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Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1993, tome 44
« Conférences de C. Dewitt-Morette, D. Foata, C. Itzykson, B.-L. Julia, J.-L. Loday, P.-A. Meyer, V. Pasquier », , exp. n° 1, p. 1-24

<http://www.numdam.org/item?id=RCP25_1993__44__1_0>

FUNCTIONAL INTEGRATION

A MULTIPURPOSE TOOL (*)

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Based on joint work with Pierre Cartier

Préambule

Je dois beaucoup à Yvonne Choquet-Bruhat. Et, s'il me fallait présenter les problèmes où elle m'a montré l'idée simple qui va au cœur du sujet, je parlerais des heures, des jours sur ... l'Analyse, les Variétés et la Physique. Mais il m'a fallu faire un choix. Pourquoi choisir l'intégration fonctionnelle? Peut-être pour marquer le cinquantième anniversaire de l'intégrale de Feynman. En réalité, en souvenir de deux dates personnelles:

En 1969, Yvonne Choquet m'invita à faire 3 conférences. Pourquoi m'invita-t-elle? Nous nous connaissions alors fort peu - encore que nous découvrîmes plus tard avoir été la même année en septième à Victor Duruy. J'ai été d'autant plus étonnée de recevoir cette invitation que je m'engourdissais alors dans une vie scientifique sclérosée par les règlements concernant les époux dans la même profession. A l'occasion de ces conférences, je revins à l'intégrale de Feynman, sujet qui m'avait intéressée vingt ans auparavant, et je repartis bon pied, bon oeil.

En 1971, Yvonne, bien que ne travaillant pas elle-même sur le sujet, réalisa qu'il me manquait un élément essentiel pour aller au-delà des idées communément acceptées, et elle me dit de lire Bourbaki, livre VI, chapitre IX, page 70 et suivantes. J'y découvrais les promesures, dans une présentation facile à généraliser pour les besoins de l'intégrale de Feynman.

(*) Ce texte reprend une prépublication de l'IHES de Novembre 1992 IHES/P/92/84

To appear in the *Proceedings of the International Colloquium in Honour of Yvonne Choquet Bruhat* Eds. M Flato, R. Kerner, A. Lichnerowicz (Kluwer academic publishers)

Yvonne n'a pas, sous son nom, de publication sur l'intégrale fonctionnelle mais elle est à l'origine de travaux sur le sujet. C'est avec grand plaisir que je lui offre aujourd'hui quelques progrès récents sur l'intégrale de Feynman. Ces progrès doivent beaucoup à Pierre Cartier qui a su utiliser les résultats des physiciens pour construire une axiomatique de l'intégrale de Feynman.

Abstract

The goal is to extract from work done by physicists during the last fifty years an axiomatic basis for functional integration which will provide simple and robust methods of calculation, in particular for integration by parts, change of variable of integration, sequential integrations. The mythical integrator $\mathcal{D}$ in physicists' equations such as $\int_\Phi \exp\left(\frac{i}{\hbar}\left(S(\varphi) - \hbar\,\langle J, \varphi\rangle\right)\right)\mathcal{D}\varphi = Z(J)$ is not unique ; but given two Banach spaces Φ and Φ', and two continuous bounded maps $\Theta : \Phi \times \Phi' \to \mathbb{C}$, and $Z : \Phi' \to \mathbb{C}$, one can choose an integrator $\mathcal{D}_{\Theta,Z}$ satisfying the equation $\int_\Phi \Theta(\varphi, J)\mathcal{D}_{\Theta,Z}\varphi = Z(J)$ and a related normed space $\mathcal{F}_{\Theta,Z}$ of functionals F on Φ integrable by $\mathcal{D}_{\Theta,Z}$. Prodistributions, white noise integrators, and, of course, Lebesgue measures fit with proposed scheme.

Functional integration

1. - Introduction

The subtitle on the French poster says "Un outil sûr et performant". The English subtitle says "A reliable and efficient tool". Let us say a "multipurpose tool". Given the title of the Colloquium, I have selected three applications of Feynman path integrals, one in Analysis, one in Differential Geometry, and one in Physics. In each case I shall only state the problem, indicate the key issues which have been solved, and give the answer.

No one will question the answers; they are clearly right; but are Feynman path integrals still a mathematical nonsense? The answer is "no" and I shall present a mathematical framework, nearly completed, which makes them not only efficient but also reliable. Pierre Cartier is the architect of this bridge from mathematics to physics.

2. - Prodistributions

Given a locally convex space $\mathbb{X}$, and a projective family of finite dimensional spaces $\{\mathbb{X}^\alpha\}$ with the usual coherence conditions which make it possible to reconstruct $\mathbb{X}$ from $\{\mathbb{X}^\alpha\}$, a promeasure μ on $\mathbb{X}$ is a *projective* family $\{\mu_\alpha\}$ of *bounded measures* on $\{\mathbb{X}^\alpha\}$ which satisfies coherence conditions adapted to the definition of $\{\mathbb{X}^\alpha\}$.

A topology on $\mathbb{X}$ defines the dual space $\mathbb{X}'$ of $\mathbb{X}$ and one can construct the family $\{\mathcal{F}\mu_\alpha\}$ of Fourier transforms of the promeasure $\{\mu_\alpha\}$. The coherence conditions satisfied by $\{\mathcal{F}\mu_\alpha\}$ are simpler to state than the coherence conditions satisfied by $\{\mu_\alpha\}$, namely.

$$(2.1)\qquad \begin{cases} \mathcal{F}\mu_\alpha(0) = \text{constant independent of } \alpha \\ \text{If } \Pi^{\alpha\beta} : \mathbb{X}^\beta \to \mathbb{X}^\alpha, \text{ then } \mathcal{F}\mu_\alpha = \mathcal{F}\mu_\beta \circ \widetilde{\Pi}_{\beta\alpha},\ \widetilde{\Pi}_{\beta\alpha} : \mathbb{X}'_\alpha \to X'_\beta \end{cases}$$

where $\widetilde{\Pi}_{\beta\alpha}$ is the transpose of $\Pi^{\alpha\beta}$.

$$\begin{array}{ccccc} \mathbb{X}' & & & & \mathbb{X} \\ \widetilde{\Pi}:=\{\widetilde{\Pi}_\alpha\}\uparrow & & & & \downarrow \{\Pi^\alpha\}=:\Pi \\ \{\mathbb{X}'_\alpha\} & & & & \{\mathbb{X}^\alpha\} \\ & \mathcal{S}'(\mathbb{X}'_\alpha) & \stackrel{\mathcal{F}}{\longleftarrow} & \mathcal{S}'(\mathbb{X}^\alpha) & \end{array}$$

Figure 1 : Prodistributions

Let

$$\tilde{\Pi} : \bigcup_{\alpha} \mathbb{X}'_{\alpha} \to \mathbb{X}' \quad \text{be defined by} \quad \tilde{\Pi}\Big|_{\mathbb{X}'_{\alpha}} := \tilde{\Pi}_{\alpha}. \tag{2.2}$$

A prodistribution $\mathcal{F}w$ is a family of Fourier transforms $\{\mathcal{F}w_{\alpha}\}$ where w_{α} is not necessarily a bounded measure, but where $\mathcal{F}w_{\alpha}$ is a continuous function on $\mathbb{X}'_{\alpha}$. Therefore there is not necessarily a promeasure w corresponding to a prodistribution $\mathcal{F}w$.

