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# INCOMPRESSIBLE HYDRODYNAMIC LIMITS OF THE BOLTZMANN EQUATION:

# A SURVEY OF MATHEMATICAL RESULTS

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The present paper reports mainly on some joint work with C. Bardos and D. Levermore: [2], [3], [4], [5].

It is classical in nonequilibrium statistical physics to derive formally the equations of compressible fluid hydrodynamics (the Euler and Navier-Stokes systems) from the Boltzmann equation. The degree of rarefaction of a flow is measured via the Knudsen number $Kn$ defined as the ratio of the mean free path of molecules to a characteristic length of the flow. Then, one expands the solution of the Boltzmann equation into a series of nonnegative powers of $Kn$: this technique has been initiated by Hilbert and later on by Chapman and Enskog . The Euler system for compressible flows corresponds to the 0th order approximation as $Kn \to 0$, whereas the Navier-Stokes system is obtained as the 1st order approximation as $Kn \to 0$. In particular, it is found that the viscosity and thermic dissipation terms in the resulting Navier-Stokes system are of order 1 in $Kn$. This circumstance makes it as difficult a problem to prove the convergence of he Hilbert or Chapman-Enskog expansions as the obtention of a solution of the Euler system. It is not surprising that the only mathematical proofs existing so far of hydrodynamic limits of the Boltzmann equation rely on the assumption of a smooth solution to the limiting Euler system: see Nishida [24], Caflisch [9], Kawashima-Matsumura-Nishida [20]. However, these techniques seem little promising since it is known that solutions of the compressible Euler system generically develop singularities in finite time: see the work by Sideris [27].

Even though there is no proof of global existence of a solution to the compressible Navier-Stokes system in arbitrary dimension (other than small perturbations of the trivial equilibria), it would certainly be a nice feature if one could obtain the Navier-Stokes system from the Boltzmann equation with the viscosity and thermal dissipation terms independent of the Knudsen number and therefore staying uniformly elliptic (nonlinear) operators as $Kn \to 0$. But this is simply impossible. The obstruction comes from the formula

$$Kn = \alpha \frac{Ma}{Re}$$

where $Ma$ is the Mach number (defined as the ratio of the bulk velocity of the flow to the speed of sound), $Re$ is the Reynolds number (defined as the ratio of the product of the bulk velocity of the flow by some characteristic length scale to the kinematic viscosiy of the fluid) and $\alpha$ is some "pure number", like $\frac{16}{\sqrt{30\pi}}$. Since the only hydrodynamic equations for which nontrivial global solutions are known to exist in arbitrary space dimension correspond to a regime where the Reynolds number stays finite, and since hydrodynamic limits of the Boltzmann equation are obtained in the limit $Kn \to 0$, the hydrodynamic regimes meeting these two requirements have a Mach number $Ma \to 0$, or, in other words, correspond to incompressible flows. (Incompressible flows are observed when the Mach number is small and the kinetic energy in the acoustic modes is small compared to that in the vortical modes: see Kainerman-Majda [21] and Bayly-Levermore-Passot [8]). This is the reason why we shall mainly study the hydrodynamic limit of the Boltzmann equation leading o the incompressible Navier-Stokes (or Stokes) equation. Besides the physical motives recalled here, there are some mathematical incentives to do so: global weak solutions of the 3 dimensional Navier-Stokes equation were constructed by J. Leray in pioneering paper that appeared in 1934 [22]. More recently (in 1990), global weak solutions of the Boltzmann equation were obtained by R. DiPerna and P.-L. Lions [14]. Both proofs are very similar in spirit, in particular due to the fact that in both cases, the mathematical theory allows certain global quantities to decay in time instead of being conserved as expected from physical conventional wisdom (like kinetic energy in the case of the Navier-Stokes equation and total energy in the case of the Boltzmann equation), or to have a dissipation rate bigger than expected (like kinetic energy in the Navier-Stokes equation and entropy in the Boltzmann equation).

## 1. Hydrodynamic scaling for the Boltzmann equation.

### 1.1. The Navier-Stokes and Boltzmann Equations.

The incompressible Navier-Stokes equations describe the evolution of the velocity field $u = u(t, x)$ of an idealized fluid over a given spatial domain in $\mathbf{R}^{\mathrm{D}}$:

$$\begin{gathered} \nabla_x \cdot u = 0\,, \\ \partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u\,, \\ u(0, x) = u^{in}(x)\,, \end{gathered} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity of the fluid. We will use the modification Leray's result for the case when the fluid is contained in a D-dimensional periodic box $\mathbf{T}^{\mathrm{D}}$; this will be stated more precisely below.

If the fluid consists of similar particles then at the kinetic level of description the state of the fluid is given by a density $F(t, x, v)$ of particle mass with position $x$ and velocity $v$ in the single particle phase space at instant $t$. If the particles interact only through a conservative interparticle force with a finite range then at low densities all but binary collisions can be neglected and the evolution of the phase space density $F$ is governed by the classical Boltzmann equation:

$$\partial_t F + v \cdot \nabla_x F = B(F, F)\,, \tag{1.2a}$$

$$F(0,x,v) = F^{in}(x,v) \geq 0\,, \tag{1.2b}$$

where the collision operator $B(F,F)$ is given by

$$B(F,F) = \iint (F_1'F' - F_1F)\, b(v_1 - v, \omega)\, d\omega\, dv_1\,.$$

The Boltzmann kernel $b(v_1 - v, \omega)$ is a nonnegative measurable function. The variable $\omega$ lies on the unit sphere $\mathbf{S}^{\mathrm{D}-1} = \{\omega \in \mathbf{R}^{\mathrm{D}} \,:\, |\omega| = 1\}$ endowed with its rotationally invariant unit measure $d\omega$. The $F$, $F_1$, $F'$ and $F_1'$ appearing in the integrand are understood to mean $F(t,x,\cdot)$ evaluated at the velocities $v$, $v_1$, $v'$ and $v_1'$ respectively, where the primed velocities are defined by

$$v' = v + \omega\,\omega\cdot(v_1 - v)\,, \qquad v_1' = v_1 - \omega\,\omega\cdot(v_1 - v)\,, \tag{1.3}$$

for any given $(v, v_1, \omega) \in \mathbf{R}^{\mathrm{D}} \times \mathbf{R}^{\mathrm{D}} \times \mathbf{S}^{\mathrm{D}-1}$. The primed and unprimed velocities denote possible velocities for a pair of particles either before or after they interact through an elastic binary collision. Conservation of momentum and energy for particle pairs during such collision is expressed as

$$v + v_1 = v' + v_1'\,, \qquad |v|^2 + |v_1|^2 = |v'|^2 + |v_1'|^2\,. \tag{1.4}$$

The Boltzmann kernel $b$ contains all the information concerning the collisional physics. It has the form

$$b(v_1 - v, \omega) = |v_1 - v|\,\Sigma(|v_1 - v|, |\mu_c|)\,, \qquad \mu_c = \frac{\omega\cdot(v_1 - v)}{|v_1 - v|}\,, \tag{1.5}$$

(where $\Sigma \geq 0$ is the specific differential cross-section).

### 1.2. Formal Structure of the Boltzmann Equation.

The formal structure of the Boltzmann equation follows from two fundamental properties of the measure $b(v_1 - v, \omega)\, d\omega\, dv_1\, dv$. First, that it is invariant under the coordinate transformations

$$(v, v_1, \omega) \mapsto (v_1, v, \omega)\,, \qquad (v, v_1, \omega) \mapsto (v', v_1', \omega)\,, \qquad (v, v_1, \omega) \mapsto (v_1', v', \omega)\,. \tag{1.6}$$

These transformations will be refered to as the collisional symmetries. Second, that it characterizes microscopic conserved quantities in the sense that for any measurable $\xi = \xi(v)$ the following statements are equivalent:

$$\begin{aligned} &(i) \quad \xi + \xi_1 - \xi' - \xi_1' = 0 \quad \text{for almost every } (v, v_1, \omega) \in \mathbf{R}^{\mathrm{D}} \times \mathbf{R}^{\mathrm{D}} \times \mathbf{S}^{\mathrm{D}-1}\,; \\ &(ii) \quad \xi = \alpha + \beta\cdot v + \gamma\tfrac{1}{2}|v|^2 \quad \text{for some } (\alpha, \beta, \gamma) \in \mathbf{R} \times \mathbf{R}^{\mathrm{D}} \times \mathbf{R}\,. \end{aligned} \tag{1.7}$$

This property will be refered to as the equilibria characterization.

Repeated application of the collisional symmetries (1.6) yields the following important identity regarding the collision operator:

$$
\begin{aligned}
-\int \xi(v)\, B(F,F)(v)\, dv &= \iiint \xi\, (F_1 F - F_1' F')\, b(v_1 - v, \omega)\, d\omega\, dv_1\, dv \\
&= \tfrac{1}{4} \iiint (\xi + \xi_1 - \xi' - \xi_1')\, (F_1 F - F_1' F')\, b(v_1 - v, \omega)\, d\omega\, dv_1\, dv
\end{aligned}
\tag{1.8}
$$

for every $\xi = \xi(v)$ and $F = F(v)$ for which the integrals make sense.

Successively setting $\xi = 1, v, \frac{1}{2}|v|^2$ in (1.8) and using the microscopic conservation laws (1.4) gives the conservation relations

$$
\int B(F,F)\, dv = 0\,, \qquad \int v\, B(F,F)\, dv = 0\,, \qquad \int \tfrac{1}{2}|v|^2 B(F,F)\, dv = 0\,, \tag{1.9}
$$

for every $F = F(v)$ for which the integrals make sense. (It can be shown that these are essentially all the conservation relations satisfied by $B(F,F)$ for all $F$).

