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Algebraic superconnections, $SU(2|1)$ and electroweak interactions (*)

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Abstract : We describe a formalism using both ideas of non commutative geometry and of Lie super-algebras to describe the geometry of Yang Mills fields and symmetry breaking

Keywords : Standard Model, Higgs fields, Lie Superalgebras, Non Commutative Geometry.

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Introduction

The purpose of this contribution is to present a new mathematical description of the phenomenon of symmetry breaking and Higgs mechanism in Physics. The traditional description uses differential geometry and involves the reduction of a principal bundle with structure group G to a sub-bundle with structure group $H \in G$. The Higgs field responsible for the reduction is related to a global section of a fiber bundle with fibers diffeomorphic with the homogeneous space G/H. This is not the road that we are going to follow: we shall describe the Higgs field as part of a "generalized connection". Of course, the very definition of what is meant by connection, in this wider sense, should be precised in the sequel. Conceptually, this description can be related to the study of symmetries of Yang-Mills on spaces that can be written locally as $M \times F$. When M and F are smooth manifolds, tools of usual differential geometry suffice. When this is not the case, for instance when F is chosen as a discrete set, tools of non-commutative geometry become relevant. One important ingredient is then the replacement of the commutative algebra of functions on Space(-Time) by a non-commutative algebra which, in the simple example that we are going to study, is just a matrix algebra of functions. Another conceptual change involves the replacement of the Dirac operator coupled to a Yang-Mills field by the Yukawa operator -interpreted as a mass operator- coupled to a Higgs field. It is rather surprising to notice that the "old" Yukawa operator provides the simplest non trivial example of a Dirac operator in non commutative geometry.

The formalism that we are going to present was, at the beginning, strongly influenced by A. Connes ideas and in particular by the article [1]. However, evolution follows different paths. The relation between the approach that will be described below and, for example, the ideas found in the book [2] are still to be clarified. A simple presentation of our ideas can be found in [3] and many comments and extensions are discussed in [4] (see also [5]). Because the mathematical tools are new, we will refrain from adopting a dogmatic style. This means that we shall only describe what we do and what we get in particular examples leaving to others (and to to the future) the work of inventing general mathematical definitions that would allow the calculations to fall in a precisely described category. Often, modelisation of physical phenomena lead to emergence of new branches of mathematics. But, most of the time, this process evolves from the study of concrete examples to the definition of an abstract framework, not the converse.

A new mathematical description of the phenomenon of symmetry breaking leads, *ipso facto*, to an alternative description of the standard model of electroweak interactions. Since it is a new description, not a new theory, physical consequences, numerical constraints *etc.* are the same as usual. The first advantage is conceptual. Indeed, many "theoretical inputs" of the standard model, that usually appear as quite artificial (for instance the mere existence of Higgs fields and the description of their self interaction via a fourth degree polynomial) can be "deduced" from the new mathematical structure. Another advantage is that a new formalism often leads to the expression of new (and useful) physical hypothesis. It is indeed obvious that a relation between physical quantities may look quite natural in one formalism and quite unnatural in another one. The full description of the Standard Model involves 18 parameters (and the minimal -and very natural- extension of the model incorporating right neutrinos involves 24 parameters). These parameters are independent and may be chosen at will. One should not forget that these parameters have also to be renormalized since one should not stop at the classical level but study the corresponding fully interacting quantum field theory. Any numerical constraint (even totally unrealistic from the physical point of view) between these independent parameters can be implemented at the quantum level and lead to a decent quantum field theory. Therfore there is no hope of deducing these values from perturbative quantum field theory alone. The value of these parameters are therefore taken from experiment (in those cases where experiment is precise enough .... This is of course rather sad. One possible belief is that a more fundamental theory (like superstrings) will allow one, someday to compute them. Another possible belief is that the collection of these (renormalized) parameters can be somehow guessed, in the sense that this collection of numbers would correspond to a precise "geometry" of Space-Time at small scale. Which kind of geometry to postulate is of course totally unknown, yet, but our hope is that the present description of symmetry breaking and the corresponding algebraico-geometrical interpretation of the Yukawa operator will be useful in this quest for uncovering the veil under which what we call "Nature" hides herself.