Does a prodistribution define an integrator ?

Is such an integrator a practical tool for Physics ?
The answer is "yes" to both questions because of the following equations :
Let P be a linear continuous map $P : \mathbb{X} \to \mathbb{Y}$. Let $\mathcal{F}w$ be a prodistribution on $\mathbb{X}$. Let $f : \mathbb{Y} \to \mathbb{C}$

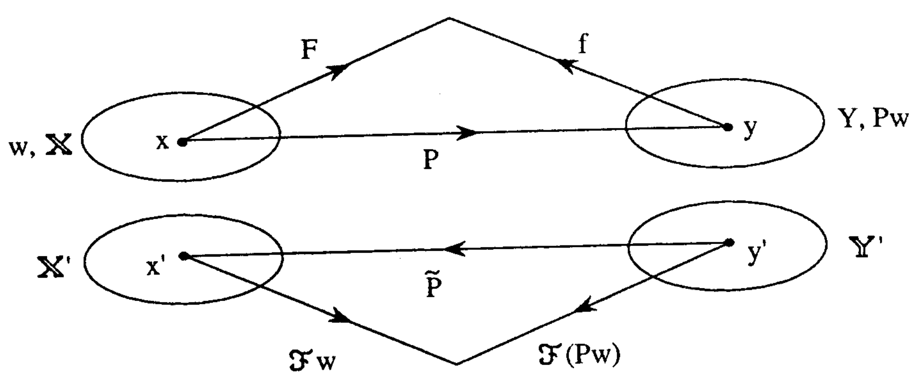

Figure 2 : Linear change of variable of integration

$$\int_{\mathbb{X}} F(x)dw(x) = \int_{\mathbb{Y}} f(y)d(Pw)(y), \tag{2.3}$$

$$\mathcal{F}(Pw) = \mathcal{F}w \circ \tilde{P}. \tag{2.4}$$

If P maps the infinite dimensional space $\mathbb{X}$ into a finite dimensional space $\mathbb{Y}$, these equations define the functional integral over $\mathbb{X}$. If $\mathbb{Y}$ is infinite dimensional, these equations define a manipulation of the functional integral.

These two equations are sufficient to solve nontrivial problems. For instance, they have been used to compute the glory scattering cross section of classical waves by black holes. The cross section can be expressed in term of a functional integral. But this functional integral cannot be computed by any of the commonly used methods: discretization, analytical continuation, WKB expansions, for the following reasons:

a) the paths take their values in a riemannian space, and discretization of the paths is ambiguous;

b) Glory scattering is a scattering process where the final momentum is antiparallel to the initial momentum. This momentum-to-momentum transition cannot be computed from a position-to-position probability amplitude because there are no plane waves in curved spacetimes. Hence the path integral has to be set up for paths taking their values in phase space. Therefore, analytical continuation of a Wiener type integral is not an option because there are no Wiener measure for the space of paths in phase space (the positivity of the Jacobi operator in configuration space does not imply the positivity of the corresponding Jacobi operator in phase space);

c) the critical points of the action are degenerate: the paths are caustic forming in phase space. Hence the WKB approximation "breaks down".

On the other hand one can use the equations (2.3) and (2.4) to compute the glory scattering cross section. A long, but unambiguous, calculation gives, in the leading order of the semiclassical approximation

$$\frac{d\sigma}{d\Omega} = 2\pi\hbar^{-1}\,|\mathbf{p}|\,B_g^2\frac{dB}{d\theta}J_{2s}^2\left(\hbar^{-1}\,|\mathbf{p}|\,B_g\sin\theta\right) \tag{2.5}$$

where Ω is the solid angle in the θ direction, $\mathbf{p}$ is the incoming momentum, $B(\theta)$ is the impact parameter of a particle scattered in the θ direction, B_g is the glory scattering impact parameter, $B_g = B(\pi)$, J_{2s} is the Bessel function of order $2s$ ($s = 0$ for a scalar particle, $s = 1$ for a photon, $s = 2$ for a graviton).

In the case of scattering by Schwarzschild Black Holes, the function $B(\theta)$ has been computed by Darwin. ($B(\theta) = M(3\sqrt{3} + 3.48\exp(-\theta)), M$ is the mass of the Black Hole in units where $G = c = 1$. If we use this function $B(\theta)$ in (2.5) the above cross section matches perfectly the numerical cross sections computed by Handler and Matzner who used the technique of partial wave decomposition, that is a totally different technique.

References to other applications of the prodistribution formalism are given below. I shall only mention the calculation of the propagator for the anharmonic oscillator

$$\frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2 + \frac{\lambda}{4} q^4 = 0, \tag{2.6}$$

because it shows that, contrary to commonly accepted ideas, the propagator is not singular in λ, but tends to the harmonic oscillator propagator when λ tends to zero.

The third application I have chosen is the expression for propagators between two points a and b of a multiply connected space M. A simple property of the path integral representation of the propagator gives :

$$|K^\alpha (b, t_b; a, t_a)| = \left| \sum_{g_i \in \pi_1(M)} \chi^\alpha (g_i) K^i (b, t_b; a, t_a) \right| \tag{2.7}$$

where K^i is the propagator obtained by summing over all the paths in the same homotopy class, and

$$\chi^\alpha : \pi_1(M) \to \mathbb{C}$$

is a unitary representation of the fundamental group $\pi_1(M)$.

Prodistributions are not necessary to derive this result. I chose this example, because once more, I have to thank Yvonne Choquet-Bruhat. I had obtained the propagator (2.7) "experimentally" by studying Schulman's calculation of the path integral for a top, as a model for a path integral for spin. I knew (as a physicist knows) that the result was correct, but I knew that my proof was not convincing, to say the least. I had the opportunity to discuss the problem with Yvonne and R. Bott during a Les Houches session. When I asked Bott to help me clean up the proof, he objected strenuously to the fact that I was combining an element of $\pi_1(M)$ with a homotopy class of paths from a to $b \neq a$. But Yvonne convinced Bott not to give up the discussion. I then realised that I had to pay more than lip service to the fact that although the groups $\pi_1(M)$ based at two different points are isomorphic, they are not *canonically* isomorphic. The proof of (2.7) was then immediate : use the principle of superposition of probability amplitudes to write K as a linear combination of all the K^i's. Determine the coefficients of this linear combination by requiring that the result be independent of the base point chosen for $\pi_1(M)$ (more precisely independent of the homotopy mesh chosen to associate g_i and K^i). The linear combination is then determined up to an overall phase factor, hence the absolute value signs in (2.7). This example shows that functional integrals reflect, as could be expected, the global properties of the manifold where the paths take their values.