If $F$ solves the Boltzmann equation (1.2a) then (1.9) implies that it satisfies local conservation laws of mass, momentum, and energy:

$$
\begin{aligned}
\partial_t \int F\, dv + \nabla_x \cdot \int v\, F\, dv &= 0\,, \\
\partial_t \int v\, F\, dv + \nabla_x \cdot \int v \otimes v\, F\, dv &= 0\,, \\
\partial_t \int \tfrac{1}{2}|v|^2 F\, dv + \nabla_x \cdot \int v \tfrac{1}{2}|v|^2 F\, dv &= 0\,.
\end{aligned}
\tag{1.10}
$$

Upon setting $\xi = \log F$ in the collision identity (1.8), Boltzmann observed that the resulting integrand is nonnegative and hence obtained the dissipation law

$$
-\int \log F\, B(F,F)\, dv = \tfrac{1}{4} \iiint \log\!\left(\frac{F_1' F'}{F_1 F}\right) (F_1' F' - F_1 F)\, b(v_1 - v) d\omega\, dv_1 dv \geq 0\,, \tag{1.11}
$$

for every $F = F(v)$ for which the integrals make sense. The equilibria of the collision operator are then characterized by using the (1.7). For any $F = F(v)$ for which the integrals make sense, the following statments are equivalent:

$$
\begin{aligned}
&(i) \quad B(F,F) = 0\,; \\
&(ii) \quad \int \log F\, B(F,F)\, dv = 0\,; \\
&(iii) \quad F = \exp\big(\alpha + \beta \cdot v + \gamma \tfrac{1}{2}|v|^2\big) \quad \text{for some } (\alpha, \beta, \gamma) \in \mathbf{R} \times \mathbf{R}^{\mathrm{D}} \times \mathbf{R}\,.
\end{aligned}
\tag{1.12}
$$

The equilibria characterized in $(iii)$ above will have finite mass, momentum, and energy density when $\gamma < 0$. In that case they can be written as $F = M(\rho, u, \theta)$, where $M(\rho, u, \theta)$ are the classical Maxwellians defined by

$$
M(\rho, u, \theta) \equiv \frac{\rho}{(2\pi\theta)^{\mathrm{D}/2}} \exp(-\tfrac{1}{2}|v - u|^2/\theta)\,, \tag{1.13}
$$

and where the density $\rho \geq 0$, the velocity $u \in \mathbf{R}^{\mathrm{D}}$, and the temperature $\theta > 0$ are determined by the relations

$$\rho = \int M(\rho, u, \theta)\, dv\,, \quad \rho u = \int v\, M(\rho, u, \theta)\, dv\,, \quad \tfrac{1}{2}\rho|u|^2 + \tfrac{\mathrm{D}}{2}\rho\,\theta = \int \tfrac{1}{2}|v|^2 M(\rho, u, \theta)\, dv\,. \tag{1.14}$$

Now, if $F$ solves the Boltzmann equation (1.2a) then the dissipation law (1.11) implies that $F$ satisfies the local entropy dissipation law

$$\begin{aligned} \partial_t \int F \log F\, dv\, + \nabla_x \cdot \int v\, F \log F\, dv \\ = -\tfrac{1}{4} \iiint \log\!\left(\frac{F_1' F'}{F_1 F}\right) (F_1' F' - F_1 F)\, b(v_1 - v) d\omega\, dv_1 dv \geq 0\,. \end{aligned} \tag{1.15}$$

### 1.3. Dimensional Analysis.

All the flows studied in the present work have a periodic structure and therefore can be considered as set on a dilated copy of the torus $\mathbf{T}^{\mathrm{D}}$. The dimensional scales of the Boltzmann initial-value problem (1.2) can be identified as follows. First, the volume of the periodic box determines a length scale $\lambda_*$ by setting

$$\int dx = \lambda_*^{\mathrm{D}}\,, \tag{1.16}$$

where here, as with all integrals, the integration is understood to be over the whole domain associated with its measure unless otherwise stated. The sides of the box $\mathbf{T}^{\mathrm{D}}$ need not be the same length; however, all these length scales are assumed to be of the same order.

Next, after a Galilean transformation to ensure that

$$\iint v\, F^{in}\, dv\, dx = 0\,, \tag{1.17$a$}$$

the initial data $F^{in}$ determines a density scale $\rho_*$ and a velocity scale $\theta_*^{1/2}$ by the relations

$$\iint F^{in}\, dv\, dx = \rho_* \lambda_*^{\mathrm{D}}\,, \qquad \iint \tfrac{1}{2}|v|^2 F^{in}\, dv\, dx = \tfrac{\mathrm{D}}{2}\rho_* \theta_* \lambda_*^{\mathrm{D}}\,. \tag{1.17$b$}$$

Associated with the initial data $F^{in}$ is an absolute (constant in space and time) Maxwellian, uniquely determined by the density $\rho_*$ and the temperature $\theta_*$ of $F^{in}$

$$M = \frac{\rho_*}{(2\pi\theta_*)^{\mathrm{D}/2}} \exp(-\tfrac{1}{2}|v|^2/\theta_*)\,. \tag{1.18}$$

(Observe that, by (1.14), $F^{in}$ and $M$ have the same moments of order less than 2). Here $\theta_*$ is related to the physical temperature $T_*$ of this equilibrium by $\theta_* = kT_*/m$, where $m$ is the single particle mass and $k$ is Boltzmann's constant.

Finally, since the Boltzmann kernel $b$ has units of reciprocal density$\times$time, it determines a timescale $\tau_*$ by

$$\iiint M_1 M\, b(v_1 - v, \omega)\, d\omega\, dv_1\, dv = \frac{\rho_*}{\tau_*}\,. \tag{1.19}$$

The finiteness of the above integral is ensured by the fact that $b$ has at most sublinear growth in the variable $v_1 - v$ in all classical physical examples. Therefore $0 < \tau_* < \infty$. This is the scale of the average time interval that particles in the equilibrium density $M$ spend traveling freely between collisions, the so-called mean free time. It is related to the length scale of the mean free path ($= \theta_*^{1/2}\tau_*$).

The initial-value problem (1.2) can then be reformulated in terms of dimensionless variables; these are introduced below adorned with hats. Dimensionless time, space, and velocity are defined by

$$t = \frac{\lambda_*}{\theta_*^{1/2}}\hat{t}\,, \quad x = \lambda_* \hat{x}\,, \quad v = \theta_*^{1/2}\hat{v}\,; \tag{1.20}$$

while a dimensionless phase space density is given by

$$F(t,x,v) = \frac{\rho_*}{\theta_*^{\mathrm{D}/2}}\hat{F}(\hat{t},\hat{x},\hat{v})\,. \tag{1.21}$$

Define the dimensionless Boltzmann kernel $\hat{b}(\hat{v}_1 - \hat{v}, \omega)$ by the relation

$$b(v_1 - v, \omega) = \frac{1}{\rho_* \tau_*}\hat{b}(\hat{v}_1 - \hat{v}, \omega)\,, \tag{1.22}$$

and set the corresponding dimensionless collision operator to be

$$\hat{B}(\hat{F},\hat{F}) = \iint (\hat{F}_1'\hat{F}' - \hat{F}_1\hat{F})\,\hat{b}(\hat{v}_1 - \hat{v}, \omega)\, d\omega\, d\hat{v}_1\,. \tag{1.23}$$

Substituting (1.20)-(1.23) into the Boltzmann equation (1.2a) and henceforth dropping all hats yields the dimensionless initial-value problem

$$\partial_t F + v\cdot\nabla_x F = \frac{1}{\epsilon}B(F,F)\,, \tag{1.24a}$$

$$F(0,x,v) = F^{in}(x,v) \geq 0\,, \tag{1.24b}$$

where $\epsilon = \theta_*^{1/2}\tau_*/\lambda_*$ is the dimensionless mean free path or Knudsen number.

### 1.4. Fluctuations about an absolute Maxwellian.

The incompressible Navier-Stokes equations are obtained with a scaling in which $F$ is considered close to $M$ in a sense that will be made more precise later. It is natural to introduce the relative density, $G(t,x,v)$, defined by $F = MG$, where the dimensionless equilibrium Maxwellian is now

$$M = \frac{1}{(2\pi)^{\mathrm{D}/2}}\exp(-\tfrac{1}{2}|v|^2)\,. \tag{1.25}$$

Recasting the initial-value problem (1.24) for $G$ yields

$$\partial_t G + v\cdot\nabla_x G = \frac{1}{\epsilon} Q(G,G)\,, \tag{1.26a}$$

$$G(0,x,v) = G^{in}(x,v) \geq 0\,, \tag{1.26b}$$

where the collision operator is now given by

$$Q(G,G) = \iint (G_1' G' - G_1 G)\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1\,. \tag{1.27}$$

This nondimensionalization has the following normalizations:

$$\int d\omega = 1\,, \qquad \int M dv = 1\,, \qquad \int |v|^2 M dv = 1\,, \qquad \int dx = 1\,, \tag{1.28}$$

associated with the domains $\mathbf{S}^{\mathrm{D}-1}$, $\mathbf{R}^{\mathrm{D}}$, and $\mathbf{T}^{\mathrm{D}}$ respectively;

$$\iint G^{in}\, M dv\, dx = 1\,, \quad \iint v\, G^{in}\, M dv\, dx = 0\,, \quad \iint \tfrac{1}{2}|v|^2 G^{in}\, M dv\, dx = \tfrac{\mathrm{D}}{2}\,, \tag{1.29}$$

associated with the initial data; and

$$\iiint b(v_1 - v, \omega)\, d\omega\, M_1 dv_1\, M dv = 1\,, \tag{1.30}$$

associated with the Boltzmann kernel.

Since $M dv$ is a positive unit measure on $\mathbf{R}^{\mathrm{D}}$, we denote by $\langle \xi \rangle$ the average over this measure of any integrable function $\xi = \xi(v)$,

$$\langle \xi \rangle = \int \xi\, M dv\,. \tag{1.31}$$

Since $d\mu \equiv b(v_1 - v, \omega)\, d\omega\, M_1 dv_1\, M dv$ is a nonnegative unit measure on $\mathbf{R}^{\mathrm{D}} \times \mathbf{R}^{\mathrm{D}} \times \mathbf{S}^{\mathrm{D}-1}$, we denote by $\langle\!\langle \Xi \rangle\!\rangle$ the average over this measure of any integrable function $\Xi = \Xi(v, v_1, \omega)$,

$$\langle\!\langle \Xi \rangle\!\rangle = \int \Xi\, d\mu\,. \tag{1.32}$$

It is easily seen that the conservation laws (1.9) now take the form:

$$\langle Q(G,G) \rangle = 0\,, \qquad \langle v\, Q(G,G) \rangle = 0\,, \qquad \langle \tfrac{1}{2}|v|^2 Q(G,G) \rangle = 0\,, \tag{1.33}$$

for every $G = G(v)$ for which the integrals make sense. If $G$ solves the Boltzmann equation (1.26a) the local conservation laws of mass, momentum, and energy (1.10) now are:

$$\begin{aligned}
\partial_t \langle G \rangle + \nabla_x \cdot \langle v\, G \rangle &= 0\,, \\
\partial_t \langle v\, G \rangle + \nabla_x \cdot \langle v \otimes v\, G \rangle &= 0\,, \\
\partial_t \langle \tfrac{1}{2}|v|^2 G \rangle + \nabla_x \cdot \langle v \tfrac{1}{2}|v|^2 G \rangle &= 0\,.
\end{aligned} \tag{1.34}$$

Integrating these over space and velocity yields the global conservation laws of mass, momentum, and energy

$$\int \langle G(t)\rangle\, dx = 1\,, \qquad \int \langle v\, G(t)\rangle\, dx = 0\,, \qquad \int \langle \tfrac{1}{2}|v|^2 G(t)\rangle\, dx = \tfrac{\mathrm{D}}{2}\,, \tag{1.35}$$

in the nondimensional form defined by (1.28) and (1.29).

