l(\Delta) - 5 - \sum_{\mathrm{codim}\Theta=1} l'(\Theta) + \sum_{\mathrm{codim}\Theta=2} l'(\Theta) l'(\Theta^*) \\ h^{1,1}(\hat{Z}_{f,\Delta}) &= h^{2,1}(\hat{Z}_{f,\Delta^*}) \\ &= l(\Delta^*) - 5 - \sum_{\mathrm{codim}\Theta^*=1} l'(\Theta^*) + \sum_{\mathrm{codim}\Theta^*=2} l'(\Theta^*) l'(\Theta). \end{aligned} \tag{4.14}$$

Here $l(\Theta)$ and $l'(\Theta)$ are the number of integral points on a face Θ of Δ and in its interior, respectively (and similarly for Θ^* and Δ^*). An l-dimensional face Θ can be represented by specifying its vertices $\mathrm{v}_{i_1}, \cdots, \mathrm{v}_{i_k}$. Then the dual face defined by $\Theta^* = \{x \in \Delta^* \mid (x, \mathrm{v}_{i_1}) = \cdots = (x, \mathrm{v}_{i_k}) = -1\}$ is a $(n - l - 1)$-dimensional face of Δ^*. By construction $(\Theta^*)^* = \Theta$, and we thus have a natural pairing between l-dimensional faces of Δ and $(n - l - 1)$-dimensional faces of Δ^*. The last sum in each of the two equations in

(2.4) is over pairs of dual faces. Their contribution cannot be associated with a monomial in the Laurent polynomial[19]. We will denote by $\tilde{h}^{2,1}$ and $\tilde{h}^{1,1}$ the expressions (4.14) without the last terms.

A sufficient criterion for the possibility to associate to a CY hypersurface in $\mathbb{P}^4[\vec{w}]$ a reflexive polyhedron is that $\mathbb{P}^4[\vec{w}]$ is Gorenstein, which is the case if $\mathrm{lcm}[w_1, \ldots, w_5]$ divides the degree d [63]. In this case we can define a simplicial, reflexive polyhedron $\Delta(\vec{w})$ in terms of the weights, s.t. $\mathbb{P}_{\Delta^*(\vec{w})} \simeq \mathbb{P}^4[\vec{w}]$. The associated n-dimensional integral convex dual polyhedron is the convex hull of the integral vectors μ of the exponents of all quasi-homogeneous monomials z^μ of degree d, shifted by $(-1, \ldots, -1)$:

$$\Delta^*(\vec{w}) := \{(x_1, \ldots, x_5) \in \mathbb{R}^5 | \sum_{i=1}^{5} w_i x_i = 0, x_i \geq -1\}. \tag{4.15}$$

Note that this implies that the origin is the only point in the interior of Δ.

If the quasihomogeneous polynomial p is Fermat, i.e. if it consists of monomials z_i^{d/w_i} $(i = 1, \cdots, 5)$, $\mathbb{P}^4[\vec{w}]$ is clearly Gorenstein, and (Δ, Δ^*) are thus simplicial. If furthermore at least one weight is one (say $w_5 = 1$) we may choose $e_1 = (1,0,0,0,-w_1)$, $e_2 = (0,1,0,0,-w_2)$, $e_3 = (0,0,1,0,-w_3)$ and $e_4 = (0,0,0,1,-w_4)$ as generators for Λ, the lattice induced from the $\mathbb{Z}^5$ cubic lattice on the hyperplane $H = \{(x_1, \ldots, x_5) \in \mathbb{R}^5 | \sum_{i=1}^{5} w_i x_i = 0\}$. For this type of models we then always obtain as vertices of $\Delta^*(\vec{w})$ (with respect to the basis $e_1, \ldots, e_4$)

$$\begin{aligned}
&\nu_1^* = (d/w_1 - 1, -1, -1, -1)\ ,\ \nu_2^* = (-1, d/w_2 - 1, -1, -1)\ ,\ \nu_3^* = (-1, -1, d/w_3 - 1, -1), \\
&\nu_4^* = (-1, -1, -1, d/w_4 - 1)\ ,\ \nu_5^* = (-1, -1, -1, -1)
\end{aligned} \tag{4.16}$$

and for the vertices of the dual simplex $\Delta(w)$ one finds

$$\begin{aligned}
&\nu_1 = (1,0,0,0)\ ,\ \nu_2 = (0,1,0,0)\ ,\ \nu_3 = (0,0,1,0)\ ,\ \nu_4 = (0,0,0,1) \\
&\nu_5 = (-w_1, -w_2, -w_3, -w_4)
\end{aligned}\ . \tag{4.17}$$

It should be clear from our description of toric geometry that the lattice points in the interior of faces of Δ of dimensions $4 > d > 0$ correspond to exceptional divisors resulting from the resolution of the Gorenstein singularities of $\mathbb{P}_{\Delta^*}$. This in turn means that the corresponding Laurent monomials in f_Δ correspond to exceptional divisors. For those CY hypersurfaces that can be written as a quasi-homogeneous polynomial constraint in $\mathbb{P}^4[\vec{w}]$, we can then give a correspondence between the monomials, which correspond (via Kodaira-Spencer deformation theory) to the complex structure moduli, and the divisors, which correspond to the Kähler moduli. The authors of ref. [64] call this the monomial-divisor mirror map. Not all deformations of the complex structure can be represented by monomial deformations, which are also referred to as algebraic deformations. We will

[19] In the language of Landau-Ginzburg theories, if appropriate, they correspond to contributions from twisted sectors.

however restrict our analysis to those. We will now describe it for Fermat hypersurfaces of degree d, following [59].

The toric variety $\mathbb{P}_{\Delta(\vec{w})}$ can be identified with

$$\begin{aligned}\mathbb{P}_{\Delta(\vec{w})} &\equiv \mathbf{H}_d(\vec{w}) \\ &= \{[U_0, U_1, U_2, U_3, U_4, U_5] \in \mathbb{P}^5 | \prod_{i=1}^{5} U_i^{w_i} = U_0^d\},\end{aligned} \tag{4.18}$$

where the variables X_i in eq.(4.13) are related to the U_i by

$$[1, X_1, X_2, X_3, X_4, \frac{1}{\prod_{i=1}^4 X_i^{w_i}}] = [1, \frac{U_1}{U_0}, \frac{U_2}{U_0}, \frac{U_3}{U_0}, \frac{U_4}{U_0}, \frac{U_5}{U_0}]. \tag{4.19}$$

Let us consider the mapping $\phi : \mathbb{P}^4[\vec{w}] \to \mathbf{H}_d(\vec{w})$ given by[20]

$$[z_1, z_2, z_3, z_4, z_5] \mapsto [z_1 z_2 z_3 z_4 z_5, z_1^{d/w_1}, z_2^{d/w_2}, z_3^{d/w_3}, z_4^{d/w_4}, z_5^{d/w_5}]. \tag{4.20}$$

Integral points in $\Delta(\vec{w})$ are mapped to monomials of the homogeneous coordinates of $\mathbb{P}^4[\vec{w}]$ by

$$\mu = (\mu_1, \mu_2, \mu_3, \mu_4) \mapsto \phi^*(X^\mu U_0) = \frac{\prod_{i=1}^4 z_i^{\mu_i d/w_i}}{\left(\prod_{i=1}^5 z_i\right)^{\sum_{i=1}^4 \mu_i - 1}} \tag{4.21}$$

Note that the Laurent polynomial f_Δ and the quasi-homogeneous polynomial $p = \phi^*(U_0 f_\Delta)$ between which the monomial-divisor map acts correspond to a mirror pair. The point at the origin of Δ is always mapped to the symmetric perturbation $z_1 \cdots z_5$ which is always present. It represents the restriction of the Kähler form of the embedding space $\mathbf{P}_{\Delta^*}$ to the hypersurface.

The situation for CY hypersurfaces in non-Gorenstein $\mathbb{P}^4[\vec{w}]$'s was discussed in [16]. Here the corresponding polyhedron is still reflexive but no longer simplicial and the associated toric variety $\mathbb{P}_\Delta$ is Gorenstein.

We have already mentioned that Fermat hypersurfaces in weighted projective space do not intersect with the singular points of the embedding space. This is no longer generally true for non-Fermat hypersurfaces. What does however hold is that hypersurfaces $Z_{f,\Delta}$ in the corresponding (Gorenstein) $\mathbb{P}_\Delta$ do not intersect singular points in $\mathbb{P}_\Delta$. The singular points of the embedding space correspond to the lattice points in the interior of faces of $\mathbb{P}_\Delta$ of codimension one. The Laurent monomials for these points thus do not correspond to complex structure deformations and we will in the following always restrict the sum in (4.13) to those lattice points of Δ which do not lie in the interior of faces of codimension one.

---

[20] In toric geometry this mapping replaces the orbifold construction for the mirror manifolds described in [8].

The final point to mention in this section is the computation of topological triple intersection numbers. They represent the classical part (field theory limit, large radius limit) of the $\langle 27^3 \rangle$ Yukawa couplings.

Given three divisors the topological triple intersection number can be computed in terms of an integral of the (by Poincaré duality) associated harmonic forms: $D_i \cdot D_j \cdot D_k = \int_X h_{D_i} \wedge h_{D_j} \wedge h_{D_k}$. In toric geometry their evaluation reduces to combinatorics. The results are scattered through the (mathematics) literature [57][65] and have been collected in [16]. We will not repeat them here.

5. Periods, Picard-Fuchs Equations and Yukawa Couplings

Now that we have learned how to construct CY manifolds and even mirror pairs we can move on towards applying mirror symmetry. This, as was mentioned before, requires the knowledge of the $\langle \overline{27}^3 \rangle$ Yukawa couplings on the mirror manifold X^* in order to compute the $\langle 27^3 \rangle$ couplings on X. But in addition we need to know (at least locally) how to map the complex structure moduli space of X^* to the Kähler structure moduli space of X. This is the so-called mirror map. Our discussion will be restricted to the neighbourhood of the large complex structure limit of X^* and the large radius limit of X which will be mapped to each other by the mirror map.

For the purposes of getting the Yukawa couplings and the mirror map the Picard-Fuchs equations play a crucial role. So let us turn to them. It is quite easy to explain what they are but harder to find them explicitly. We will start with the easy part.