The problems I have mentioned were solved one at a time, the space of integrable functionals on $\mathbb{X}$ had not been identified.

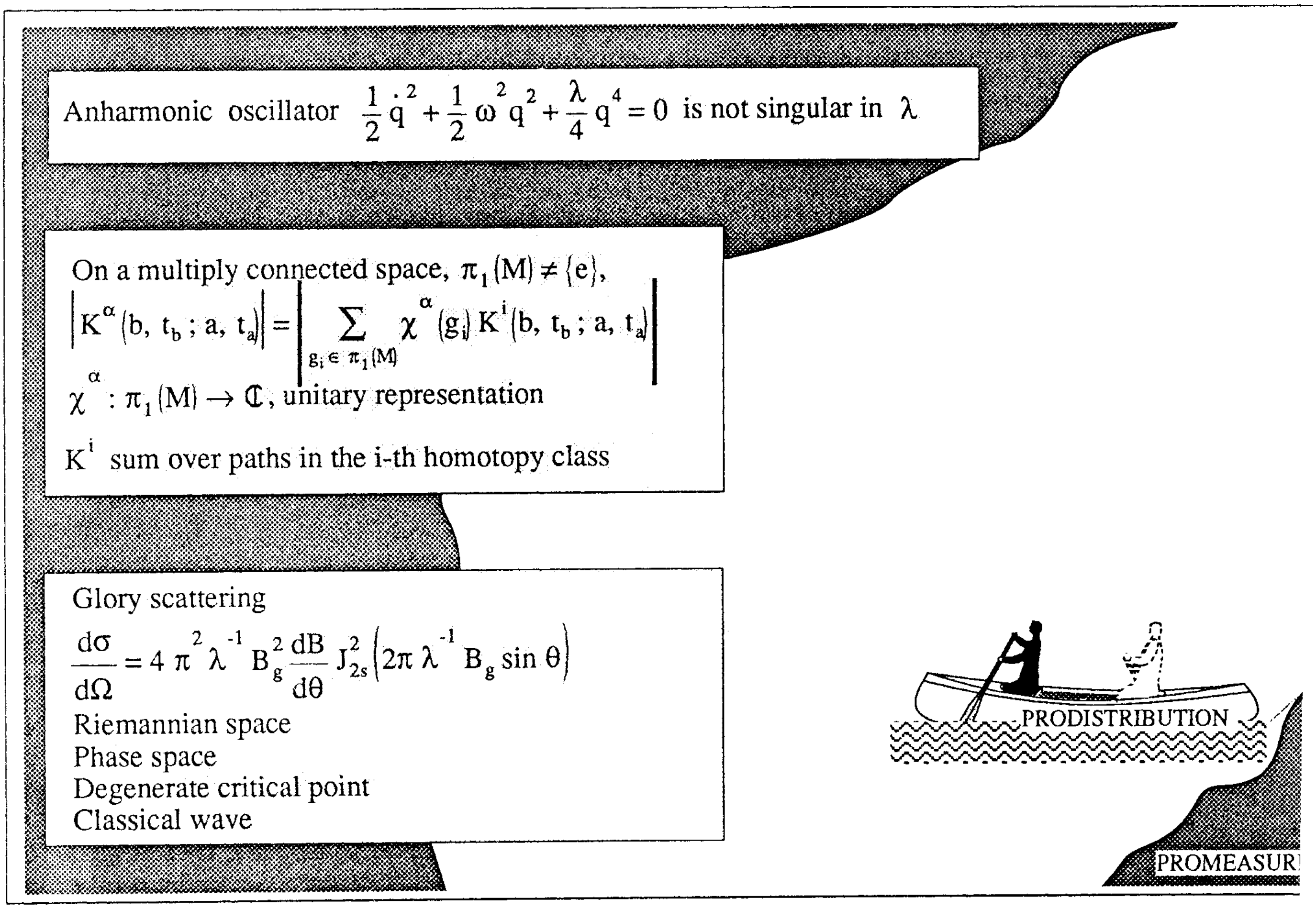

Figure 3 : Examples

3. - The Cartier Bridge

Pierre Cartier is surveying the work which has been done by physicists and by mathematicians on Feynman path integrals (see references below). Using the material at hand, he builds a bridge, now nearly completed, from mathematics to physics to make functional integration a robust tool ; in particular, a user friendly tool for

integration by parts

linear change of variable of integration

successive integrals (generalized Fubini theorem)

The bridge pillars are labelled by roman numerals, as stone pillars used to be; the roman numerals are repeated in the numbering of the equations.