The most important feature of the Boltzmann equation to study fluctuations about an absolute Maxwelian is the notion of relative entropy. The relative entropy of the distribution $F$ with respect to the absolute Maxwellian state $M$ is defined as

$$H(G) = \int \langle G \log G - G + 1\rangle\, dx\,, \tag{1.36}$$

with $G = F/M$. This choice of $H$ is based on the fact that its integrand is a nonnegative strictly convex function of $G$ with a minimum value of 0 at $G = 1$. Thus for any $G$,

$$H(G) \geq 0\,, \qquad \text{and} \qquad H(G) = 0 \quad \text{iff} \quad G = 1\,. \tag{1.37}$$

The relative entropy provides a natural measure of the proximity of $F$ to the reference equilibrium $M$.

Now, the local entropy dissipation law (1.15) takes the form

$$\begin{aligned} &\partial_t \langle G \log G - G + 1\rangle + \nabla_x \cdot \langle v\,(G \log G - G + 1)\rangle \\ &\qquad = -\tfrac{1}{4}\frac{1}{\epsilon} \left\langle\!\!\left\langle \log\left(\frac{G_1' G'}{G_1 G}\right) (G_1' G' - G_1 G) \right\rangle\!\!\right\rangle \leq 0\,. \end{aligned} \tag{1.38}$$

Integrating this over space and time gives the global entropy equality

$$H(G(t)) + \frac{1}{\epsilon}\int_0^t R(G(s))\, ds = H(G^{in})\,, \tag{1.39}$$

where $R(G)$ is the entropy dissipation rate functional

$$R(G) = \int \tfrac{1}{4} \left\langle\!\!\left\langle \log\left(\frac{G_1' G'}{G_1 G}\right) (G_1' G' - G_1 G) \right\rangle\!\!\right\rangle dx\,. \tag{1.40}$$

### 1.5. Global Existence Theory.

*The Leray Theory.*

Besides the original 1934 article [22], the Leray theory has been presented in an considerable number of books and review articles. We shall only mention the monographs by Constantin-Foias [12] and that of J.-L. Lions [23] and recall the basic facts of this theory.

In [22], J. Leray constructs a so-called "turbulent solution" of the Navier-Stokes equation (1.1). Let us first introduce the functional spaces $\mathcal{H}$ and $\mathcal{V}$, defined by

$$
\begin{aligned}
\mathcal{H} &= \left\{ w \in L^2(dx; \mathbf{R}^{\mathrm{D}}) \,:\, \nabla_x \cdot w = 0 \,, \int w \, dx = 0 \right\}, \\
\mathcal{V} &= \left\{ w \in \mathcal{H} \,:\, \int |\nabla_x w|^2 \, dx < \infty \right\}.
\end{aligned}
\tag{1.41}
$$

A "turbulent solution" of the Navier-Stokes equation is a vector field $u = u(t,x) \in C([0,\infty); w\text{-}\mathcal{H}) \cap L^2_{loc}(dt; \mathcal{V})$ * such that

— the Navier-Stokes equation holds in the following weak sense: for every $w \in \mathcal{H} \cap C^1(\mathbf{T}^{\mathrm{D}})$

$$
\begin{aligned}
\int w \cdot u(t_2) \, dx - \int w \cdot u(t_1) \, dx - \int_{t_1}^{t_2} \int \nabla_x w : (u \otimes u) \, dx \, dt& \\
= -\nu \int_{t_1}^{t_2} \int \nabla_x w : \nabla_x u \, dx \, dt \,,&
\end{aligned}
\tag{1.42}
$$

for every $0 \leq t_1 < t_2$;

— the energy dissipation inequality (henceforth referred to as the Leray energy inequality):

$$
\int \tfrac{1}{2} |u(t)|^2 dx + \int_0^t \int \nu |\nabla_x u|^2 dx \, dt' \leq \int \tfrac{1}{2} |u^{in}|^2 dx \,, \tag{1.43}
$$

The Leray theory asserts that given any $u^{in} \in \mathcal{H}$, there exists a "turbulent solution $u$ which is initially such that the initial condition in (1.1) holds:

$$
u(0, \cdot) = u^{in} \,.
$$

The Leray theory does assert neither regularity nor uniqueness of the solution. Henceforth, we shall abandon the original (but somewhat misleading) term of "turbulent" and refer to those solutions as the "Leray solutions" of the Navier-Stokes equation. Although Leray's original article was written for the whole space $\mathbf{R}^{\mathrm{D}}$, modifying its argument to the $\mathbf{T}^{\mathrm{D}}$ case involves no supplementary difficulty.

*The DiPerna-Lions Theory.*

The theory of R. DiPerna and P.-L. Lions [14] (modified slightly for the periodic box) gives the existence of global weak solutions to the entire class of normalized Boltzmann initial-value problems

$$
(\partial_t + v \cdot \nabla_x) \Gamma(G) = \frac{1}{\epsilon} \frac{1}{N(G)} Q(G, G) \,, \tag{1.44a}
$$

$$
G(0, x, v) = G^{in}(x, v) \,, \tag{1.44b}
$$

* The notation $w$-$E$ designates the space $E$ equipped with its weak topology.

where the normalization $N(G) > 0$ satisfies $(1+Z)/N(Z) \leq C$ over $Z > 0$ for some constant $C < \infty$ and where $\Gamma'(z) = 1/N(z)$. They showed that if $G$ is a weak solution of (1.44) for one such $N(G)$ then it is a weak solution for all such $N(G)$. Such solutions they called renormalized solutions of the Boltzmann initial-value problem (1.26).

More specifically, given any initial data in the entropy class $\{G^{in} \geq 0 : H(G^{in}) < +\infty\}$ that satisfies the initial normalizations (1.29), there exists at least one weak solution of (1.44) in $C([0,\infty); w\text{-}L^1(Mdv\,dx))$ with

$$\begin{aligned} &\frac{1}{N(G)} Q^-(G,G) \in L^\infty(dt; L^1(Mdv\,dx))\,, \\ &\frac{1}{N(G)} Q^+(G,G) \in L^1_{loc}(dt; L^1(Mdv\,dx))\,, \end{aligned} \tag{1.45}$$

where $Q^-$ and $Q^+$ are the source and sink components of the collision operator (1.27)

$$\begin{aligned} Q^+(G,G) &= \iint G_1' G'\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1\,, \\ Q^-(G,G) &= \iint G_1 G\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1\,. \end{aligned} \tag{1.46}$$

Here, to say $G$ is a weak solution of (1.44) means that it is initially equal to $G^{in}$ and that it satisfies the normalized Boltzmann equation (1.44a) in the sense that for every $\chi \in L^\infty(Mdv; C^1(\mathbf{T}^{\mathrm{D}}))$ and every $0 \leq t_1 < t_2 < \infty$ it satisfies

$$\begin{aligned} \int \langle \Gamma(G(t_2))\,\chi\rangle\, dx - \int \langle \Gamma(G(t_1))\,\chi\rangle\, dx - \int_{t_1}^{t_2}\int \langle \Gamma(G)\, v\cdot\nabla_x \chi\rangle\, dx\, dt& \\ = \frac{1}{\epsilon}\int_{t_1}^{t_2}\int \langle \frac{1}{N(G)} Q(G,G)\,\chi\rangle\, dx\, dt\,,& \end{aligned} \tag{1.47}$$

It also satisfies the global entropy inequality

$$H(G(t)) + \frac{1}{\epsilon}\int_0^t R(G(s))\, ds \leq H(G^{in})\,, \tag{1.48}$$

the local conservation law of mass

$$\partial_t \langle G\rangle + \nabla_x\cdot\langle v\, G\rangle = 0\,, \tag{1.49}$$

the global conservation law of momentum

$$\int \langle v\, G(t)\rangle\, dx = 0\,, \tag{1.50}$$

and the global energy inequality

$$\int \langle \tfrac{1}{2}|v|^2 G(t)\rangle\, dx \leq \tfrac{\mathrm{D}}{2}\,. \tag{1.51}$$

for every $t > 0$.

The finiteness of the entropy is enough to insure the integrability of the conserved densities. However, similarly to the Leray theory the DiPerna-Lions theory does not assert the local conservation of momentum (see 1.34), the global conservation of energy (see 1.35), or the global entropy equality (1.39); nor does it assert the regularity or the uniqueness of the solution.

## 2. Navier-Stokes Scalings and Main Results.

### 2.1. Navier-Stokes Scalings.

In order to obtain a hydrodynamic limit of the Boltzmann equation, one has to study solutions of the Boltzmann equation (1.26a) as the Knudsen number $\epsilon$ defined in Section 1.3 tends to zero.

Then, as explained in the introduction above, in order to obtain the Navier-Stokes equation in the limit, the Mach number should be taken $O(\epsilon)$. This is achieved by two supplementary ansatz:

— One takes first a longer time scale $\tau'$ than the one defined in (1.26a). This longer time scale is defined in such a way that the velocity scale $\lambda_*/\tau'$ is the scale of the bulk velocity corresponding to the rotational modes, whereas the velocity scale of the microscopic velocities $v$ is the speed of sound $\theta_*$. Setting the Mach number of order $\epsilon$ means that

$$\frac{(\lambda_*/\tau')}{\theta_*} = \epsilon$$

or, in other words, that

$$\tau' = \frac{\lambda_*^2}{\theta_* \tau_*}\,.$$

The Boltzmann equation is therefore rewritten with this new time scale:

$$\epsilon\,\partial_t G_\epsilon + v\cdot\nabla_x G_\epsilon = \frac{1}{\epsilon} Q(G_\epsilon, G_\epsilon)\,. \tag{2.1}$$

— Here is an example of a distribution of molecules such that the ratio of its bulk velocity to the corresponding speed of sound is small of order $\epsilon$:

$$F_\epsilon(x,v) = M(\rho_*, \epsilon u(x), \theta_*)\,.$$

Expanding this as $\epsilon \to 0$ show that

$$F_\epsilon(t,x,v) = M(\rho_*, 0, \theta_*)\left(1 + \epsilon\frac{u(t,x)\cdot v}{\theta_*} + O(\epsilon^2)\right)\,.$$

More generally, the distributions of interest for the Navier-Stokes limit are of the form

$$F_\epsilon(t,x,v) = M(\rho_*, 0, \theta_*)(1 + \epsilon g_\epsilon(t,x,v))\,. \tag{2.2}$$

One therefore sets $F_\epsilon(0,x,v) = F_\epsilon^{in}$ of the form (2.2) and shows that the form (2.2) is maintained for all time with $g_\epsilon(t,x,v) = O(1)$ uniformly in $t \geq 0$.