We know from the discussion in section two that the dimension of the third cohomology group is $\dim(H^3) = b_3 = 2(h^{2,1} + 1)$. Furthermore we know that the unique holomorphic three form Ω depends only on the complex structure. If we take derivatives with respect to the complex structure moduli, we will get elements in $H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3}$. Since b_3 is finite, there must be linear relations between derivatives of Ω of the form $\mathcal{L}\Omega = d\eta$ where $\mathcal{L}$ is a differential operator with moduli dependent coefficients. If we integrate this equation over an element of the third homology group H_3, i.e. over a closed three cycle, we will get a differential equation $\mathcal{L}\Pi_i = 0$ satisfied by the periods of Ω. They are defined as $\Pi_i(a) = \int_{\Gamma_i} \Omega(a)$. $\Gamma_i \in H_3(X, \mathbb{Z})$ and we have made the dependence on the complex structure moduli explicit. In general we will get a set of coupled linear partial differential equations for the periods of Ω. These equations are called Picard-Fuchs equations. In case we have only one complex structure modulus (i.e. $b_3 = 4$) one gets just one ordinary (in fact, hypergeometric) differential equation of order four. For the general case, i.e. if $b_3 \geq 1$, we will describe below how to set up a complete system of Picard-Fuchs equations.

For a detailed discussion of ordinary differential equations we recommend the book by Ince [66]. For the general case we have profited from the book by Yoshida [67]. There exists a vast mathematical literature on PF systems and the theory of complex moduli spaces; main results are collected in [68]. Two results which are of relevance for our discussion are that the global monodromy is completely reducible and that the PF equations have only regular singularities. The first result enables us to consider only a subset of the periods by treating only the moduli corresponding to $\tilde{h}^{2,1}$ out of $h^{2,1}$ moduli. The second result means that the PF equations are Fuchsian and we can use the theory developped for them.

A systematic, even though generally very tedious procedure to get the Picard-Fuchs equations for hypersurfaces in $\mathbb{P}[\vec{w}]$, is the reduction method due to Dwork, Katz and Griffiths. As shown in ref. [69] the periods $\Pi_i(a)$ of the holomorphic three form $\Omega(a)$ can be written as[21]

$$\Pi_i(a) = \int_{\Gamma_i} \Omega(a) = \int_\gamma \int_{\Gamma_i} \frac{\omega}{p(a)}, \quad i = 1, \ldots, 2(h^{2,1}+1). \tag{5.1}$$

Here

$$\omega = \sum_{i=1}^{5} (-1)^i w_i z_i dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_5 ; \tag{5.2}$$

Γ_i is an element of $H_3(X, \mathbb{Z})$ and γ a small curve around the hypersurface $p = 0$ in the 4-dimensional embedding space. a_i are the complex structure moduli, i.e. the coefficients of the perturbations of the quasi-homogeneous polynomial p. The fact that $\Omega(a)$ as defined above is well behaved is demonstrated in [70].

The observation that $\frac{\partial}{\partial z_i}\left(\frac{f(z)}{p^r}\right)\omega$ is exact if $f(z)$ is homogeneous with degree such that the whole expression has degree zero, leads to the partial integration rule, valid under the integral ($\partial_i = \frac{\partial}{\partial z_i}$):

$$\frac{f\partial_i p}{p^r} = \frac{1}{r-1}\frac{\partial_i f}{p^{r-1}} \tag{5.3}$$

In practice one chooses a basis $\{\varphi_k(z)\}$ for the elements of the local ring $\mathcal{R} = \mathbb{C}[z_1, \ldots, z_{n+1}]/(\partial_i p)$. From the Poincaré polynomial associated to p [7] one sees that there are $(1, \tilde{h}^{2,1}, \tilde{h}^{2,1}, 1)$ basis elements with degrees $(0, d, 2d, 3d)$ respectively. The elements of degree d are the perturbing monomials. One then takes derivatives of the expressions $\pi_k = \int \frac{\varphi_k(z)}{p^{n+1}}$ ($n = \deg(\varphi_k)/d$) w.r.t. the moduli. If one produces an expression such that the numerator in the integrand is not one of the basis elements, one relates it, using the equations $\partial_i p = \ldots$, to the basis and uses (5.3). This leads to the so called Gauss-Manin system of first order differential equations for the π_k which can be rewritten as a system of partial differential equations for the period. These are the Picard-Fuchs equations. In fact, the Picard-Fuchs equations just reflect the structure of the local ring and expresses the relations between its elements (modulo the ideal). It thus depends on the details of the ring how many equations and of which order comprise a complete system of Picard-Fuchs equations. To see this, let us consider a model with $\tilde{h}^{2,1} = 2$, i.e. we have two monomials at degree d : φ_1 and φ_2. Since the dimension of the ring at degree $2d$ is two, there must be one relation (modulo the ideal) between the three combinations φ_1^2, $\varphi_1\varphi_2$, φ_2^2. Multiplying this relation by φ_1 or φ_2 leads to two independent relations at degree three. Since the dimension of the ring at degree three is one, there must be one further relation at degree three. The system of Picard-Fuchs equations for models with $\tilde{h}^{2,1} = 2$ thus consists of one

[21] Again, here and below we only treat the case of hypersurfaces in a single projective space. Complete intersections in products of projective spaces are covered in [17].

second and one third order equation [22]. For models with $\tilde{h}^{2,1} > 2$, a general statement is no longer possible. For instance, for $\tilde{h}^{2,1} = 3$, the simplest case is a system of three equations of second order. The three relations at degree $2d$ then generate all relations at degree $3d$. There are however cases where this is not the case and one has to add extra relations at degree three, leading to third order equations. Examples of this type are presented in [16].

It is clear from the discussion that the Picard-Fuchs equations we have obtained only contain those complex structure moduli for which there exists a monomial perturbation. Also, the method outlined above applies only to manifolds in projective spaces and not to manifolds embedded in more general toric varieties.

We will now describe an alternative and often more efficient way to obtain the Picard-Fuchs equations using the toric data of the hypersurfaces. The general method has been outlined, in the context of mirror symmetry, in [58] and is based on the generalized hypergeometric system of Gelf'and, Kapranov and Zelevinsky (GKZ) [71]. We will not decribe it in its generality here, since for our purposes the following simplified treatment is sufficient.

The way we will proceed is to compute one of the periods, the so-called fundamental period [19][58] directly and then set up a system of partial differential equations satisfied by this period. This system is the GKZ hypergeometric system. It is not quite yet the PF system since its solution space is larger than that of the PF system, which are the periods of the holomorphic three form, of which there are $2(\tilde{h}^{2,1}+1)$. (As mentioned before, we are only able to treat a subset of the $2(h^{2,1}+1)$ periods.) However the monodromy acts reducibly on the (larger) space of solutions and the periods are a subset on which it acts irreducibly. It is often (sometimes?) easy, starting with the GKZ system, to find a reduced system of differential equations which is then the PF system. This will be explained in the example that we will treat below.

Before showing how to compute the fundamental period and how to extract the GKZ generalized hypergeometric system, we first have to discuss the correct choice of coordinates on moduli space. We have already mentioned that we will only discuss the mirror map in the neighbourhood of the large complex structure limit. The large complex structure limit corresponds to the point in moduli space where the periods have maximal unipotent monodromy [72]. This in particular means that the characteristic exponents of the PF equations are maximally degenerate. What this means is the following: if we make a power series ansatz for the solution of the PF equations one gets recursion relations for the coefficients. However, the condition that the lowest powers vanish, gives polynomial equations (of the order of the differential equations) for the (characteristic) exponents of the lowest order terms of the power series [23]. At the point of large complex structure the common zeros of these polynomials are all equal, and in fact, by a suitable moduli

---

[22] In these considerations we use the fact that the homogeneous subspace of $\mathcal{R}$ of degree nd is generated by the elements of degree d and furthermore that taking a derivative w.r.t. to the modulus parameter a_i produces one power of the corresponding monomial φ_i in the numerator of the period integral.

[23] In the case of ordinary differential equations this polynomial equation is called indicial equations; see e.g. [66].

dependent rescaling of the period (this constitutes a choice of gauge, c.f. below) they can be chosen to be zero. According to the general theory of Frobenius we then get, in a neighbourhood of large complex structure, one power series solutions and the other solutions contain logarithms. We will have more to say about these solutions later. The problem now consists of finding the correct variables in which we can write a power series expansion with these properties.

To find these variables it is necessary to introduce the so-called lattice of relations. Among the $5+\tilde{h}^{2,1}$ integer points $\nu_0,\ldots,\nu_{\tilde{h}^{2,1}+5}$ (ν_0 is the origin) in Δ which do not lie in the interior of faces of codimension one, i.e. those points to which we associate Laurent monomials in (4.13), there are relations of the form $\sum_i l_i v_i = 0,\, l_i \in \mathbb{Z}$. The vectors l generate the $\tilde{h}^{2,1}$ dimensional lattice of relations. In this lattice one defines a cone, the so-called Mori cone, whose minimal set of generators we denote by l^α, $\alpha = 1,\ldots,\tilde{h}^{2,1}$. (They are also a basis of the lattice of relations.) This cone is in fact the same as the one mentioned in our brief discussion about the Kähler condition of the resolution of singularities. We then define the extended vectors $(\bar{l}^\alpha) \equiv (-\sum_i l_i^\alpha, \{l_i^\alpha\}) \equiv (l_0^\alpha, \{l_i^\alpha\})$. In terms of the parameters appearing in the Laurent polynomial (4.13) the large complex structure limit is defined to be the point $u_1 = \ldots = u_{\tilde{h}^{2,1}} = 0$ in complex structure moduli space with $u_\alpha \equiv a^{\bar{l}^\alpha} \equiv \prod a_i^{\bar{l}_i^\alpha}$.

A systematic method to find the generators of Mori's cone has been reviewed in [16] where the construction of the upper convex piecewise linear functions was explained. An equivalent way[24] is as follows. Consider a particular 'triangulation' (i.e. decomposition into four-simplices with apex at the origin) of Δ with lattice points $\nu_0,\ldots,\nu_{\tilde{h}^{2,1}+5}$. Each four-simplex is then specified by four vertices (in addition to the origin ν_0). We then take any pair of four simplices which have a common three simplex and look for integer relations $\sum n_i \nu_i = 0,\, n_i \in \mathbb{Z}$ of the five non-trivial vertices such that the coefficients of the two vertices which are not common to both four-simplices are positiv. This provides a set of relations. There will be $\tilde{h}^{2,1}$ independent relations in terms of which the others can be expanded with non-negative integer coefficients. They constitute the basis of Mori's cone and define the coordinates in the neighbourhood of large complex structure.