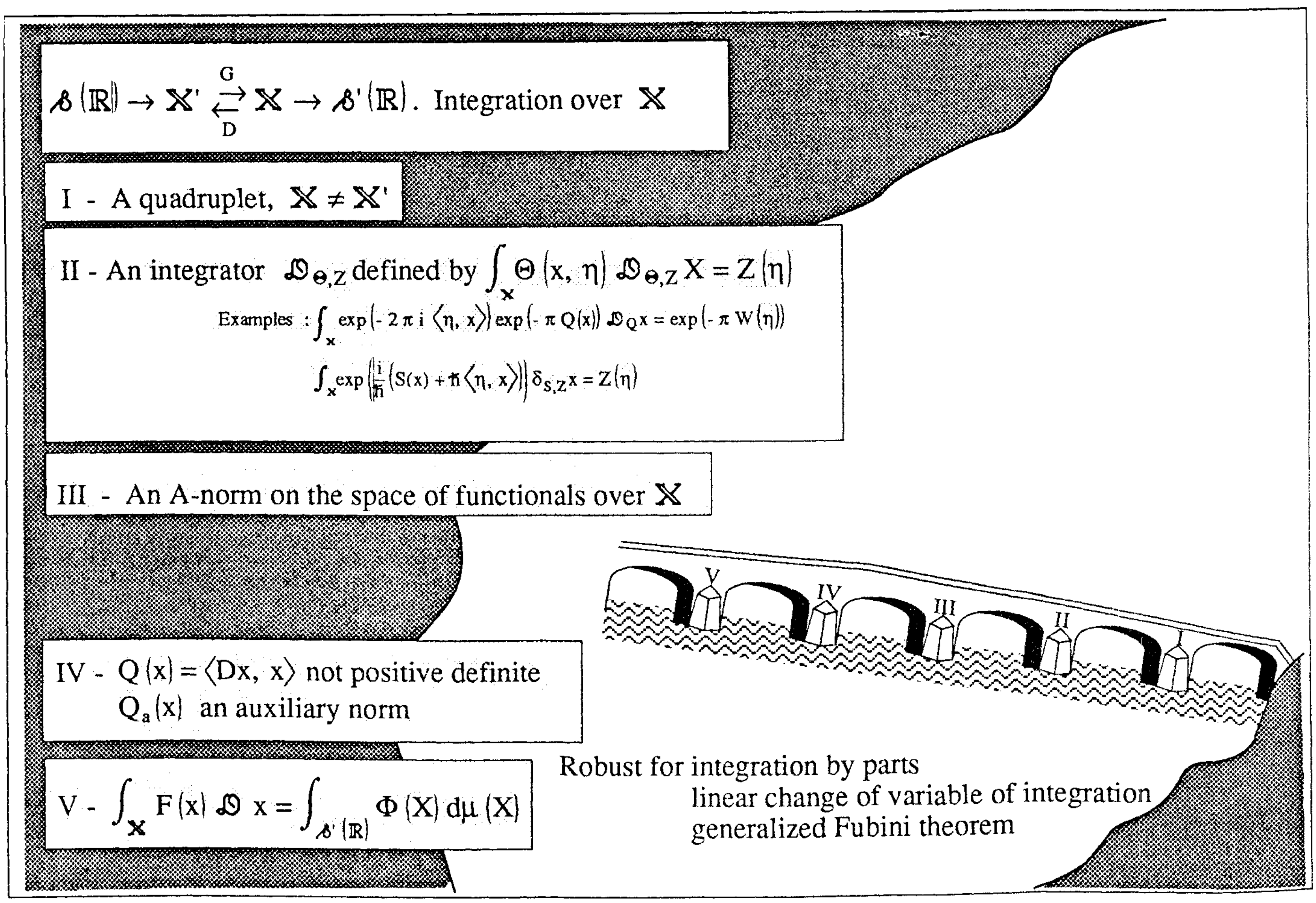

Figure 4 : The Cartier Bridge.

Pillar I : In broad outline, let $\mathbb{X}$ and $\mathbb{X}'$ be two Banach spaces in duality making together with the Schwartz spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ the quadruplet

$$\mathcal{S}(\mathbb{R}) \hookrightarrow \mathbb{X}' \underset{D}{\overset{G}{\rightleftarrows}} \mathbb{X} \hookrightarrow \mathcal{S}'(\mathbb{R}). \tag{3.I_1}$$

D is a differential operator acting on $\mathbb{X}$, and G is its kernel (Green function).

$$DG = \mathbb{1} \quad , \quad GD = \mathbb{1}. \tag{3.I_2}$$

The boundary conditions which define a unique kernel G are encoded in the space $\mathbb{X}$. For instance, if D is a second order differential operator on the space $\mathbb{X}$ of paths $x : [t_a, t_b] \to \mathbb{R}^d$ which vanish at t_a and t_b, $G^{\alpha\beta}(t,s)$ is the unique kernel of D which vanishes for either t or s equal to t_a or t_b.

In the case $\mathbb{X} = \mathbb{X}'$, the quadruplet reduces to a Gelfand triplet. Although the mathematics of the quadruplet is, to a large extent, similar to the mathematics of the triplet, the physics is considerably different. The physicist working with a Lagrangian needs $D : \mathbb{X} \to \mathbb{X}'$ where $\mathbb{X}'$ is not necessarily equal to $\mathbb{X}$.

Pillar II : Physicists want the action, integral of the Lagrangian, to appear explicitly. They need a path integral of the following type.

$$\int_{\mathbb{X}} \exp\left(\frac{i}{\hbar} S(x)\right) \mathcal{D}x \tag{3.II_1}$$

rather than an integral of the type

$$\int_{\mathbb{X}} \exp\left(-\frac{i}{\hbar} \int V(x(t))dt\right) dw(x). \tag{3.II_2}$$

This cannot be accomplished by a universal "Lebesgue" measure $\mathcal{D}x$ and Cartier has proposed the following framework. Let $\mathcal{D}_{\Theta,Z}$ be an integrator on $\mathbb{X}$ characterized by an equation

$$(3.II_3) \qquad \int_{\mathbb{X}} \Theta(x,\eta)\mathcal{D}_{\Theta,Z}x = Z(\eta) \quad \text{for} \quad \eta \in \mathbb{Y};$$

here $\mathbb{X}$ and $\mathbb{Y}$ are two Banach spaces, $\Theta(x,\eta)$ and $Z(\eta)$ are scalar valued functions dictated by the problem under consideration.
A simple example : with the notation of pillar I, if $\mathbb{Y} = \mathbb{X}'$, and $Q(x) = \langle Dx, x\rangle$, and $W(\eta) = \langle \eta, G\eta\rangle$,

$$(3.II_4) \qquad \int_{\mathbb{X}} \exp\left(-2\pi i \langle \eta, x\rangle\right)\exp(-\pi Q(x))\mathcal{D}_Q x = \exp(-\pi W(\eta))$$

defines an integrator $\mathcal{D}_Q$. Given the relationship between Q and W, the subscript Q is sufficient to characterize the integrator.

We note that if $\mathbb{X} = \mathbb{R}^\nu$ and $Q(x) = \sum_{i=1}^{\nu} \left(x^i\right)^2$, then $\mathcal{D}_Q x$ is the ordinary Lebesgue measure namely $dx^1 \cdots dx^\nu$.

An example from quantum physics : Let φ be a field and J a source; then the action S and the generating functional $Z(J)$ introduced by Schwinger define an integrator $\mathcal{D}_{S,Z}$ by:

$$(3.II_5) \qquad \int_{\mathbb{X}} \exp\left(\frac{i}{\hbar}\left(S(\varphi) + \hbar\langle J, \varphi\rangle\right)\right)\mathcal{D}_{S,Z}\varphi = Z(J).$$

The important point is to note that there is no universal $\mathcal{D}\varphi$ even if we abbreviate $\mathcal{D}_{\Theta,Z}\varphi$ by $\mathcal{D}\varphi$ when the context is clear.