The $U(1) \times U(1)$ example

The Dirac-Yukawa operator

In the simplest non-trivial case, the Dirac-Yukawa operator coupled to a generalized connection and acting on the field Ψ will be written as

$$
\begin{aligned}
\not{\mathcal{D}} &= \begin{pmatrix} \not\partial & \mu \\ \mu & \not\partial \end{pmatrix} + g \begin{pmatrix} i\gamma^\mu L_\mu & \phi/\sqrt{2} \\ \overline{\phi}/\sqrt{2} & i\gamma^\mu R_\mu \end{pmatrix} \\
\Psi &= \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} (\frac{1-\gamma_5}{2})\psi \\ (\frac{1+\gamma_5}{2})\psi \end{pmatrix}
\end{aligned}
\tag{1}
$$

Here $\not\partial = \gamma^\mu \partial/\partial x^\mu$ denotes the free (and massless) Dirac operator defined in flat Minkowski space, μ is an arbitrary positive real number (to be interpreted as a mass), g is an arbitrary real number (to be interpreted as a coupling constant), ϕ is a complex scalar field, L_μ and R_μ are the components of two 1-forms (to be interpreted in terms of $U(1)$-connections fields). Also, γ_5 denotes the chirality operator of the complexified Clifford algebra (the dimension of the underlying manifold is indeed even), ψ_L and ψ_R are half spinors and $\psi = \psi_L + \psi_R$ is a Dirac spinor.

The fermionic lagrangian may be defined as

$$
\begin{aligned}
\mathcal{L}_{Fermion} &= \overline{\Psi}\not{\mathcal{D}}\,\Psi \\
&= \overline{\psi}\gamma^\mu\partial_\mu\psi + i\frac{g}{\sqrt{2}}\overline{\psi}\not{P}\psi - i\frac{g}{\sqrt{2}}\overline{\psi}\gamma_5\not{Z}\psi + m\overline{\psi}\psi + \frac{g}{\sqrt{2}}(\overline{\psi}_L\phi\psi_R + \overline{\psi}_R\overline{\phi}\psi_L)
\end{aligned}
\tag{2}
$$

This describes one massive Dirac fermion of mass $m = \mu$ with two bosonic fields $P = \frac{(L+R)}{\sqrt{2}}$ and $Z = \frac{(L-R)}{\sqrt{2}}$. The mixing angle that comes out naturally is equal to $\pi/4$. If one sets

$\phi' = \mu + \phi$ and express $\mathcal{L}_{Fermion}$ in terms of ϕ', the massive term $m\overline{\psi}\psi$ disappears. By rescaling μ and ϕ, *i.e.* , by replacing them by $k\mu$ and $k\phi$, with $k \in R^+$, we would get a mass term with $m = k\mu$ and a coefficient in front of the Yukawa interaction term equal to $\frac{gk}{\sqrt{2}} = \frac{gm}{\mu\sqrt{2}}$ (so that it appears as proportionnal to the fermionic mass). We could have also introduced two coupling constants g_1 and g_2 (one for L_μ and one for R_μ). In this case, the mixing angle between the two fields would have been arbitrary. Notice that the constant μ in the expression of the Dirac-Yukawa operator appears as a discrete analogue of $\not\partial$ and that the scalar field ϕ appears as a discrete analogue of the gauge fields L_μ and R_μ. Therefore, the second term in the expression of the Dirac-Yukawa operator appears as expressiong the coupling of Ψ to a generalized connection incorporating scalar fields (0-forms) and gauge fields (1-forms). Let us consider the associative algebra $\mathcal{C}$ of 2×2 matrices generated by $\Omega_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\Omega_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. We call $I = \Omega_+\Omega_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Y = \Omega_-\Omega_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This associative algebra appears as a kind of discrete analogue of the Clifford algebra since the second term of the Dirac-Yukawa operator can be considered as an element of the tensor product of two associative algebras: the usual Clifford algebra and $\mathcal{C}$. Matrices Ω_+ and Ω_- are then an analogue of the γ^μ. This associative algebra $\mathcal{C}$ is also Z_2 graded (the Ω's being odd by definition). Of course, it is enough to know the odd part to know the whole algebra. Whenever one has a Z_2-graded associative algebra, one may construct the corresponding Lie super-algebra by using graded commutators. In the present case, the situation is slightly too simple to exhibit generic features but it is clear that we get in this way is the Lie super-algebra $SU(1|1)$. This emergence of a Lie super-algebra comes only from the fact that we have an associative Z_2-graded algebra to start with and has nothing to do with supersymmetries (and we shall never try to gauge a Lie super-algebra).

