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On the Definition of 2-Category of 2-Knots

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Content

1. Introduction

The aim of this paper is to define a 2-category of 2-knots in 4-dimensional Euclidean space. The categorical approach to knots and tangles in $\mathbb{R}^3$ introduced in [Tu], [Ye] plays an important role in the construction of 3-dimensional topological quantum field theories (TQFT's) based on the theory of quantum groups (see [RT1], [RT2]). The category of tangles consists of objects which are finite subsets of $\mathbb{R}$ considered up to isotopy in $\mathbb{R}$ and morphisms which are isotopy classes of tangles in $\mathbb{R}^2 \times [0,1]$. Each tangle has several (≥ 0) bottom endpoints lying in $\mathbb{R} = \mathbb{R} \times 0 \times 0$ and several top endpoints lying in $\mathbb{R} = \mathbb{R} \times 0 \times 1$. Such a tangle is regarded as a morphism from the set of its bottom endpoints in the set of its top endpoints. For instance, links in $\mathbb{R}^3$ are just endomorphisms of the empty subset of $\mathbb{R}$. The composition of morphisms is defined by attaching one tangle on the top of another one and compressing the result into $\mathbb{R}^2 \times [0,1]$. This category of tangles admits a number of useful modifications. For instance, one may consider oriented tangles, framed tangles, colored tangles, etc. It is important that the category of tangles admits a purely algebraic description (in terms of generators and relations or in terms of universal properties). It is this fact which allows to use the category of tangles in 3-dimensional TQFT's.

To extend these ideas to surfaces in $\mathbb{R}^4$ it is natural to involve the notion of 2-category. A 2-category is a category provided with so-called 2-morphisms. More precisely, for any two (ordinary) morphisms $f : X \to Y, g : X \to Y$ we have a class of 2-morphisms "acting" from f into g. The 2-morphisms are subject to two composition operations $\circ$ and $\star$. The composition $\star$ transforms a pair of 2-morphisms $f \Rightarrow g$ and $g \Rightarrow h$ into a 2-morphism $f \Rightarrow h$. The composition $\circ$ transforms a pair of 2-morphisms $f \Rightarrow g$ and $f' \Rightarrow g'$ with $source(f') = source(g') = target(f) = target(g)$ into a 2-morphism $f'f \Rightarrow g'g$. For more details see Section 2.

The 2-categories seem to be adequate for an algebraic description of surfaces in $\mathbb{R}^4$. The idea is to use as 2-morphisms the isotopy classes of surfaces in $\mathbb{R}^2 \times [0,1] \times [0,1]$ interpolating between a tangle in $\mathbb{R}^2 \times [0,1] \times 0$ and a tangle in $\mathbb{R}^2 \times [0,1] \times 1$. The compositions $\circ$ and $\star$ of such 2-morphisms are obtained by attaching one such surface on the top of another one along the third or forth coordinate respectively.

2-categories of surfaces in $\mathbb{R}^4$ were first considered by J.E.Fisher [F]. Despite the simplicity of the underlying ideas, this approach meets an important difficulty. Namely, to define the composition $\star$ one has to glue a surface in $\mathbb{R}^2 \times [0,1] \times [0,1]$ to a surface in $\mathbb{R}^2 \times [0,1] \times [1,2]$ along isotopic tangles in $\mathbb{R}^2 \times [0,1] \times 1$. To perform this gluing one has to fix an isotopy in question. In general, different isotopies give rise to topologically different results which leads to absence of a correctly defined composition $\star$. The problem is due to the fact that the space of tangles isotopic to a given one may have a non-trivial fundamental group. Note that a similar problem does not come up in the lower dimension because for any $n \geq 0$ the space of n-point subsets of $\mathbb{R}$ is contractible.

To circumvent this problem we have to change the definition of 1-morphisms. We use as 1-morphisms the diagrams of tangles in $\mathbb{R} \times [0,1]$ considered up to a certain equivalence relation. We look for a relation such that the space of diagrams equivalent to a given one is simply-connected. There are different equivalence relations satisfying this condition. For instance the identity relation (geometric coincidence of diagrams) obviously satisfy this condition. However, the resulting 2-categories are too large and can not be described in algebraic terms. Thus, our aim is to find an equivalence relation satisfying the condition above but leading to a not excessively large 2-category, possibly admitting a purely algebraic description. Here we propose such an equivalence relation. For each diagram we consider the set A (resp. B) of points of local maximum (resp. minimum) of the projection $\mathbb{R} \times [0,1] \to [0,1]$ restricted to the diagram. Both these sets are ordered in accordance with the values of the projection. We consider the equivalence relation generated by ambient isotopies of diagrams in $\mathbb{R} \times [0,1]$ preserving the order both in A and B. In other words, we consider those isotopies which never exchange the levels of two points of local maximum or two points of local minimum. Our main result affirms that the space of diagrams equivalent to the given one is simply-connected. This leads to a 2-category of surfaces in $\mathbb{R}^4$ which seems to be suitable for an algebraic study. Similar to the classical setting this 2-category admits various modifications involving oriented, framed, and colored tangles and surfaces.

The results of this paper and related results were reported by the first author at the 56-th RCP in May 1993. This talk also included a discussion of the relationships between our work and the tetrahedron equations (see [Za], [M-S], [Kh], [F], [K-V], [K-S]).

