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Heat Kernel : rencontre entre physiciens et mathématiciens

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HEAT KERNEL: RENCONTRE ENTRE PHYSICIENS ET MATHEMATICIENS*)

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ABSTRACT

The heat kernel for the general background is calculated and applied to loop diagrams of field theory.

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As I am going to convince my dear colleagues, heat kernel theory is a place where physicists and mathematicians make their *rendez-vous* with surprising permanence.

By heat kernel, I mean a generalized function which is a solution of the following equations

$$\frac{\partial K(s)}{\partial s} = HK(s) , \qquad K(s)|_{s=0} = \delta(x,y)$$

or, for short,

$$K(s) = \ell^{sH}\delta(x,y)$$

or, for even shorter,

$$\ell^{sH} .$$

Here H is a certain differential operator which for most of the applications that I am going to discuss can be taken as a second-order operator

$$H = g^{\mu\nu}\nabla_\mu\nabla_\nu + Q$$

with suitably defined covariant derivative ∇_μ and potential Q (see below); but an important limitation is that the second derivatives must be contracted with an invertible and positive-definite matrix:

$$\det g^{\mu\nu} \neq 0, \quad g^{\mu\nu}\xi_\mu\xi_\nu > 0 \qquad \forall\, \xi_\mu \neq 0 .$$

In fact, one often needs the operator H to be of a more general form, but there are special reduction methods by which a more general problem boils down to the case above. For these methods I refer to A.O. Barvinsky and G.A. Vilkovisky, *Physics Reports* **119** (1985) 1.

The operator H is understood as acting in a linear space of fields

$$H\phi(x) = \phi'(x)$$

and ϕ may be any collection of physical fields

$$\phi(x) = (\varphi(x), A_\mu(x), \psi(x), h_{\mu\nu}(x), \ldots) ,$$

- scalar, vector, spinor, tensor, any, defined on a given manifold. The matrix contracting second derivatives in the operator H is also defined on this manifold, and its inverse will be regarded as a metric of this manifold:

$$g^{\mu\nu}g_{\nu\alpha} = \delta^\mu_\alpha , \quad ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu .$$

The ∇_μ is a covariant derivative with respect to an arbitrary connection, acting on the set of fields ϕ, and since ϕ is multicomponent, the operator H has a matrix structure. I shall denote

$$g^{\mu\nu}\nabla_\mu\nabla_\nu \equiv \Box , \quad Q + \frac{1}{6}RI \equiv P$$

so that

$$H = \Box + (P - \frac{1}{6}RI)$$

where R is the Ricci scalar of the metric $g_{\mu\nu}$, and I is the unit matrix. I do not include a term linear in the derivatives since it can always be removed by a redefinition of the connection, and the above redefinition of the potential term is merely a matter of convenience. The potential matrix P is arbitrary except that for a calculation more serious than those I shall discuss, one may need the operator H to be negative semi-definite.

Thus there are three independent inputs in the operator H:

$$g_{\mu\nu}\ , \quad \nabla_\mu\ , \quad P$$

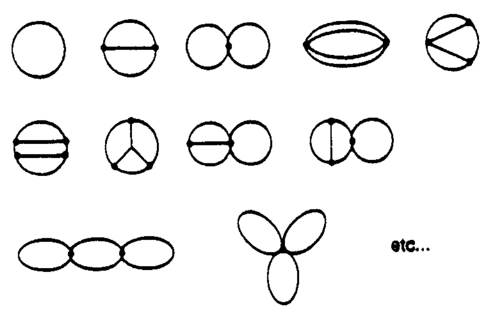
- the metric contracting the second derivatives, the connection which defines the covariant derivative, and the potential matrix. They may be regarded as external, or background, fields. To each corresponds its field strength, or curvature. There is the Riemann curvature associated with $g_{\mu\nu}$, the commutator curvature associated with ∇_μ, and the potential P which is its own curvature:

$$R_{\alpha\beta\mu\nu}\ , \quad \mathcal{R}_{\alpha\beta} \equiv [\nabla_\alpha, \nabla_\beta]\ , \quad P\ .$$

The background is trivial if these three curvatures vanish, and any reasonable quantity vanishing for a trivial background can only be built out of these three curvatures and their covariant derivatives ∇.

Something should be said in advance about the manifold on which the fields above are defined, and I shall suppose this is a non-compact, trivial topology, 2ω-dimensional space with an asymptotically Euclidean metric. This choice is explained by the fact that I am more interested in the effects induced by the curvatures than in the effects induced by the topology, but this is already a question of applications and one's interests. There is nothing in the methods below that would make them inapplicable to compact manifolds (with or without boundaries) although the final results may look different.

The heat kernel has, of course, many applications, but I speak on behalf of a class of physicists who are interested in it because they adore drawing diagrams like these:

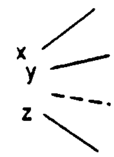


In these diagrams, each line is a kernel of the $1/H$ operator:

$$\frac{x \qquad y}{\qquad\qquad} = \frac{-1}{H}\,\delta(x,y)$$

and each vertex implies that i) a number of covariant derivatives act on endpoints of the lines which join at this vertex, ii) these endpoints are next made coincident, and iii) the result is integrated over the whole of the manifold with a certain weight function which is a local function of the curvatures already contained in the H operator:

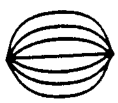
$$= \int dx\, f(x)\, [(\nabla^x \ldots \nabla^x)\, (\nabla^y \ldots \nabla^y) \ldots (\nabla^z \ldots \nabla^z)$$



$$]_{x=y=\ldots=z}\,,$$

$$f(x) \quad = f(R_{\mu\nu\alpha\beta}(x),\ \mathcal{R}_{\alpha\beta}(x),\ P(x))\,.$$

Thus, for example, a diagram with two vertices and n lines is the integrated product of n factors, like this

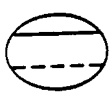


$$= \int dx\, dy\ f(x) f(y) \times$$

$$\times [\nabla \ldots \nabla \tfrac{-1}{H}\delta(x,y) \nabla \ldots \nabla] \ldots [\nabla \ldots \nabla \tfrac{-1}{H}\delta(x,y) \nabla \ldots \nabla]$$

where the covariant derivatives act on both sides of the kernel of the $1/H$ operator.

Up to an additive constant each diagram is a (non-local) functional of the curvatures contained in the H operator:



$$= \text{const} \ + F[R_{\mu\nu\alpha\beta},\ \mathcal{R}_{\mu\nu},\ P\ ]$$

except that all the diagrams above are, of course, ill-defined. This is because they all contain loops, which means that the generalized functions like

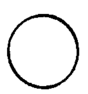
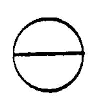
$$\frac{1}{H}\ \delta(x,y)$$

will appear at coincident points making ill-defined products like

$$[\frac{1}{H}\ \delta(x,y)]\ [\frac{1}{H}\ \delta(x,y)] = \ \text{bad}\ .$$

The derivatives in the vertices (if any) make them even worse.

These diagrams are not, however, completely meaningless, and many physicists made their careers (some even got Nobel prizes) by making sense of them. Why these diagrams are so loved is because a certain combination of them with quite definite coefficients and quite definite prescriptions for vertices plays the rôle of an action functional for physical fields:

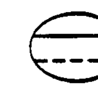
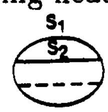
$$\Gamma[R_{\mu\nu\alpha\beta},\ \mathcal{R}_{\mu\nu},\ P] = \ \text{const} + \Gamma_{tree} - \frac{1}{2}$$  $$- \frac{1}{12}$$  $$- \frac{1}{8}$$  $$- \frac{1}{48}$$  $$+ \ldots\ .$$

This is the action of the fields whose curvatures are represented here, and it takes into account all quantum vacuum contributions of these fields (Γ_{tree} is the classical action). All predictions of quantum field theory are contained in this single functional (the effective action) as coefficients of various terms in the curvatures. With reasonably simple rules to make sense of loop diagrams, these predictions are sometimes experimentally confirmed with tremendous accuracy. The external, or background, fields which now appear as arguments of the effective action are in fact not external and not background; they are exact vacuum expectation values of quantum fields which can be obtained by applying the least-action principle to Γ (along with certain rules for going over from Euclidean to Lorentzian [1]) and solving the resultant field equations.