Application : Ward-Takahashi anomalies. (See also (3. VI.3)). It has often been expected that the symmetries of the generating functional $Z(J)$ are the same as the symmetries of the classical action $S(\varphi)$. Originally, the Ward-Takahashi identities expressed invariance properties of $Z(J)$ under transformations leaving $S(\varphi)$ invariant, assuming $\mathcal{D}\varphi$ (no reference to S or Z) to be invariant. When "anomalous" terms were found in the Ward-Takahashi identities, Fujikawa showed that, in the case of chiral transformations, the so called anomalous term was simply the determinant of the linear map $M : \varphi \mapsto M\varphi$, defining the chiral transformation. i.e.

$$(3.II_6) \qquad \mathcal{D}_{\Theta,Z}(M\varphi) = \operatorname{Det} M \cdot \mathcal{D}_{\Theta,Z}(\varphi) \neq \mathcal{D}_{\Theta,Z}(\varphi).$$

Pillar III : Norms $\| \ \|$ on $\mathbb{X}$, and norms $\| \ \|_A$ on the space $\mathcal{F}$ of integrable functionals over $\mathbb{X}$ with respect ot $\mathcal{D}_{\Theta,Z}$.

Sobolev norms on $\mathbb{X}$ are usually suitable [1]. For instance

$$(3.III_1) \qquad \mathbb{X} = W_2^1 \ , \ \ \mathbb{X}' = W_2^{-1}$$

where W_2^m, also labelled H^m, is the space of square integrable functions whose partial derivatives of order $\leq m$ (in the sense of distributions) are also square integrable. Its dual W_2^{-m} is the space of distributions made of derivatives of order $\leq m$ of square integrable functions.

In the case $\mathbb{X} = \mathbb{X}'$, Albeverio and Høegh-Krohn have proposed a space of integrable functionals which are Fourier transforms of bounded measures on $\mathbb{X}$. Adapting their suggestions to the integrators defined by the second pillar we consider (as a minimal choice) the space of integrable functionals to be the space $\mathcal{F}_{\Theta,Z}$ ($\mathcal{F}$ for Feynman) of functions F such that [2]

$$(3.III_2) \qquad F(x) = \int_{\mathbb{Y}} \Theta(x,\eta) d\mu(\eta), \quad F \in \mathcal{F}_{\Theta,Z} \quad \text{(abbreviated to } \mathcal{F}\text{)}$$

where μ is a bounded measure on the Banach space $\mathbb{Y}$, possibly complex. This equation does not imply that there is a one-to-one correspondence between μ and F. It does not mean either that, given F, one needs to identify μ in order to compute $\int_{\mathbb{X}} F(x)\mathcal{D}x$. It only means that we can write

$$(3.III_3) \qquad \int_{\mathbb{X}} F(x)\mathcal{D}x = \int_{\mathbb{X}} \mathcal{D}x \int_{\mathbb{Y}} \Theta(x,\eta) d\mu(\eta)$$

[1] For instance if one requires that the action of the system be finite.

[2] For simplicity we assume the functions Θ on $\mathbb{X} \times \mathbb{Y}$ and Z on $\mathbb{Y}$ to be bounded and continuous.

$$= \int_{\mathbb{Y}} d\mu(\eta) \int_{\mathbb{X}} \Theta(x,\eta)\mathcal{D}x = \int_{\mathbb{Y}} Z(\eta)d\mu(\eta)$$

and it suggests a norm on $\mathcal{F}$

$$(3.III_4) \qquad \|F\|_A := \min_{\mu} \int_{\mathbb{Y}} |Z(\eta)|\, d\,|\mu|\,(\eta).$$

Although μ is not necessarily defined by $(3.\mathrm{III}_2)$, we can prove in many cases that $\int_{\mathbb{X}} F(x)\mathcal{D}x$ is well defined: assume that there exists a family $\{\lambda_n\}$ of Borel measures [3] on $\mathbb{X}$ such that

$$(3.III_5) \qquad Z(\eta) = \lim_{n=\infty} \int_{\mathbb{X}} \Theta(x,\eta)d\lambda_n(x),$$

then

$$(3.III_6) \qquad \int_{\mathbb{Y}} Z(\eta)d\mu(\eta) = \lim_{n=\infty} \int_{\mathbb{X}} d\lambda_n(x) \int_{\mathbb{Y}} \Theta(x,\eta)d\mu(\eta)$$
$$= \lim_{n=\infty} \int_{\mathbb{X}} d\lambda_n(x)F(x).$$

Neither the measure μ in $(3.\mathrm{III}_2)$ nor the family $\{\lambda_n\}$ in $(3.\mathrm{III}_5)$ are uniquely defined. But $(3.\mathrm{III}_5)$ which is independent of the choice made for μ and $(3.\mathrm{III}_2)$ which is independent of the choice made for $\{\lambda_n\}$ lead to the equality $(3.\mathrm{III}_6)$.

Pillar IV : The axiomatic formulation of functional integrals summarized in the pillar II by $\int_{\mathbb{X}} \Theta(x,\eta)\mathcal{D}_{\Theta,Z}x = Z(\eta)$ would be nearly useless to physicists if it did not include indefinite quadratic forms. For instance, one may wish to compute $(3.\mathrm{III}_4)$ in terms of a Fresnel integrator, $\exp(-\pi Q(x))\mathcal{D}_Q x$ with $Q(x) = i\,\langle Dx, x\rangle$, where D is the Jacobi operator defined by the action S on $\mathbb{X}$. Therefore one needs an axiomatic framework which includes the d'Alembertian

$$(3.IV_1) \qquad D = \eta^{\mu\nu}\partial_\mu\partial_\nu.$$

[3] $\mathbb{X}$ is a separable complete metric space, hence Borel subsets are defined. The λ_n are not necessarily bounded but we assume that $|\lambda_n|(B)$ is finite for $B \subset \mathbb{X}$ bounded.

That is, one needs to redefine the norm $\|F\|_A = \int |Z(\eta)| d|\mu|(\eta)$ when $Z(\eta) = \exp(-\pi W(\eta))$ with W not positive definite. Following the Gupta-Bleuler strategy, we introduce an auxiliary quadratic form Q_a positive definite [4], $Q_a(x) > 0$ for $x \neq 0$, to define an auxiliary norm $\| \ \|_a$ on $\mathbb{X}$ which defines a norm on $\mathbb{X}'$ and a norm on $\mathcal{F}$.

This can be obtained by generalizing the well-known *Sylvester decomposition* of quadratic forms. Namely, we consider real-valued quadratic forms Q on infinite-dimensional space $\mathbb{X}$ with the following property:

There exists a decomposition $\mathbb{X} = \mathbb{X}_1 \oplus \mathbb{X}_2$ into a direct sum such that

$$Q(x) = Q_1(x_1) - Q_2(x_2) \tag{3.IV$_2$}$$

for $x = x_1 + x_2$, where x_1 is in $\mathbb{X}_1$ and x_2 in $\mathbb{X}_2$. Moreover Q_1 and Q_2 are positive definite quadratic froms, and define $\mathbb{X}_1$ and $\mathbb{X}_2$ as (real) Hilbert spaces.