More generally we consider a sequence of solutions $G_\epsilon$ to the scaled Boltzmann equation (2.1) in the form

$$G_\epsilon = 1 + \epsilon^m g_\epsilon\,. \tag{2.3}$$

As $\epsilon$ tends to zero, the leading behavior of the fluctuations $g_\epsilon$ is formally consistent with the incompressible Navier-Stokes equations (1.1) when $m = 1$, and with the Stokes equations (the linearization of (1.1)) when $m > 1$. We make this more precise below.

Setting (2.3) into (2.1) and Taylor expanding the collision operator gives

$$\epsilon\, \partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon + \frac{1}{\epsilon} L g_\epsilon = \epsilon^{m-1} Q(g_\epsilon, g_\epsilon)\,, \tag{2.4}$$

where $L$, the linearized collision operator, is given by

$$Lg \equiv -2\, Q(1,g) = \iint (g + g_1 - g' - g_1')\, b\, d\omega\, M_1 dv_1\,. \tag{2.5}$$

Repeated application of the collisional symmetries (1.6) yields the identity

$$\begin{aligned} \langle \xi\, Lg\rangle &= \langle\!\langle \xi\,(g + g_1 - g' - g_1')\rangle\!\rangle \\ &= \tfrac{1}{4}\langle\!\langle (\xi + \xi_1 - \xi' - \xi_1')(g + g_1 - g' - g_1')\rangle\!\rangle\,, \end{aligned} \tag{2.6}$$

for every $\xi = \xi(v)$ and $g = g(v)$ for which the integral makes sense. This shows that $L$ is formally self-adjoint and has a nonnegative Hermitian form. These properties ensure that $L$ has a self-adjoint extension to the Hilbert space $L^2(M dv)$ with the inner product $\langle f\, g\rangle$. Furthermore, using the equilibria characterization (1.7), it can be shown that for any $g = g(v)$ in the form domain of $L$, the following statements are equivalent:

$$\begin{aligned} &(i) \quad Lg = 0\,; \\ &(ii) \quad g = \alpha + \beta\cdot v + \gamma\tfrac{1}{2}|v|^2 \quad \text{for some } (\alpha,\beta,\gamma) \in \mathbf{R}\times\mathbf{R}^{\mathrm{D}}\times\mathbf{R}\,. \end{aligned} \tag{2.7}$$

This characterizes $\mathrm{N}(L)$, the nullspace of $L$, as the set obtained by linearizing $(iii)$ of (1.12) about $(\alpha, \beta, \gamma) = (0, 0, 0)$, the so-called infinitesimal Maxwellians.

In studying the formal incompressible Navier-Stokes limit of the Boltzmann equation, one finds a special role is played by the functions $\phi(v) \in \mathbf{R}^{\mathrm{D}\times\mathrm{D}}$ and $\psi(v) \in \mathbf{R}^{\mathrm{D}}$ that are the unique solutions to the equations

$$L\phi(v) = B(v) = v\otimes v - \tfrac{1}{\mathrm{D}}|v|^2 I\,, \qquad L\psi(v) = A(v) = \tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}+2}{2}\,, \tag{2.8}$$

which are orthogonal to $\mathrm{N}(L)$; henceforth $\phi$ and $\psi$ will always refer to these functions. The main formal result of [3] is the following. We refer to the appendix where we show that the tensor $\phi(v)$ and the vector $\psi(v)$ are proportional to $v\otimes v - \frac{1}{\mathrm{D}}|v|^2 I$ and $\frac{1}{2}|v|^2 - \frac{\mathrm{D}+2}{2}$ respectively.

**Theorem 2.1.** Let $G_\epsilon(t, x, v)$ be a sequence of nonnegative solutions to the scaled Boltzmann equation (2.1) such that, when it is written according to formula (2.2), the sequence $g_\epsilon$ converges in the sense of distributions and almost everywhere to a function $g$ as $\epsilon$ tends to zero. Furthermore, assume that the moments

$$\langle g_\epsilon\rangle\,, \qquad \langle v\, g_\epsilon\rangle\,, \qquad \langle v\otimes v\, g_\epsilon\rangle\,, \qquad \langle v|v|^2 g_\epsilon\rangle\,,$$

$$\langle \phi \otimes \phi\, g_\epsilon \rangle\,, \qquad \langle \phi\, Q(g_\epsilon, g_\epsilon) \rangle\,, \qquad \langle \psi \otimes \psi\, g_\epsilon \rangle\,, \qquad \langle \psi\, Q(g_\epsilon, g_\epsilon) \rangle$$

converge in the sense of distributions to the corresponding moments

$$\langle g \rangle\,, \qquad \langle v\, g \rangle\,, \qquad \langle v \otimes v\, g \rangle\,, \qquad \langle v |v|^2 g \rangle\,,$$

$$\langle \phi \otimes \phi\, g \rangle\,, \qquad \langle \phi\, Q(g, g) \rangle\,, \qquad \langle \psi \otimes \psi\, g \rangle\,, \qquad \langle \psi\, Q(g, g) \rangle\,,$$

and that all formally small terms in $\epsilon$ vanish. Then the limiting form of $g$ is that of an infinitesimal Maxwellian,

$$g = \rho + u \cdot v + \theta \left( \tfrac{1}{2} |v|^2 - \tfrac{\mathrm{D}}{2} \right), \tag{2.9}$$

where the velocity $u$ satisfies the incompressibility relation, while the density and temperature fluctuations, $\rho$ and $\theta$, satisfy the Boussinesq relation:

$$\nabla_x \cdot u = 0\,, \qquad \nabla_x (\rho + \theta) = 0\,. \tag{2.10}$$

Moreover, the functions $\rho$, $u$ and $\theta$ are weak solutions of the equations

$$\partial_t u + \nabla_x p = \nu \Delta_x u\,, \quad \partial_t \theta = \kappa \Delta_x \theta\,, \quad \text{if } m > 1\,. \tag{2.11}$$

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = \nu \Delta_x u\,, \quad \partial_t \theta + u \cdot \nabla_x \theta = \kappa \Delta_x \theta\,, \quad \text{if } m = 1\,; \tag{2.12}$$

In these equations the coefficients $\nu$ and $\kappa$ are given by

$$\nu = \tfrac{1}{(\mathrm{D}-1)(\mathrm{D}+2)} \langle \phi : L\phi \rangle\,, \qquad \kappa = \tfrac{2}{\mathrm{D}(\mathrm{D}+2)} \langle \psi \cdot L\psi \rangle\,. \tag{2.13}$$

In the sequel, we shall refer to (2.11) as the Stokes system and to (2.12) as the Navier-Stokes system. The momentum equations in these systems shall be referred to as the Stokes equation and the Navier-Stokes equation respectively.

Theorem 2.1 can be viewed as a counterpart of the expansions of Hilbert and Chapman-Enskog where the Mach number is small of the same order as the Knudsen number. In a recent article, DeMasi, Esposito and Lebowitz [13] have constructed a family of solutions of the Boltzmann equation (2.1) of the form (2.3) with

$$g_\epsilon(t, x, v) = u(t, x) \cdot v + O(\epsilon)$$

where $u = u(t, x)$ is a smooth solution of the Navier-Stokes equation. Unfortunately, it is still unknown whether the Navier-Stokes equation has smooth solutions (other than those close to some constant state). The strategy in [13] goes back to Caflisch [9] who proved the hydrodynamic limit of the Boltzmann equation leading to the compressible Euler equation in the regime of (local) smooth solutions.

The strategy followed in the work of Bardos-Golse-Levermore [4] is different and based only on the estimates bearing on physical quantities (like conservation of mass, momentum or energy, and the entropy inequality).

**2.2. The Program.**

Let $G_\epsilon \geq 0$ be a sequence of DiPerna-Lions renormalized solutions to the scaled Boltzmann initial-value problem

$$\epsilon\,\partial_t G_\epsilon + v\cdot\nabla_x G_\epsilon = \frac{1}{\epsilon} Q(G_\epsilon, G_\epsilon)\,, \tag{2.14a}$$

$$G_\epsilon(0,x,v) = G_\epsilon^{in}(x,v) \geq 0\,. \tag{2.14b}$$

For any given DiPerna-Lions normalization $N(Z)$, the associated normalized Boltzmann equation is

$$(\epsilon\,\partial_t + v\cdot\nabla_x)\Gamma(G_\epsilon) = \frac{1}{\epsilon}\frac{1}{N(G_\epsilon)} Q(G_\epsilon, G_\epsilon)\,, \tag{2.15}$$

with $\Gamma(Z)$ is related to $N(Z)$ by $\Gamma'(Z) = 1/N(Z)$. The associated DiPerna-Lions entropy inequality is

$$H(G_\epsilon(t)) + \frac{1}{\epsilon^2}\int_0^t R(G_\epsilon(s))\,ds \leq H(G_\epsilon^{in})\,. \tag{2.16}$$

Assume that the initial data $G_\epsilon^{in}$ satisfies the normalizations (1.29) and the entropy bound

$$H(G_\epsilon^{in}) \leq C^{in}\epsilon^{2m}\,, \tag{2.17}$$

for some fixed $C^{in} > 0$ and $m \geq 1$. Moreover, assume that the initial data has the form $G_\epsilon^{in} = 1 + \epsilon^m g_\epsilon^{in}$ where

$$g_\epsilon^{in} \to u^{in}\cdot v \tag{2.18}$$

in $L^1(M dv\, dx)$ as $\epsilon$ tends to zero, where $u^{in} \in \mathcal{H}$.

Consider the sequence $g_\epsilon$ as defined by the relation $G_\epsilon = 1 + \epsilon^m g_\epsilon$ as $\epsilon$ tends to zero. The DiPerna-Lions entropy inequality (2.16) and the entropy bound (2.17) are consistent with this order of fluctuation about the equilibrium $G = 1$. Given the formal result contained in Theorem 1.1, it is natural to ask whether, and in what sense, one has the limits

$$g_\epsilon \to u\cdot v\,, \tag{2.19a}$$

$$\langle v\, g_\epsilon\rangle \to u\,, \tag{2.19b}$$

where $u \in C([0,\infty); w\text{-}\mathcal{H}) \cap L^2_{loc}(dt; \mathcal{V})$ is a solution of the Stokes equation (2.11) when $m > 1$, or else a Leray solution of the Navier-Stokes system (2.12) when $m = 1$.

While this program is not yet complete, we present significant partial results in this paper. It is clear that completion of the program may require a better knowlege of properties of the DiPerna-Lions solutions. For example, in order to obtain the dynamical equation for $u$, we shall assume that the local momentum conservation law is satisfied.

### 2.3. Main results.

*The Normalized Boltzmann Equation.*

We choose to work with a DiPerna-Lions normalization of the Boltzmann equation in the form

$$N_\epsilon = N(G_\epsilon) = \tfrac{2}{3} + \tfrac{1}{3}G_\epsilon = 1 + \tfrac{1}{3}\epsilon^m g_\epsilon \,, \tag{2.20}$$

One reason for this choice is such that formally $N_\epsilon \to 1$ as $\epsilon$ tends to zero; thus, the normalizing factor will convieniently disappear from all algebraic expressions considered in this limit. Of course, our main results are independent of this particular choice of normalization.