There is a very general and clean way to separate here good from bad, which is an old idea due to Schwinger. This is to write each $\frac{1}{H}$ operator as the integral of the heat kernel

$$-\frac{1}{H} = \int_0^\infty \ell^{sH} ds$$

and do this for each line of each diagram. Then each diagram will become a multiple integral over times of the same diagram but with lines being heat kernels instead of $\frac{1}{H}$'s:

 $= \int_0^\infty \dots \int_0^\infty ds_1 ds_2 \dots$  ,

$$\underset{x \qquad\quad y}{\overset{s}{\overline{\qquad\qquad}}} = \ell^{sH}\delta(x,y)\ .$$

Now heat kernels will appear at coincident points and make products like

$$[\ell^{sH}\delta(x,y)]\ [\ell^{sH}\delta(x,y)] = \text{ good}$$

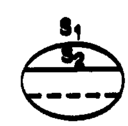
but this is absolutely harmless, and if there are derivatives in the vertices, they also spoil nothing. Loop diagrams with heat kernels are well defined. The problem does not disappear, of course; it shifs to the time integrals at their *lower* limits, which are now divergent.

One advantage of dealing with the heat kernel is that calculations can be done once for any number of dimensions. It has been noticed by L.S. Brown, *Phys.Rev.* **D15** (1977) 1469 and L.S. Brown and J.P. Cassidy, *Phys.Rev.* **D15** (1977) 2810 that, when the heat kernel is calculated in terms of the curvatures, the manifold dimension 2ω enters *explicitly* only the overall factor of the form

$$\ell^{sH}\delta(x,y) \propto \frac{1}{(4\pi s)}\omega\ .$$

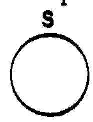
This explicit dependence can be conveniently used to regularize the time integrals at their lower limits by an analytic continuation in ω.

Thus the central problem becomes a calculation of loop diagrams with heat kernels:



1) See: A.O. Barvinsky and G.A. Vilkovisky, *Nucl.Phys.* **B282** (1987) 163.

of which the simplest is the one-loop diagram



$$= \int dx \; [\ell^{sH} \delta(x,y)]_{x=y}$$

which is just the trace of the heat kernel.

Here an essential remark should be made. There is, of course, a lot of work done on the heat kernel for specific backgrounds: given metric, given connection, given potential. For example, for symmetric spaces, the heat kernel can sometimes be done exactly. But the problem is that the effective action is to be used really as an *action*, which means it will be *varied*, that is we need it as a *functional*. Heat kernel and loop diagrams with the heat kernel should, therefore, be calculated for the *general* background; no specific results can be used. This cannot, of course, be done exactly (except one special case in two dimensions) but the point is to work out regular approximation techniques good enough to reproduce various physical effects.

Now, as far as calculations for the general background are concerned, it turns out that not much work has been done and not many regular methods are known: just two (to the best of my knowledge).

<u>Method No 1</u>

consists of writing down the following early-time expansion of the heat kernel:

$$\ell^{sH}\delta(x,y) = \frac{1}{(4\pi s)^{\omega}} D^{1/2}(x,y) \; \exp(-\frac{\sigma(x,y)}{2s}) \; \sum_{n=0}^{\infty} s^n a_n(x,y) \; .$$

Here

$$\begin{aligned} \sigma(x,y) \quad &= \tfrac{1}{2} \text{ (geodetic distance between } x \text{ and } y \text{ in the metric } g_{\mu\nu})^2 \; , \\ D(x,y) \quad &= \det \tfrac{\partial^2 \sigma(x,y)}{\partial x \partial y} \end{aligned}$$

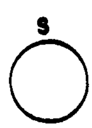
(this is the so-called Van Vleck-Morette determinant), and the coefficients $a_n(x,y)$ of the power series in time are calculable.

One way to arrive at the small-time expansion is to write down the path integral for the heat kernel and do this path integral by the saddle-point technique; the ansatz above is then obtained immediately. But one can also insert the expansion above in the heat equation and get everything from the equation itself.

When the early-time expansion is used in the simplest loop diagram - the trace of the heat kernel, then, since

$$\sigma(x,x) = 0 \; , \quad D^{1/2}(x,x) = \sqrt{g(x)} \; ,$$

one obtains



$$= \frac{1}{(4\pi s)^{\omega}} \sum_{n=0}^{\infty} s^n \int dx \sqrt{g} \; tr \; a_n(x,x)$$

where tr is the matrix trace [the coefficients $a_n(x,x)$ have the matrix structure of the operator $\hat{H}$]. Thus what remains is only the a_n coefficients at coincident points. These can be calculated in a regular way [2] in terms of the curvatures:

$$
\begin{aligned}
a_0(x,x) &= I\ , \\
a_1(x,x) &= P\ , \\
a_2(x,x) &= \tfrac{1}{180}(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - R_{\mu\nu}R^{\mu\nu} + \Box R)I + \tfrac{1}{2}P^2 + \tfrac{1}{12}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \tfrac{1}{6}\Box P\ ,
\end{aligned}
$$

and so on.

This is my example of how physicists and mathematicians make their *rendez-vous.* Here is the list of names (far from complete) associated with the early-time expansion of the heat kernel and/or its applications:

Hadamard (1921)

Minakshisundaram	Fock
McKean and Singer	Schwinger
Seeley	DeWitt
Gilkey	't Hooft and Veltman
	Avramidi
...	...

Left column lists mathematicians, right column lists physicists. Although they have everything in common except the language, mathematicians remember only mathematicians, and physicists remember only physicists. At the top is Hadamard who is out of competition. Physicists call the technique of the early-time expansion the Schwinger-DeWitt technique and the coefficients a_n the DeWitt coefficients. Mathematicians change a_n to b_n and call them the Minakshisundaram-Seeley coefficients. (In addition, names vary from paper to paper. The "Esperanto" version coined by Gary Gibbons is HAMIDEW coefficients, which means Hadamard-Minakshisundaram-DeWitt.) P.B. Gilkey of the left column was the first to calculate $a_3(x,x)$ (*J.Diff.Geom.* **10** (1975) 601). I.G. Avramidi of the right column was the first to calculate $a_4(x,x)$ (*Teor.Mat.Fiz.* **79** (1989) 219). A recent work of T.P. Branson and P.B. Gilkey, *Comm.Part.Diff.Eq.* **15** (1990) 245, contains an exhaustive calculation of surface contributions to $a_2(x,x)$ (for compact spaces with boundaries).