Let us now turn to the computation of the fundamental period. In the language of toric geometry the period integrals are written as [25]

$$\Pi_i(a) = \int_{\Gamma_i} \frac{a_0}{f(a,X)} \frac{dX_1}{X_1} \wedge \cdots \wedge \frac{dX_4}{X_4} \tag{5.4}$$

with $\Gamma_i \in H_4((\mathbb{C}^*)^4 \setminus Z_f)$ and $f(a,X)$ the Laurent polynomial (4.13). We get the fundamental period if we choose the cycle $\Gamma = \left\{(X_1,\ldots,X_4) \in \mathbb{C}^4 \middle| |X_i| = 1\right\}$, expand the integrand in a power series in $1/a_0$ and evaluate the integral using the residue formula.

[24] which we have learned from V. Batyrev; see also [73].

[25] This in fact differs from the period defined before by a factor of a_0, which we have included for later convenience. This corresponds to a redefinition of $\Omega \to a_0\Omega$.

Straightforward computation gives

$$w_0(a) = \sum_{\substack{\mu_i \geq 0 \\ \sum \mu_i \nu_i = 0}} \frac{(\sum \mu_i)!}{\prod \mu_i!} \frac{\prod_i a_i^{\mu_i}}{a_0^{\sum \mu_i}} \tag{5.5}$$

i.e. the sum is over a subset of the lattice of relations. Since the generators of Mori's cone are a basis of the lattice of relations we can in fact express the relations to be summed over in the form $\sum n_\alpha l^\alpha$ and sum over the n_α. If we then introduce the variables u_α, the fundamental period becomes[26]

$$w_0(u) = \sum_{\{n_\alpha\}} \frac{(-\sum_\alpha l_0^\alpha n_\alpha)!}{\prod_{i>0}(\sum_\alpha l_i^\alpha n_\alpha)!} \prod_\alpha u_a^{n_\alpha} \equiv \sum_n c(n) u^n \tag{5.6}$$

where the sum is over those n_α which leave the arguments of the factorials non-negative. It is now straightforward to set up a system of $\tilde{h}^{2,1}$ partial differential equations which are satisfied by $w_0(u)$. Indeed, the coefficients $c(n)$ satisfy recursion relations of the form $p_\beta(n_\alpha, n_\beta + 1)c(n_\alpha, n_\beta + 1) - q_\beta(n_\alpha, n_\beta)c(n_\alpha, n_\beta) = 0$. Here p_α and q_α are polynomials of their respective arguments. The recursion relations translate to linear differential operators

$$\mathcal{L}_\beta = p_\beta(\Theta_\alpha, \Theta_\beta) - u_\beta q_\beta(\Theta_\alpha, \Theta_\beta) \tag{5.7}$$

where we have introduced the logarithmic derivatives $\Theta_\alpha = u_\alpha \frac{d}{du_\alpha}$. The order of the operator $\mathcal{L}_\beta$ equals the sum of the positive (or negative) components of $\bar{l}^\beta$. The system of linear differential equations $\mathcal{L}_\beta w(u) = 0$ is the generalized hypergeometric system of GKZ. Since it is related to the polyhedron Δ, it is also called the Δ-hypergeometric system.

The hypergeometric systems that one gets this way are not generic, but rather (semi-non)resonant in the language of [71] in which case the monodromy acts no longer irreducibly. It is in general not straightforward to extract the PF system from the GKZ system, but in simple cases the operators $\mathcal{L}_\alpha$ factorize $\mathcal{L}_\alpha = \ell_\alpha \mathcal{D}_\alpha$ and the $\mathcal{D}_\alpha$ form the complete PF system. The general situation, which might require an extension of the Δ-hypergeometric system, has been discussed in [16] and [17].

In any case, when the dust settles, one has a system of PF operators of the form

$$\mathcal{D}_\alpha = p_\alpha(\Theta) + \sum_\beta f_{\alpha\beta}(u) q_{\alpha\beta}(\Theta) \tag{5.8}$$

where $p_\alpha, q_{\alpha\beta}$ and $f_{\alpha\beta}$ are polynomials with $f_{\alpha\beta}(0) = 0$ and p_α is homogeneous and p_α and $q_{\alpha\beta}$ are of the same degree. The homogeneity of $p_\alpha(\Theta)$ follows from the characterization of the large complex structure by the requirement that the characteristic exponents of the PF differential equations should be maximally degenerate and the gauge choice which gives a power series solution that starts with a constant.

[26] Here and below we will use the following notation: for a multi-index $n = (n_1, \ldots, n_N)$ we define $|n| = n_1 + \ldots + n_N$, $u^n = u_1^{n_1} \cdots u_N^{n_N}$ and also $n! = \prod n_i!$.

Note that the terms in (5.8) of top degree in the Θ correspond to relations between monomials of the same degree modulo the ideal, whereas the lower order terms correspond to terms in the ideal. This comment applies if we work with quasi-homogeneous polynomials.

A necessary condition for a set of period equations to be complete is that there be $2(\tilde{h}^{2,1}+1)$ degenerate characteristic exponents at $z=0$. The polynomial ring

$$\mathcal{R} = \mathbb{C}[\Theta_1, \cdots, \Theta_{\tilde{h}^{2,1}}]/\{p_a(\Theta)\} \tag{5.9}$$

then has $(1, \tilde{h}^{2,1}, \tilde{h}^{2,1}, 1)$ elements at degrees (in Θ) $(0,1,2,3)$. This in particular means that the symbols of $\mathcal{D}_\alpha$ generate the ideal of symbols [27]. This observation is important for the determination of the singular locus of the PF equations. But let us first explain what is being said [67].

The PF operators $\mathcal{D}_\alpha$ define a (left) ideal I in the ring of differential operators[28], i.e. $\mathcal{D} \in I \Leftrightarrow \mathcal{D}w = 0$. The symbol of a partial linear differential operator $\mathcal{D}$ in k variables of order m, i.e. $\mathcal{D} = \sum_{|p|\le m} a_p(u)\left(\frac{d}{du}\right)^p$, $(|p| = p_1 + \cdots + p_k)$ is defined as $\sigma(\mathcal{D}) = \sum_{|p|=m} a_p(u)\xi_1^{p_1}\cdots\xi_k^{p_k}$. $\xi_1,\ldots,\xi_k$ is a coordinate system on the fiber of the cotangent bundle T^*U at $z=0$. The ideal of symbols is then $\sigma(I) = \{\sigma(\mathcal{D})|\mathcal{D} \in I\}$. The singular locus is $S(I) = \pi(\mathrm{Ch}(I) - U \times \{0\})$ where the characteristic variety $\mathrm{Ch}(I)$ is the subvariety in T^*U given by the ideal of symbols (i.e. given by $\sigma(\mathcal{D}) = 0, \mathcal{D} \in I$) and π the projection along the fiber, i.e. setting $\xi = 0$. The singular locus is also the discriminant of the CY hypersurface, i.e. the locus in moduli space where the manifold fails to be transverse [29]. We will demonstrate the method introduced here in the example in section seven.

Let us now turn to the discussion of the remaining solutions of the PF equations in the neighbourhood of the point $u=0$. Due to the fact that the characteristic exponents are all degenerate, the fundamental period is the only power series solution. The other $(2\tilde{h}^{2,1}+1)$ solutions contain logarithms of u. We will now show that there are $\tilde{h}^{2,1}$ solutions with terms linear in logarithms, the same number of solutions with parts quadratic and one solution with a part cubic in logarithms. This corresponds to the grading of the ring $\mathcal{R}$ (eq.(5.9)).

Extending the definition of $x! = \Gamma(x+1)$ to $x \in \mathbb{R}$, and that of the coefficients $c(n+\rho)$ in (5.6) for arbitrary values of $\tilde{h}^{2,1}$ parameters ρ_α, we define the power series

$$w_0(u,\rho) = \sum c(n+\rho)u^{n+\rho}. \tag{5.10}$$

Clearly, setting $\rho = 0$ gives the fundamental period. By the method of Frobenius the logarithmic solutions are obtained by taking linear combinations of derivatives $D_\rho =$

[27] This is a feature of PF systems and not generally true for generalized hypergeometric systems. A counterexample can be found in ref. [67].

[28] Here we work in a neighbourhood U of the origin of a coordinate system on $\mathbb{C}^{\tilde{h}^{2,1}}$.

[29] An alternative way to determine the discriminant of hypersurfaces in $\mathbb{P}_\Delta$ was given in [49].

$\sum \frac{1}{(2\pi i)^n} \frac{b_n}{n!} \partial_\rho^n$ of $w_0(u,\rho)$, evaluated at $\rho = 0$. As $[\mathcal{D}_\alpha, \partial_{\rho_\alpha}] = 0$ it is then sufficient to check

$$D_\rho \left(\mathcal{D}_\alpha w_0(u,\rho) \right)\big|_{\rho=0} = 0, \ \forall \alpha \tag{5.11}$$

to establish $D_\rho w_0(u,\rho)|_{\rho=0}$ as a solution. By consideration of the explicit form of the series (5.10) one can show that the conditions for vanishing of the constant terms in (5.11)

$$D_\rho (p_\alpha(\rho) c(0,\rho) u^\rho)\big|_{\rho=0} = 0, \ \forall \alpha, \tag{5.12}$$

are in fact also sufficient. A moment's thought shows that the following construction of the operators D_ρ is valid. We consider the ideal $\mathcal{I}$ in the polynomial ring $\mathbb{C}[\Theta]$, generated by the $p_\alpha(\Theta)$. We endow $\mathbb{C}[\Theta]$ with a natural vector space structure with the normalized monomials as orthonormal basis. We can then define $\mathcal{I}^\perp$ which consists of the (homogeneous) polynomials orthogonal to the elements in $\mathcal{I}$. If we denote a homogeneous element in $\mathcal{I}^\perp$ by $\sum b_n \theta^n$ then the D_ρ are simply

$$D_\rho^{(|n|)} = \sum \frac{1}{(2\pi i)^{|n|}} \frac{b_n}{n!} \partial_\rho^n. \tag{5.13}$$

Here the sum is over the n_i such that $\sum n_i = |n|$. The corresponding solutions contain up to $|n|$ powers of logarithms. Note that since the PF equations are always at least of order two, the solutions linear in logarithms are $w_i(u) = D_i^{(1)} w_0(u,\rho)\big|_{\rho=0} = \frac{1}{2\pi i} \frac{\partial}{\partial \rho_i} w_0(u,\rho)\big|_{\rho=0}$ $(i = 1, \ldots, \tilde{h}^{2,1})$. In section six we will show that as a result of mirror symmetry, the operators $D^{(3)}$ and $D^{(2)}$ are of the form $D^{(3)} = -\frac{1}{3!} \frac{1}{(2\pi i)^3} \sum \kappa^0_{ijk} \partial_{\rho_i} \partial_{\rho_j} \partial_{\rho_k}$ and $D_i^{(2)} = \frac{1}{2} \frac{1}{(2\pi i)^2} \sum \kappa^0_{ijk} \partial_{\rho_j} \partial_{\rho_k}$ with κ^0 the topological triple couplings in a basis to be introduced there.