The generalized connection

Rather than working with the Clifford algebra, we define the generalized connection in terms of 0-forms and 1-form as the antihermitian matrix

$$\mathcal{A} = \begin{pmatrix} L & i\mu^{-1}\phi \\ i\mu^{-1}\overline{\phi} & R \end{pmatrix} \tag{3}$$

Here L and R are dimensionless one-forms so that L_μ and R_μ defined by $L = L_\mu dx^\mu$ and $R = R_\mu dx^\mu$ have dimension of a mass. $\mathcal{A}$ is an element of a Z_2-graded associative and differential algebra that is constructed as the graded tensor product of two graded associative and differential algebras. The first is the algebra of 2×2 complex matrices. Its Z_2-grading is defined as follows: even elements are diagonal matrices. The differential of a matrix a is defined as $da = i[\eta, a]_S$, where $[.,.]_S$ denotes the graded commutator and where $\eta = \cos\gamma\,\tau_1 + \sin\gamma\,\tau_2$. Here γ denotes an arbitrary phase factor and $\tau_{1,2}$ are Pauli matrices. For instance, chosing $\eta = \tau_1$ leads to

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, da = i \begin{pmatrix} a_{21} + a_{12} & a_{22} - a_{11} \\ a_{11} - a_{22} & a_{21} + a_{12} \end{pmatrix} \tag{4}$$

It is easy to check that d is a graded derivation. The second graded differential associative algebra (that is even graded commutative in this case) is the algebra of differential forms. When x is a homogeneous element in a Z_2-graded algebra –*i.e.* when its grading is well defined– we call ∂x its Z_2-grading. The graded tensor product is defined as

$$(a \otimes B) \odot (a' \otimes B') = (-1)^{\partial B \partial a'} (a \cdot a') \otimes (B \wedge B') \tag{5}$$

and the differential as

$$d(a \otimes B) = da \otimes B + (-1)^{\partial a} a \otimes dB \tag{6}$$

For arbitrary matrices X and X' of differerential forms,

$$X = \begin{pmatrix} A & C \\ D & B \end{pmatrix}, X' = \begin{pmatrix} A' & C' \\ D' & B' \end{pmatrix} \tag{7}$$

one gets

$$X \odot X' = \begin{pmatrix} A \wedge A' + (-1)^{\partial C} C \wedge D' & C \wedge B' + (-1)^{\partial A} A \wedge C' \\ D \wedge A' + (-1)^{\partial B} B \wedge D' & B \wedge B' + (-1)^{\partial D} D \wedge C' \end{pmatrix} \tag{8}$$

and

$$dX = \begin{pmatrix} dA + i(e^{i\gamma} C + e^{-i\gamma} D) & -dC - ie^{-i\gamma}(A - B) \\ -dD + ie^{i\gamma}(A - B) & dB + i(e^{i\gamma} C + e^{-i\gamma} D) \end{pmatrix} \tag{9}$$

It is easy to check that d is a graded derivation (for the total Z_2 grading). For more details, cf [3].

Generalized curvature and bosonic lagrangian

From the expression of the generalized connection $\mathcal{A}$ given previously and from the covariant derivative $\nabla = d + \mathcal{A}$, one gets $\mathcal{F} = \nabla^2 = d\mathcal{A} + \mathcal{A} \odot \mathcal{A}$. Explicitely, one finds

$$\begin{aligned} \mathcal{F}_{11} &= F^L - \mu^{-2}(\mu(e^{i\gamma}\phi + \overline{e^{i\gamma}\phi}) + \phi\overline{\phi}) \\ \mathcal{F}_{12} &= -i\mu^{-1}(\nabla\phi + \mu e^{-i\gamma}(L - R)) \\ \mathcal{F}_{21} &= -i\mu^{-1}(\nabla\overline{\phi} - \mu e^{i\gamma}(L - R)) \\ \mathcal{F}_{22} &= F^R - \mu^{-2}(\mu(e^{i\gamma}\phi + \overline{e^{i\gamma}\phi}) + \overline{\phi}\phi) \end{aligned} \tag{10}$$

with

$$\begin{aligned} \nabla\phi &= d\phi + L\phi - \phi R \\ \nabla\overline{\phi} &= \overline{\nabla\phi} = d\overline{\phi} - \overline{\phi} L + R\overline{\phi} \end{aligned} \tag{11}$$

The Yang-Mills action itself is defined as

$$\mathcal{L} = ||\mathcal{F}||^2 \doteq Tr < \overline{\mathcal{F}}, \mathcal{F} >= ||\mathcal{F}_{11}||^2 + ||\mathcal{F}_{12}||^2 + ||\mathcal{F}_{21}||^2 + ||\mathcal{F}_{22}||^2 \tag{12}$$

where $\overline{\mathcal{F}}$ denotes the hermitian conjugate of $\mathcal{F}$. The symbol $< .,. >$ refers to a global scalar product in the exterior algebra. Clearly p-forms and q-forms are orthogonal whenever $p \neq q$ and the scalar product in teh space of 1-forms is directly defined *via* the space-time metric.