The problem is, however, that the early-time expansion of the heat kernel is incapable of producing most of the physical effects. It is good for obtaining renormalizations and anomalies. At the one-loop level these are governed by $a_1(x,x)$ in two dimensions

2) There is more than one way to do this calculation. The most straightforward one is to be found in B.S. DeWitt, Dynamical Theory of Groups and Fields (Gordon and Breach 1965). Some other techniques save time but do not allow one to calculate quantities like $[\nabla^x \dots \nabla^x a_n(x,y)]_{x=y}$ which are also needed for loop diagrams. For a review of the diagrammatic technique based on the early-time expansion of the heat kernel, see: A.O. Barvinsky and G.A. Vilkovisky, *Physics Reports* **119** (1985) 1.

and $a_2(x,x)$ in four dimensions . It can also be used to get local vacuum polarization by massive fields provided that all local curvature invariants are small in comparison to the respective powers of the mass parameter:

$$\frac{(\nabla\ldots\nabla R\cdot\cdot)\ \ldots\ (\nabla\ldots\nabla R\cdot\cdot)}{m^N} \ll 1\ .$$

But one cannot go beyond that, and in a massless theory the applicability of the early-time technique is even more limited. To see this, recall that the original loop diagrams involve time integrals, in particular

$$\bigcirc = \int_0^\infty \frac{ds}{s}\ \overset{s}{\bigcirc}\ .$$

If the loop with the heat kernel is expanded as above, one obtains

$$\bigcirc = \sum_{n=0}^{\infty}\left(\int_0^\infty \frac{ds}{s}\,\frac{s^n}{(4\pi s)^\omega}\right)\int dx\sqrt{g}\ \ tr\ a_n(x,x)\ ,$$

and, beginning with some n, the time integral becomes divergent at the *upper* limit. For several first n it diverges also at the lower limit (in fact, there is no alive place in this integral) but, as mentioned above, the divergence at the lower limit is the defect of the loop itself (disease of the theory) whereas the divergence at the upper limit simply means that we have done the heat kernel badly: the early-time expansion cannot be integrated term by term.

So now I go over to a better technique for the heat kernel which will correspond to a summation of the Schwinger-DeWitt series.

Method No 2

This is my own contribution with A.O. Barvinsky. The starting idea is to make a partial summation of the Schwinger-DeWitt series by summing all terms with a given power of the curvature and any number of derivatives [3)]. Symbolically, the structure of DeWitt's coefficients is as follows:

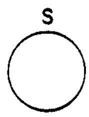
$$\begin{aligned}
a_2(x,x) &= \Box\, R\cdot\cdot + R^2\cdot\cdot \\
a_3(x,x) &= \Box^2\, R\cdot\cdot + R\cdot\cdot\Box\, R\cdot\cdot + R^3\cdot\cdot \\
a_4(x,x) &= \Box^3\, R\cdot\cdot + R\cdot\cdot\Box^2\, R\cdot\cdot + R\cdot\cdot\Box\, R^2\cdot\cdot + R^4\cdot\cdot \\
\cdots\cdots &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

and the proposal is to sum up terms in each vertical sequence of this table: first all terms linear in $R\cdot\cdot$, next quadratic in $R\cdot\cdot$, and so on. The result will be an expansion of the (trace of the) heat kernel in powers of the curvatures but this expansion will be non-local

3) Such a summation was first discussed in G.A. Vilkovisky, article in Quantum Theory of Gravity, ed. S.M. Christensen (Hilger 1984) and was subsequently actually realized to second order in the curvature by I.G. Avramidi, *Yad.Fiz.* **49** (1989) 1185.

because a summation of an infinite number of derivatives will result in non-local form factors.

Of course, I am not going to do any explicit summation. There is a regular technique to do that for which I refer to A.O. Barvinsky and G.A. Vilkovisky, *Nucl.Phys.* **B282** (1987) 163, *Nucl.Phys.* **B333** (1990) 471, *Nucl.Phys.* **B333** (1990) 512. Here I shall not go into any details of the procedure but merely present the result. To third order in the curvature, the result for the trace of the heat kernel is this:



$$
\begin{aligned}
= \; & \tfrac{1}{(4\pi s)^{\omega}} \int dx \, \sqrt{g} \, tr\Big\{ I + sP + s^2 [R_{\mu\nu} f_1(-s\,\Box) R^{\mu\nu} I + \\
& + R \; f_2(-s\,\Box) R I + P \; f_3(-s\,\Box) R + \\
& + P \; f_4(-s\,\Box) P + \mathcal{R}_{\mu\nu} f_5(-s\,\Box) \mathcal{R}^{\mu\nu}] + \\
& + s^3 \textstyle\sum_{i=1}^{11} F_i(-s\,\Box_1, -s\,\Box_2, -s\,\Box_3) \Re_1 \Re_2 \Re_3(i) + \\
& + s^4 \textstyle\sum_{i=12}^{25} F_i(-s\,\Box_1, -s\,\Box_2, -s\,\Box_3) \Re_1 \Re_2 \Re_3(i) + \\
& + s^5 \textstyle\sum_{i=26}^{28} F_i(-s\,\Box_1, -s\,\Box_2, -s\,\Box_3) \Re_1 \Re_2 \Re_3(i) + \\
& + s^6 F_{29}(-s\,\Box_1, -s\,\Box_2, -s\,\Box_3) \Re_1 \Re_2 \Re_3(29) + 0(\Re^4) \Big\} \; .
\end{aligned}
$$

Here terms of zeroth and first order in the curvature are local and coincide with DeWitt's $a_0(x,x)$ and $a_1(x,x)$. Second-order terms are represented by five quadratic structures of the form

$$\Re \; f(-s\,\Box) \Re$$

with non-local form factors which are functions of the operator

$$-s\,\Box \equiv \xi \; .$$

The sixth possible structure

$$R_{\alpha\beta\mu\nu} f(-s\,\Box) R^{\alpha\beta\mu\nu}$$

is absent. The Riemann tensor never makes its appearance in the present technique; it gets expressed (in a non-local way) through the Ricci tensor by the use of the Bianchi identities (see the references above). The five functions f_1, f_2, f_3, f_4, f_5 are all expressed through the basic second-order form factor

$$f(\xi) = \int_{\alpha \geq 0} d^2\alpha \; \delta(1 - \alpha_1 - \alpha_2) \exp(-\alpha_1 \alpha_2 \xi) = \int_0^1 d\alpha \; \ell^{-\alpha(1-\alpha)\xi}$$

as follows

$$
\begin{aligned}
f_1(\xi) &= \frac{f(\xi)-1+\frac{1}{6}\xi}{\xi^2}\,, \\
f_2(\xi) &= \frac{1}{8}\Big[\frac{1}{36}f(\xi)+\frac{1}{3}\,\frac{f(\xi)-1}{\xi}-\frac{f(\xi)-1+\frac{1}{6}\xi}{\xi^2}\Big]\,, \\
f_3(\xi) &= \frac{1}{12}f(\xi)+\frac{1}{2}\,\frac{f(\xi)-1}{\xi}\,, \\
f_4(\xi) &= \frac{1}{2}f(\xi)\,, \\
f_5(\xi) &= -\frac{1}{2}\frac{f(\xi)-1}{\xi}\,.
\end{aligned}
$$

Calculation of the third-order terms has been completed only recently (A.O. Barvinsky, Yu.V. Gusev, V.V. Zhytnikov and G.A. Vilkovisky, *Nucl.Phys.*, to be published). There are initially forty cubic structures but only twenty nine of them are independent. Among the latter, eleven contain no derivatives

$$
\begin{aligned}
\Re_1\Re_2\Re_3(1) &= P_1P_2P_3 \\
\Re_1\Re_2\Re_3(2) &= \mathcal{R}_1{}^{\mu}{}_{\alpha}\mathcal{R}_2{}^{\alpha}{}_{\beta}\mathcal{R}_3{}^{\beta}{}_{\mu} \\
\Re_1\Re_2\Re_3(3) &= \mathcal{R}_1^{\mu\nu}\mathcal{R}_{2\mu\nu}P_3 \\
\Re_1\Re_2\Re_3(4) &= R_1R_2P_3 \\
\Re_1\Re_2\Re_3(5) &= R_1^{\mu\nu}R_{2\mu\nu}P_3 \\
\Re_1\Re_2\Re_3(6) &= P_1P_2R_3 \\
\Re_1\Re_2\Re_3(7) &= R_1\mathcal{R}_2^{\mu\nu}\mathcal{R}_{3\mu\nu} \\
\Re_1\Re_2\Re_3(8) &= R_1^{\alpha\beta}\mathcal{R}_{2\alpha}{}^{\mu}\mathcal{R}_{3\beta\mu} \\
\Re_1\Re_2\Re_3(9) &= R_1R_2R_3 \\
\Re_1\Re_2\Re_3(10) &= R^{\mu}_{1\alpha}R^{\alpha}_{2\beta}R^{\beta}_{3\mu} \\
\Re_1\Re_2\Re_3(11) &= R_1^{\mu\nu}R_{2\mu\nu}R_3
\end{aligned}
$$