To summarize the discussion above we collect all the solutions to the PF equations into the period vector:

$$\Pi(z) = \begin{pmatrix} w_0(u) \\ D_i^{(1)} w_0(u,\rho)|_{\rho=0} \\ D_i^{(2)} w_0(u,\rho)|_{\rho=0} \\ D^{(3)} w_0(u,\rho)|_{\rho=0} \end{pmatrix}. \tag{5.14}$$

The final point we want to discuss in this section is how to compute the $\langle \overline{27}^3 \rangle$ Yukawa couplings as functions of the complex structure moduli. In the next section we will relate them, via the mirror map, to the $\langle 27^3 \rangle$ couplings on the mirror manifold.

We have already given an expression for the $\langle \overline{27}^3 \rangle$ couplings in eq.(2.2). There is in fact a more convenient (for what is going to come) way to write the same expression in terms of bilinears of the (derivatives of the) periods of Ω [74] [75]. For this purpose we introduce an integral basis of $H^3(X,\mathbb{Z})$ with generators α_i and β^j $(i,j = 0, \ldots, h^{2,1})$ which are dual to a canonical homology basis (A^i, B_j) of $H_3(X,\mathbb{Z})$ with intersection numbers $A^i \cdot A^j = B_i \cdot B_j = 0$, $A^i \cdot B_j = \delta^i_j$. Then

$$\int_{A^j} \alpha_i = \int_X \alpha_i \wedge \beta^j = -\int_{B_i} \beta^j = \delta_i^j \tag{5.15}$$

with all other pairings vanishing. This basis is unique up to $Sp(2(h^{2,1}+1),\mathbb{Z})$ transformations.

A complex structure on X is now fixed by choosing a particular 3-form as the holomorphic (3,0) form Ω. It may be expanded in the above basis of $H^3(X,\mathbf{Z})$ as $\Omega = z^i\alpha_i - \mathcal{F}_i\beta^i$ where $z^i = \int_{A^i}\Omega$, $\mathcal{F}_i = \int_{B_i}\Omega$ are periods of Ω. As shown in [76] and [77] the z^i are local complex projective coordinates for the complex structure moduli space, i.e. we have $\mathcal{F}_i = \mathcal{F}_i(z)$. The coordinates z^i are called special projective coordinates. They are related to the special coordinates of eq.(3.6), say in a patch where $z^0 \neq 0$, i.e. $\lambda_i = \frac{z^i}{z^0}$. Under a change of complex structure Ω, which was pure (3,0) to start with, becomes a mixture of $(3,0)$ and $(2,1)$, i.e. $\frac{\partial}{\partial z^i}\Omega \in H^{(3,0)} \oplus H^{(2,1)}$. In fact [26] $\frac{\partial\Omega}{\partial z^i} = k_i\Omega + b_i$ where $b_i \in H^{(2,1)}$ is related to elements in $H^1(M,T_X)$ via Ω and k_i is a function of the moduli but independent of the coordinates of X. One immediate consequence is that $\int\Omega\wedge\frac{\partial\Omega}{\partial z^i} = 0$. Inserting the expression for Ω in this equation, one finds $\mathcal{F}_i = \frac{1}{2}\frac{\partial}{\partial z^i}(z^j\mathcal{F}_j)$, or $\mathcal{F}_i = \frac{\partial\mathcal{F}}{\partial z^i}$ with $\mathcal{F} = \frac{1}{2}z^i\mathcal{F}_i(z)$, $\mathcal{F}(\mu z) = \mu^2\mathcal{F}(z)$. From $\frac{\partial^2}{\partial z^i\partial z^k}\Omega \in H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)}$ it immediately follows that also $\int\Omega\wedge\frac{\partial^2}{\partial z^i\partial z^j}\Omega = 0$. In fact, this is already a consequence of the homogeneity of $\mathcal{F}$. Finally, $\frac{\partial^3}{\partial z^i\partial z^j\partial z^k}\Omega \in H^{(3,0)} \oplus H^{(2,1)} \oplus H^{(1,2)} \oplus H^{(0,3)}$ and one easily finds $\int\Omega\wedge\frac{\partial^3}{\partial z^i\partial z^j\partial z^k}\Omega = \frac{\partial^3}{\partial z^i\partial z^j\partial z^k}\mathcal{F} = (z^0)^2\frac{\partial^3}{\partial\lambda_i\partial\lambda_j\partial\lambda_k}F$ where $\mathcal{F} = (z^0)^2F$ (cf. (3.6)); here $i,j,k = 1,\ldots,h^{2,1}$. If one computes the $(0,3)$ part of $\partial^3\Omega$ explicitly [75], one recovers indeed the couplings $\bar{\kappa}_{ijk}$ in (2.2). From the discussion above it also follows that under a change of coordinates $\lambda_i \to \tilde{\lambda}_i(t)$ the Yukawa couplings transform homogeneously and thus $\bar{\kappa}_{ijk} = \int\Omega\wedge\partial_i\partial_j\partial_k\Omega$ holds in any coordinate system, whereas it can be written as the third derivative of the prepotential only in special coordinates. If we redefine $\Omega \to \frac{1}{z_0}\Omega$, the periods are $(1,\lambda_i,\frac{\partial}{\partial\lambda_i}F, 2F - \lambda_i\frac{\partial}{\partial\lambda_i}F)$. We also note that $K = -\ln\int\Omega\wedge\bar{\Omega}$, which is easily shown to be in agreement with (3.6), up to a Kähler transformation.

We are now ready to link the Yukawa couplings to the PF equations. In inhomogeneous coordinates λ_i the Yukawa couplings are

$$\bar{\kappa}_{ijk} = \int\Omega\wedge\frac{\partial^3}{\partial\lambda_i\partial\lambda_j\partial\lambda_k}\Omega = \sum_{l=0}^{\bar{h}^{2,1}}(z^l\partial_i\partial_j\partial_k\mathcal{F}_l - \mathcal{F}_l\partial_i\partial_j\partial_k z^l) \tag{5.16}$$

where z^l and $\mathcal{F}_l$ are periods of Ω (in a symplectic basis, i.e. for the particular choice of cycles A^l, B_l as specified above).

We now define

$$\begin{aligned} W^{(k_1,\cdots,k_d)} &= \sum_l[z^l\partial_{\lambda_1}^{k_1}\cdots\partial_{\lambda_d}^{k_d}\mathcal{F}_l - \mathcal{F}_l\partial_{\lambda_1}^{k_1}\cdots\partial_{\lambda_d}^{k_d}z^l] \\ &:= \sum_l(z^l\partial^{\mathbf{k}}\mathcal{F}_l - \mathcal{F}_l\partial^{\mathbf{k}}z^l) \ . \end{aligned} \tag{5.17}$$

In this notation, $W^{(\mathbf{k})}$ with $\sum k_i = 3$ describes the various types of Yukawa couplings and $W^{(\mathbf{k})} \equiv 0$ for $\sum k_i = 0,1,2$.

If we now write the Picard-Fuchs differential operators in the form

$$\mathcal{D}_\alpha = \sum_{\mathbf{k}} f_\alpha^{(\mathbf{k})}\partial^{\mathbf{k}} \quad , \tag{5.18}$$

then we immediately obtain the relation

$$\sum_{\mathbf{k}} f_\alpha^{(\mathbf{k})} W^{(\mathbf{k})} = 0 \quad . \tag{5.19}$$

Further relations are obtained from operators $\partial_{\lambda_\beta}\mathcal{D}_\alpha$. If the system of PF differential equations is complete, it is sufficient for deriving linear relations among the Yukawa couplings and their derivatives, which can be integrated to give the Yukawa couplings up to an overall normalization. In the derivation, we need to use the following relations which are easily derived

$$\begin{aligned}
W^{(4,0,0,0)} &= 2\partial_{\lambda_1} W^{(3,0,0,0)} \\
W^{(3,1,0,0)} &= \frac{3}{2}\partial_{\lambda_1} W^{(2,1,0,0)} + \frac{1}{2}\partial_{\lambda_2} W^{(3,0,0,0)} \\
W^{(2,2,0,0)} &= \partial_{\lambda_1} W^{(1,2,0,0)} + \partial_{\lambda_2} W^{(2,1,0,0)} \\
W^{(2,1,1,0)} &= \partial_{\lambda_1} W^{(1,1,1,0)} + \frac{1}{2}\partial_{\lambda_2} W^{(2,0,1,0)} + \frac{1}{2}\partial_{\lambda_3} W^{(2,1,0,0)} \\
W^{(1,1,1,1)} &= \frac{1}{2}(\partial_{\lambda_1} W^{(0,1,1,1)} + \partial_{\lambda_2} W^{(1,0,1,1)} + \partial_{\lambda_3} W^{(1,1,0,1)} + \partial_{\lambda_4} W^{(1,1,1,0)}) \quad .
\end{aligned} \tag{5.20}$$

By symmetry the above relations exhaust all possibilities.

We have now described all the calculation that have to be done on the mirror manifold $\hat{X}^*$, namely the computation of the couplings $\bar{\kappa}_{ijk}(\hat{X}^*)$[30]. In the next section we show how to go back to the manifold $\hat{X}$ on which we want to compute the $\langle 27^3 \rangle$ Yukawa couplings.

6. Mirror Map and Applications of Mirror Symmetry

The question now arises whether mirror symmetry is merely a hitherto unknown mathematical curiosity or whether it can also be used as a practical tool in string theory. The demonstration that this is indeed the case will be attempted in what follows. We will show how mirror symmetry allows for the computation of the otherwise difficult (if not impossible) to get Kähler moduli dependence of the $\langle 27^3 \rangle$ Yukawa couplings. The methods developped can, in principle, be applied to any CY hypersurface and incorporates the dependence of those Kähler moduli which correspond to toric divisors. Given a general model it is however in practice technically rather cumbersome to actually carry the program through. However, for models with few moduli it can and has been done successfully. A simple but non-trivial example will be given in the following section. But before turning to it, we still need a few ingredients.