We thank D. Bennequin for a helpful discussion concerning the proof of Theorem 3.4.

2. Two-categories

2.1. The notion of 2-category. The categories we are interested in are strictly associative and have strict units. That is why some authors, see for example [K-V], name them *strict 2-categories*. We omit the adjective "strict" and call them *2-categories*.

Let us recall the definition.

Definition. A 2-category $\mathcal{A}$ is a collection of :

(a) three sets

$$\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$$

whose elements are called respectively *objects*, *1-morphisms* and *2-morphisms*;

(b) four maps

$$s_0, t_0 : \mathcal{A}_1 \to \mathcal{A}_0,$$

$$s_1, t_1 : \mathcal{A}_2 \to \mathcal{A}_1,$$

s_0, s_1 and t_0, t_1 are called respectively *source* and *target maps*;

(c) two maps

$$I_0 : \mathcal{A}_0 \to \mathcal{A}_1, I_1 : \mathcal{A}_1 \to \mathcal{A}_2,$$

their images $I_0(A), A \in \mathcal{A}_0$ and $I_1(u), u \in \mathcal{A}_1$ are called *identity morphisms* and denoted respectively by 1_A and 1_u;

(d) a composition operation for 1-morphisms

$$(u, v) \mapsto u \circ v$$

defined when u, v verify the condition $t_0(v) = s_0(u)$;

(e) two composition operations for 2-morphisms

$$(F, G) \mapsto F \circ G,$$

$$(F, G) \mapsto F \star G,$$

the first is defined for any 2-morphisms F, G such that $t_0 s_1(G) = s_0 t_1(F)$, the second for 2-morphisms with $t_1(G) = s_1(F)$.

It is required that

(1) $t_0 s_1 = t_0 t_1, s_0 t_1 = s_0 s_1$;

(2) $\mathcal{A}_0, \mathcal{A}_1, s_0, t_0, \circ$ form a category (called the *underlying category of* $\mathcal{A}$) with 2-sided identities $1_A, A \in \mathcal{A}_0$, i.e. the composition $\circ$ of 1-morphisms is associative and the equalities $1_B \circ u = u = u \circ 1_A$ are verified for any 1-morphism u with $s_0(u) = A, t_0(u) = B$;

(3) $\mathcal{A}_1, \mathcal{A}_2, s_1, t_1, \star$ form a category with 2-sided identities $1_u, u \in \mathcal{A}_1$, i.e. the composition $\star$ of 2-morphisms is associative and the equalities $1_v \star F = F = F \star 1_u$ are verified for any 2-morphism F with $s_1(F) = u, t_0(F) = v$;

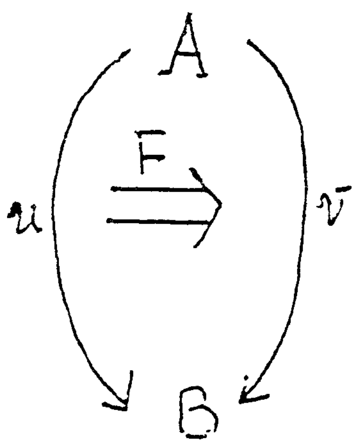
(4) $\mathcal{A}_0, \mathcal{A}_2, s_0 t_1, t_0 s_1, \circ$ form a category with 2-sided identities $1_u, u = 1_A, A \in \mathcal{A}_0$, i.e. the composition $\circ$ of 2-morphisms is associative and the equalities $1_v \circ F = F = F \circ 1_u$ are verified for $u = 1_A, v = 1_B$ and any 2-morphism F with $s_0 t_1(F) = A, t_0 s_1(F) = B$;

(5) for any 1-morphisms u, v such that $t_0(v) = s_0(u)$ there is the equality $1_{u \circ v} = 1_u \circ 1_v$;

(6) for any 2-morphisms F, G such that $t_0 s_1(G) = s_0 t_1(F)$ there are the following identities

$$F \circ G = (F \circ 1_{t_1(G)}) \star (1_{s_1(F)} \circ G) = (1_{t_1(F)} \circ G) \star (F \circ 1_{s_1(G)}).$$

As traditionaly, we use symbols $u : A \to B$ and $A \xrightarrow{u} B$ to notate a 1-morphism $u \in \mathcal{A}_1$ with $A = s_0(u)$ and $B = t_0(u)$. A 2-morphism F with $s_1(F) = u, t_1(F) = v$ is notated by the symbol $F : u \Rightarrow v$ and the pictures