fourteen contain two derivatives

$$
\begin{aligned}
\Re_1\Re_2\Re_3(12) &= \mathcal{R}_1^{\alpha\beta}\nabla^{\mu}\mathcal{R}_{2\mu\alpha}\nabla^{\nu}\mathcal{R}_{3\nu\beta} \\
\Re_1\Re_2\Re_3(13) &= \mathcal{R}_1^{\mu\nu}\nabla_{\mu}P_2\nabla_{\nu}P_3 \\
\Re_1\Re_2\Re_3(14) &= \nabla_{\mu}\mathcal{R}_1^{\mu\alpha}\nabla^{\nu}\mathcal{R}_{2\nu\alpha}P_3 \\
\Re_1\Re_2\Re_3(15) &= R_1^{\mu\nu}\nabla_{\mu}R_2\nabla_{\nu}P_3 \\
\Re_1\Re_2\Re_3(16) &= \nabla^{\mu}R_1^{\nu\alpha}\nabla_{\nu}R_{2\mu\alpha}P_3 \\
\Re_1\Re_2\Re_3(17) &= R_1^{\mu\nu}\nabla_{\mu}\nabla_{\nu}P_2P_3 \\
\Re_1\Re_2\Re_3(18) &= R_{1\alpha\beta}\nabla_{\mu}\mathcal{R}_2^{\mu\alpha}\nabla_{\nu}\mathcal{R}_3^{\nu\beta} \\
\Re_1\Re_2\Re_3(19) &= R_1^{\alpha\beta}\nabla_{\alpha}\mathcal{R}_2^{\mu\nu}\nabla_{\beta}\mathcal{R}_{3\mu\nu} \\
\Re_1\Re_2\Re_3(20) &= R_1\nabla_{\alpha}\mathcal{R}_2^{\alpha\mu}\nabla^{\beta}\mathcal{R}_{3\beta\mu} \\
\Re_1\Re_2\Re_3(21) &= R_1^{\mu\nu}\nabla_{\mu}\nabla_{\lambda}\mathcal{R}_2^{\lambda\alpha}\mathcal{R}_{3\alpha\nu} \\
\Re_1\Re_2\Re_3(22) &= R_1^{\alpha\beta}\nabla_{\alpha}R_2\nabla_{\beta}R_3 \\
\Re_1\Re_2\Re_3(23) &= \nabla^{\mu}R_1^{\nu\alpha}\nabla_{\nu}R_{2\mu\alpha}R_3 \\
\Re_1\Re_2\Re_3(24) &= R_1^{\mu\nu}\nabla_{\mu}R_2^{\alpha\beta}\nabla_{\nu}R_{3\alpha\beta} \\
\Re_1\Re_2\Re_3(25) &= R_1^{\mu\nu}\nabla_{\alpha}R_{2\beta\mu}\nabla^{\beta}R^{\alpha}_{3\nu}
\end{aligned}
$$

three contain four derivatives

$$
\begin{aligned}
\Re_1\Re_2\Re_3(26) &= \nabla_{\alpha}\nabla_{\beta}R_1^{\mu\nu}\nabla_{\mu}\nabla_{\nu}R_2^{\alpha\beta}P_3 \\
\Re_1\Re_2\Re_3(27) &= \nabla_{\alpha}\nabla_{\beta}R_1^{\mu\nu}\nabla_{\mu}\nabla_{\nu}R_2^{\alpha\beta}R_3 \\
\Re_1\Re_2\Re_3(28) &= \nabla_{\mu}R_1^{\alpha\lambda}\nabla_{\nu}R^{\beta}_{2\lambda}\nabla_{\alpha}\nabla_{\beta}R_3^{\mu\nu}
\end{aligned}
$$

and one contains six derivatives

$$\Re_1\Re_2\Re_3(29) = \nabla_\lambda\nabla_\sigma R_1^{\alpha\beta}\nabla_\alpha\nabla_\beta R_2^{\mu\nu}\nabla_\mu\nabla_\nu R_3^{\lambda\sigma} \ .$$

These twenty nine structures form a complete basis of non-local invariants of third order in the curvature. (Ten of them are purely gravitational, and with gravity switched off there are six.)

The third-order form factors F in the expressions

$$F(-s\,\Box_1,\ -s\,\Box_2,\ -s\,\Box_3)\ \Re_1\Re_2\Re_3$$

are functions of three arguments

$$-s\,\Box_1 \equiv \xi_1, \quad -s\,\Box_2 = \xi_2, \quad -s\,\Box_3 = \xi_3 \ ,$$

and it is assumed that $\Box_1$ acts on the curvature with the label 1, $\Box_2$ acts on the curvature with the label 2, $\Box_3$ acts on the curvature with the label 3. All the twenty nine functions F are expressed through the basic third-order form factor

$$\begin{aligned}&F(\xi_1,\xi_2,\xi_3) = \textstyle\int_{\alpha\geq 0} d^3\alpha\ \delta(1-\alpha_1-\alpha_2-\alpha_3)\times\\ &\times\exp(-\alpha_1\alpha_2\xi_3 - \alpha_2\alpha_3\xi_1 - \alpha_1\alpha_3\xi_2)\end{aligned}$$

and the basic second-order form factor $f(\xi)$ introduced above. The coefficients of these expressions are rational functions with a universal denominator

$$\Delta = \xi_1^2 + \xi_2^2 + \xi_3^2 - 2\xi_1\xi_2 - 2\xi_1\xi_3 - 2\xi_2\xi_3$$

raised to a certain power. To give an idea of what these functions look like, I shall reproduce here the two simplest, the form factors of the structures $i = 1$ and $i = 2$:

$$\begin{aligned}F_1(\xi_1,\xi_2,\xi_3) =&\ \ \tfrac{1}{3}F(\xi_1,\xi_2,\xi_3)\ ,\\ F_2(\xi_1,\xi_2,\xi_3) =&\ \ [\tfrac{4}{\Delta^2}\,(\xi_3^3 + 2\xi_1\xi_2\xi_3 - 2\xi_1^2\xi_2) + \tfrac{4\xi_1\xi_2\xi_3}{3\Delta^3}\,(3\xi_3^3 + 2\xi_1\xi_2\xi_3 - 6\xi_1^2\xi_2)]\ \times\\ &\times F(\xi_1,\xi_2,\xi_3) + \tfrac{8\xi_1\xi_2\xi_3}{\Delta^3}(2\xi_2\xi_3 - \xi_3^2 + \xi_1^2 - \xi_2^2)f(\xi_1) +\\ &+\tfrac{4\xi_1}{\Delta^2}\,(2\xi_2\xi_3 - 2\xi_1\xi_2 - 2\xi_1\xi_3 - \xi_2^2 + 3\xi_1^2 - \xi_3^2)\ \times\\ &\times\tfrac{f(\xi_1)-1}{\xi_1} - \tfrac{2}{\xi_1-\xi_2}\,(\tfrac{f(\xi_1)-1}{\xi_1} - \tfrac{f(\xi_2)-1}{\xi_2})\ .\end{aligned}$$

The remaining twenty seven functions F are of a similar form [4] but take pages.

---

4) The form factors F, when taken separately, should be explicitly symmetrized according to the symmetry (if any) of their respective curvature structures $\Re_1\Re_2\Re_3$, whereas, when multiplied by these structures, they get symmetrized automatically. Thus, for example, the expressions above for F_1 and F_2 should be made completely symmetric in ξ_1, ξ_2, ξ_3.