In the previous section we have shown how to compute, via the PF equations, the $\langle \overline{27}^3 \rangle$ couplings on a Calabi-Yau space given as a hypersurface of a toric variety. If we now also want to compute the $\langle 27^3 \rangle$ couplings, we proceed as follows. We go to the mirror

[30] In fact, if we are only interested in the couplings $\kappa_{ijk}(\hat{X})$ we do not need the $\bar{\kappa}_{ijk}(\hat{X}^*)$ explicitly, as will become clear below.

manifold $\hat{X}^*$ and compute the $\langle \overline{27}^3 \rangle$ couplings there and then use mirror symmetry to go back to $\hat{X}$. What this requires is to find the map from the complex structure moduli space with coordinates u_α, $(\alpha = 1, \ldots, \tilde{h}^{2,1}(\hat{X}^*))$ to the Kähler structure moduli space on $\hat{X}$ with coordinates t_i, $(i = 1, \ldots, \tilde{h}^{1,1}(\hat{X}) = \tilde{h}^{2,1}(\hat{X}^*))$. The mirror hypothesis then states that the two Yukawa couplings transform into each other under this transformation. What one has to take into account is the transformation properties of the Yukawas under coordinate transformations: for $t_i \to \tilde{t}_i(t)$ they transform as $\kappa_{ijk}(t) = \frac{\partial \tilde{t}_l}{\partial t_i} \frac{\partial \tilde{t}_m}{\partial t_j} \frac{\partial \tilde{t}_n}{\partial t_k} \tilde{\kappa}_{lmn}(\tilde{t}(t))$. Another point to consider is the normalization of the periods and consequently also of the Yukawas which are quadratic in the periods. In fact, a change of Kähler gauge[31] $\Omega \to f(u)\Omega$ results in an change of the Yukawa couplings $\bar{\kappa}_{ijk} \to f^2(u)\bar{\kappa}_{ijk}$. The gauge we choose is such that the fundamental period is one, i.e. $f(u) = 1/w_0(u)$. The $\tilde{h}^{2,1}$ solutions linear in logarithms are then

$$t_i(z) = \frac{w_i(u)}{w_0(u)}. \tag{6.1}$$

They serve as the coordinates on the Kähler moduli space on $\hat{X}$ in the neighbourhood of infinite radius which is obtained for $\mathrm{Im}(t_i) \to \infty$ (recall that $\mathrm{Im}(t_i)$ are the real moduli). Equations (6.1) define the mirror map. This coordinate choice can be identified with the special coordinates of special geometry[32] [77][78]. As discussed in [79], in these coordinates the Picard-Fuchs differential equations can be written in the form

$$\sum_{i=1}^{k} \partial_j \partial_p (K_r^{-1})^{li} \partial_i \partial_r \, \Pi(t) = 0, \tag{6.2}$$

where $K_{ijk} = \partial_i \partial_j \partial_k F$ is derived from the prepotential $\mathcal{F} = w_0^2 F$ $(\partial_i = \frac{\partial}{\partial t_i})$. This system of fourth order equations can be rewritten as a system of linear differential equations, the Gauss-Manin system in special coordinates. The solutions of (6.2) are easily written down in terms of F: $\Pi(t) = (1, t_i, \partial_i F, 2F - t_i \partial_i F)$. Note that these are the periods in the canonical basis discussed in section five, after going to inhomogeneous coordinates. The mirror conjecture now states that $F(t)$ can also be identified with the prepotential for the Kähler structure moduli of the manifold X.

The Yukawa couplings are then

$$\kappa_{ijk}(t) = \partial_{t_i} \partial_{t_j} \partial_{t_k} F(t) = \frac{1}{w_0(u(t))^2} \frac{\partial u_\alpha(t)}{\partial t_i} \frac{\partial u_\beta(t)}{\partial t_j} \frac{\partial u_\gamma(t)}{\partial t_k} \bar{\kappa}_{\alpha\beta\gamma}(u(t)) \tag{6.3}$$

Here $\bar{\kappa}_{\alpha\beta\gamma}$ are the Yukawa couplings on the mirror manifold which we showed how to compute in the previous section. In order to espress the couplings κ_{ijk} in terms of the Kähler moduli t_i, we have to invert the expressions $t_i(u) = \frac{w_i(u)}{w_0(u)} = \frac{1}{2\pi i} \log(u_i) + O(u)$

---

[31] This corresponds to a Kähler transformation of the Kähler potential $K = -\log\left(\int \Omega \wedge \bar{\Omega}\right)$ of moduli space.

[32] These are the so-called flat coordinates of the associated topological field theory.

which leads to expressions of the form $u_i = q_i(1+O(q))$. Here we have defined $q_j = e^{2\pi i t_j}$. This then provides the Yukawa couplings as a power series in the variables q_j.

We now want to compare this with the general form for these Yukawa couplings given in eq.(2.3). For this we introduce the (multi) degree of the curve C, which is defined as $n_i = \int_C h_i \in \mathbb{Z}$ for $h_i \in H^{(1,1)}(X, \mathbb{Z})$. For the integral over the Kähler form we then get $\int_C J = \sum t_i n_i$. In terms of the degrees and the variables q_i the Yukawa coupling is

$$\kappa_{ijk} = \kappa^0_{ijk} + \sum_{\{n_i\}} \frac{N(\{n_i\}) n_i n_j n_k \prod_l q_l^{n_l}}{1 - \prod_l q_l^{n_l}} \tag{6.4}$$

where $N(\{n_i\})$ are integers. In the simplest case of isolated non-singular rational curves C, they give the number of curves at degree $\{n_i\}$. More generally they have to be interpreted as Euler numbers of a suitably compactified moduli space of holomorphic maps of degree $\{n_i\}$ from $\mathbb{P}^1$ (the genus zero world-sheet) to the CY manifold.

Before turning to the example in the next section, we want to say a few words about the prepotential. If we introduce homogeneous coordinates on the Kähler structure moduli space of $\hat{X}$ via $t_i = z^i/z^0$, then the most general ansatz for the prepotential $\mathcal{F}(z) = (z^0)^2 F(t)$ which respects homogeneity, is

$$\begin{aligned}\mathcal{F} &= \frac{1}{6}\kappa^0_{ijk}\frac{z^i z^j z^k}{z^0} + \frac{1}{2}a_{ij}z^i z^j + b_i z^i z^0 + \frac{1}{2}c(z^0)^2 + \mathcal{F}_{\text{inst.}} \\ &= (z^0)^2\Big(\frac{1}{6}\kappa^0_{ijk}t_i t_j t_k + \frac{1}{2}a_{ij}t_i t_j + b_i t_i + \frac{1}{2}c + F_{\text{inst.}}(t)\Big)\end{aligned} \tag{6.5}$$

We have split the prepotential into the classical intersection part and the instanton part ($F_{\text{inst.}}$); κ^0_{ijk} are the classical intersection numbers. The constants a_{ij}, b_i and c do not enter the Yukawa couplings $\kappa_{ijk}(t) = \partial_i\partial_j\partial_k F(t)$. Their real parts are also irrelevant for the Kähler potential. There is a continuous Peccei-Quinn symmetry $t_i \to t_i + \alpha_i$, α_i real [33], which is broken by instanton corrections to discrete shifts [80]. Requiring this symmetry in the absence of instanton corrections gives $\mathrm{Im}(a_{ij}) = \mathrm{Im}(b_i) = 0$.

From the function $\mathcal{F}$, viewed as the pre-potential for the Kähler structure moduli space on X, we construct the vector $(z^0, z^i, (\partial\mathcal{F}/\partial z^i), (\partial\mathcal{F}/\partial z^0)) \equiv z^0\Pi(t)$ with

$$\Pi(t) = \begin{pmatrix} 1 \\ t_i \\ \frac{1}{2}\kappa^0_{ijk}t_j t_k + a_{ij}t_j + b_i + \partial_i(F_{\text{inst.}}) \\ -\frac{1}{6}\kappa^0_{ijk}t_i t_j t_k + b_i t_i + c + O(e^{2\pi i t}) \end{pmatrix} \tag{6.6}$$

The mirror conjecture now says that this is the same as the period vector (5.14). Comparing the last components of these two vectors, using that $\log z_j = 2\pi i t_j + O(t^2)$ we verify that, up

[33] Under constant shifts of $\mathrm{Re}(t_i)$ the sigma-model action (1.1) changes according to $\Delta S \sim \int_\Sigma d^2 z b_{i\bar{j}}(\phi)(\partial\phi^i\bar{\partial}\phi^{\bar{j}} - \bar{\partial}\phi^i\partial\phi^{\bar{j}}) = \int_{\Phi(\Sigma)} b_{i\bar{j}} d\phi^i \wedge d\phi^{\bar{j}} = \int_{\Phi(\Sigma)} b$. For $\Phi(\Sigma)$ topologically trivial $b = da$ and $\Delta S = 0$.

to an overall normalization, the coefficients of the operator $D_\rho^{(3)}$ are indeed the topological couplings. We also conclude that the fully instanton corrected couplings κ_{ijk} are given by the concise expressions

$$\kappa_{ijk}(t) = \partial_{t_i}\partial_{t_j} \frac{D_k^{(2)} w_0(u(t),\rho)\big|_{\rho=0}}{w_0(u(t))} \tag{6.7}$$

Let us finally turn to the constants a_{ij}, b_i and c in the prepotential. From what has been said we see that they can be expressed in terms as the expansion coefficient $c(0,\rho)$ in eq.(5.6): $c = D^{(3)}c(0,\rho)\big|_{\rho=0}$, $b_i = D_i^{(2)}c(0,\rho)\big|_{\rho=0}$ and $a_{ij} = 0$. One finds

$$\begin{aligned}
\frac{\partial}{\partial\rho_\beta}c(0) &= -(l_0^\beta + \sum_{i>0} l_i^\beta)\Gamma'(1) \equiv 0 \\
\frac{\partial}{\partial\rho_\beta}\frac{\partial}{\partial\rho_\gamma}c(0) &= \frac{\pi^2}{6}\Big(l_0^\beta l_0^\gamma - \sum_{i>0} l_i^\beta l_i^\gamma\Big) \\
\frac{\partial}{\partial\rho_\alpha}\frac{\partial}{\partial\rho_\beta}\frac{\partial}{\partial\rho_\gamma}c(0) &= 2\Big(l_0^\alpha l_0^\beta l_0^\gamma + \sum_{i>0} l_i^\alpha l_i^\beta l_i^\gamma\Big)\zeta(3)
\end{aligned} \tag{6.8}$$