This being so, put

$$Q_a(x) = Q_1(x_1) + Q_2(x_2) \tag{3.IV$_3$}$$

with x, x_1, x_2 as above. Then Q_a is a positive definite quadratic from on $\mathbb{X}$, hence the norm $\|x\|_a = Q_a(x)^{1/2}$ under which $\mathbb{X}$ is complete.

The decomposition mentioned in (3.IV$_2$) is not unique but two different auxiliary norms $\| \ \|_a$ and $\| \ \|_{a'}$ are related by inequalities of the form

$$\alpha \|x\|_{a'} \leq \|x\|_a \leq \beta \|x\|_{a'} \tag{3.IV$_4$}$$

(for some constants α, β in $\mathbb{R}^+$) hence either norm can be used on $\mathbb{X}$.

[4] The case of a semidefinite quadratic form is not excluded but requires special consideration. The case of a degenerate quadratic form has already been investigated in the context of prodistributions.

Pillar V : On the first pillar, let the pair $\mathbb{X}, \mathbb{X}'$ in the quadruplet (3.I_1),

$$\mathcal{S}(\mathbb{R}) \hookrightarrow \mathbb{X}' \underset{D}{\overset{G}{\rightleftarrows}} \mathbb{X} \hookrightarrow \mathcal{S}'(\mathbb{R}), \tag{3.V_1}$$

consist of Hilbert spaces and the quadratic form $Q(x) := \langle Dx, x \rangle$ be positive definite. Then $\mathbb{X}$ is of measure zero with respect to the white noise measure on $\mathcal{S}'(\mathbb{R})$. Moreover powerful theorems have been derived for the space of Hida distributions defined on $\mathcal{S}'(\mathbb{R})$. So why work with the pair $\mathbb{X}, \mathbb{X}'$? Two reasons :

- The structure (G, D) does not exist on the pair $\mathcal{S}(\mathbb{R}), \mathcal{S}'(\mathbb{R})$, indeed $G : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ does not have an inverse.

- We shall show that, if $Fe^{-\pi Q}$ is in the space $\mathcal{F}$ of Feynman integrable functions, then

$$\int_{\mathbb{X}} F(x) \exp(-\pi Q(x)) \mathcal{D}_Q x = \int_{\mathcal{S}'(\mathbb{R})} \widetilde{F}(X) dw_Q(X), \tag{3.V_2}$$

where the Fourier transform of dw_Q and the Fourier transform of $\exp(-\pi Q(x))\mathcal{D}_Q x$ evaluated at $\eta \in \mathcal{S}(\mathbb{R})$ are identical, and where $\widetilde{F}$ is the extension of F from $\mathbb{X}$ to $\mathcal{S}'(\mathbb{R})$ constructed as follows.

Let $\{e_i\}$ and $\{\varepsilon^i\}$ be two orthonormal bases on $\mathbb{X}$ and $\mathbb{X}'$, respectively, such that $\varepsilon^i \in \mathcal{S}(\mathbb{R})$; assume

$$De_i = \varepsilon^i \quad , \quad \langle e_i, \varepsilon^j \rangle = \delta_i^j. \tag{3.V_3}$$

Let us define

$$P_N : \mathcal{S}'(\mathbb{R}) \to \mathbb{X} \ \text{ by } \ P_N X = \sum_{i=1}^{N} \langle X, \varepsilon^i \rangle \cdot e_i; \tag{3.V_4}$$

then the limit

$$\widetilde{F}(X) := \lim_{N=\infty} F(P_N X) \tag{3.V_5}$$

exists a.s. for w_Q and defines the extension of F from $\mathbb{X}$ to $\mathcal{S}'(\mathbb{R})$.

The privileged extension of F to $\widetilde{F}$ is analogous to the extension of a continuous function defined on the rationals to a continuous function defined on $\mathbb{R}$.

We can summarize ($3.V_2$) and ($3.II_4$) in one formula

$$(3.V_6) \qquad \int_{\mathbb{X}} F(x) \exp\left(-2\pi i \langle \eta, x \rangle\right) \exp(-\pi Q(x)) \mathcal{D}_Q x$$

$$= \int_{\mathcal{S}'(\mathbb{R})} \widetilde{F}(X) \exp\left(-2\pi i \langle \eta, X \rangle\right) dw_Q(X),$$

for $\eta \in \mathcal{S}(\mathbb{R})$.

Pillar VI : Expectation values of operators. Functional integrals can be used to compute expectation values of "time ordered" products of operators. When the Cartier Bridge is completed, pillar VI will present functional integration in quantum physics as a generalisation of the relationship between the Schrödinger and the Heisenberg quantizations. It will give another justification of the well known formulae :

$$(3.VI_1) \qquad \int_{\mathbb{X}_{ba}} \exp\left(\frac{i}{\hbar}\left(S(\varphi) + \hbar \langle J, \varphi \rangle\right)\right) \mathcal{D}_{\Theta,Z}(\varphi)$$

$$(3.VI_2) \qquad \begin{aligned} &= \left\langle \Psi_b \middle| \mathcal{T} \exp\left(-\tfrac{i}{\hbar}\textstyle\int_{t_a}^{t_b} H\,dt\right) \middle| \Psi_a \right\rangle && \text{(Dirac bracket)} \\ &=: \langle \Psi_b, t_b | \Psi_a, t_a \rangle_J && \text{(Feynman bracket)} \end{aligned}$$

and

$$(3.VI_3) \qquad \int_{\mathbb{X}_{ba}} F(\varphi) \exp\left(\frac{i}{\hbar}\left(S(\varphi) + \hbar \langle J, \varphi \rangle\right)\right) \mathcal{D}_{\Theta,Z}(\varphi) = \langle \Psi_b, t_b | \mathcal{T} F(\varphi) | \Psi_a, t_a \rangle_J$$

with the following notation

- $\mathbb{X}_{ba}$ is the space of paths [or histories] with given values at times t_a and t_b corresponding respectively to the states $|\Psi_a\rangle$ and $|\Psi_b\rangle$

- H is the hamiltonian corresponding to the action $S(\varphi) + \hbar \langle J, \varphi \rangle$.

- $\mathcal{T}$ is a map from the space $\mathcal{F}$ of bounded functionals on $\mathbb{X}_{b,a}$ to the space of bounded operators on the Hilbert space of states of the system,

$$(3.VI_4) \qquad \mathcal{T} : \mathcal{F} \to \mathcal{B}(\mathcal{H})$$

which time-orders products of operators. For instance, in quantum mechanics,

$$(3.VI_5) \qquad \mathcal{T}\varphi_1(t)\varphi_2(s) = \theta(t-s)\widehat{\varphi}_1(t)\widehat{\varphi}_2(s) + \theta(s-t)\widehat{\varphi}_2(s)\widehat{\varphi}_1(t)$$

where $\theta(t-s) = 1$ for $t > s$, $\theta(t-s) = 0$ for $t < s$, $\theta(t-s)$ undefined for $t = s$.