Given this choice, we then choose to write the normalized Boltzmann equation (2.15) as

$$\epsilon\,\partial_t\gamma_\epsilon + v\cdot\nabla_x\gamma_\epsilon = \frac{1}{\epsilon^{m+1}}\frac{Q(G_\epsilon, G_\epsilon)}{N_\epsilon}\,, \tag{2.21}$$

where we have introduced $\gamma_\epsilon$ by

$$\gamma_\epsilon = \frac{1}{\epsilon^m}\Gamma(G_\epsilon) = \frac{3}{\epsilon^m}\log(1 + \tfrac{1}{3}\epsilon^m g_\epsilon)\,. \tag{2.22}$$

Since $\gamma_\epsilon$ formally behaves like $g_\epsilon$ for small $\epsilon$, it should be thought of as the normalized form of the fluctuations $g_\epsilon$.

*Implications of the Entropy Inequality.*

The first objective of the paper is to characterize the limiting form of the fluctuations $g_\epsilon$; the formal argument indicated that this should have the form of an infinitesimal Maxwellian (2.9).

It will also be of interest to study the rescaled collision integrand defined by

$$q_\epsilon = \frac{1}{\epsilon^{m+1}}(G'_{\epsilon 1}G'_\epsilon - G_{\epsilon 1}G_{\epsilon 1})\,. \tag{2.23}$$

One observes that the entropy and dissipation rate can be recast as

$$H(G_\epsilon) = \int \langle h(\epsilon^m g_\epsilon)\rangle\,dx\,, \qquad R(G_\epsilon) = \int \left\langle\!\!\left\langle \tfrac{1}{4}\frac{1}{\epsilon^2} r\Big(\frac{\epsilon^{m+1}q_\epsilon}{G_{\epsilon 1}G_\epsilon}\Big)G_{\epsilon 1}G_\epsilon \right\rangle\!\!\right\rangle dx\,, \tag{2.24}$$

where the integrands are written in terms of the convex functions

$$h(z) = (1+z)\log(1+z) - z\,, \qquad r(z) = z\log(1+z)\,.$$

Since $h(z) = O(z^2)$ and $r(z) = O(z^2)$ as $z \to 0$, one easily sees that $H(G_\epsilon)$ and $R(G_\epsilon)$ asymptotically behave almost like $L^2$ norms of $g_\epsilon$ and $q_\epsilon$ respectively as $\epsilon$ tends to zero. Using this observation, the entropy bound (2.16)-(2.17) results in the following statement.

**The Infinitesimal Maxwellian Form.** Let the family $g_\epsilon = g_\epsilon(t,x,v)$ satisfy the entropy inequality and bound (2.16)-(2.17) (where $g_\epsilon$ and $G_\epsilon$ are related through (2.3)). Then

1) the family $(1+|v|^2)\,g_\epsilon$ is relatively compact in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(M dv\, dx))$;
2) the family $(1+|v|^2)\,q_\epsilon/N_\epsilon$ is relatively compact in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(d\mu\, dx))$;
3) any convergent subsequence of $g_\epsilon$ as $\epsilon \to 0$ has a limit $g$ of the form of an infinitesimal Maxwellian,

$$g = \rho + u\cdot v + \theta\left(\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2}\right), \tag{2.25}$$

with $(\rho, u, \theta) \in L^\infty_{loc}(dt; L^2(dx; \mathbf{R}\times\mathbf{R}^{\mathrm{D}}\times\mathbf{R}))$.

It is remarkable that the statement above does not involve the fact that $g_\epsilon$ will eventually represent fluctuations of the number density in the Boltzmann equation; the only features of the Boltzmann equation used in these result are the entropy and entropy dissipation bounds resulting from the entropy inequality and bound (2.16)-(2.17).

*Implications of the Normalized Boltzmann Equation.*

Let $G_\epsilon$ be a family of renormalized solution of the Boltzmann initial-value problem (2.14) with initial data satisfying the entropy bound (2.16)-(2.17). Let $G_\epsilon = 1 + \epsilon^m g_\epsilon$. As a consequence of the above subsection, we may assume that $g_\epsilon$ converges to $g$ in $w\text{-}L^1_{loc}(dt; w\text{-}L^1((1+|v|^2)M dv\, dx))$, that $q_\epsilon/N_\epsilon$ converges to $q$ in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(d\mu\, dx))$, and that $g$ has the form of an infinitesimal Maxwellian (2.25).

The limit of the normalized Boltzmann equation (2.14a) reads:

$$v\cdot\nabla_x g = \iint q\, b(v_1 - v, \omega)\, d\omega\, M_1 dv_1\,. \tag{2.26}$$

Using the collisional symmetries (1.6) yields the following relations.

**The Incompressibility and Boussinesq Relations.**

$$\nabla_x\cdot u = 0\,, \qquad \nabla_x(\rho+\theta) = 0\,. \tag{2.27}$$

One of the most remarkable features of the incompressible Navier-Stokes scalings is that the DiPerna-Lions entropy inequality (2.16) transforms into a variant of the Leray energy inequality (1.43) as $\epsilon$ tends to zero.

Before going further in this direction, we introduce the notion of "entropic convergence" that is the right topology to study sequences of fluctuations of number densities in the Boltzmann equation. A sequence of fluctuations $g_\epsilon$ is said to converge entropicly of order $m$ to $g$ if and only if

$$g_\epsilon \to g \text{ in } w\text{-}L^1(M dv\, dx)\,, \quad \text{and} \quad \lim_{\epsilon\to 0}\int \left\langle \frac{1}{\epsilon^{2m}} h(\epsilon^m g_\epsilon)\right\rangle dx = \tfrac{1}{2}\int \langle g^2\rangle\, dx\,. \tag{2.28}$$

We will then consider the entropy inequality (2.16) multiplied by $\epsilon^{-2m}$ and take the limits in the resulting inequality as $\epsilon$ tends to zero, to obtain the following result.

**The Leray Energy Inequality.** Let $(\rho^{in}, u^{in}, \theta^{in}) \in L^2(dx; \mathbf{R} \times \mathbf{R}^{\mathrm{D}} \times \mathbf{R})$ and define the infinitesimal Maxwellian $g^{in}$ in $L^2(M dv\, dx)$ by the formula

$$g^{in} = \rho^{in} + v \cdot u^{in} + (\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2})\theta^{in} . \tag{2.29}$$

Suppose that $G_\epsilon^{in} = 1 + \epsilon^m g_\epsilon^{in} \geq 0$ such that $g_\epsilon^{in} \to g^{in}$ entropicly of order $m$ for some $m \geq 1$. Let $G_\epsilon \geq 0$ be a sequence of renormalized solutions of the scaled Boltzmann initial-value problem (2.14) and let $g_\epsilon$ and $q_\epsilon$ be the corresponding sequences of fluctuations and scaled collision integrands. Let $g$ and $q$ be limits of the sequences $g_\epsilon$ and $q_\epsilon/N_\epsilon$ in $w\text{-}L^1_{loc}(dt; w\text{-}L^1(M dv\, dx))$ and $w\text{-}L^1_{loc}(dt; w\text{-}L^1(d\mu\, dx))$ respectively, Then $g$ has the form of an infinitesimal Maxwellian (2.25), where $\rho \in L^\infty(dt; L^2(dx))$, $u \in L^2(dt; \mathcal{V})$, and $\nabla_x \theta \in L^2(dt; L^2(dx))$ satisfy the inequality

$$\begin{aligned} \tfrac{1}{2}\int |\rho(t)|^2 + |u(t)|^2 + \tfrac{\mathrm{D}}{2}|\theta(t)|^2 dx + \int_0^t \int \tfrac{1}{2}\nu \, |\nabla_x u + (\nabla_x u)^T|^2 + \tfrac{\mathrm{D}+2}{2}\kappa \, |\nabla_x \theta|^2 dx\, ds \\ \leq \tfrac{1}{2}\int |\rho^{in}|^2 + |u^{in}|^2 + \tfrac{\mathrm{D}}{2}|\theta^{in}|^2 dx . \end{aligned} \tag{2.30}$$

The proof is based essentially on the convexity of the integrands of both the entropy $H$ and the entropy dissipation rate $R$, and on the collisional symmetries (1.6).

*The Stokes Limit.*

So far, the local conservation laws associated to the Boltzmann equation have not been used. However, the only local conservation law known to be satisfied by all renormalized solutions of the Boltzmann equation is that of mass (1.49). In order to formulate the hydrodynamic limits (which are obviously based on the fundamental principle of dynamics), we are consistently led to restrict our attention to sequences of renormalized solutions $G_\epsilon$ of the scaled Boltzmann initial-value problem (2.14) such that the following assumption holds:

**(H0).** The solutions $G_\epsilon$ satisfy the local momentum conservation law:

$$\partial_t \langle v\, G_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle v \otimes v\, G_\epsilon \rangle = 0 . \tag{2.31}$$

Whether renormalized solutions of (2.14) generally satisfy (H0) is still an open problem.

The Stokes equation will be obtained as the limiting form of the above local momentum conservation law as $\epsilon$ tends to zero when the parameter $m$ in the defining the scale of the fluctuation is greater than 1 (see the entropy bound (2.17) on the initial data). But, in order to take the small $\epsilon$ limit in the local momentum conservation law, it is essential to control the high velocity tails of the quantities involved. High velocities being generated by the collision operator, it is therefore little surprising that controlling the high velocity tails can be achieved by some assumptions bearing on the Boltzmann kernel $b$. We shall therefore make the following assumption:

**(H1).** The Boltzmann kernel $b$ is that of a cut-off hard potential (see Cercignani [11] for this definition) such that the two following inequalities hold

$$(|\phi(v)| + |\phi(v_1)|)\, b(v_1 - v, \omega) \leq C\,(1 + |v|^2 + |v_1|^2)\,, \tag{2.32a}$$

$$(1 + |v|^2) \leq C\,(1 + |\phi(v)|)^2\,, \tag{2.32b}$$

where $\phi = \phi(v)$ is the matrix valued function defined by (2.8).

Assumption (H1) is certainly satisfied by Maxwell potentials. In that case the key observation is that the entries of the matrix $\phi$ are eigenfunctions of the linearized collision operator $L$ (see [11]); both inequalities in (H1) then follow from (2.8).

Our main result concerning the Stokes limit is the following.