However one is free to introduce unrelated scaling factors r_0, r_1 and r_2 in the definition of scalar products of 0-forms, 1-forms, and 2-forms. The factor r_2 can be reabsorbed in a global rescaling of the bosonic lagrangian and one gets

$$\mathcal{L} = -\frac{1}{4}\left((F^L_{\mu\nu})^2 + (F^R_{\mu\nu})^2\right) + 2r_1^2\overline{D_\nu\phi}D^\nu\phi + 2r_0^2(\mu(e^{i\gamma}\phi + \overline{e^{i\gamma}\phi}) + \phi\overline{\phi})^2 \tag{13}$$

with

$$\begin{aligned} D_\nu\phi &= \nabla_\nu\phi + \mu e^{-i\gamma}(L_\nu - R_\nu) \\ D_\nu\overline{\phi} &= \overline{D_\nu\phi} = \nabla_\nu\overline{\phi} - \mu e^{i\gamma}(L_\nu - R_\nu) \end{aligned} \tag{14}$$

It is convenient to introduce coupling constant $\mathcal{L} = 1/g^2\,\|\mathcal{F}\|^2$ and to rescale the Yang-Mills fields by setting $iL = L^{old}/g$ and $iR = R^{old}/g$. We introduce a factor i so that L and R are hermitian and also rescale the scalar field in order to get a conventional kinetic energy term in the Lagrangian. We therefore obtain a $U(1)\times U(1)$ Yang-Mills action with a symmetry breaking Higgs potential. The potential is already shifted onto an absolute minimum, no further shift is necessary. As it is clear from the above expressions, the freedom of choice for γ in the definition of the derivation d on the algebra of 2×2 matrices amounts to choose the position of the vacuum (the origin) on the circle of minima of the potential. The gauge field $L-R$ becomes massive since a term $(L-R)^2$ appears in the Lagrangian. To get a kinetic term that is diagonal in the dynamical variables, one redefines the fields as follows

$$Z = (R-L)/\sqrt{2}\ P = (L+R)/\sqrt{2} \tag{15}$$

Because of the freedom in the definition of the scalar product in the space of differential forms (constants r_0, r_1), the mass of the Higgs particle associated with the field ϕ and the mass of the Z are not related (and are not related with the mass of the lepton). However, one can play with the idea of imposing a "natural" scalar product ... Also, the back-shifted lagrangian is unvariant under $U(1)\times U(1)$ and gauge freedom alone allows one to rescale differently the gauge fields, *i.e.* to introduce two independent coupling constants, thus destroying the value $\theta = \pi/4$ found for the mixing angle. This is however quite unnatural, considering the way we obtained the lagrangian.

Generalizations

The approach followed in the previous section can be carried out as soon as we have a Z_2-graded differential associative algebra to start with. One can then, in turn, build the corresponding Lie super-algebra generated by graded commutators. The previous construction rests on the structure given by an associative algebra and the corresponding Lie super-algebra is only a by-product. However, for model building purposes, it is convenient to start from the data given by a Lie super-algebra and one of its representations. The reason is that finite dimensional Lie super-algebras have been classified and their representations are known. Our building receipe can therefore be described as follows. 1) Choose a Lie super-algebra 2) Choose a representation (reducible or not). 3) Consider the odd generators in this representation (call them Ω) and build the *associative* algebra generated by these Ω's. 4) This algebra is Z_2 graded and one can give it the structure of a graded differential algebra by mimicking the construction explained previously. If its dimension is even ($2N \times 2N$ matrices) one can can consider it has a 2×2 matrix whose elements are blocks. If its dimension is odd, one can embedd it in even dimension by adding a line and a raw of zeros. The definition of the Z_2-grading and of the differential then exactly follows the description given in the last section (provided we think in terms of 2×2 block-matrices). If the dimension is even, one defines the covariant differential as $\nabla = d + \mathcal{A}$ and gets, as before, the curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A} \odot \mathcal{A}$. If the dimension is odd, the problem is that the d of a matrix that has a last line and last column filled with zeros does not have the same structure. One has then to define the covariant derivative as $\nabla = p\, d + \mathcal{A}$ where p is the projector that projects back onto the space of matrices of interest (we suppose that $\mathcal{A} = p\,\mathcal{A}\,p$. For instance, in the case of 3×3 matrices embedded into 4×4 matrices, one takes $p = diag(1,1,1,0)$. Then $\mathcal{F} = \nabla^2$ is not given by the previous equation but by $\mathcal{F} = d\mathcal{A} + \mathcal{A} \odot \mathcal{A} + p\, dp\, dp$. This is similar to what happens when one performs differential geometric calculations in a smooth manifold while using a description that is not intrinsic but that uses explicitely some kind of embedding. In all cases anyway, one gets as structure group the group whose Lie algebra corresponds to the even part of the Lie super-algebra we started with. The difference between the choice of one representation or another is reflected in the pattern of quantum numbers and mixing angles that emerges.