An interesting observation is now that for all hypersurfaces that we have treated explicitly, the following relations hold:

$$\begin{aligned}
\chi(\hat{X}) = \int_{\hat{X}} c_3 &= \frac{1}{3}\kappa^0_{\alpha\beta\gamma}(l_0^\alpha l_0^\beta l_0^\gamma + \sum_{i>0} l_i^\alpha l_i^\beta l_i^\gamma) \\
\int_{\hat{X}} c_2 \wedge h_\alpha &= \frac{1}{2}\kappa^0_{\alpha\beta\gamma}(l_0^\beta l_0^\gamma - \sum_{i>0} l_i^\beta l_i^\gamma)
\end{aligned} \tag{6.9}$$

For the case of singular hypersurfaces we have no proof of these relations. For non-singular complete intersections in products of projective spaces equivalent formulas are derived and proven in [16]. Above results lead to the following expressions for the constants b_i and c:

$$b_i = \frac{1}{24}\int_{\hat{X}} c_2 \wedge h_i \qquad c = \frac{1}{(2\pi i)^3}\chi(\hat{X})\xi(3) \tag{6.10}$$

We thus find that c, being imaginary, is the only relevant contribution of a, b, c to the Kähler potential. In fact the value of c reproduce the expected contribution from the σ model loop calculation. Moreover (6.10) reproduces the values for b and c of all examples where the prepotential is derived by specifying the integral basis[12][13]. We therefore conjecture that the prepotential (6.5) describes in general the Yukawa couplings *and* the Kähler metric for $\mathcal{M}_{h^{2,1}}(\hat{X}^*)$ and therefore by mirror hypothesis also on $\mathcal{M}_{h^{1,1}}(\hat{X})$ in the region of convergence of the large complex (Kähler) structure expansion.

We now have to comment on the topological couplings $\kappa^0_{\alpha\beta\gamma}$. The indices refer to the coordinates t_α, which, via the mirror map (6.1), are related to the u_α. In the large radius limit we have $t_\alpha = \frac{1}{2\pi i}\log(u_\alpha)$. The coordinates u_α are monomials in the perturbation parameters in the Laurent polynomial f_Δ. These parameters are in one-to-one correspondence with (exceptional) divisors on $\hat{X}$. To the coordinates u_α we thus have to associate linear combinations of divisors, or, equivalently, of harmonic (1, 1) forms. For Fermat hypersurfaces this is done in the following way. If $h_J, h_{D_1}, \ldots, h_{D_n}$, ($n = \tilde{h}^{2,1}(\hat{X}^*) - 1 = \tilde{h}^{1,1}(\hat{X}) - 1$) are the harmonic (1, 1) forms corresponding to the basis $a_0, \ldots, a_n$, the forms corresponding to the basis $u_1, \ldots, u_{\tilde{h}^{2,1}}$ are,

$$h_J = h_1 \quad h_{D_i} = \sum_\alpha l^\alpha_{i+5} h_\alpha \tag{6.11}$$

7. A Two Moduli Example: Hypersurface in $\mathbb{P}^4[2,2,2,1,1]$

After all the rather formal discussions, we now want to present an example for which we choose the degree eight Fermat CY hypersurface in $\mathbb{P}^4[2,2,2,1,1]$ defined by

$$p_0(z) = z_1^4 + z_2^4 + z_3^4 + z_4^8 + z_5^8 = 0 \tag{7.1}$$

We will end up with the various Yukawa couplings between the to 27-plets of E_6, in particular their dependence on the Kähler moduli. This will also provide the number of rational curves at all degrees (instanton numbers).

Being a Fermat model, both Δ and Δ^* are simplicial and their corners can be easily written down using (4.17). Constructing the remaining lattice points in Δ and Δ^* and applying formulas (4.14) gives $h^{1,1}(\hat{X})$=2 and and $h^{2,1}(\hat{X}) = 86$ and the reversed numbers for the mirror partner. Note however that $\tilde{h}^{2,1}(\hat{X}) = 83$. This corresponds to the fact that we can only incorporate 83 complex structure deformations as monomial perturbations of the quasi-homogeneous polynomial p_0. In fact, one readily writes down the most general quasi-homogeneous polynomial of degree eight. It has 105 monomials. Using the freedom of homogeneous coordinate redefinitions, which provide 22 parameters, we are left with 83 possible monomial deformations which we might choose to be the degree eight elements of the ring $\mathcal{R} = \frac{\mathbb{C}[z_1,\ldots,z_5]}{\{\partial_i p_0\}}$. An easy way to see this is that if we make infinitesimal homogeneous coordinate transformations p_0 changes (to first order) by terms in the ideal.

We also understand the number of Kähler moduli. The model has a singular $\mathbb{Z}_2$ curve C and the exceptional divisor is $C \times \mathbb{P}^1$. We thus have $h^{1,1} = 2$, one of the forms coming from the embedding space and the second from the exceptional divisor. Since we have a singular curve we expect a lattice point on a face of Δ of dimension one (i.e. on an edge). This point is $\nu_6 = (-1,-1,-1,0) = \frac{1}{2}(\nu_5 + \nu_4)$. Via the monomial-divisor map it corresponds to the perturbation $z_4^4 z_5^4$. The Laurent polynomial for the mirror is thus

$$f_\Delta(X) = a_0 - \left(a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + \frac{a_5}{X_1^2 X_2^2 X_3^2 X_4} + \frac{a_6}{X_1 X_2 X_3}\right) \tag{7.2}$$

where we may use the freedom to rescale the variables X_i and the polynomial to set $a_1 = \ldots = a_5 = 1$. Equivalently we can write the homogeneous polynomial $p(z) = p_0(z) - a_0 z_1 \cdots z_5 - a_6 z_4^4 z_5^4$.

There are two independent relations between the lattice points which can be represented by the following two generators of the lattice of relations: $l^{(1)} = (-4, 1, 1, 1, 0, 0, 1)$ and $l^{(2)} = (0, 0, 0, 0, 1, 1, -2)$. They do in fact generate the Mori cone and thus define the coordinates in the neighbourhood of the large complex structure point: $u_1 \equiv u = \frac{a_1 a_2 a_3 a_6}{a_0^4}$ and $u_2 \equiv v = \frac{a_4 a_5}{a_6^2}$. The fundamental period is

$$w_0(u, v) = \sum c(n, m) u^n v^m \tag{7.3}$$

with

$$c(n, m) = \frac{(4n)!}{(n!)^3 (m!)^2 (n - 2m)!}$$

The Picard-Fuchs equations are then found to be $\mathcal{D}_{1,2}\omega(u, v) = 0$ with

$$\begin{aligned} \mathcal{D}_1 &= \Theta_u^2(\Theta_u - 2\Theta_v) - 4u(4\Theta_u + 3)(4\Theta_u + 2)(4\Theta_u + 1) \\ \mathcal{D}_2 &= \Theta_v^2 - v(2\Theta_v - \Theta_u + 1)(2\Theta_v - \Theta_u) \end{aligned} \tag{7.4}$$

where $\mathcal{L}_1 = \Theta_u \mathcal{D}_1$ and $\mathcal{L}_2 = \mathcal{D}_2$.

From the Picard-Fuchs equations we can read off the symbols $\sigma(\mathcal{D}_1)$ and $\sigma(\mathcal{D}_2)$ which generate the ideal of symbols:

$$\begin{aligned} \sigma(\mathcal{D}_1) &= u^2 \xi_u^2 \left(u\xi_u(1 - 4^4 u^2) - 2v\xi_v\right) \\ \sigma(\mathcal{D}_2) &= v^2 \xi_v^2 - v(2v\xi_v - u\xi_u)^2 \end{aligned} \tag{7.5}$$

To get the discriminant we have to look for simultaneous solutions of $\sigma(\mathcal{D}_1) = \sigma(\mathcal{D}_2) = 0$ other than $\xi_u = \xi_v = 0$, which leads to the characteristic variety $\mathrm{Ch}(I)$. Setting $\xi_u = \xi_v = 0$ then gives the discriminant. It is straightforward to verify that $\mathrm{dis}(\hat{X}^*) = \Delta_1\Delta_2\Delta_3\Delta_4$ with

$$\Delta_1 = (1 - 512u + 65536u^2 - 262144u^2 v), \Delta_2 = (1 - 4v), \ \Delta_3 = u, \ \Delta_4 = v \tag{7.6}$$

being its irreducible components.

The PF equations also determine the Yukawa couplings (up to an overall multiplicative constant). From the PF equations we derive three third order equations with operators $\Theta_u\mathcal{D}_1$, $\Theta_v\mathcal{D}_1$ and $\mathcal{D}_2$ which provide three linear relations between the four different Yukawa couplings. We can thus express all of them in terms of one for which we derive two linear first oder differential equations which can be integrated. The final result is:

$$\begin{aligned} K_{uuu} &= c\frac{1}{\Delta_1 \Delta_3^3}, \quad K_{uuv} = c\frac{1 - 256u}{2\Delta_1\Delta_3^2\Delta_4} \\ K_{uvv} &= c\frac{512u - 1}{\Delta_1\Delta_2\Delta_3\Delta_4}, \quad K_{vvv} = c\frac{1 - 256u + 4v - 3072uv}{2\Delta_1(\Delta_2\Delta_4)^2} \end{aligned} \tag{7.7}$$

where c is an integration constant which will be fixed below. These are the $\langle \overline{27}^3 \rangle$ couplings on the manifold $\hat{X}^*$. We now perform the mirror map to compute the $\langle 27^3 \rangle$ couplings on

$\hat{X}$. To construct the variables $t_i = \frac{w_i(u)}{w_0(u)}$ we need the solutions to the PF equations which are linear in logarithms of the variables. Following our discussion in section five we write

$$
\begin{aligned}
w_i(u) &= \frac{1}{2\pi i}\frac{\partial}{\partial\rho_i} w_0(u,\rho) \\
&= \frac{1}{2\pi i}\frac{\partial}{\partial\rho_i}\sum_{n,m}\frac{\Gamma(4(n+\rho_1)+1)}{\Gamma^3(n+\rho_1+1)\Gamma^2(m+\rho_2+1)\Gamma(n-2m+\rho_1-2\rho_2+1)}u^{n+\rho_1}v^{m+\rho_2}\Big|_{\rho_1=\rho_2=0} \\
&= w_0(u)\log u_i + \tilde{w}_i(u)
\end{aligned}
\tag{7.8}
$$

where

$$
\tilde{w}_i(u) = \frac{1}{2\pi i}\sum_{m,n} d_i(n,m)u^n v^m \tag{7.9}
$$

with ($\psi = \Gamma'/\Gamma$)

$$
\begin{aligned}
d_1(n,m) &= \{4\psi(4n+1) - 3\psi(n+1) - \psi(n-2m+1)\}c(n,m) \\
d_2(n,m) &= \{-2\psi(m+1) + 2\psi(n-2m+1)\}c(n,m)
\end{aligned}
\tag{7.10}
$$

We can also write down the remaining solutions of the PF equations. The ideal $\mathcal{I}$ is generated by $I_1 = \Theta_u^2(\Theta_u - 2\Theta_v)$ and $I_2 = \Theta_v^2$, so that a basis of $\mathcal{I}^\perp$ is $\{1, \Theta_u, \Theta_v, \Theta_u^2, \Theta_u\Theta_v, 2\Theta_u^3 + \Theta_u^2\Theta_v\}$. The elements at degrees zero and one lead, via (5.13), to the periods w_0 and $w_{1,2}$ already given above. The elements at degrees two and three give the remaining solutions, with up to two and three powers of logarithms, respectively.