**Strong Stokes Limit Theorem.** Let $u^{in} \in \mathcal{H}$. Define the infinitesimal Maxwellian $g^{in}$ by

$$g^{in} = u^{in}\cdot v\,. \tag{2.33}$$

Let $G_\epsilon^{in} = 1 + \epsilon^m g_\epsilon^{in} \geq 0$ be any sequence such that $g_\epsilon^{in} \to g^{in}$ entropicly of order $m$ for some $m > 1$. Let $G_\epsilon = 1 + \epsilon^m g_\epsilon \geq 0$ be any corresponding sequence of renormalized solutions of the scaled Boltzmann initial-value problem (2.14). Then

$$g_\epsilon(t) \to u(t)\cdot v \qquad \text{entropicly of order } m \text{ for almost every } t > 0\,, \tag{2.34}$$

where $u(t)$ is the unique solution of the Stokes initial-value problem

$$\partial_t u + \nabla_x p = \nu \Delta_x u\,, \qquad \nabla_x\cdot u = 0\,, \tag{2.35a}$$

$$u(0, x) = u^{in}(x)\,, \tag{2.35b}$$

with the viscosity $\nu$ given by formula (2.13). Moreover, the normalized scaled collision integrands converge strongly to $q$:

$$\frac{q_\epsilon}{N_\epsilon} \to \left(\nabla_x u + (\nabla_x u)^T\right) : \Phi \qquad \text{in } L^1_{loc}(dt; L^1((1 + |v|^2)d\mu\, dx))\,, \tag{2.36}$$

where $\Phi = \frac{1}{4}(\phi_1 + \phi - \phi_1' - \phi')$ and $\phi$ is given by (2.8).

A key step in its proof is the following compactness result: that any consistently scaled sequence of DiPerna-Lions solutions has a subsequence whose velocity moments converges weakly to a solution of the Stokes equation.

*The Time-Discretized Navier-Stokes Limit.*

The scaling leading to the nonlinear Navier-Stokes equation corresponds to the case $m = 1$ in the entropy bound (2.17) defining the amplitude of the initial data.

For various reasons discussed below, we have not been able to prove the exact analogue of the Stokes Limit Theorem in the case where $m = 1$. The main simplification we have to concede is to study time-discretized analogues of the evolution equations above. The time-discretized scaled Boltzmann equation is

$$\epsilon \frac{G_\epsilon - G_\epsilon^{in}}{\Delta t} + v \cdot \nabla_x G_\epsilon = \frac{1}{\epsilon} Q(G_\epsilon, G_\epsilon) ; \tag{2.37}$$

it is an implicit time-discretization of the scaled Boltzmann equation (2.14a). Throughout this paper, we shall always set the time step $\Delta t = 1$. With the same definitions as in (2.20), (2.22), the normalized Boltzmann equation reads:

$$\epsilon \frac{g_\epsilon - g_\epsilon^{in}}{N_\epsilon} + v \cdot \nabla_x \gamma_\epsilon = \frac{1}{\epsilon^2} \frac{Q(G_\epsilon, G_\epsilon)}{N_\epsilon} . \tag{2.38}$$

The DiPerna-Lions theory can be transposed to this new problem without significant change. The form of the entropy inequality is however somewhat different:

$$H(G_\epsilon) + J(G_\epsilon^{in}, G_\epsilon) + \frac{1}{\epsilon^2} R(G_\epsilon) \leq H(G_\epsilon^{in}) , \tag{2.39}$$

where $J(G_\epsilon^{in}, G_\epsilon)$ is the relative entropy of $G_\epsilon^{in}$ with respect to $G_\epsilon$ which is given by

$$J(G_\epsilon^{in}, G_\epsilon) = \int \langle G_\epsilon^{in} \log\left(\frac{G_\epsilon^{in}}{G_\epsilon}\right) - G_\epsilon^{in} + G_\epsilon \rangle \, dx . \tag{2.40}$$

The corresponding time-discretized Navier-Stokes equation reads

$$u + \nabla_x \cdot (u \otimes u) + \nabla_x p = \nu \Delta_x u + u^{in} , \qquad \nabla_x \cdot u = 0 . \tag{2.41}$$

In any dimension, for every $u^{in}$ in $\mathcal{H}$, this equation has a solution in $\mathcal{V}$ that satisfies the Leray energy inequality:

$$\int |u|^2 \, dx + \int \nu |\nabla_x u|^2 \, dx \leq \int u^{in} \cdot u \, dx . \tag{2.42}$$

In dimension D = 2, 3, 4, any solution of the time-discretized Navier-Stokes equation in $\mathcal{V}$ satisfies the equality in (2.42) ([1])

For a sequence of initial data for (2.37) chosen to satisfy the entropy bound

$$H(G_\epsilon^{in}) \leq C^{in} \epsilon^2 ,$$

analogs of the results implied by the evolution entropy inequality and the evolution Boltzmann equation hold.

([1]) We are grateful to F. Murat who brought this particular point to our attention.

For the same reason as in the previous subsection, in order to derive the Navier-Stokes limit, it has been necessary to assume the local momentum conservation law for the renormalized solutions $G_\epsilon$ of (2.37) considered:

**(H0').** $G_\epsilon$ satisfies the time-discretized local momentum conservation law

$$\langle v\, G_\epsilon\rangle + \frac{1}{\epsilon}\nabla_x \cdot \langle v \otimes v\, G_\epsilon\rangle = \langle v\, G_\epsilon^{in}\rangle \,.$$

Most of the proof of the Stokes Limit Theorem can be reproduced in the case where $m = 1$. It becomes therefore essential to control quadratic nonlinearities at high velocities. To this end, we have been led to introduce the supplementary assumption

**(H2).** the family $(1 + |v|^2)g_\epsilon^2/N_\epsilon$ is relatively compact in $w\text{-}L^1(Mdv\, dx)$.

The term $g_\epsilon^2/N_\epsilon$ somehow measures the difference between the entropy bound (2.16) and an $L^2$ bound on $g_\epsilon$. We have been able to prove the following partial result in this direction

1) the family $g_\epsilon^2/N_\epsilon$ is bounded in $L^\infty(dt; L^1(Mdv\, dx))$;

2) as $\epsilon$ tends to zero, the family

$$|v|^2 \frac{g_\epsilon^2}{N_\epsilon} = O\left(\log\left(\frac{1}{\epsilon|\log(\epsilon)|}\right)\right)\,, \qquad \text{in } L^\infty(dt; L^1(Mdv\, dx)).$$

This result is enough to achieve the Stokes limit (in the case where $m > 1$); however, assumption (H2) is needed to achieve the Navier-Stokes limit in the case where $m = 1$.

**The Strong Navier-Stokes Limit Theorem.** Assume (H0'), (H1), (H2) and $\mathrm{D} \leq 4$. Let $u^{in} \in \mathcal{H}_v$ and define the initial Maxwellian by

$$g^{in} = u^{in} \cdot v\,.$$

Let $G_\epsilon^{in} = 1 + \epsilon g_\epsilon^{in}$ be any sequence of initial data such that $g_\epsilon^{in}$ converges to $g^{in}$ entropicly, and $G_\epsilon$ a family of renormalized solutions of the corresponding time discretized Boltzmann equation (2.38). Then the family $g_\epsilon$ is relatively compact in $w\text{-}L^1((1 + |v|^2)Mdv\, dx)$ and any of its sequential limit points $g$ is of the form

$$g = u \cdot v\,,$$

where $u \in \mathcal{V}_v$ is a weak solution of the time discretized Navier-Stokes equation

$$u + \nabla_x \cdot (u \otimes u) + \nabla_x p = \nu \Delta_x u + u^{in}\,, \qquad \nabla_x \cdot u = 0\,,$$

with the viscosity $\nu$ given by the formula (2.12). Moreover, the normalized scaled collision integrands converge strongly to $q$:

$$\frac{q_\epsilon}{N_\epsilon} \to q = (\nabla_x u + (\nabla_x u)^T) : \Phi\,, \qquad \text{in } w\text{-}L^1((1 + |v|^2 + |v_1|^2)d\mu dx).$$

A weaker statement than this one (but more general since it applies to any space dimension) can be found in [4]. However, we shall not state it here since the only dimensions of physical interest are $\mathrm{D} = 2$ and $\mathrm{D} = 3$.

## 3. Method of Proof.

The implications of the entropy inequality all rely on very specific properties enjoyed by both the functions $h$ and $r$ defined in Section 2.3: let $h^*$ and $r^*$ denote the Legendre transforms of $h$ and $r$ respectively. Then

$$h^*(y) = O(e^y), \quad r^*(y) = O(e^y), \tag{3.1}$$

as $y \to +\infty$, and $h^*$ and $r^*$ have superquadratic homogeneity. Moreover, $h$ and $r$ satisfy the reflection inequality

$$h(|z|) \leq h(z), \quad r(|z|) \leq r(z), \tag{3.2}$$

for all $z > -1$.

These properties are used to obtain the infinitesimal Maxwellian form. Indeed, using the Young inequality with (3.2) and the superquadratic homogeneity of $h^*$ shows that

$$\tfrac{1}{4}(1+|v|^2)|g_\epsilon| \leq \frac{1}{\alpha} h^*(\tfrac{1}{4}(1+|v|^2)) + \frac{\alpha}{\epsilon^{2m}} h(\epsilon g_\epsilon)$$

for any $\alpha > 0$. Observe that

$$\int h^*(\tfrac{1}{4}(1+|v|^2))\, M dv < +\infty$$

in view of (3.1); using then the entropy bound (2.16) (2.17) shows that $(1+|v|^2)g_\epsilon$ is bounded in $L^1_{loc}(dt; L^1(M dv\, dx))$. The weak compactness asserted in point 1) of "The Infinitesimal Maxwellian Form" is then obtained by letting $\alpha$ converge to zero. Point 2) is obtained mutatis mutandis, exchanging $h$ and $g_\epsilon$ with $r$ and $q_\epsilon$. Point 3) follows from points 1) and 2) after some technicalities.