The $SU(2) \times U(1)$ example

The Lagrangian describing electroweak interactions (the standard model) can be recovered by following the previous method. One may start with the Lie super-algebra $SU(2|1)$. Its use in weak interactions was advocated long ago (*cf.* [6-8]) but its meaning was not correctly recognized since many physicists tried (without succes) to gauge it. Needless to say, this is of course not what we intend to do. In all cases, (*i.e.* for different choices of -graded- representations), the emerging bosonic lagrangian will be the same and is the Lagrangian of the Standard Model (*cf.* [4] for details). The only gauge invariance of the theory is described by the group $SU(2) \times U(1)$. In all cases the graded sum of weak hypercharges vanishes (this means physically that the average electric charge of left handed particles is equal to the average electric charge of right handed particles). Let us consider the associative algebra $\mathcal{C}$. It is generated by four elements that we call $\Omega_\pm$ and $\Omega'_\pm$. We set

$$\{\Omega_\pm, \Omega_\pm\} = \{\Omega_\pm, \Omega_\mp\} = \{\Omega'_\pm, \Omega'_\pm\} = \{\Omega'_\pm, \Omega'_\mp\} = 0 \tag{16}$$

and call

$$\begin{aligned} Y &= \{\Omega_+, \Omega'_-\} + \{\Omega_-, \Omega'_+\} \\ 2I_3 &= \{\Omega_+, \Omega'_-\} - \{\Omega_-, \Omega'_+\} \\ \sqrt{2} I_\pm &= \{\Omega_\pm, \Omega'_\pm\} \end{aligned} \tag{17}$$

We also set $Q = \{\Omega_+, \Omega'_-\} = I_3 + Y/2$. The generator Y is called weak hypercharge and Q is the electic charge. In all the cases that we consider next (except in the last paragraph) the Lie super-algebra generated by $\Omega_\pm$, $\Omega'_\pm$, $I_\pm$, I_3 and Y is isomorphic with $SU(2|1)$. The associative algebra generated by the Ω's is of course bigger. It may be of interest to notice that, in any representation of this Lie super-algebra, the Ω's satisfy two non trivial polynomial relations: the first is of degree 4 and the other of degree 6. These two relations are related to the existence of Casimir operators [9] of degree 2 and 3 in $SU(2|1)$. Being of degree 2 or 3 in the generators of the Lie super-algebra means indeed being of degree 4 or 6 in terms of the matrices Ω since even generators of $SU(2|1)$ are themselves expressed as expressions of degree two in terms of the odd ones.

Elementary leptons.

The Ω matrices are described by

$$P\begin{pmatrix} \mathbf{0}_{2\times 2} & \begin{pmatrix} \Omega'_- \\ -\Omega'_+ \end{pmatrix} \\ (\,\Omega_+ \quad \Omega_-\,) & 0 \end{pmatrix} P^{-1} = \begin{pmatrix} \mathbf{0}_{2\times 2} & \begin{pmatrix} \Omega'_-/\alpha \\ -\Omega'_+/\alpha \end{pmatrix} \\ (\,-\alpha\Omega_+ \quad \alpha\Omega_-\,) & 0 \end{pmatrix} \tag{18}$$

Here $P = diag(1,1,\alpha)$ where α is an arbitrary constant. The notation means that, in order to get the expression of the generator Ω_+, for example, one replace the symbol Ω_+ by 1 in the previous matrix and the others by 0. In the present case, one gets $Y = diag(-1,-1,-2)$ and $Q = diag(-1,0,-1)$. This corresponds to the (one of the two) fundamental representations $[\ell]$ of $SU(2|1)$ which is of dimension 3. It is non-typical (the number of right-handed fields is not equal to the number of and left-handed fields). Under $SU(2)\times U(1)$ we have the branching rule