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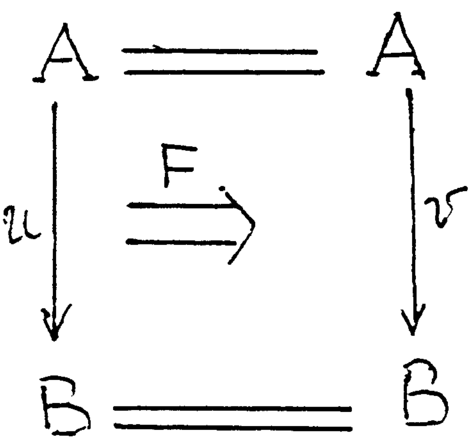
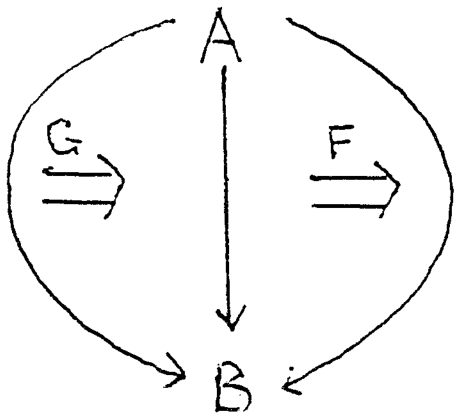


where $s_0(u) = A, t_0(v) = B$. The figures



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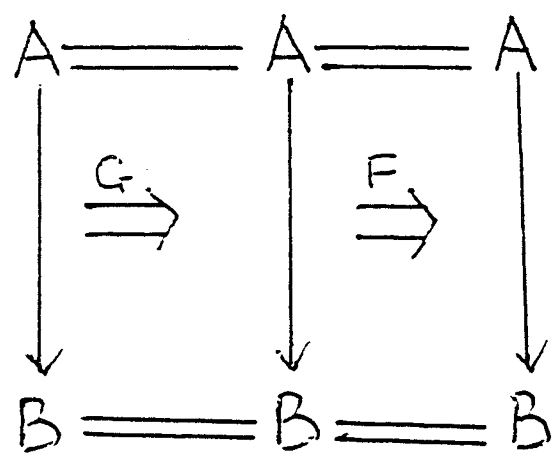
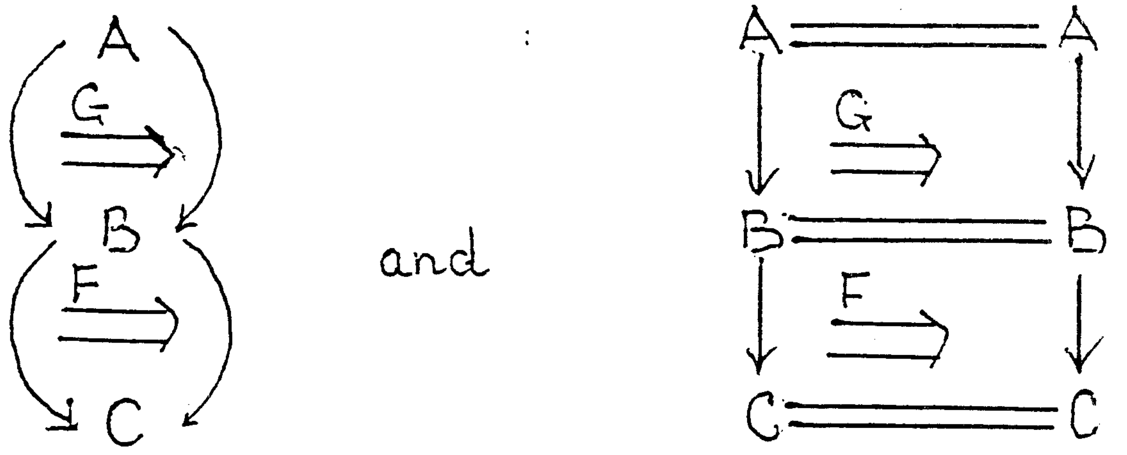
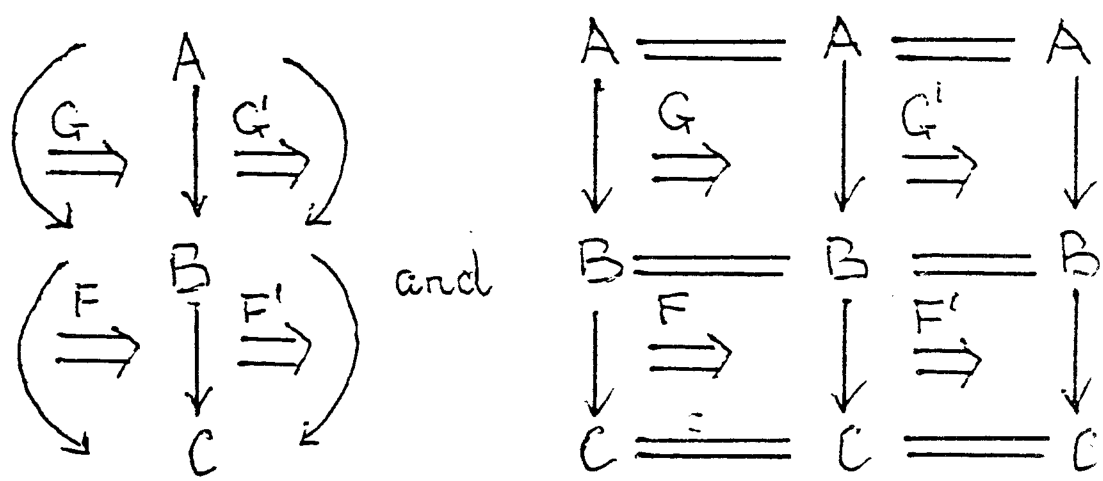


serves to indicate the composite $F \star G$ and the figures



serves to indicate $F \circ G$.

Remark finally that at the situation indicated at the following pictures



$(F' \star F) \circ (G' \star G) = (F' \circ G') \star (F \circ G)$, as it follows from the definition of 2-category, and thus one can give the unique meaning to composites presented by this and similar plain cellular and polygonal decompositions.