There are several powerful checks of the result above. First, by expanding the form factors at small values of their arguments,

$$\begin{aligned} f(-s\,\Box) &= 1 + \tfrac{1}{6}\, s\,\Box + \dots\,, \\ F(-s\,\Box_1, -s\,\Box_2, -s\,\Box_3) &= \tfrac{1}{2} + \tfrac{1}{24}\,(s\,\Box_1 + s\,\Box_2 + s\,\Box_3) + \dots \end{aligned}$$

one should reproduce the Schwinger-DeWitt expansion up to terms cubic in the curvature. This has really been checked for all the known $a_n(x,x)$ including $a_4(x,x)$. Second, by putting

$$P = \frac{1}{6}RI, \quad \mathcal{R}_{\mu\nu} = 0, \quad R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R, \quad tr\ I = 1,\ \omega = 1$$

which corresponds to a conformal invariant scalar field in two dimensions, and doing the integral

$$\bigcirc = \int_0^\infty \frac{ds}{s}\ \overset{s}{\bigcirc}\ ,$$

one should obtain the effective action precisely of the form

$$-\frac{1}{2}\ \bigcirc\ = \frac{1}{96\pi}\int d^2x\sqrt{g}R\frac{1}{\Box}R$$

with all cubic terms vanishing! This has also been checked, and the mechanism of this vanishing has been revealed. Third, by specializing to a conformal invariant field in four dimensions, one should be able to reproduce the conformal anomaly proportional to $a_2(x,x)$. It really reproduces! After varying the effective action, taking the trace and restoring the Riemann tensor, all non-local terms vanish, and the correct local anomaly emerges.

Most important, however, is the late-time behaviour of the heat kernel. Having the Schwinger-DeWitt series summed as above, one may ask if this improves the convergence of the time integral at the upper limit. The late-time behaviour of the form factors has been obtained to all orders in the curvature (*Nucl.Phys.* **B333** (1990) 471). In particular,

$$f(-s\,\Box) = -\tfrac{1}{s}\,\tfrac{2}{\Box} + 0(\tfrac{1}{s^2}), \quad s \to \infty$$

$$F(-s\,\Box_1,\ -s\,\Box_2,\ -s\,\Box_3) = \tfrac{1}{s^2}\left(\tfrac{1}{\Box_1\Box_2} + \tfrac{1}{\Box_2\Box_3} + \tfrac{1}{\Box_1\Box_3}\right) + 0(\tfrac{1}{s^3}), \quad s \to \infty\,.$$

The result is that, to all orders in the curvature except zeroth, the behaviour of the trace of the heat kernel at large s is

$$\overset{s}{\bigcirc} \propto \frac{s}{(4\pi s)^\omega}\,, \quad s \to \infty\,, \quad \Re \neq 0$$

and the coefficient of this asymptotic behaviour [5] is calculable:

$$\begin{aligned} \overset{s}{\bigcirc} = \tfrac{s}{(4\pi s)^\omega}\int dx\sqrt{g}\ tr\{&P - P\tfrac{1}{\Box}P - \tfrac{1}{2}\mathcal{R}_{\mu\nu}\tfrac{1}{\Box}\mathcal{R}^{\mu\nu} - \tfrac{1}{6}R_{\mu\nu}\tfrac{1}{\Box}R^{\mu\nu}I + \\ &+ \tfrac{1}{18}R\tfrac{1}{\Box}RI + \tfrac{1}{3}P\tfrac{1}{\Box}R + \tfrac{1}{3}\left(\tfrac{1}{\Box_1\Box_2} + \tfrac{1}{\Box_2\Box_3} + \tfrac{1}{\Box_1\Box_3}\right) P_1P_2P_3 + \dots\}\,, \quad s \to \infty\,. \end{aligned}$$

---

5) This power asymptotic behaviour is characteristic of a non-compact manifold. For a compact manifold it will be replaced by the exponential behaviour

$$\overset{s}{\bigcirc} \propto \exp(-\lambda_{min}s)\,, \quad s \to \infty$$

The convergence of the integral

$$\bigcirc = \int_0^\infty \frac{ds}{s} \overset{s}{\bigcirc}$$

at the upper limit now becomes a criterion of analyticity of the effective action in the curvature. For manifolds of dimensions

$$2\omega > 2$$

the integral above converges at the upper limit at each order in the curvature. Therefore, the effective action in four dimensions is always analytic in the curvature whereas in two dimensions, $\omega = 1$, it is generally not. The conformal scalar field in two dimensions is an exceptional case where the second-order form factors appear in the combination

$$[\frac{1}{2} f_1(\xi) + f_2(\xi) + \frac{1}{6} f_3(\xi) + \frac{1}{36} f_4(\xi)] \propto \frac{1}{\xi^2}\,, \quad \xi \to \infty$$

for which the leading asymptotic terms $1/\xi$ cancel and the expansion at large ξ begins with $1/\xi^2$. Since, in addition, the expansion in the curvature terminates in this case at the second order, a finite result is obtained as presented above. From a purely formal standpoint, there is nothing wrong about a non-conformal invariant field in two dimensions except that even the non-local expansion above cannot be integrated term by term. This may only mean that a further summation is required which will result in an effective action both non-local and non-analytic.

Since the proof of the pudding is in the eating, several illustrations will not be out of place.

To begin with, let us apply the technique above to quantum electrodynamics. In this case the Riemann curvature vanishes, the commutator curvature is proportional to the Maxwell tensor $F_{\mu\nu}$, and the potential curvature contains the spinor field ψ. The following result is then obtained for the one-loop effective action to third order in the curvature (A.A. Ostrovsky and G.A. Vilkovisky, *J.Math.Phys.* **29** (1988) 702):

$$\begin{aligned}\Gamma_{\text{one-loop}} = \; & \tfrac{\ell^2}{(4\pi)^2} \int d^4x \bar{\psi}(x) \Big[m\big(-\tfrac{3}{2-\omega} + 3\log m^2 + 3\mathrm{C} - 4 - 3\log 4\pi + \\ & + \tfrac{m^2-H}{H} + \tfrac{m^2-H}{H}\,\big(2 - \tfrac{m^2-H}{H}\big) \log \tfrac{m^2}{m^2-H}\big) + (\gamma^\mu \nabla_\mu + m) \times \\ & \times \big(-\tfrac{1}{2-\omega} + \log m^2 + \mathrm{C} - 2 - \log 4\pi - \tfrac{m^2-H}{H} + \tfrac{m^2-H}{H}\,\big(2 + \tfrac{m^2-H}{H}\big) \log \tfrac{m^2}{m^2-H}\big) + \\ & + \tfrac{\ell}{2m} \sigma F \big(-\big(1 + \tfrac{m^2-H}{H}\big)\,\big(1 + 2\tfrac{m^2-H}{H}\big) + 2\tfrac{m^2-H}{H}\big(1 + \tfrac{m^2-H}{H}\big)^2 \log \tfrac{m^2}{m^2-H}\big) + \\ & + \{\gamma^\mu \nabla_\mu + m,\; \tfrac{\ell}{2m^2} \sigma F\}\,\big(2\big(1 + \tfrac{m^2-H}{H}\big) - \big(1 + \tfrac{m^2-H}{H}\big)^2\,\big(2 + \tfrac{m^2-H}{H}\big) \log \tfrac{m^2}{m^2-H}\big) + \\ & + 0(2-\omega)\Big] \psi(x) + 0(F^2) + 0(\psi^4) + 0(\nabla F)\end{aligned}$$

---

and then one should be able to obtain the minimum eigenvalue λ_{min} of the operator $(-H)$ as a non-local expansion in powers of the curvature (or a deviation of the curvature from the reference one). I am grateful to Gary Gibbons for discussing this point.