$$[\ell] \longrightarrow (I = \frac{1}{2})_{y=-1} \oplus (I=0)_{y=-2} \tag{19}$$

This describes therefore a left doublet (e_L, ν_L) and a right singlet (e_R). The other fundamental representation gives hypercharges $Y = diag(1,1,2)$ and therefore describes the corresponding antiparticles. Actually, it is natural to explicitely add both contributions to get the leptonic Lagrangian [4]. The generalized connection reads

$$\mathcal{A} = \begin{pmatrix} \mathbf{L} & i\mu^{-1}\Phi \\ i\mu^{-1}\overline{\Phi} & \mathbf{R} \end{pmatrix}, \; \Phi = \begin{pmatrix} \phi_0 & 0 \\ \phi_+ & 0 \end{pmatrix} \tag{20}$$

where $\mathbf{L}$ and $\mathbf{R}$ anti-hermitian 2×2 matrices. Therefore

$$\mathcal{A} = \begin{pmatrix} \frac{i}{\sqrt{2}}\overrightarrow{\tau}\overrightarrow{W} + \frac{i}{\sqrt{6}}\mathbb{1}W_8 & \frac{i}{\mu}\begin{pmatrix} \phi_0 & 0 \\ \phi_+ & 0 \end{pmatrix} \\ \frac{i}{\mu}\begin{pmatrix} \overline{\phi}_0 & \overline{\phi}_- \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} +\frac{2i}{\sqrt{6}}W_8 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \tag{21}$$

We denoted by W_8 the abelian gauge field associated with the $U(1)$ gauge field because of the formal analogy with Gell-Mann matrices. Notice that $\mathcal{A}$ contains a line and a raw of zeros, but it is convenient to keep them in order to use the results of the previous section. One then computes the curvature $\mathcal{F}$ as it was explained previously and obtains the expression of the bosonic lagrangian of the Standard Model. Massive gauge fields are, here again, described by a term $(L-R)^2$, *i.e.* , one gets a term proportionnal to $(W_1^2 + W_2^2 + (W_3 - \frac{1}{\sqrt{3}}W_8)^2)$. To diagonalize the kinetic term for the gauge fields, one sets

$$\begin{aligned} Z &= -\cos\theta\, W_3 + \sin\theta\, W_8 \\ P &= \sin\theta\, W_3 + \cos\theta\, W_8 \\ \theta &= \frac{\pi}{6} \end{aligned} \tag{22}$$

One can also define as usual $W_\pm = \frac{1}{\sqrt{2}}(W_1 \pm iW_2)$. Notice finally that the mass term for leptons comes only from the neutral Higgs field so that the mass matrix is $\mu(\Omega_+ + \Omega'_-)$.

Quarks.

The calculations are very similar. Ω matrices are given by

$$\mu P \cdot \begin{pmatrix} 0_{2\times 2} & \begin{pmatrix} \frac{2}{3}\Omega_+ & \frac{1}{3}\Omega'_+ \\ \frac{2}{3}\Omega_- & -\frac{1}{3}\Omega'_- \end{pmatrix} \\ \begin{pmatrix} \Omega'_- & \Omega'_+ \\ -\Omega_- & \Omega_+ \end{pmatrix} & 0_{2\times 2} \end{pmatrix} \cdot P^{-1} \tag{23}$$

Here P is the matrix $diag(1,1,\alpha,\beta)$ and the two arbitrary constants that enter its expression come from the arbitrariness in the normalization of scalar products between the representation spaces $I = 0$ and $I = 1/2$ of $SU(2)$. Two representations differing by the action of P are equivalent but not unitarily equivalent. The above matrices Ω generate an algebra $\mathcal{C}$ of 4×4 matrices acting on a graded vector space $[q]$ of dimension 4. One gets here $Y = diag(1/3, 1/3, 4/3, -2/3)$ and $Q = diag(2/3, -1/3, 2/3, -1/3)$ which is the right pattern for quarks. Again, notice that $\Sigma_L Y = \Sigma_R Y$. This corresponds to the (one of the two) fundamental representations of $SU(2|1)$ which is of dimension 4. It is typical (the number of right-handed fields is equal to the number of and left-handed fields, here 2). $\mathcal{C}$ contains $SU(2|1)$ and in particular an homomorphic image of the Lie algebra of $SU(2) \times U(1)$. Under this Lie algebra we have the branching rule (with $y = 1/3$)

$$[q] \longrightarrow (I = \frac{1}{2})_y \oplus (I = 0)_{y-1} \oplus (I = 0)_{y+1} \tag{24}$$

This therefore describes a doublet of left-handed quarks along with their right-handed partners. The calculation to be carried out in the bosonic sector is analoguous to the one done previously. The only difference is that the natural mixing angle turns out to be defined by $tan^2\theta = 9/11$. More generally it is easy to show that, with the gauge group $SU(2) \times U(1)$ and for given I_3 and Y generators, one gets $\tan^2\theta = TrI_3^2/Tr(Y/2)^2$. There is also a corresponding representation describing antiquarks and it is natural to explicitely add both contributions to get the Lagrangian for quarks [4].