2.2. Examples. We sketch several elementary examples. For more examples and a detailed discussion, see [K-V].

(A). At the basic example coming from the theory of categories $\mathcal{A}_0$ consists of all categories[1], $\mathcal{A}_1$ of all functors between categories and $\mathcal{A}_2$ of all natural

[1] There are several well known ways to overcome the difficulty produced by the fact that

transformations of functors. Here, to define the composition $\star$ of two natural transformations

$$T = \{T_L : \varphi(L) \to \psi(L) | L \in \mathcal{L}\} : \varphi \to \psi,$$

$$S = \{S_L : \psi(L) \to \eta(L) | L \in \mathcal{L}\} : \psi \to \eta,$$

where φ, ψ and η are functors from $\mathcal{L}$ to $\mathcal{M}$ $(\mathcal{L}, \mathcal{M} \in \mathcal{A}_0)$, one takes

$$(S \star T)_L = S_L \circ T_L.$$

To define the composition $\circ$ of two natural transfromations

$$U = \{U_L : \varphi(L) \to \psi(L) | L \in \mathcal{L}\} : \varphi \to \psi,$$

$$T = \{T_M : \varphi'(M) \to \psi'(M) | M \in \mathcal{M}\} : \varphi' \to \psi',$$

where φ, ψ are functors from $\mathcal{L}$ to $\mathcal{M}$ and φ', ψ' are functors from $\mathcal{M}$ to $\mathcal{N}$ $(\mathcal{L}, \mathcal{M}, \mathcal{N} \in \mathcal{A}_0)$, one takes

$$(T \circ U)_L = \psi'(U_L) \circ T_{\varphi(L)}.$$

(B). Another typical example comes from the homotopy theory. There, considering points of an arbitrary topological space X as objects, continuous pathes as 1-morphisms and homotopies of pathes as 2-morphisms one supplies them easily by a structure of 2-category. Two types of composition of 2-morphisms correspond to two types of decomposition of a rectangle in 2 halfs: they are decompositions produced by vertical and horizontal pastings. Indeed if, as usually, a path is defined to be a map $\xi : I \to X$ and a homotopy to be a map $H : I \times I \to X$, then the resulting 2-category will be not strict and will not satisfy axioms enumerated in 2.1 : thus, in particular, composition of 1-morphisms will be not associative. To obtain a true strict 2-category it is sufficient to identify pathes and, correspondingly, homotopies differing by reparametrization.

(C). A simple example more closed to our needs is provided by scanned surfaces in $\mathbb{R}^4$. This is a category $\mathcal{S}$ such that $\mathcal{S}_0$ is formed by finite subsets

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they don't constitute a set.

of $\mathbb{R}^2$, $\mathcal{S}_1$ by one-dimensional proper submanifolds of $\mathbb{R}^2 \times \Delta$, Δ being an arbitrary non-fixed closed interval in $\mathbb{R}$, and $\mathcal{S}_2$ by two-dimensional proper submanifolds of $\mathbb{R}^2 \times \Delta \times \Delta'$, Δ' being also an arbitrary closed interval. Here, to overcome a difficulty analogous to that of the preceeding example, we define the composition of 1-morphisms $t \subset \mathbb{R}^2 \times \Delta$ and $t' \subset \mathbb{R}^2 \times \Delta'$ only if the end-point of Δ is the beginning-point of Δ' (and, certainly, if in addition the target of t coincides with the source of t'). Taking the same precautions at the definition of compositions of 2-morphisms we get a 2-category in the sense of 2.1 (i.e. a strict one).

3. Spaces of tangle-diagrams

3.1. Systems of arc and loops. In what follows we denote the closed strip $\mathbb{R}^1 \times [0,1]$ by B and its boundary components $\mathbb{R}^1 \times \{1\}$ and $\mathbb{R}^1 \times \{0\}$ by respectively B_1 and B_0.

We call a compact subset L of B a *regular system of arcs and loops* if it verifies the following conditions: L is a 1-dimensional smooth submanifold of B except in a finite number of points, each exceptional point is a transversal double point and lies in the interior of B, each boundary point of L belongs to $B_0 \cup B_1$, and L is nowhere tangent to $B_0 \cup B_1$. It is clear that any regular system of arcs and loops can be presented as the image of a proper immersion of a disjoint union of intervals and circles.

A regular system L of arcs and loops is said to be *generic* if the projection $B = \mathbb{R}^1 \times [0,1] \to [0,1]$ restricted to L is a Morse function whose critical values are pairwise distinct and neither critical point is a double point of L. If the projection $B \to [0,1]$ restricted to L is a Morse function except in a finite number of points which are simple degenerated critical points (i.e. the third derivative of the function is $\neq 0$ at these points, while the first and the second ones are equal to 0) and if, in addition, neither two local maximum values or two local minimum ones are equal, the system L is said to be *weakly generic*.

3.2. Tangle-diagrams. A *1-diagram* (or a tangle-diagram) is a regular system of arcs and loops in $B = \mathbb{R}^1 \times [0,1]$ equipped with an overcrossing-undercrossing mark at each double point; the mark serves to distinguish the, so called, upper and the lower branches[2]. A 1-diagram is said to be *generic*

[2]Such a definition is originated by the fact that any 1-diagram can be represented as the

if the underlying system of arcs and loops is generic. If the system is weakly generic, then the diagram is called *weakly generic* too.