The time ordering operator $\mathcal{T}$ on $\mathcal{F}$ is defined by the functional integral (3.VI$_3$) , and not by equations such as (3.VI$_5$) which can be ambiguous at equal times. Therefore, time ordering commutes with differentiation

$$(3.VI_6) \qquad \frac{d}{dt}\langle\, |\mathcal{T}\varphi(t)\varphi(s)|\,\rangle = \left\langle \left|\mathcal{T}\frac{d\varphi}{dt}\varphi(s)\right| \right\rangle$$

or in field theory

$$(3.VI_7) \qquad \frac{\partial}{\partial x^\mu}\langle\, |\mathcal{T}\varphi(x)\varphi(y)|\,\rangle = \left\langle \left|\mathcal{T}\frac{\partial}{\partial x^\mu}\varphi(x)\varphi(y)\right| \right\rangle$$

Application : Quantum Nœther's theorems.

Nœther's theorems apply to classical currents, i.e. currents which are functional of fields satisfying the Euler-Lagrange equations. There is no reason to expect that they apply to expectation values of time ordered product of currents : the left hand side of (3.VI$_3$) is, for φ a field, an integral over *all* fields with given values on an initial and a final spacelike surface. The Nœther theorems are the classical limits of "quantum Nœther's theorems" which are consequences of (3.VI$_3$) : Indeed, under a change of variable of integration $\varphi \mapsto \psi$, it follows from the defining equation of the integration $\mathcal{D}_{\Theta,Z}$ that

$$(3.VI_8) \qquad 0 = \int \Theta(\varphi, J)\mathcal{D}_{\Theta,Z}(\varphi) - \int \Theta(\psi, J)\mathcal{D}_{\Theta,Z}(\psi).$$

In case of a linear map, $\psi = M\varphi$, it follows from (3.II$_6$) that

$$(3.VI_9) \qquad 0 = \int ((\mathrm{Det}\ \mathbb{1})\Theta(\varphi, J) - (\mathrm{Det}\ M)\Theta(M\varphi, J))\, \mathcal{D}_{\Theta,Z}(\varphi),$$

and, if $\Theta(\varphi, J) = \exp\left(\frac{i}{\hbar}(S(\varphi) + \hbar \langle J, \varphi\rangle)\right)$,

$$(3.VI_{10}) \qquad 0 = \int \Bigg(\exp\left(\frac{i}{\hbar}S(\varphi) + i\langle J, \varphi\rangle + \text{trace } \ln \mathbb{1}\right) - \exp\left(\frac{i}{\hbar}S(M\varphi) + i\langle J, M\varphi\rangle + \text{trace } \ln M\right)\Bigg)\mathcal{D}_{\Theta,Z}(\varphi).$$

Consider the linear map

$$(3.VI_{11}) \qquad (M\varphi)(x) = \varphi(x) + \theta(x)\varphi(x),$$

where θ is a continuous map on the domain of φ ; it follows from (3.VI$_{10}$) that

$$(3.VI_{12}) \qquad 0 = \int \exp\left(\frac{i}{\hbar}S(\varphi) + i\langle J, \varphi\rangle\right) \frac{\delta}{\delta\theta}\left(\frac{i}{\hbar}S(M\varphi) + i\langle J, M\varphi\rangle + \text{trace } \ln M\right)\Bigg|_{\theta=0} \mathcal{D}_{\Theta,Z}(\varphi).$$

In terms of time ordered product of operators (3.VI$_3$), this equation says

$$(3.VI_{13}) \qquad 0 = \left\langle \mathcal{T}\frac{\delta}{\delta\theta}\left(\frac{i}{\hbar}S(M\varphi) + i\langle J, M\varphi\rangle + \text{trace } \ln M\right)\right\rangle_J .$$

Repeated functional derivatives of (3.VI$_{13}$) with respect to J give the correct Ward-Takahashi identities. At $J = 0$ equation (3.VI$_{13}$) gives

$$(3.VI_{14}) \qquad \left\langle \mathcal{T}\frac{\delta}{\delta\theta}\left(\frac{i}{\hbar}S(M\varphi)\right)\right\rangle = -\frac{\delta}{\delta\theta}\text{trace } \ln M.$$

Trace ln M is known as the anomaly function - but should not be unexpected.

The action $S(\varphi) = \int L(\varphi, \varphi_{,\mu})dx$, therefore it follows from (3.VI$_{14}$) that

$$(3.VI_{15}) \qquad \left\langle \mathcal{T} \int \left(\frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial \varphi_{,\mu}} \right) \right) \delta\varphi + \frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial \varphi_{,\mu}} \delta\varphi \right) dx \right\rangle$$

$$= -i\hbar \text{ trace } \ln M.$$

Nœther's theorem says that, the current density

$$j^\mu := \left. \frac{\partial L}{\partial \varphi_{,\mu}} \delta\varphi \right|_{\varphi = \varphi_{\text{cl}}}$$

evaluated at solutions φ_{cl} of the Euler-Lagrange equation satisfies the equation

$$j^\mu{}_{,\mu} = 0$$

for actions invariant under the transformation $\varphi \mapsto \varphi + \delta\varphi$. Even if we assume (see pillar VII) that

$$\left\langle \mathcal{T} \int \left(\frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial \varphi_{,\mu}} \right) \right) dx \right\rangle = 0,$$

equation (3.VI$_{15}$) says only that

$$\lim_{\hbar=0} \left\langle \mathcal{T} \int j^\mu{}_{,\mu}(x)dx \right\rangle = 0.$$

Pillar VII : Renormalization.

We applied to two cases the defining equation of the integrator $\mathcal{D}_{\Theta,Z}$

$$(3.VII_1) \qquad \int_{\mathbb{X}} \Theta(\varphi, J)\mathcal{D}_{\Theta,Z}\varphi = Z(J).$$

a) The gaussian case:

$$(3.VII_2) \qquad \begin{cases} \Theta(\varphi, J) = \exp(-\pi Q(\varphi) - 2\pi i \langle J, \varphi \rangle) \\ Z(J)/Z(0) = \exp(-\pi W_Q(J)) \end{cases}$$

where the second functional derivative of Q is the inverse of the second functional derivative of W_Q :

$$(3.VII_3) \qquad \frac{\delta^2 Q(\varphi)}{\delta\varphi^2}\frac{\delta^2 W_Q(J)}{\delta J^2} = DG = \mathbb{1}.$$

b) The quantum case:

$$(3.VII_4) \qquad \begin{cases} \Theta(\varphi, J) = \exp\left(\frac{i}{\hbar}(S(\varphi) + \hbar\,\langle J, \varphi\rangle)\right) \\ Z(J)/Z(0) =: \exp\left(\frac{i}{\hbar}W(J)\right). \end{cases}$$

We shall now analyze the relationship between the action $S(\varphi)$, the generating functional $Z(J)/Z(0)$ and the integrator $\mathcal{D}_{\Theta,Z}$. The generating functional encodes the effect of the interaction stated by the action $S(\varphi) + \hbar\,\langle J, \varphi\rangle$ where φ may be an abbreviation for several interacting fields, or a self-interacting field. Experiments measure $Z(J)$ directly or indirectly. In this sense we refer to $Z(J)$ as an experimental quantity. The effect of the interactions can, in a number of cases, show up as a change in the constants used to describe the non interacting fields. One says that the (bare) constants have been renormalized (by the interactions). The ratio of the constants before and after the interactions may be infinite if we model the interactions by local actions. Various regularization techniques can be used to evaluate them; important as they are in checking the theory against experiments, they should not be given logical precedence over renormalization.