There is more about the entropy inequality. As already said, $h(z) = O(z^2)$ near $z = 0$. Therefore, if $g = g(x, v)$ is a function such that

$$\frac{1}{\epsilon^{2m}} \int \langle h(\epsilon^m g) \rangle \, dx \leq C$$

for all positive $\epsilon$, then $g \in L^2(M dv\, dx)$ and

$$\frac{1}{\epsilon^{2m}} \int \langle h(\epsilon^m g) \rangle \, dx \to ||g||_{L^2(M dv\, dx)}$$

as $\epsilon \to 0$. In other words, for *a single* function $g$, the entropy bound (2.17) bearing on $G = 1 + \epsilon^m g$ is equivalent to a $L^2$ bound; however, the same is obviously not true for *a sequence of functions* indexed by $\epsilon$. Since the Leray existence theory for the Navier-Stokes equation is naturally posed in $L^2$, there is a definite interest in understanding the defect between (2.17) and a $L^2$ bound on $g_\epsilon$. This is done via the decomposition

$$g_\epsilon = a_\epsilon + \epsilon^m b_\epsilon \,,$$

where

$$a_\epsilon = \frac{g_\epsilon}{N_\epsilon}, \quad b_\epsilon = \tfrac{1}{3}\frac{g_\epsilon^2}{N_\epsilon}.$$

The entropy bound ensures that the family $g_\epsilon^2/N_\epsilon$ is bounded in $L^\infty(dt; L^1(Mdv\,dx))$, whence the family $a_\epsilon$ is bounded in $L^\infty(dt; L^2(Mdv\,dx))$. The defect between (2.17) and a $L^2$ bound is therefore concentrated on $b_\epsilon$. The best result we have been able to prove so far on $b_\epsilon$ is the following one:

$$\|\,|v|^2 \frac{g_\epsilon^2}{N_\epsilon}\,\|_{L^\infty(dt;L^1(Mdv\,dx))} = O\Big(\log\Big(\frac{1}{\epsilon|\log\epsilon|}\Big)\Big). \tag{3.3}$$

This bound is obtained from the entropy inequality and the Young inequality applied to the function $h \circ s^{-1}$, where

$$s(z) = \tfrac{1}{3}\frac{z^2}{1+\frac{1}{3}z}.$$

As a consequence, if $m > 1$, (3.3) shows that $g_\epsilon - a_\epsilon$ converges to zero in $L^1_{loc}(dt; L^1((1+|v|^2 Mdv\,dx))$ as $\epsilon \to 0$. But (3.3) yields not enough information if $m = 1$, and we have to make assumption (H2) to achieve the Navier-Stokes limit.

So far, we have mostly dwelled on the consequences of the control on the entropy $H(G_\epsilon)$ obtained from (2.16) (2.17). However, there is a perfect symmetry in the roles played by the entropy control and the dissipation control on $R(G_\epsilon)$ obtained from (2.16)-(2.17), essentially because of the similarities between the functions $h$ and $r$ defined above. The reader is referred to [4] for more detailed information on this point: in particular, the dissipation control yields a control on the collision operator somewhat more accurate than the one used by DiPerna-Lions in [14].

The next thing of importance is the relation between the Leray inequality and the entropy inequality (H theorem) of DiPerna-Lions. From now on, our attention is restricted to subsequences such that $g_\epsilon$ and $q_\epsilon/N_\epsilon$ converge in the sense described in Section 2 (2.3, The Infinitesimal Maxwellian Form, point 3)) to respectively $g$ and $q$. The first step in obtaining the result stated in Section 2 (2.3) is the inequality

$$\int \tfrac{1}{2}\langle g^2\rangle(t)\,dx + \int_0^t \int \langle\!\langle q^2 \rangle\!\rangle(s)\,dxds \le \int \tfrac{1}{2}\langle g^{in\,2}\rangle(t)\,dx \tag{3.4}$$

which holds by convexity and weak limits as soon as the initial data $g_\epsilon^{in}$ converge entropicly of order $m$ (see (2.28) for the definition of this notion). The subtlety here is that $q$ is not exactly known in terms of $g$, however the limit of the normalized Boltzmann equation provides almosts as valuable information about $q$. The key is then to observe that (2.26) results in

$$\langle\!\langle \phi\, q \rangle\!\rangle = \nu\left(\nabla_x u + (\nabla_x u)^T\right), \qquad \langle\!\langle \psi\, q \rangle\!\rangle = \tfrac{D+2}{2}\kappa\,\nabla_x\theta,$$

whence the Leray type inequality (2.30) follows by projecting the dissipation term in (3.4) above on the subspace spanned by the entries of $\phi$ and $\psi$. A complete proof is to be found in [4].

The heart of the matter is the way the hydrodynamic equations are derived from the Boltzmann equation. Formal derivations are due to Bardos-Golse-Levermore [2], [3]. Here however, more care should be exerted since the Boltzmann equation only holds in the renormalized sense. As is said above, the incompressibility and Boussinesq relations (2.27) are obtained from the limit of the normalized Boltzmann equation (2.26) simply by using the collisional symmetries (1.6) and the conservation relations (1.33). Then, one uses assumption (H0) and the resulting conservation of momentum:

$$\partial_t \langle v g_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle v \otimes v \, g_\epsilon \rangle = 0 .$$

Observe that the above conservation law can be recast in the form

$$\partial_t \langle v g_\epsilon \rangle + \frac{1}{\epsilon} \nabla_x \cdot \langle L(\phi) \, g_\epsilon \rangle = \nabla_x \tfrac{1}{\mathrm{D}} \frac{1}{\epsilon} \langle |v|^2 g_\epsilon \rangle . \tag{3.5}$$

The question of the asymptotics of the gradient

$$\nabla_x \tfrac{1}{\mathrm{D}} \frac{1}{\epsilon} \langle |v|^2 g_\epsilon \rangle$$

is of no interest so far; this term will disappear upon integration of (3.5) against divergence free test vector fields, as is classical in incompressible hydrodynamics. So far, one has

$$\langle v g_\epsilon \rangle \to \langle v g \rangle = u \, ,$$

in $w$-$L^\infty(dt; w$-$L^1(dx))$. It remains to identify the limit of

$$\frac{1}{\epsilon} \langle v \otimes v \, g_\epsilon \rangle .$$

To do this, observe first that

$$\frac{1}{\epsilon} \langle v \otimes v \, g_\epsilon \rangle = \langle \phi \, \frac{1}{\epsilon} L(g_\epsilon) \rangle . \tag{3.6}$$

Then, one has

$$\begin{aligned} \frac{1}{\epsilon} L(g_\epsilon) &= \frac{1}{\epsilon} \Big(1 - \frac{1}{N_\epsilon}\Big) L(g_\epsilon) \epsilon^{m-1} \frac{Q(g_\epsilon, g_\epsilon)}{N_\epsilon} - \frac{1}{\epsilon^{m+1}} \frac{Q(G_\epsilon, G_\epsilon)}{N_\epsilon} \\ &= \frac{1}{\epsilon} \Big(1 - \frac{1}{N_\epsilon}\Big) L(g_\epsilon) \epsilon^{m-1} \frac{Q(g_\epsilon, g_\epsilon)}{N_\epsilon} - \iint \frac{q_\epsilon}{N_\epsilon} b(v_1 - v, \omega) d\omega \, M_1 dv_1 \, . \end{aligned} \tag{3.7}$$

It is quite easy to show that

$$\langle \phi \, \frac{1}{\epsilon} \Big(1 - \frac{1}{N_\epsilon}\Big) L(g_\epsilon) \rangle \to 0$$

in $w$-$L^\infty(dt; w\text{-}L^1(dx))$, and, in the case where $m > 1$, that

$$\langle \phi\, \epsilon^{m-1} \frac{Q(g_\epsilon, g_\epsilon)}{N_\epsilon} \rangle \to 0$$

in $w$-$L^\infty(dt; w\text{-}L^1(dx))$, using the weak compactness properties of $g_\epsilon$ and $q_\epsilon$ (see points 1) and 2) in Section 2 (2.3) "The Infinitesimal Maxwellian Form") and the decomposition $g_\epsilon = a_\epsilon + b_\epsilon$ introduced above. Then, since it is assumed that $q_\epsilon / N_\epsilon \to q$ (in the sense of point 2) in "The Infinitesimal Maxwellian Form", Section 2 (2.3)), one has, in view of the limiting Boltzmann equation (2.26) the following convergence:

$$\lim_{\epsilon \to 0} \langle \phi \iint \frac{q_\epsilon}{N_\epsilon} b(v_1 - v, \omega) d\omega\, M_1 dv_1 \rangle = \langle \phi\, v \cdot \nabla_x g \rangle$$

$$= \langle \phi \otimes L(\phi) \rangle : \tfrac{1}{2}(\nabla_x u + (\nabla_x u)^T)\,. \tag{3.8}$$

Then, it follows from the various symmetries of the tensor $L(\phi)$ and from the incompressibility condition (2.27) that

$$\nabla_x \cdot (\langle \phi \otimes L(\phi) \rangle : \tfrac{1}{2}(\nabla_x u + (\nabla_x u)^T)) = \nu \Delta_x u$$

where $\nu$ is given by (2.13). The Stokes equation follows immediatly in the case where $m > 1$. However, more technical refinements are needed to arrive at the form of the Stokes Limit Theorem stated above, the most important one being the fact that the Leray inequality (1.43) is an *equality* for the Stokes equation. This helps greatly in identifying the limit (2.36) and the entropic convergence (2.34).

What remains to be treated is the case where $m = 1$. The main difference with the case $m > 1$ is the understanding of the limit of

$$\langle \phi \frac{Q(g_\epsilon, g_\epsilon)}{N_\epsilon} \rangle$$

as $\epsilon \to 0$. Observe that this is the only instance of a genuinely nonlinear term in the theory described so far. The key is to prove that $g_\epsilon \to g$ pointwise as $\epsilon \to 0$. This is done in the following way:

— One first shows that $\frac{1}{\epsilon} L(g_\epsilon)$ converges to zero in $L^1_{loc}(dt; L^1((1 + |\phi|) M dv\, dx))$; this step is one of the most technical ones in the reference [4]. However, its meaning is particularly simple: the distance between $g_\epsilon$ and the nullspace of $L$ tends to zero as $\epsilon \to 0$. In other words,

$$g_\epsilon - (\langle g_\epsilon \rangle + \langle v\, g_\epsilon \rangle \cdot v + \frac{1}{2\mathrm{D}} \langle (|v|^2 - \mathrm{D})\, g_\epsilon \rangle (|v|^2 - \mathrm{D})) \to 0\,. \tag{3.9}$$

— By point 2) in Section 2 (2.3) "The Infinitesimal Maxwellian Form", one has that

$$\epsilon \partial_t \gamma_\epsilon + v \cdot \nabla_x \gamma_\epsilon$$

is relatively compact in $L^1_{loc}(dt; L^1((1+|v|^2)M dv\, dx))$. Using this piece of information plus point 1) in Section 2 (2.3) "The Infinitesimal Maxwellian Form", it follows that

$$\langle \chi(v) g_\epsilon \rangle \to \langle \chi(v) g \rangle\,, \tag{3.10}$$

in $w$-$L^1_{loc}(dt; L^1(dx))$ for any function $\chi = \chi(v)$ such that

$$\chi(v) = O(|v|^2)$$

when $|v| \to \infty$. This *strong convergence in the variable $x$* is a quite straightforward consequence of the Velocity Averaging Method due to Golse-Lions-Perthame-Sentis [17]. However, this strong convergence cannot be extended to the $t$ dependence because of the long time scaling; this is explained in more details in [4]. This difficulty has been resolved in other examples of nonlinear hydrodynamic limits involving diffusions, like the Rosseland approximation in Radiative Transfer (see Bardos-Golse-Perthame-Sentis [6]) and the drift-diffusion approximation of the Boltzmann equation of semi-conductors (see Golse-Poupaud [19]). In this particular case, the difficulty appears more formidable and is related to the way the acoustic modes disappear in the asymptotic above. This is the very reason why we have been bound to the time-discretized Navier-Stokes limit so far. The desired pointwise convergence of $g_\epsilon$ to $g$ follows from (3.9) and (3.10).