3.3. Spaces of diagrams. The set of 1-diagrams supplied with the C^∞ topology is an infinite dimensional topological space. This space has an infinite number of connected components: 1-diagrams belong to the same component if and only if they are isotopic[3].

Let us denote the space of all 1-diagrams by D, the subspace of the generic diagrams by D^0 and the subspace of the weakly generic diagrams by D'. Remark, that isotopic generic or weakly generic 1-diagrams may belong to different components of D^0 and respectively D'. For example, 3 diagrams shown at Figure 1 belong to 3 different components of D^0 and to 2 different components of D' (the second diagram and the third one are from the same component of D').

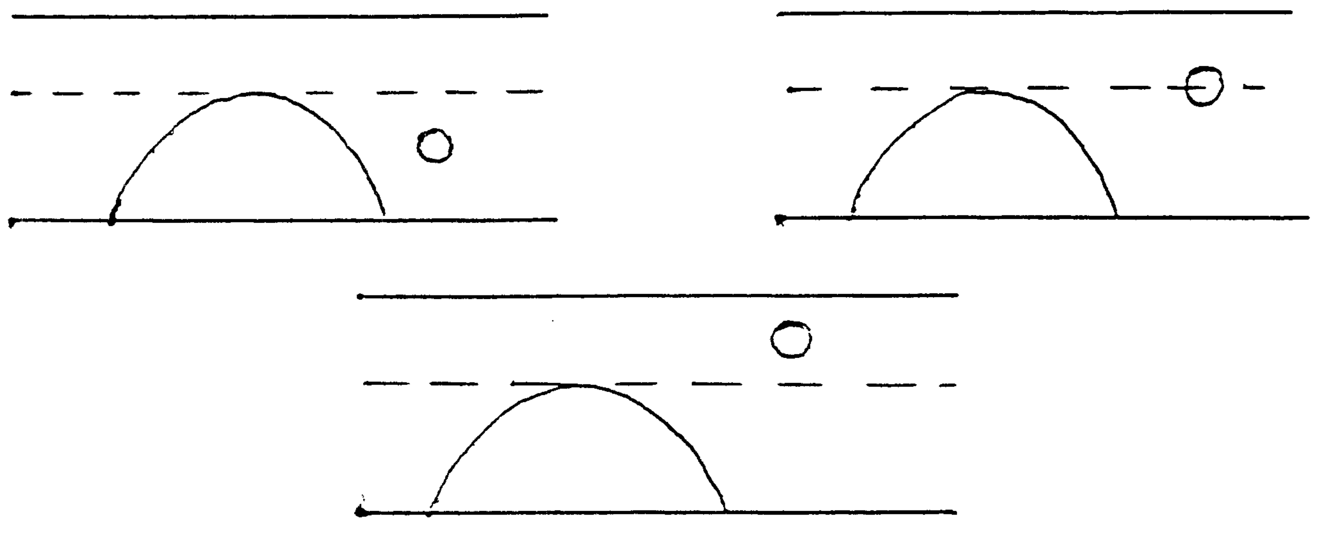

FIGURE 1

3.4. Theorem. *Each component of D' is simply connected.*

This result is the key point at the construction of 2-category of 2-knots.

projection of a proper imbedding of a disjoint union of intervals and circles into $B \times \mathbb{R}$, cf. 3.4

[3]In other words, we consider the group of C^∞ ambient isotopies supplied with the C^∞ topology and introduce in the set of 1-diagrams the minimal topology for which the action of the group of ambient isotopies is continuous.

Remark that the space D^0 has the same property and thus can replace D' in what follows. We prefere to employ D' because it has a less number of components and thus simplifies the category under construction. Contrary to D' and D^0, the space D is not simply connected.

Our proof of the theorem looks as follows.

Take a general loop in D'. It is represented by a family of weakly generic 1-diagrams $L_\alpha, \alpha \in S^1$ in B. The graph of the family is the image of a fiber-to-fiber map $F : l \times S^1 \to B \times S^1$, where $F|_{l\times\{\alpha\}}, \alpha \in S^1$ is a parametrization of L_α, l being a disjoint sum of segments and circles. This map is a proper immersion.

The composition F' of F with the projection $B \times S^1 = \mathbb{R} \times [0,1] \times S^1 \to [0,1] \times S^1$ has, in general, critical points. The critical point set C_e of F' is a union of a finite number of disjoint circles in $l \times S^1$. The restriction of F' to C_e is a topological imbedding locally. The points where it is locally not a smooth imbedding are *pleats* of $F' : l \times S^1 \to [0,1] \times S^1$. At all other points the image $F'(C_e)$ is transversal to fibers of $[0,1] \times S^1 \to S^1$.

Each component s of C_e which is not transversal to the fibers of $[0,1]\times S^1 \to S^1$ contains two and only two pleats $b(s), d(s)$ and bounds in $l \times S^1$ a disc $C(s)$. One pleat, $b(s)$, corresponds to a birth of a maximum $M_s(t)$ and a minimum $m_s(t)$ at a regular point of some L_α, the other one, $d(s)$ corresponds to their death at a regular point of some L_β. The disc is formed by the arc of $L_t, t \in [\alpha, \beta]$ bounded by $M_s(t)$ and $m_s(t)$.