We cannot, a priori, choose the experimental generating functional, but we can assume the following property,

$$(3.VII_5) \qquad \int_{\mathbb{X}} \frac{\delta}{\delta\varphi}\left(\exp\left(\frac{i}{\hbar}\left(S(\varphi) + \hbar\,\langle J, \varphi\rangle\right)\right)\right)\mathcal{D}_{\Theta,Z}\varphi = 0,$$

and derive its consequences. From (3.VII$_5$) and (3. VI$_3$), it follows that

$$(3.VII_6) \qquad \left\langle \text{out}\left|\mathcal{T}\frac{1}{\hbar}\frac{\delta S}{\delta\varphi}\right|\text{in}\right\rangle_J = -J$$

where the $|\text{in}\rangle$ and $|\text{out}\rangle$ state are the ones encoded in $\mathbb{X}$. On the other hand, having in mind an equation reminiscent of (3.VI$_3$), we note that

$$(3.VII_7) \qquad \frac{1}{\hbar}\frac{\delta W(J)}{\delta J} = \int \varphi \Theta(\varphi, J) \mathcal{D}_{\Theta,Z}\varphi = \langle \text{out} \, |\varphi| \, \text{in} \rangle_J =: \overline{\varphi}.$$

Hence the Legendre transform $\Gamma(\overline{\varphi})$ of $W(J)$,

$$(3.VII_8) \qquad \Gamma(\overline{\varphi}) = W(J) - \hbar \langle J, \overline{\varphi} \rangle$$

known as the *effective action*, is such that

$$(3.VII_9) \qquad \frac{1}{\hbar}\frac{\delta \Gamma}{\delta \overline{\varphi}} = -J;$$

and

$$(3.VI_{10}) \qquad \left\langle \text{out} \left| \mathcal{T} \frac{\delta S}{\delta \varphi} \right| \text{in} \right\rangle_J = \frac{\delta \Gamma}{\delta \overline{\varphi}}.$$

Moreover, the second functional derivative of the effective action is the inverse of the second functional derivative of W

$$(3.VII_{11}) \qquad \frac{\delta^2 \Gamma(\overline{\varphi})}{\hbar \delta \overline{\varphi}^2} \frac{\delta^2 W(J)}{\hbar \delta J^2} = \mathbb{1}.$$

This equation can be compared with (3.VII$_3$). If the effective action $\Gamma(\overline{\varphi})$ were equal to the action $S(\varphi)$, the quantum case would be a gaussian case. Therefore we can consider the gaussian case as a non interacting quantum case, since it is a case where effective action and bare action are identical.

We note also the following consequence of (3.VII$_{4,8,9}$) :

$$(3.VII_{12}) \qquad \exp\left(\frac{i}{\hbar}\Gamma(\overline{\varphi})\right) = \frac{1}{Z(0)} \int \exp\left(\frac{i}{\hbar}\left(S(\varphi) + \left\langle \frac{\delta \Gamma(\overline{\varphi})}{\delta \overline{\varphi}}, \overline{\varphi} - \varphi \right\rangle\right)\right) \mathcal{D}_{\Theta,Z}\varphi.$$

4. - Conclusion

Given a space $\mathbb{X}$ of paths $x : [t_a, t_b] \to \mathbb{M}^d$, one can either work with

$$x \in \mathbb{X} \qquad \text{or with} \quad \{x(t_1), \cdots, x(t_N)\} \in \mathbb{M}^d \times \cdots \times \mathbb{M}^d \quad (N \text{ factors}).$$

Working with $\mathbb{X}$ is admittedly more delicate than working with $\mathbb{M}^d$, but much simpler than working on $\lim_{N=\infty} \mathbb{M}^d \times \cdots \times \mathbb{M}^d$ (N factors). To mention but a few examples:

- An integrator is simpler than the family of its finite dimensional distributions.
- A linear change of variable in $\mathbb{X}$ is simpler than its discretized version on $\mathbb{M}^d \times \mathbb{M}^d \times \cdots \times \mathbb{M}^d$.
- An integration by parts is simpler on $\mathbb{X}$ than on $\mathbb{M}^d \times \cdots \times \mathbb{M}^d$ etc ...

The progress accomplished in recent years can be traced to the shift from $\mathbb{M}^d \times \mathbb{M}^d \times \cdots \times \mathbb{M}^d$ (arbitrary number of factors) to $\mathbb{X}$. This can be seen, not only in the work reported here, but in the major progress due to P. Krée, P. Malliavin, P.A. Meyer, the White Noise team, in particular, T. Hida, H.-H. Kuo, L. Streit, M. de Faria, J. Pottshoff and Khandekar.

Functional integrals are more than solutions of partial differential equations with a chosen set of boundary conditions. In particular, they provide information on the global properties of $\mathbb{M}^d$ the target space of the paths. They provide representations of expectation values of time ordered products of operators. Pierre Cartier and I plan to test the proposed axiomatic on a variety of problems to make sure that it covers, at least, the known applications of functional integrals.

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6. The Cartier bridge The construction called here the "Cartier Bridge" will be published in a Note aux Compte-Rendus de l'Académie des Sciences by P. Cartier and C. DeWitt-Morette. In the years to come the authors will write a book on functional integration and test the proposed axiomatic on many problems of physical interest.

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ACKNOWLEDGMENTS

The recent developments reported here have been worked out during visits of Cécile DeWitt-Morette and Pierre Cartier at several institutions. Cécile DeWitt-Morette at the Institut des Hautes Etudes Scientifiques at Bures-sur-Yvette, at the Département de Mathématique et d'Informatique de l'Ecole Normale Supérieure à Paris, and at the Universidade da Madeira in Funchal ; Pierre Cartier at the Center for Relativity of the University of Texas at Austin. Both of us have appreciated the excellent working conditions and the help of colleagues and staff which have made our collaboration fruitful.

Our collaboration has been made possible by the NATO collaborative Research Grant CRG 910101.