The above analysis shows that

$$\langle \phi \, \frac{Q(g_\epsilon, g_\epsilon)}{N_\epsilon} \rangle \to \langle \phi \, Q(g,g) \rangle = \tfrac{1}{2} \langle L(\phi)\, g^2 \rangle$$

(see the formal paper [3] for the last equality). Inserting the infinitesimal Maxwellian form (2.9) in the above equality shows that

$$\lim_\epsilon \langle \phi \, \frac{Q(g_\epsilon, g_\epsilon)}{N_\epsilon} \rangle = u \otimes u\,.$$

For the sake of being complete, we wish to mention two other kinds of proof for the incompressible hydrodynamic limits of the Boltzmann equation. Those proofs work in the regime of regular solutions for the Navier-Stokes equation. Existence of such solutions is known only in two cases: for small (in some sense) initial data or in finite time for arbitrary big initial data. One of those proofs is due to DeMasi-Esposito-Lebowitz [13] and is based on a rigorous analysis of the Hilbert expansion "à la Caflisch" [9]; this proof works locally in time for arbitrary big regular initial data (of course, the time of regularity for the Navier-Stokes equation gives the limit of validity for this expansion to approximately solve the Boltzmann equation). Another proof is due to Bardos-Ukai and mimicks the earlier proof of compressible hydrodynamic limit due to Kawashima-Matsumura-Nishida [20]. The difference with the DeMasi-Esposito- Lebowitz result lies in the choice of the initial data. The Bardos-Ukai result allows any small enough regular initial data having the proper asymptotic as $\epsilon \to 0$ and is global in time. On the contrary, the DeMasi-Esposito-Lebowitz proof shows that, given any regular initial data for the Navier-Stokes equation, one can find in its vicinity (of the order of $\epsilon$) an initial data corresponding to a solution of the Boltzmann equation having the expected asymptotic behavior as $\epsilon \to 0$ in finite time, as explicited above.

## Appendix A.

**Theorem A.0.** There exists two functions $a$ and $b$ : $[0,+\infty[\to \mathbf{R}$ such that the tensor field $\phi$ and vector field $\psi$ defined in (2.8) satisfy:

$$\phi(v) = a(|v|)\left(v \otimes v - \frac{|v|^{\mathrm{D}}}{\mathrm{D}} I\right) ,$$

$$\psi(v) = b(|v|)v(|v|^2 - \mathrm{D} - 2) .$$

Although the result stated in Theorem A.0 is considered classical in Kinetic Theory, classical treatises fail to give a complete proof of it. The following argument is a streamlined version of an argument due to L. Desvillettes and the author [31].

**Lemma A.1.** For all $R \in O_{\mathrm{D}}(\mathbf{R})$, the vector field $\psi$ defined by (2.6) on $\mathbf{R}^{\mathrm{D}}$ satisfies:

$$\psi(RV) = R\psi(V) . \tag{A.1}$$

Moreover, the tensor field $\phi$ defined by (2.6) on $\mathbf{R}^{\mathrm{D}}$ satisfies the following properties:

i) for all $V$ in $\mathbf{R}^{\mathrm{D}}$ , $\phi(V)$ is a traceless symetric tensor;

ii) for all isometry $R \in O_{\mathrm{D}}(\mathbf{R})$,

$$\phi(RV) = R\phi(V)R^{-1} \tag{A.2}$$

(where the right hand side of (A.2) denotes a matrix product).

**Proof .** We note that, according to the invariance of $L$ under $O_{\mathrm{D}}(\mathbf{R})$

$$L(\psi \circ R) = (Lpsi) \circ R = A \circ R = RA = R(L(\psi)) = L(R\psi) = L(\psi \circ R) . \tag{A.3}$$

Moreover,

$$\int_{\mathbf{R}^{\mathrm{D}}} \psi(RV) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV = 0 \iff \int_{\mathbf{R}^{\mathrm{D}}} \psi(V) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV = 0 \iff$$

$$R\int_{\mathbf{R}^{\mathrm{D}}} \psi(V) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV = 0 = \int_{\mathbf{R}^{\mathrm{D}}} R\psi(V) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV = 0. \tag{A.4}$$

The Fredholm alternative implies that the system of equations

$$Lp = RA\,, \tag{A.5}$$

$$\int_{\mathbf{R}^{\mathrm{D}}} p(V) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV = 0\,, \quad 1 \le i \le \mathrm{D} \tag{A.6}$$

has a unique solution: hence (A.1).

Next we consider $\phi$. Notice that

$$L(Tr\,\phi) = Tr\,(L(\phi)) = TrB = 0\,, \tag{A.7}$$

$$L(\phi - \phi^T) = L\phi - (L\phi)^T = B - B^T = 0\,, \tag{A.8}$$

$$\int_{\mathbf{R}^{\mathrm{D}}} (Tr\,\phi)(V) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV = Tr\,\{\int_{\mathbf{R}^{\mathrm{D}}} \phi(V) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV\} = 0\,, \tag{A.9}$$

for all $1 \le i \le \mathrm{D}$. The same uniqueness argument as above applied to (A.7)-(A.9) shows that $Tr\,B' = 0$. Then

$$\int_{\mathbf{R}^{\mathrm{D}}} (\phi - \phi^T)(V) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV =$$

$$\int_{\mathbf{R}^{\mathrm{D}}} \phi(V) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV - \{\int_{\mathbf{R}^{\mathrm{D}}} \phi(V) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV\}^T = 0\,, \tag{A.10}$$

so that, again by the uniqueness argument applied to (A.8)-(A.10), one gets $\phi - \phi^T = 0$: this proves i).

Finally, we prove ii). As in the case of $A'$

$$L(\phi \circ R) = L(R\phi R^{-1}) \tag{A.11}$$

and

$$\int_{\mathbf{R}^{\mathrm{D}}} \phi(RV) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV = 0 \iff \int_{\mathbf{R}^{\mathrm{D}}} \phi(V) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV = 0 \iff$$

$$R\{\int_{\mathbf{R}^{\mathrm{D}}} \phi(V) \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV\}R^{-1} = 0 = \int_{\mathbf{R}^{\mathrm{D}}} R\phi(V)R^{-1} \begin{pmatrix} 1 \\ V_i \\ |V|^2 \end{pmatrix} e^{-\frac{|V|^2}{2}} dV = 0\,. \tag{A.12}$$

The uniqueness argument applied to the system (A.11)-(A.12) implies that ii) holds. //

The next lemma characterizes vector fields satisfying relation (A.1)

**Lemma A.2.** Let $D \geq 2$ and $s : \mathbf{R}^D \to \mathbf{R}^D$ be a vector field such that for all $R \in O_D(\mathbf{R})$

$$s \circ R = Rs\,. \tag{A.13}$$

Then, there exists $t : \mathbf{R}^+ \to \mathbf{R}$ such that

$$\forall x \in \mathbf{R}^D\,, \quad s(x) = t(|x|)x\,. \tag{A.14}$$

**Proof .** Let $x \in \mathbf{R}^D \setminus \{0\}$ and let $O_x$ be the stabilizer of $x$ under the action of $O_D(\mathbf{R})$, ie. the subgroup of $O_D(\mathbf{R})$ consisting precisely of the isometries of $\mathbf{R}^D$ leaving $x$ invariant. Equation (A.13) implies that

$$\forall R \in O_x\,, \qquad Rs(x) = s(x) \tag{A.15}$$

and hence

$$\forall R \in O_x\,, \qquad RP_x s(x) = P_x s(x) \tag{A.16}$$

where $P_x$ is the orthogonal projection on $(\mathbf{R}x)^\perp \subset \mathbf{R}^D$. Since $D \geq 2$, $O_x \simeq O_{D-1}(\mathbf{R})$ (with $\mathbf{R}^{D-1}$ identified with $(\mathbf{R}x)^\perp \subset \mathbf{R}^D$). But, for $D \geq 2$, $O_{D-1}(\mathbf{R})$ contains at least one isometry having no fixed point in $(\mathbf{R}x)^\perp$ other than 0 (take $y \mapsto -y$). Hence $s(x)$ is colinear to $x$: there exists a scalar function $\lambda$ of $x$ such that $s(x) = \lambda(x)x$. Then, (A.13) shows that $\lambda \circ R = \lambda$ for all $R \in O_D(\mathbf{R})$: therefore, $\lambda$ depends only on the euclidian norm of $x$. //

The next lemma characterizes tensor fields satisfying i) and ii) in Lemma A.1.

**Lemma A.3.** Let $D \geq 2$ and $m : \mathbf{R}^D \to M_D(\mathbf{R})$ be a tensor field such that for all $R \in O_D(\mathbf{R})$

$$m \circ R = RmR^{-1}\,. \tag{A.17}$$

Assume moreover that

$$\forall x \in \mathbf{R}^D\,, \qquad m(x) = m(x)^T\,, \quad Tr\, m(x) = 0\,. \tag{A.18}$$

Then, there exists $n : \mathbf{R}^+ \to \mathbf{R}$ such that

$$\forall x \in \mathbf{R}^D, \quad m(x) = n(|x|)\{x \otimes x - \frac{|x|^2}{D} Id\}. \tag{A.19}$$

**Proof .** Let $x \in \mathbf{R}^D \setminus \{0\}$. Equation (A.17) shows that

$$\forall R \in O_x\,, \quad Rm(x) = m(x)R\,. \tag{A.20}$$

But, since $m(x)$ is symmetric and real, it is reducible to diagonal form. (A.20) shows that each eigenspace of $m(x)$ must be stable by all isometries in $O_x$. Hence, $m(x)$ can only have $\mathbf{R}x$, $(\mathbf{R}x)^\perp$ or $\mathbf{R}^{\mathrm{D}}$ as eigenspaces (indeed, if $\mathrm{D} > 2$, $O_x$ acts transitively on $(\mathbf{R}x)^\perp$ and if $\mathrm{D} = 2$, the statement above is trivial). Therefore there exists two scalar functions of $x$ denoted by $\lambda$ and $\mu$ such that

$$m(x) = \lambda(x)I + \mu(x)xx^T . \tag{A.21}$$

(In other words, $m(x)$ must be a linear combination of the orthogonal projection on $\mathbf{R}x$ and the identity). But the traceless condition (A.18) shows that

$$\mathrm{D}\lambda(x) + |x|^2\mu(x) = 0 .$$

This tranforms (A.21) into

$$m(x) = \mu(x)(xx^T - \tfrac{1}{\mathrm{D}}|x|^2 I) . \tag{A.22}$$

But then, (A.17) shows clearly that $\mu \circ R = \mu$, or, in other words, that $\mu$ depends only on the euclidian norm of $x$. //

The three lemmas above imply Theorem A.0.

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