All pleats can be eliminated by induction: there is a loop in D' homotopic to the given one, $L_\alpha, \alpha \in S^1$, and having 2 pleats less. Indeed, the elimination is made by an algorithm which selects a minimal pair of pleats, puts the image of critical and double points in an elimination position and eliminates the pair of pleats by a standard procedure.

After elimination of pleats, each component of C_e, as well as each component of the set C_d formed by the double points of diagrams, becomes embedded in $[0,1] \times S^1$ by F' and transversal to fibers. Images of different components of $C_e \cup C_d$ can intersect one another. Again by some algorithmic procedure, these intersection points can be eliminated by pairs. After that the loop is easily made constant.

3.5. From diagrams to tangles. In what follows we denote the closed 3-dimensional strip $\mathbb{R}^2 \times [0,1] = B \times \mathbb{R}$ by K and its boundary components

$\mathbb{R}^2 \times \{1\} = B_1 \times \mathbb{R}$ and $\mathbb{R}^2 \times \{0\} = B_0 \times \mathbb{R}$ by respectively K_1 and K_0.

By a *tangle* we mean a proper 1-dimensional compact smooth submanifold of K. A tangle Γ is said to be *regular*, if: (a) the standard projection $\phi : \Gamma \to B = \mathbb{R}^1 \times [0,1]$ along the second axis of $\mathbb{R}^2$ is a proper immersion, (b) the immersion has only finite number of multiple points each being a transveral double crossing and lying in $B \setminus (B_0 \cup B_1)$. If, in addition, the composition of the projections $\theta : \Gamma \to \mathbb{R}^1 \times [0,1] \to [0,1]$ is a Morse function whose critical values are pairwise distinct and neither critical point is a double point of the immersion, the tangle is called a *generic tangle*.

Thus the image $\phi(\Gamma)$ of a generic tangle is a generic system of arcs and loops and thus the tangle gives rise, in a canonical way, to a 1-diagram. The tangle is generic if and only if its 1-diagram is generic.

Finally, we call a *weakly generic tangle* a regular tangle whose 1-diagram is weakly generic.

Lemma. *Tangles having the same 1-diagram are isotopic. Moreover, tangles with isotopic 1-diagrams are isotopic.*

Both assertions concern only regular tangles, because by definition a tangle presented by a 1-diagram is regular.

Let us supply the set of regular tangles with the C^∞ topology and denote this space by $\tilde{D}$. The subspace of generic and weakly generic tangels are denoted respectively by $\tilde{D}^0$ and $\tilde{D}'$.

Theorem. *Each component of $\tilde{D}'$ is simply connected.*

Remark that since D^0 is simply connected too, each component of $\tilde{D}^0$ is also simply connected.

4. Two-category of 2-knots

In this section we construct a 2-category of 2-knots in $\mathbb{R}^4$. We denote it by $\mathcal{F}$.

4.1. Objects and 1-morphisms. Let us start with a definition of the underlying 1-category.

Objects of $\mathcal{F}$ are finite subsets of $\mathbb{R}^1$ considered up to isotopies in $\mathbb{R}^1$. Since the only isotopy invariant is the number points, the set of objects, $\mathcal{F}_0$, can be identified with $\mathbb{N}$.

A *1-morphism* of $\mathcal{F}$ is a weakly generic 1-diagram considered up to isotopies in the class of weakly generic diagrams. In other words, a 1-morphism is a connected component of D', see 3.3, i.e. $\mathcal{F}_1 = \pi_0(D')$.

The *source* $s_0(u)$ of a 1-morphism u presented by a 1-diagram d is defined to be $d \cap B_0$. Similarly, $t_0(u) = d \cap B_1$.

Further, to any object A of $\mathcal{F}$ one puts in correspondance the *identity morphism* I_A presented by the diagram $d = A \times [0,1]$. Finally, to define the composition $u \circ v$ of 1-morphisms u, v such that $t_0(v) = s_0(u)$, present them respectively by 1-diagrams c, d, put d in the first one-third $\mathbb{R}^1 \times [0, 1/3]$ of the strip B and c at the last one-third $\mathbb{R}^1 \times [2/3, 1]$ and then fill the middle one-third by an isotopy between the sets constituting the target of d and the source of c, see Figure 2.