where only terms quadratic in ψ and linear in $F_{\mu\nu}$ are retained, and, moreover, terms in derivatives of $F_{\mu\nu}$ are discarded. Here

$$\begin{aligned} m^2 - H &= -(\gamma^\mu \nabla_\mu - m)\,(\gamma^\mu \nabla_\mu + m)\ , \\ \sigma F &\equiv \sigma^{\mu\nu} F_{\mu\nu}\ , \quad \sigma^{\mu\nu} = \tfrac{i}{2}[\gamma^\mu, \gamma^\nu]\ , \end{aligned}$$

$\{,\}$ is the anticommutator, and C is the Euler constant. The renormalization boils down to deleting the local terms

$$\begin{aligned} -\Gamma_{\text{counter}} = \ & \tfrac{\ell^2}{(4\pi)^2} \int d^4x \bar{\psi}(x) \Big[m(- \tfrac{3}{2-\omega} + 3\log m^2 + 3\mathrm{C} - 4 - 3\log 4\pi) + \\ & + (\gamma^\mu \nabla_\mu + m)\,(- \tfrac{1}{2-\omega} + \log m^2 + \mathrm{C} - 2 - \log 4\pi) \Big] \psi(x) \end{aligned}$$

proportional to the terms of the classical action

$$\Gamma_{\text{tree}} = - \int d^4x \bar{\psi}(x) (\gamma^\mu \nabla_\mu + m)\, \psi(x)\ .$$

The above expression is highly non-local but, when the renormalized effective action

$$\Gamma_{\text{ren}} = \Gamma_{\text{tree}} + \Gamma_{\text{one-loop}} + \Gamma_{\text{counter}} + 0(e^4)$$

is restricted to the mass shell, all logarithms become suppressed, all non-local terms vanish, and there remains only [6)]

$$\begin{aligned} \Gamma_{\text{ren}}|_{\text{mass shell}} \ = \ & - \int d^4x \bar{\psi}(x) (\gamma^\mu \nabla_\mu + m + \tfrac{\ell^2}{(4\pi)^2}\, \tfrac{\ell}{2m} \sigma F) \psi(x) + \\ & + 0(F^2) + 0(\psi^4) + 0(\nabla F) + 0(e^4) \end{aligned}$$

with Schwinger's value for the anomalous magnetic moment of the electron.

Another application that has been considered is a derivation of the Hawking effect in two and four dimensions. This is already an essentially non-local effect and, therefore, boundary conditions imposed by the choice of a quantum state become important. The gravitational collapse problem in its simplest quantum setting [7)] is a problem about the expectation values of quantum fields in the so-called relative standard in-vacuum state [8)].

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6) This manifestly covariant calculation nowhere encounters the infra-red divergences because objects like n-point functions or vertex functions, meaningless in gauge theory, are never considered (see a discussion in the above reference). The effective action method has also purely technical advantages over the method of Green functions because it deals only with diagrams without external lines. This is especially important for theories like gravity where the calculation of n-point functions turns into an algebraic nightmare. In addition, the number of diagrams at higher orders becomes considerably reduced, which is important even for electrodynamics where the precision of experiment requires already four-loop calculations. For an extension of the effective action method to higher-loop orders, see A.O. Barvinsky and G.A. Vilkovisky, in Quantum Field Theory and Quantum Statistics, eds. I.A. Batalin, C.J. Isham and G.A. Vilkovisky (Hilger 1987) Vol. 1.

7) G.A. Vilkovisky, Lectures at the University of Texas at Austin (1989); Class. and Quantum Gravity (1992, to be published).

For this state, the rules for calculating the expectation values of currents are as follows [9]: i) consider the Euclidean effective action and do the loops, ii) vary this action by applying the variational rules of finite matrices to operators, iii) in the Euclidean current thus obtained go over to the Lorentzian signature and replace everywhere the Euclidean Green functions by the retarded Green functions. WARNING: never make this replacement for propagators in the loops, and never use this rule for the effective action itself.

In two dimensions, the contribution of a massless scalar loop to the gravitational effective action is

$$\Gamma_{\text{one loop}} = \frac{1}{96\pi}\int d^2x\sqrt{g}R\frac{1}{\Box}R$$

(see above). By applying the rules above to this action one arrives at the following expression for the expectation value of the energy-momentum tensor in the in-vacuum state:

$$\begin{aligned}\langle \text{in, vac}|T_{\mu\nu}|\text{in, vac}\rangle = {} & \tfrac{1}{48\pi}\{-2\nabla_\mu\nabla_\nu G^{ret}R+ \\ & +(\nabla_\mu G^{ret}R)\,(\nabla_\nu G^{ret}R) - g_{\mu\nu}[2R+ \\ & +\tfrac{1}{2}(\nabla_\alpha G^{ret}R)\,(\nabla^\alpha G^{ret}R)]\}\end{aligned}$$

where

$$\Box\, G^{ret} = -1$$

and G^{ret} is the retarded Green function. Since this is an exact expression, the toy may be instructive. For the purpose of deriving the Hawking effect it has been considered by V.P. Frolov and G.A. Vilkovisky, in Proc. 1981 Moscow Seminar on Quantum Gravity, eds. M.A. Markov and P.C. West (Plenum 1983).

There is no classical gravitational dynamics in two dimensions but one may take for one's space-time the two-dimensional section

$$ds^2 = 2\Psi du dv\ , \quad \Psi = \Psi(u,v) < 0$$

of a four-dimensional spherically symmetric space-time

$$ds^2_{(4)} = 2\Psi du dv + r^2 d\Omega^2\ , \quad r = r(u,v) \geq 0\ .$$

Here $r^2 d\Omega^2$ is the line element of a two-sphere of an area $4\pi r^2$, and null co-ordinates u, v are introduced in the fixed-angle section. The four-dimensional metric is supposed to be asymptotically flat at past null infinity

$$\mathcal{I}^- : r \to \infty\ , \quad v = \text{const}$$

and future null infinity

$$\mathcal{I}^+ : r \to \infty\ , \quad u = \text{const}$$

8) B.S. DeWitt, in Relativity, Groups and Topology II, eds. B.S. DeWitt and R. Stora (North Holland 1984).

9) A.O. Barvinsky and G.A. Vilkovisky, *Nucl.Phys.* **B282** (1987) 163.

(see the Figure). If, in addition, this metric satisfies the classical Einstein equations with a matter source, then

$$\begin{aligned} \left.\frac{d^2r}{d\lambda^2}\right|_{v=\text{const}} &= -4\pi r\ T^{\text{source}}_{\mu\nu}\ (\nabla^\mu v)\ (\nabla^\nu v)\ , \\ \left.\frac{d^2r}{d\lambda^2}\right|_{u=\text{const}} &= -4\pi r\ T^{\text{source}}_{\mu\nu}\ (\nabla^\mu u)\ (\nabla^\nu u)\ , \end{aligned}$$

which are the constraint equations on null surfaces (and the cornerstones of the singularity theorems). Here λ is an affine parameter along the null geodesic $v=$ const or $u=$ const:

$$\left.\frac{1}{\Psi}\ \frac{d\lambda}{du}\right|_{v=\text{const}} = -1\ , \quad \left.\frac{1}{\Psi}\ \frac{d\lambda}{dv}\right|_{u=\text{const}} = -1\ .$$

The null co-ordinates u, v are defined up to an arbitrary transformation of the form

$$u \to f_1(u)\ , \quad v \to f_2(v)\ .$$

Two specific sets of such co-ordinates will be employed that become asymptotically Cartesian at $\mathcal{I}^-$ and $\mathcal{I}^+$ respectively (generally, they exist only locally):

$$\begin{aligned} ds^2 = &\ \ 2\Psi_- du_- dv_-\ , \quad \Psi_-|_{\mathcal{I}^-} = -1\ , \quad r|_{\mathcal{I}^-} = \frac{v_- - u_-}{\sqrt{2}}\ , \\ ds^2 = &\ \ 2\Psi_+ du_+ dv_+\ , \quad \Psi_+|_{\mathcal{I}^+} = -1\ , \quad r|_{\mathcal{I}^+} = \frac{v_+ - u_+}{\sqrt{2}}\ , \\ u_+ = &\ \ u_+(u_-)\ , \quad v_+ = v_+(v_-)\ , \quad \Psi_+ = \Psi_- \frac{du_-}{du_+}\ \frac{dv_-}{dv_+}\ . \end{aligned}$$

Since the energy-momentum tensor of in-vacuum is retarded, it vanishes at the past null infinity

$$\langle \text{in, vac}|T_{\mu\nu}|\text{in, vac}\rangle|_{\mathcal{I}^-} = 0$$

which means there is no incoming radiation. Its possible non-vanishing at the future null infinity would signal an outgoing radiation originating from the vacuum. The collapsing source would then lose energy at a rate proportional to the energy flux through $\mathcal{I}^+$:

$$-\frac{dE}{d\tau} = \frac{1}{2}\langle \text{in, vac}|T_{\mu\nu}|\text{in, vac}\rangle \nabla^\mu v_+ \nabla^\nu v_+|_{\mathcal{I}^+}\ .$$

Here τ is the proper time of an observer at $r=\infty$ who registers the outgoing radiation.