The topological triple couplings, i.e. the infinite radius limit of the $\langle 27^3\rangle$ couplings on $\hat{X}$, are, using the rules given in [16] and an obvious notation, $\kappa^0 = 8J^3 - 8JD^2 - 16D^3$. Here J and D refer to the divisors which correspond to the two lattice points ν_0 and ν_6, respectively with the associated moduli parameters a_0 and a_6. To transform to the divisors whose associated moduli parameters are u and v, we have to take linear combinations (c.f. (6.11)) and define $D_1 = J$ and $D_2 = \frac{1}{2}(J - D)$. In this basis the triple intersection numbers are $\kappa^0 = 8D_1^3 + 4D_1^2 D_2$. They will be used to normalize the couplings $\langle 27^3\rangle$. Ratios of the couplings in the latter basis can also be read off from the elements of $\mathcal{I}^\perp$: $D_i^{(2)} = \frac{1}{2}\kappa^0_{ijk}\partial_{\rho_j}\partial_{\rho_k}$ and $D^{(3)} = -\frac{1}{6}\kappa^0_{ijk}\partial_{\rho_i}\partial_{\rho_j}\partial_{\rho_k}$.

Now we are almost done. What is left to do is to make the coordinate transformation from the complex structure moduli space of $\hat{X}^*$ with coordinates u, v to the Kähler structure moduli space of $\hat{X}$ with coordinates t_1, t_2 and to go to the gauge $w_0 = 1$, i.e. to divide the Yukawa couplings by $(w_0(z(t)))^2$ and express them in terms of $q_i = e^{2\pi i t_i}$ (this involves inversion of power series). At lowest orders this leads to the expansions

$$
\begin{aligned}
\kappa_{111} &= \kappa^0_{111} + N(1,0)\,q_1 + (N(1,0) + 8N(2,0))\,q_1^2 + N(1,1)\,q_1 q_2 + O(q^3) \\
&= 8 + 640\,q_1 + 80896\,q_1^2 + 640\,q_1 q_2 + O(q^3) \\
\kappa_{112} &= \kappa^0_{112} + N(1,1)\,q_1 q_2 + O(q^3) = 4 + 640\,q_1 q_2 + O(q^3) \\
\kappa_{122} &= \kappa^0_{122} + N(1,1)\,q_1 q_2 + O(q^3) = 640\,q_1 q_2 + O(q^3) \\
\kappa_{222} &= \kappa^0_{222} + N(0,1)q_2 + (N(0,1) + 4N(0,2))\,q_2^2 + N(1,1)\,q_1 q_2 + O(q^3) \\
&= 4\,q_2 + 4\,q_2^2 + 640\,q_1 q_2 + O(q^3)
\end{aligned}
\tag{7.11}
$$

from which we can read off the numbers of instantons at lowest degrees. Results at higher degrees can be found in refs.[15] and [16][34].

This completes our discussion of this model, which served as a demonstration of the techniques outlined in earlier sections. These techniques have been applied for models with up to three moduli in [16] and [17] and can easily be extended to even more moduli. The hard part seems to be to set up the PF equations. It is not always as easy as in the example above (cf.[16]).

8. Conclusions and Outlook

We have tried to convey an idea of the main concepts necessary to understand recent developments of Calabi-Yau compactification of string theory. One of the main advances in the past few years has been the use of mirror symmetry to compute Yukawa couplings. The information necessary to get the Kähler potential, which is of course essential in order to normalize the fields and hence the Yukawa couplings, is also contained in the Picard-Fuchs equations. This has been done explicitly for models with $h^{1,1}=1$ in refs.[12] and [13], and for a few models with $h^{1,1}=2$ in [15]. We have conjectured that, at least in the vicinity of the large radius limit, we have constructed quite generally the correct prepotential from which one can get the Kähler metric. The analysis presented here was restricted to the region in moduli space close to the large complex structure and large radius. In the references just cited, this has been extended to the whole moduli space. Since one expects the internal dimensions, or, equivalently, the vacuum expectation values of the moduli, to be of order one (in units of $1/\alpha'$), one needs expressions for the Yukawa couplings which are valid in this range.

Even though there has been considerable progress towards the computation of phenomenologically relevant couplings in strings on Calabi-Yau manifolds, there is still a lot of work left to do. The computation of the Yukawa couplings involving the E_6 singlets is one of them. Also, as long as we have no information on the value of the moduli, the Yukawa couplings are not yet fixed. A potential for the moduli might be generated by non-perturbative string effects. At present there is no hope to compute them. Some information of their possible form can be obtained from studying functions of the moduli with the correct transformation behaviour under duality transformations. However, for general CY compactifications the duality groups are not known, and even if so, one still has to face the task to construct functions of the moduli which are candidates for a non-perturbative potential. One would expect that in general this will not lead to a unique answer. Also, most of the things said here seem to be restricted to the symmetric (2,2) theories. An open problem is the treatment of more general string vacua. We hope to be able, in the not so far future, to report some progress on some of these issues, may be even at the same occasion.

[34] Note that in ref. [16], the degrees of the rational curve are defined with respect to the basis h_J, h_D as $n_J=\int_C h_J$ and $n_D=\int_C h_D$ whereas here we define them with respect to the basis h_{D_1}, h_{D_2} as $(n_1,n_2)=(\int_C h_{D_1}, \int_C h_{D_2})$.

Acknowledgement: S.T. would like the organizers for the opportunity to present our (and others') results.

References

[1] For introductions, see e.g. M. Green, J. Schwarz and E. Witten, *Superstring Theory*, Vols. I and II, Cambridge University Press, 1986; D. Lüst and S. Theisen, *Lectures on String Theory*, Springer Lecture Notes in Physics, Vol. 346, 1990; M. Kaku, *Introduction to Superstrings*, Springer 1988

[2] E. Verlinde, H.Verlinde, *Lectures on String Perturbation Theory*, in *Superstrings 88*, Trieste Spring School, Ed. Greene et al. World Scientific (1989) 189

[3] C. Callan, D. Friedan, E. Martinec and M. Perry, Nucl. Phys. B262 (1985) 593, A. Sen, Phys. Rev. D32 (1985) 2102 and Phys. Rev. Lett. 55 (1985) 1846

[4] D. Nemeschanski and A. Sen, Phys. Lett. 178B (1986) 365

[5] For review, see A. Giveon, M. Porrati and E. Rabinovici, *Target Space Duality in String Theory*, hep-th 9401139, to be published in Phys. Rep.

[6] L. Dixon in *Proc. of the 1987 ICTP Summer Workshop in High Energy Physics and Cosmology*, Trieste, ed. G. Furlan et al., World Scientific

[7] W. Lerche, C. Vafa and N. Warner, Nucl. Phys. B324 (1989) 427

[8] B. Greene and M. Plesser, Nucl. Phys. B338 (1990) 15

[9] *Essays on Mirror Manifolds* (ed. S.-T. Yau), Int. Press, Hong Kong, 1992

[10] R. Dijkgraaf, E. Verlinde and H. Verlinde, Utrecht preprint THU-87/30; A. Shapere and F. Wilczek Nucl. Phys. B320 (1989) 669; A. Giveon, E. Rabinovici, G. Veneziano, Nucl. Phys. B322 (1989) 169

[11] S. Ferrara and S. Theisen, *Moduli Spaces, Effective Actions and Duality Symmetry in String Compactifications*, in Proc. of the Third Hellenic School on Elementary Particle Physics, Corfu 1989, Argyres et al.(eds), World Scientific 1990

[12] P. Candelas, X. De la Ossa, P. Green and L. Parkes, Nucl. Phys. B359 (1991) 21

[13] D.Morrison, *Picard-Fuchs Equations and Mirror Maps for Hypersurfaces*, in *Essays on Mirror Manifolds* (ed. S.-T. Yau), Int. Press, Hong Kong, 1992; A. Klemm and S. Theisen, Nucl. Phys. B389 (1993) 153; A. Font, Nucl. Phys. B391 (1993) 358

[14] A. Libgober and J. Teitelbaum, *Duke Math. Journ., Int. Res. Notices* 1 (1993) 29; A. Klemm and S. Theisen, *Mirror Maps and Instanton Sums for Complete Intersections in Weighted Projective Space*, preprint LMU-TPW 93-08

[15] P. Candelas, X. de la Ossa, A. Font, S. Katz and D. Morrison, *Mirror Symmetry for Two Parameter Models I*, preprint CERN-TH.6884/93

[16] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror Symmetry, Mirror Map and Applications to Calabi-Yau Hypersurfaces*, to appear in Comm. Math. Phys.