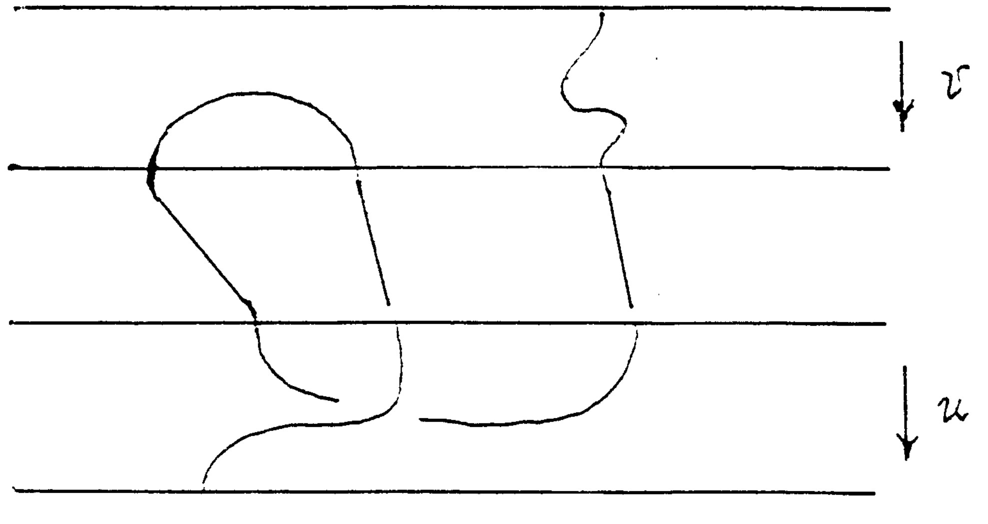

FIGURE 2. Composition of two 1-morphisms

Correctness of the last definition follows from the fact that the configuration space of finite subsets of given cardinality in $\mathbb{R}$ is contractible (cf. 4.2, where the similar arguments used to verify correctness of definitions concerning 2-morphisms compositions are given in all details).

Standard arguments[4] show that the morphisms $1_A, A \in \mathcal{F}_0$, are 2-sided identities with respect to the composition $\circ$ of 1-morphisms and the com-

[4]The same ones which are used in the construction of the fundamental group.

position of 1-morphisms is associative. In particular, we conclude that *the collection* $(\mathcal{F}_0, \mathcal{F}_1, s, t, \circ)$ *forms a category.*

4.2. Two-morphisms. In the category under construction, $\mathcal{F}$, two-morphisms are presented by surfaces in $K \times [0,1] = \mathbb{R}^2 \times [0,1]^2$ satisfying certain boundary conditions.

To precise the definition, let us consider a compact smooth proper 2-dimensional submanifold Σ of $K \times [0,1]$. The boundary of such a surface is decomposed in 4 parts:

$$i(\Sigma) = \Sigma \cap (K_0 \times [0,1]), e(\Sigma) = \Sigma \cap (K_1 \times [0,1]),$$

and

$$s(\Sigma) = \Sigma \cap (K \times \{0\}), t(\Sigma) = \Sigma \cap (K \times \{1\}).$$

Such a surface Σ is said to be *regular*, if

(1) $i(\Sigma), e(\Sigma), s(\Sigma)$ and $t(\Sigma)$ are proper 1-submanifolds of, respectively, $K_0 \times [0,1], K_1 \times [0,1], K \times \{0\}$ and $K \times \{1\}$;
(2) $s(\Sigma)$ and $t(\Sigma)$ are weakly generic tangles in $K = K \times \{0\} = K \times \{1\}$;
(3) the image of $i(\Sigma)$, as well as of $e(\Sigma)$, under the projection $K_0 \times [0,1] = K_1 \times [0,1] = \mathbb{R}^2 \times [0,1] \to B = \mathbb{R}^1 \times [0,1]$ forgetting the second coordinate of $\mathbb{R}^2$ is a graph of an isotopy of a finite subset of $\mathbb{R}^1$. [5]

By definition: each *2-morphism* of the category $\mathcal{F}$ is represented by a regular surface; two surfaces Σ_1, Σ_2 represent the same 2-morphism if an only if they are isotopic in the class of regular surfaces.

Further, a 2-morphism F being represented by a regular surface Σ, the corresponding 1-morphisms $s_1(F)$ and $t_1(F)$ are realized by the 1-diagrams of tangles $s(\Sigma)$ and $t(\Sigma)$. To define the *unity 2-morphism* 1_u, where u is a 1-morphism, we represent u by the 1-diagram of a weakly generic tangle γ and take 1_u to be the cylinder $\gamma \times [0,1]$.

Consistence of the preceeding definitions is trivial. And it is clear that the maps s, t defined above verify the relations $t_0 s_1 = t_0 t_1, s_0 t_1 = s_0 s_1$.

The remaining 2 definitions are definitions of compositions of 2-morphisms.

[5]That is the case if the images of $i(\Sigma)$ and $e(\Sigma)$ are proper 1-submanifolds of $B = \mathbb{R}^1 \times [0,1]$ and the projection $\mathbb{R}^1 \times [0,1] \to [0,1]$ restricted to these submanifolds has no critical points.

First, let us define the *composition* $\circ$. Assume that F, G are 2-morphisms such that $t_0 s_1(G) = s_0 t_1(F)$. Then there exists a finite set $A \subset \mathbb{R}$ and regular surfaces Σ_1, Σ_2 representing 2-morphisms F and G and such that

$$e(\Sigma_2) = (A \times \{r\}) \times \{1\} \times [0,1],$$

$$i(\Sigma_1) = (A \times \{r\}) \times \{0\} \times [0,1].$$

Put Σ_2 in the first half of $K \times [0,1] = K_0 \times [0,1] \times [0,1]$ by the map

$$K_0 \times [0,1] \times [0,1] \to K_0 \times [0,1/2] \times [0,1], (x,y,z,t) \mapsto (x,y,z/2,t)$$

and Σ_1 in the second half by the map

$$K_0 \times [0,1] \times [0,1] \to K_0 \times [1/2,1] \times [0,1], (x,y,z,t) \mapsto (x,y,(1+z)/2,t).$$

Their union Σ, smoothed if necessary along their common boundary, represents a 2-morphism. This morphism is independent on the choice of axulliary realizations and that is, by definition, $F \circ G$.