Since in two dimensions

$$\Box \equiv \frac{2}{\Psi}\partial^2_{uv}\ , \quad R = \frac{2}{\Psi}\ \partial^2_{uv}\log|\Psi| = \Box\ \log|\Psi|\ ,$$

one finds that, up to boundary conditions,

$$\frac{1}{\Box}\ R = \log\ |\Psi|\ ,$$

and the retarded solution is, obviously,

$$G^{\text{ret}}R = -\log\ |\Psi_-|\ , \quad \Psi_-|_{\mathcal{I}^-} = -1\ .$$

As a result, the vacuum energy-momentum tensor at $\mathcal{I}^+$ takes the form

$$\langle \text{in, vac}|T_{\mu\nu}|\text{in, vac}\rangle \nabla^\mu v_+ \nabla^\nu v_+|_{\mathcal{I}^+} = \frac{1}{48\pi}\left\{2\frac{d^2}{du_+^2}\ \log\ |\Psi_-| + (\frac{d}{du_+}\log\ |\Psi_-|)^2\right\}|_{\mathcal{I}^+}\ .$$

Furthermore, since

$$\Psi_- = \Psi_+ \frac{du_+}{du_-}\frac{dv_+}{dv_-}\ ,$$

one finds that

$$G^{\text{ret}}R|_{\mathcal{I}^+} = \log\ |\frac{du_-}{du_+}| + \text{const}\ ,$$

and the whole effect depends only on the form of the function $u_+(u_-)$ which characterizes the dynamics of a given gravitational field:

$$\langle \text{in, vac}|T_{\mu\nu}|\text{in, vac}\rangle \nabla^\mu v_+ \nabla^\nu v_+|_{\mathcal{I}^+} = \frac{1}{48\pi}\{-2\frac{d^2}{du_+^2}\ \log\ |\frac{du_-}{du_+}| + (\frac{d}{du_+}\log\ |\frac{du_-}{du_+}|)^2\}\ .$$

Note that the first term in this expression, which is linear in the curvature, has an indefinite sign, whereas the second term, quadratic in the curvature, is manifestly non-negative.

The event horizon $\mathcal{H}$, if there is one in the metric, is an outgoing null geodesic $u = \text{const}$ along which $r = \text{const}$. In null co-ordinates that cover the horizon (u_-, v_- in particular [10]) this should correspond to some finite value of u, say $u_- = u_-^0$:

$$\mathcal{H}:\quad u_- = u_-^0\ ,\quad \frac{dr}{d\lambda}\bigg|_{u_-=u_-^0} = 0\ .$$

However, if the metric solves the classical constraints above, then, outside the collapsing source,

$$\frac{d^2r}{d\lambda^2}\bigg|_{u=\text{const}} \equiv 0$$

and, therefore,

$$\frac{dr}{d\lambda}\bigg|_{u=\text{const}} = F(u)$$

where $F(u)$ is a function only of u. At $u_- = u_-^0$ this function should have a zero, and, by analyticity of the chart (u_-, v_-) at the horizon, this will generally be a simple zero:

$$F(u)|_{\mathcal{H}} = \alpha(u_- - u_-^0) + 0(u_- - u_-^0)^2\ ,\quad \alpha \neq 0\ .$$

On the other hand, in the domain $u_- < u_-^0$, where the charts (u_-, v_-) and (u_+, v_+) overlap, one has identically

$$F(u) = -\frac{1}{\Psi_-}\frac{\partial r}{\partial v_-} = -\frac{1}{\Psi_+}\frac{du_-}{du_+}\frac{\partial r}{\partial v_+}\ ,$$

---

10) In the collapse problem, a frame defined by conditions at $\mathcal{I}^-$ is co-moving with the source.

and, since this is a function only of u,

$$F(u) = F(u)|_{\mathcal{I}^+} = \frac{du_-}{du_+}\left(-\frac{1}{\Psi_+}\frac{\partial r}{\partial v_+}\right)\Big|_{\mathcal{I}^+} = \frac{1}{\sqrt{2}}\frac{du_-}{du_+} .$$

The result is a limiting behaviour of the conversion derivative at the horizon:

$$\frac{du_-}{du_+}\Big|_{\mathcal{H}} = \sqrt{2}\alpha(u_- - u_-^0) + 0(u_- - u_-^0)^2$$

which shows in particular that

$$|u_+|\Big|_{\mathcal{H}} \to \infty ,$$

and

$$\frac{du_-}{du_+}\Big|_{\mathcal{H}} = \ell^{-\sqrt{2}|\alpha u_+|}$$

(u_+ is synchronized with the proper time of an external observer at $r = \infty$). The constant α is directly related to the mass of the collapsing source. For the Schwarzschild metric

$$|\alpha| = \frac{1}{4M} .$$

When the limiting behaviour of du_-/du_+ is inserted in the above expression for the vacuum radiation flux, it turns out that, as the horizon is approached, the indefinite component of the radiation dies out and, asymptotically at late times, there remains only a positive constant flux:

$$\langle\text{in, vac}|T_{\mu\nu}|\text{in, vac}\rangle\nabla^\mu v_+\nabla^\nu v_+\Big|_{\substack{\mathcal{I}^+\\ u_+\to\infty}} = \frac{(\sqrt{2}\alpha)^2}{48\pi} , \quad -\frac{dE}{d\tau}\Big|_{\tau\to\infty} = \frac{\pi}{12}\frac{1}{(8\pi M)^2} .$$

This corresponds to the thermal radiation of massless bosons at the Hawking temperature $1/8\pi M$.

The life in four dimensions is more difficult technically but more easy physically. At least, the energy conservation law will be a consequence of the equations and not an evening prayer. And there will be equations (to begin with!), not just their right-hand sides. In four dimensions, the contribution of a massless scalar loop to the gravitational effective action is given by an expansion in powers of the curvature, and this expansion does not terminate:

$$\Gamma_{\text{one-loop}} = \frac{1}{32\pi^2}\int d^4x\sqrt{g}\Big[\frac{1}{60}R_{\mu\nu}(\gamma(-\Box) - \tfrac{16}{15})R^{\mu\nu} - \tfrac{1}{180}R(\gamma(-\Box) - \tfrac{37}{30})R +$$

$$+ \textstyle\sum_i \Gamma_i(-\Box_1, -\Box_2, -\Box_3)\,\Re_1\Re_2\Re_3(i) + 0(R^4_{\mu\nu})\Big] .$$

Here the third-order form factors

$$\Gamma_i(-\Box_1, -\Box_2, -\Box_3)$$

are obtained by integrating over s the respective form factors in the heat kernel, and the sum is over ten purely gravitational cubic structures i = 9, 10, 11, 22, 23, 24, 25, 27, 28, 29. The only second-order form factor is of the form

$$\gamma(-\Box) = \log\left(-\frac{\Box}{\mu^2}\right)$$

where $\mu^2 > 0$ is a parameter describing the ultra-violet renormalization arbitrariness.