A more general example.

We now discuss a more general example incorporating color degrees of freedom, right neutrinos and family mixing in both quark and leptonic sectors. This corresponds to an algebra of 48×48 matrices acting on a graded vector space of dimension $48 = 24 + 24$ with an equal number of "left" and "right" dimensions. We shall not give here the full matrix structure for the matrices Ω but just mention that calculation of Y and Q and of the branching rules show that it describes three families of quarks $[q_1]$, $[q_2]$, $[q_3]$ coming in three colors with mixing between generations (we use the notation $[q_1]\Subset[q_2]\Subset[q_3]$ to indicate that it should correspond to a reducible (but indecomposable) representation of $SU(2|1)$ (*cf.* [10]). The restriction of the Ω matrices to the corresponding subspace have the following structure (each block refers to a 4×4 matrix) :

$$P \cdot \begin{pmatrix} A & B & B \\ 0 & A & B \\ 0 & 0 & A \end{pmatrix} \cdot P^{-1} \tag{25}$$

where P is an invertible matrix, involving constants α, β, *etc.* and blocks A and B are given by

$$A = \begin{pmatrix} \mathbf{0}_{2\times 2} & \begin{pmatrix} \frac{2}{3}\Omega_+ & \frac{1}{3}\Omega'_+ \\ \frac{2}{3}\Omega_- & -\frac{1}{3}\Omega'_- \end{pmatrix} \\ \begin{pmatrix} \Omega'_- & \Omega'_+ \\ -\Omega_- & \Omega_+ \end{pmatrix} & \mathbf{0}_{2\times 2} \end{pmatrix} \quad ; \quad B = \begin{pmatrix} \mathbf{0}_{2\times 2} & \begin{pmatrix} \Omega_+ & -\Omega'_+ \\ \Omega_- & \Omega'_- \end{pmatrix} \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} \end{pmatrix} \tag{26}$$

We also want to describe three extended families of leptons, each family being denoted by $([\ell]\Subset[1])$ where $[1]$ refers to the trivial representation and descibes a right-handed neutrino (with zero hypercharge and isospin) which has interactions with the other leptons via Yukawa couplings as well as a direct coupling to its left-handed partner. The restriction of the Ω matrices to the corresponding subspace reads

$$\begin{pmatrix} \mathbf{0}_{2\times 2} & \begin{pmatrix} \Omega'_-/\alpha \\ -\Omega'_+/\alpha \end{pmatrix} \begin{pmatrix} \varepsilon\Omega_- \\ \varepsilon\Omega_+ \end{pmatrix} \\ \begin{matrix} (-\alpha\Omega_+ & \alpha\Omega_-) \\ (\ 0 & 0\ ) \end{matrix} & \mathbf{0}_{2\times 2} \end{pmatrix} \tag{27}$$

Moreover the several extended leptonic families may interact via these neutrinos. What we get at the end is therefore a description of 24 elementary particles schematized as

$$3([q_1]\Subset[q_2]\Subset[q_3]) \oplus ([\ell_1]\Subset[1])\Subset([\ell_2]\Subset[1])\Subset([\ell_3]\Subset[1]) \tag{28}$$