[17] S. Hosono, A. Klemm, S. Theisen and S.-T. Yau, *Mirror Symmetry, Mirror Map and Applications to Complete Intersection Calabi-Yau Spaces*, preprint LMU-TPW-94-03 (to appear)

[18] P. Aspinwall, B. Greene and D. Morrison, *Space-Time Topology Change: the Physics of Calabi-Yau Moduli Space*, preprint IASSNS-HEP-93-81; *Calabi-Yau Moduli Space, Mirror Manifolds and Space-Time Topology Change in String Theory*, preprint IASSNS-HEP-93-38; *The Monomial-Divisor Mirror Map*, preprint IASSNS-HEP-93-43

[19] P. Berglund, P. Candelas, X. de la Ossa, A. Font, T. Hübsch, D. Jancic and F. Quevedo, *Periods for Calabi-Yau and Landau-Ginsburg Vacua*, preprint CERN-TH.6865/93; P. Berglund, E. Derrich, T. Hübsch and D. Jancic, *On Periods for String Compactification*, preprint HUPAPP-93-6

[20] P. Berglund and S. Katz, *Mirror Symmetry for Hypersurfaces in Weighted Projective Space and Topological Couplings*, preprint IASSNS-HEP-93-65

[21] P. Candelas, G. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258 (1985) 46

[22] Easily accessible introductions to Calabi-Yau manifolds are G. Horowitz, *What is a Calabi-Yau Space?* in *Unified String Theories*, M. Green and D. Gross, editors, World Scientific 1986 and P. Candelas, *Introduction to Complex Manifolds*, Lectures at the 1987 Trieste Spring School, published in the proceedings. For a rigorous mathematical treatment we refer to A.L. Bessis, *Einstein Manifolds*, Springer 1987; P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons 1978; K. Kodaira, *Complex Manifolds and Deformation of Complex Structures*, Springer 1986

[23] A good and detailed introduction to Calabi-Yau manifolds is the book by T. Hübsch, *Calabi-Yau Manifolds*, World Scientific 1991

[24] S.T. Yau, Proc. Natl. Acad. Sci. USA 74 (1977), 1798; Comm. Pure Appl. Math. 31 (1978) 339

[25] K. Kodaira, L. Nirenberg and D. C. Spencer, Ann. Math. 68 (1958) 450

[26] G. Tian, in *Mathematical Aspects of String Theory*, ed. S. T. Yau, World Scientific, Singapore (1987)

[27] A. Strominger, Phys. Rev. Lett. 55 (1985) 2547, and in *Unified String Theory*, eds. M. Green and D. Gross, World Scientific 1986; A. Strominger and E. Witten, Comm. Math. Phys. 101 (1985) 341

[28] L. Dixon, D. Friedan, E. Martinec and S, Shenker, Nucl. Phys. B282 (1987) 13; S. Hamidi and C. Vafa, Nucl. Phys. B279 (1987) 465; J. Lauer, J. Mas and H. P. Nilles Nucl. Phys. B351 (1991) 353; S. Stieberger, D. Jungnickel, J. Lauer and M. Spalinski, Mod. Phys. Lett. A7 (1992) 3859; J. Erler, D. Jungnickel, M. Spalinski and S. Stieberger, Nucl. Phys. B397 (1993) 379

[29] D. Gepner, Phys. Lett. 199B (1987) 380 and Nucl. Phys. B311 (1988) 191

[30] J. Distler and B. Greene, Nucl. Phys. B309 (1988) 295

[31] M. Dine, N. Seiberg, X-G. Wen and E. Witten, Nucl. Phys. B278 (1987) 769, Nucl. Phys. B289 (1987) 319

[32] S. Ferrara, D. Lüst, A. Shapere and S. Theisen, Phys. Lett. 225B (1989) 363

[33] P. Aspinwall and D. Morrison, Comm. Math. Phys. 151 (1993) 245

[34] E. Witten, Comm. Math. Phys. 118 (1988) 411

[35] T. Banks, L. Dixon, D. Friedan and E. Martinec, Nucl. Phys. B284 (1988) 613

[36] M. Ademollo et al., Nucl. Phys. B11 (1976) 77 and Nucl. Phys. B114 (1976) 297 and Phys. Lett. 62B (1976) 105

[37] A. Schwimmer and N. Seiberg, Phys. Lett. 184B (1987) 191

[38] M. Dine and N. Seiberg, Nucl. Phys. B301 (1988) 357

[39] E. Witten, Nucl. Phys. B202 (1982) 253; R. Rohm and E Witten, Ann. Phys. **170** (1986) 454

[40] P. Candelas, E. Derrick and L. Parkes, Nucl. Phys. B407 (1993) 115

[41] R. Schimmrigk, Phys. Rev. Lett. 70 (1993) 3688 and *Kähler manifolds with positive first Chern class and mirrors of rigid Calabi-Yau manifolds*, preprint BONN-HE 93-47

[42] N. Seiberg, Nucl. Phys. B303 (1988) 286

[43] S. Cecotti, S. Ferrara and L. Girardello, Int. J. Mod. Phys. A4 (1989) 2475; S. Ferrara, Nucl. Phys. (Proc. Suppl.) 11 (1989) 342

[44] P. Candelas, P. Green and T. Hübsch, *Connected Calabi-Yau compactifications*, in Strings '88, J. Gates et al. (eds), World Scientific 1989 and Nucl. Phys. B330 (1990) 49; P. Candelas, T. Hübsch and R. Schimmrigk, Nucl. Phys. B329 (1990) 582

[45] L. Dixon, V. Kaplunovsky and J. Louis, Nucl. Phys. B329 (1990) 27

[46] B. de Wit, P.G. Lauwers, R. Philippe, S.Q. Su and A. van Proeyen, Phys. Lett. 134B (1984) 37; B. de Wit and A. van Proeyen, Nucl. Phys. B245 (1984) 89; J.P. Derendinger, S. Ferrara, A. Masiero and A. van Proeyen, Phys. Lett. 140B (1984) 307; B. de Wit, P.G. Lauwers and A. van Proeyen, Nucl. Phys. B255 (1985) 569; E. Cremmer, C. Kounnas, A. van Proeyen, J.P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, Nucl. Phys. B250 (1985) 385

[47] M. Kreuzer and H. Skarke, Nucl. Phys. B388 (1993) 113; A. Klemm and R. Schimmrigk, Nucl. Phys. B411 (1994) 559

[48] P. Candelas, A. Dale, A. Lütken and R. Schimmrigk, Nucl. Phys. B298 (1988) 493; P. Candelas, A. Lütken and R. Schimmrigk, Nucl. Phys. B306 (1989) 105

[49] V. Batyrev and D. van Straten, *Generalized Hypergeometric Functions and Rational Curves on Calabi-Yau Complete Intersections in Toric Varieties*, preprint 1992

[50] I. Dolgachev, *Weighted Projective Varieties* in *Lecture Notes in Mathematics* 956 Springer-Verlag (1992) 36

[51] A. R. Fletcher, *Working with Weighted Complete Intersections*, Max-Planck-Institut Series, No. 35 (1989) Bonn

[52] D. G. Markushevich, M. A. Olshanetsky and A. M. Perelomov, Comm. Math. Phys. 111 (1987) 247

[53] J. Erler and A. Klemm, Comm. Math. Phys. 153 (1993) 57

[54] L. Dixon, J. Harvey, C. Vafa and E. Witten, Nucl. Phys. B261 (1985) 678 and Nucl. Phys. B274 (1986) 285

[55] S.S. Roan and S. T. Yau, Acta Math. Sinica (NS) 3 (1987) 256; S.S. Roan, J. Diff. Geom. 30 (1989) 523

[56] M. Kreuzer and H. Skarke, Nucl. Phys. B405 (1993) 305; M. Kreuzer, Phys. Lett. 314B (1993) 31; A. Niemeyer, Diplom Thesis, TU-München, 1993

[57] T. Oda, *Convex Bodies and Algebraic Geometry: an Introduction to the Theory of Toric Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. 15, Springer Verlag 1988; V.I. Danilov, Russian Math. Surveys, 33 (1978) 97; M. Audin, *The Topology of Torus Actions on Symplectic Manifolds*, Progress in Mathematics, Birkhäuser 1991; W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies, Princeton University Press 1993; V.I. Arnold, S. M. Gusein-Zade and A.N. Varchenko, *Singularities of Differntial Maps*, Birkhäuser 1985, Vol. II, Chapter 8

[58] V. Batyrev, Duke Math. Journal 69 (1993) 349

[59] V. Batyrev, Journal Alg. Geom., to be published

[60] V. Batyrev, *Quantum Cohomology Rings of Toric Manifolds*, preprint 1992

[61] V. Batyrev and D. Cox, *On the Hodge Structure of Projective Hypersurfaces in Toric Varieties*, preprint 1993

[62] F. Hirzebruch, Math. Annalen 124 (1951) 77

[63] M. Beltrametti and L. Robbiano, Expo. Math. 4 (1986) 11.

[64] D.Aspinwall, B.Greene and D.Morrison, Phys.Lett. 303B (1993) 249 and *The Monomial-Divisor Mirror Map*, preprint IASSNS-HEP-93/43

[65] S.-S. Roan: *Topological Couplings of Calabi-Yau Orbifolds*, Max-Planck-Institut Serie No. 22 (1992), to appear in *J. of Group Theory in Physics*

[66] E.L. Ince, *Ordinary Differential Equations*, Dover 1956

[67] M. Yoshida, *Fuchsian Differential Equations*, Braunschweig (1987) Vieweg

[68] P. A. Griffiths, Bull. Amer. Math. Soc. 76 (1970) 228

[69] P. Griffiths, Ann. of Math. 90 (1969) 460

[70] P. Candelas, Nucl. Phys. B298 (1988) 458

[71] I.M.Gel'fand, A.V.Zelevinsky and M.M.Kapranov, Func. Anal. Appl. 28 (1989) 12 and Adv. Math. 84 (1990) 255

[72] D. Morrison, *Where is the Large Radius Limit*, preprint IASSNS-HEP-93/68 and *Compactifications of Moduli Spaces Inspired by Mirror Symmetry*, preprint DUK-M-93-06

[73] M. Reid, *Decomposition of Toric Morphisms*, in Progress in Mathematics 36, M. Artin and J. Tate, eds, Birkhäuser 1983

[74] S. Ferrara and A. Strominger, in the Proceedings of the Texas A & M Strings '89 Workshop; ed. R. Arnowitt et al., World Scientific 1990

[75] P. Candelas, P. Green and T. Hübsch Nucl. Phys. B330 (1990) 49; P. Candelas and X.C. de la Ossa, Nucl. Phys. B355 (1991) 455

[76] R. Bryant and P. Griffiths, *Some observations on the Infinitesimal Period Relations for Regular Threefolds with Trivial Canonical Bundle*, in Progress in Mathematics 36, M. Artin and J. Tate, eds; Birkhäuser, 1983

[77] A. Strominger, Comm. Math. Phys. 133 (1990) 163

[78] L. Castellani, R. D'Auria and S. Ferrara, Phys. Lett. 241B (1990) 57 and Class. Quantum Grav. 7 (1990) 1767; S. Ferrara and J. Louis, Phys. Lett. 278B (1992) 240

[79] A. Ceresole, R. D'Auria, S. Ferrara, W. Lerche and J. Louis, Int. J. Mod. Phys. A8 (1993) 79

[80] X.-G. Wen and E. Witten, Phys. Lett. 166B (1986) 397