To verify its independence on the choice made at the definition of $\circ$, let take another representations Σ_1', Σ_2' of F and G with

$$e(\Sigma_2') = (A' \times \{r'\}) \times \{1\} \times [0,1],$$

$$i(\Sigma_1') = (A' \times \{r'\}) \times \{0\} \times [0,1].$$

Since Σ_1', Σ_2' are isotopic in the class of regular surfaces to Σ_1, Σ_2, the composed surfaces Σ' and Σ are also isotopic in the class of regular surfaces. Moreover, reparametrizing the composed isotopy connecting Σ' and Σ one can get an isotopy (in the class of regular surfaces) leading from Σ' to a regular surface Σ'' built from 3 parts: the image of Σ_2 under the map $K_0 \times [0,1] \times [0,1] \to K_0 \times [0,1/4] \times [0,1], (x,y,z,t) \mapsto (x,y,z/4,t)$, the graph $A_{t,\tau} \times \{\tau\} \times \{t\}, \tau \in [1/4,3/4], t \in [0,1]$ of a loop at the loop space $Map(S^1, Emb(A,\mathbb{R}))$ and the image of Σ_1 under the map $K_0 \times [0,1] \times [0,1] \to \mathbb{R}^2 \times [3/4,1] \times [0,1], (x,y,z,t) \mapsto (x,y,(3+z)/4,t)$. The configuration space $Emb(A,\mathbb{R})$ is contractible and thus any loop of loops of this space is homotopic to the trivial loop. Hence the regular surface Σ'' is isotopic to Σ in the class of regular surfaces. From where the required independence follows.

Now, let us define the *composition* $\star$. Assume that F, G are 2-morphisms such that $t_1(G) = s_1(F)$. Then there exists a weakly generic (and even generic) tangle γ and regular surfaces Σ_1, Σ_2 representing 2-morphisms F and G and such that $t(\Sigma_2) = s(\Sigma_1) = \gamma$. Put Σ_2 in the left half of $K \times [0,1]$ by the map

$$K \times [0,1] \to K \times [0,1/2], (x,y,z,t) \mapsto (x,y,z,t/2)$$

and Σ_1 in the other half by the map

$$K \times [0,1] \to K \times [1/2,1], (x,y,z,t) \mapsto (x,y,z,(1+t)/2.$$

Their union Σ, smoothed if necessary along their common boundary, represents a 2-morphism. This morphism is independent on the choice of axulliary realizations and that is, by definition, $F \star G$.

To verify its independence on the choice made at the definition of $\star$, let take another representations Σ_1', Σ_2' of F and G with $t(\Sigma_2') = s(\Sigma_1') = \gamma'$. The same arguments as at the preceding verification show that, since Σ_1', Σ_2' are isotopic (in the class of regular surfaces) to Σ_1, Σ_2, the composed surface Σ' is isotopic (in the class of regular surfaces) to a regular surface Σ'' built from 3 parts: the image of Σ_2 under the map $K \times [0,1] \to K \times [0,1/4], (x,y,z,t) \mapsto (x,y,z,t/4)$, the graph of a loop of $\tilde{D}'$ and the image of Σ_1 under the map $K \times [0,1] \to K \times [3/4,1], (x,y,z,t) \mapsto (x,y,z,(3+t)/4)$. By Theorem 3.5, each component of $\tilde{D}'$ is simply connected and thus any loop of this space is homotopic to the trivial loop. Hence Σ'' is isotopic to Σ in the class of regular surfaces. From where the required independence follows.

4.3. Theorem. *The collection* $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2, s_0, t_0, s_1, t_1, \circ, \star)$ *is a 2-category.*

References

[F] John E.Fischer, Jr., *2-categories and 2-knots*, Preprint (1993).

[K-V] M.M.Kapranov, V.A.Voevodsky, *Braided monoidal 2-categories, 2-vector spaces and Zamolodchikov tetrahedra equations*, Preprint (1991).

[K-S] D.Kazhdan, Y.Soibelman, *Representations of the quantized function algebras, 2-categories and Zamolodchikov tetrahedra equation*, Preprint (1993).

[Kh] V.M.Kharlamov, *Movements of straight lines and the tetrahedron equations*, Publ. Dipart. Mat. Univ. Pisa (1992).

[M-S] Yu.I.Manin, V.V.Schechtman, *Arrangements of hyperplanes, higher braid groups and higher Bruhat orders*, Adv. Studies in Pure Math. **17** (1989), 289–308.

[RT1] N.Reshetikhin, V.Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), 1–26.

[RT2] N.Reshetikhin, V.Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991), 547–598.

[Tu] V.Turaev, *Operator invariants of tangles and R-matrices*, Izvestia Ac. Sci. USSR **53** (1989); English transl. in Math. USSR Izvestia **35** (1990), 411–444.

[Ye] D.N.Yetter, *Markov algebras and Braids*, AMS Contemp. Math. **78** (1988), 705–730.

[Z] A.B.Zamolodchikov, *Tetrahedron equations and integrable systems in three-dimensional space*, J.E.T.P. **52** (1980), 325–336.

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