Here again, the symbol $\Subset$ refers to the existence of reducible but indecomposable (*i.e.* not fully reducible) representations. Here again, the definition of the matrices Ω incorporate a number of (arbitrary) parameters that can be interpreted later in terms of masses and mixing matrices for quarks and leptons. Although we have no room for this discussion here, we want to point out that such indecomposable representations imply (in the case of quarks) the existence of a non diagonal Y generator. The corresponding (unwanted) contributions may be cancelled by taking into account the contribution of a representation describing the anti-fermions. The number of parameters of the model is now 24 (another 24!) since, on the top of the usual parameters of the Standard Model, we have the three masses of the (Dirac) neutrinos and the four parameters entering the Kobayashi-Maskawa like matrix for leptons. One may notice that for a family of particles incorporating leptons and anti-quarks, one gets $\Sigma_L Y^3 = \Sigma_R Y^3$ as it should (cancellation of the anomalies). The value found for the Weinberg mixing angle is now $tan^2\theta = 3/5$ –use the general formula given previously– *i.e.* $sin^2\theta = 3/8$. The existence or not of right neutrinos do not modify this calculation. As already mentionned in the introduction, this value 3/8 is "natural" in the present approach but cannot be justified on the grounds of gauge invariance alone that would allow for arbitrary rescalings of the different components of the non-simple Lie group $SU(2) \times U(1)$. Notice also that the $12 + 4 + 4 = 20$ parameters describing masses and mixing parameters are encoded in the matrix $\Omega_+ + \Omega'_-$. From the present point of view, this matrix (directly related to the matrix of Yukawa coupling constants in the usual approach) appears as part of a kind of discrete Dirac operator whose modulus is related to particle masses and whose phase is related to the Kobayashi-Maskawa-Cabibbo mixing coefficients. Since the value of these parameters lies anyway beyond quantum field theory (their renormalized values may be postulated at will without harming the corresponding quantum field theory), it is tempting to hope that these physical values (that are still unknown, because of experimental incertainties and because it is not obvious how to reconstruct the previous operator from the usually tabulated quantities) characterizing the structure of Space-Time at very small distances should correspond to a mathematically natural (meaning here aesthetic) discrete Dirac operator describing an exceptionnally simple "non-commutative" -or discrete- geometry. This hope motivates the search for educated ansatz concerning the structure of this operator. For instance, in the case of only two generations of quarks, postulating a very natural ansatz leads to a calculation [4] of the Cabibbo angle in good agreement with experiment. One finds

$$|\theta_c| = \frac{1}{2}\left[\arcsin\frac{2\sqrt{m_d m_s}}{m_d + m_s} - \arcsin\frac{2\sqrt{m_u m_c}}{m_u + m_c}\right] \approx \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}} \tag{29}$$

This may be taken as a good indication that something simple has still to be discovered in the physical and mathematical structures of the Standard Model.

A non physical example

Incorporating color degrees of freedom is done by considering a separated $SU(3)$ gauge group (or a separated associative algebra of 3×3 matrices). Some new ideas in this direction recently appeared ([2]). We shall not discuss them here. However, in order to prevent possibly wrong extrapolations and also for the sake of illustrating again some general features of our construction, we consider the following example (that will turn out to have nothing to do with particle physics and the Standard Model). Let us start with the Lie super-algebra $SU(3|2)$ or better, consider its odd generators along with the associative algebra $\mathcal{C}$ they generate. It acts on a vector space of dimension $5 = 3 + 2$ (this could describe for instance a fermionic multiplet containing three left-handed fermions and two right-handed ones). The emerging gauge group is $G = SU(3) \times SU(2) \times U(1)$ with gauge fields respectly denoted by W_α, with $\alpha \in \{1, 2, \ldots, 8\}$, A_i, with $i \in \{1, 2, 3\}$ and a field U_{15}. Again, by adding a line and a column of zeros, one can consider the corresponding matrices as 2×2 matrices, each element being a 3×3 block. Spontaneous breaking of the symmetry leads from $SU(3) \times SU(2) \times U(1)$ to a $SU(2) \times U(1)$ subgroup whose embedding is specified below. Indeed, a subset of the gauge fields becomes massive (there is a $(L-R)^2$ term in the lagrangian). It is easy to see that one gets a mass term proportionnal to

$$(W_1 - A_1)^2 + (W_2 - A_2)^2 + (W_3 - A_3)^2 + \Sigma W_{\alpha=4}^7 W_\alpha^2 + (W_8 - W_{15}/\sqrt{15})^2 \tag{30}$$

This gives four –unmixed– massive gauge fields, three massive gauge fields –with a mixing angle of $\pi/4$– and one massive gauge field –with a mixing angle defined by $tan\theta = \frac{1}{\sqrt{5}}$. Four gauge fields –corresponding to an $SU(2) \times U(1)$ subgroup – stay massless, namely $\frac{W_i + A_i}{\sqrt{2}}$, $i \in \{1, 2, 3\}$ and the abelian field $\frac{1}{\sqrt{6}}W_8 + \frac{5}{\sqrt{6}}W_{15}$.

Needless to say –and despite of the emergence of a familiar structure group– the pattern of symmetry breaking exhibited in this last example has nothing to do with the one usually associated with the description of strong and electrowek interactions!

Remarks

The fact that it is possible to write the whole lagrangian (bosonic as fermionic) describing gauge theories with symmetry breaking in terms of very simple structures using non-commutative geometry suggests that an analoguous reformulation could be usefull at the quantum field theory level (quantum effective action). This is clearly to be investigated. We hope that the present exposition will trigger new ideas and research in these "non-commutative directions".

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