Even for varying an action with operator functions [11] the latter should be disentangled through the inverse operators. It is the spectral representation that serves this purpose. The spectral representation is also used to implement boundary conditions and is, generally, a means of making the non-local equations tractable. For the second-order form factor this representation is, obviously,

$$\log(-\frac{\Box}{\mu^2}) = \int_0^\infty dm^2 \,(\frac{1}{m^2+\mu^2} - \frac{1}{m^2-\Box}) \,.$$

For the spectral forms of the third-order form factors see A.O. Barvinsky and G.A. Vilkovisky, *Nucl.Phys.* **B333** (1990) 512. The equations for the gravitational field in the in-vacuum state (with only the spin-0 vacuum particles taken into account [12]) are then obtained in the form

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = &\ -8\pi\ T^{\rm source}_{\mu\nu} - 8\pi\langle \text{in, vac}|T_{\mu\nu}|\text{in, vac}\rangle \ , \\ \langle \text{in, vac}|T_{\mu\nu}|\text{in, vac}\rangle\Big|_{\text{spin}-0} = &\ \beta\{\tfrac{1}{9}\ (\nabla_\mu\nabla_\nu - g_{\mu\nu}\ \Box)R + [\tfrac{16}{15} - \log(-\tfrac{\Box_{\rm ret}}{\mu^2})]\times \\ &\times(\ \Box\ R_{\mu\nu} - \tfrac{1}{6}\ g_{\mu\nu}\ \ \Box\ R - \tfrac{1}{3}\nabla_\mu\nabla_\nu R)\} + 0(R^2_{\mu\nu}) \end{aligned}$$

where

$$\begin{aligned} \beta &= 2\times\tfrac{1}{32\pi^2}\times\tfrac{1}{60}\ , \\ \log(-\tfrac{\Box_{\rm ret}}{\mu^2}) &= \textstyle\int_0^\infty dm^2(\frac{1}{m^2+\mu^2} - G^{ret}(m^2))\ , \\ (\ \Box - m^2)G^{ret}(m^2) &= -1\ , \end{aligned}$$

and $G^{ret}(m^2)$ is the retarded massive Green function.

A systematic study of non-local gravitational equations (for a spherically symmetric in-state) started only recently. So far only the terms linear in the curvature in the above equations have been analysed (A.G. Mirzabekian and G.A. Vilkovisky, to be published). The following expression was obtained for the non-local kernel that governs these terms:

$$\begin{aligned} -\log(-\tfrac{\Box_{\rm ret}}{\mu^2})\ J = &\ \tfrac{1}{r}\textstyle\int_\infty^0 d\bar r\frac{\bar r}{\bar r+r}\ \bar J\Big|_{\text{path 1}} + \\ &+\tfrac{1}{r}\textstyle\int_0^r d\bar r\ \ \log|\mu(\bar r - r)|\frac{d}{d\bar r}\ (\bar r\bar J)\Big|_{\text{path 2}} + \\ &+\tfrac{1}{r}\textstyle\int_\infty^r d\bar r\ \ \log|\mu(\bar r - r)|\frac{d}{d\bar r}\ (\bar r\bar J)\Big|_{\text{path 3}} + \\ &+2CJ + 0(R_{\mu\nu})\ . \end{aligned}$$

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11) It is important that only functions of commuting operators appear in the heat kernel and effective action.

12) Since the gravitational interaction is universal, all vacuum particles existing in nature should contribute to the right-hand side of this equation. Furthermore, at higher-loop orders, all the details of their couplings will matter. To make the equations model-independent, one may use the approach proposed in G.A. Vilkovisky, Class. and Quantum Gravity (1992, to be published).

Here J is an arbitrary spherically symmetric scalar [13], r refers to an observation point, and the expression involves the integrals along three null paths 1, 2, 3 shown in the Figure ($\bar{r}$ and $\bar{J}$ are restricted to the integration paths). The null paths 1 and 2, taken together, form a single light ray which starts at $\mathcal{I}^-$ and comes to an observation point; the null path 3 is the other such ray.

The equation of the energy balance in four dimensions is

$$-\frac{dE}{d\tau} = \frac{1}{2} 4\pi r^2 \langle \text{in, vac}|T_{\mu\nu}|\text{in, vac}\rangle \nabla^\mu v_+ \nabla^\nu v_+ \Big|_{\mathcal{I}^+}$$

with the same notation as before except that now it follows from the field equations and E is the Bondi mass at $\mathcal{I}^+$:

$$E = \frac{r}{2}(1 - g^{\mu\nu}\nabla_\mu r \nabla_\nu r)\Big|_{\mathcal{I}^+} .$$

The total mass stored in space-time (and measured at spatial infinity I^0) is, on the contrary, conserved:

$$\begin{aligned} M &= \tfrac{r}{2}(1 - g^{\mu\nu}\nabla_\mu r \nabla_\nu r)\Big|_{I^0} , \\ \tfrac{dM}{d\tau} &= 0 . \end{aligned}$$

The vacuum stress tensor violates the dominant energy condition [14], but one can argue that the total mass M in the solution of the effective equations should remain positive and equal to the initial mass of the collapsing source. Indeed, since M is conserved, it can be calculated on a spacelike hypersurface taken in the remote past. On the other hand, since the vacuum contribution to the full energy-momentum tensor is retarded, it should vanish in the remote past. The total mass should, therefore, remain unaffected by quantum corrections to the equations. This very important fact has really been checked to first order in the curvature, and for the Bondi mass the following result was obtained:

$$\begin{aligned} E &= M + \tfrac{2\sqrt{2}}{3}\pi\beta \, \tfrac{d}{du_+} \textstyle\int_{\mathcal{I}^-}^{\mathcal{I}^+} dr \; rR + 0(R^2_{\mu\nu}) , \\ -\tfrac{dE}{d\tau} &= -\tfrac{2}{3}\pi\beta \, \tfrac{d^2}{du_+^2} \textstyle\int_{\mathcal{I}^-}^{\mathcal{I}^+} dr \; rR + 0(R^2_{\mu\nu}) \end{aligned}$$

where β is the coefficient of the vacuum stress in the equations above [15], and the expressions involve the integral of the Ricci scalar along the full world line of a radial light ray which starts at $\mathcal{I}^-$ and comes to a given point of $\mathcal{I}^+$ (this is the path 1 $\bigcup$ path 2 in the Figure, when the observation point is brought to $\mathcal{I}^+$).

The latter result for the radiation flux to first order in the curvature is very similar to the component linear in the curvature of the radiation flux obtained in two dimensions.

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13) For a tensor the form of the kernel is different.

14) This being the source of all hopes.

15) The renormalization parameter μ in the equations does not, of course, affect the result since a change in this parameter would affect only local terms of the equations, which vanish at infinity.

In the same way as in two dimensions, this first-order radiation of indefinite sign dies out as the horizon is approached [16], and the Hawking effect sits in the terms quadratic in the curvature. These terms involve ten triple form factors [17]

$$\Gamma_i(-\Box_1, -\Box_2, -\Box_3)$$

and are presently under study (A.G. Mirzabekian and G.A. Vilkovisky, work in progress).

The non-local expansion of the heat kernel is thus much better than the Schwinger-DeWitt expansion. As we have seen, it is already capable of producing finite vacuum polarization effects and effects of particle creation. But the validity of this technique is also limited. Although it already allows the curvature to fluctuate however rapidly, it does not allow the curvature to be large in the absolute magnitude. The condition is, roughly speaking, that the curvature should be much smaller than its gradient

$$\Re^2 \ll \nabla\nabla\Re \, .$$

If you think, for example, of the collapse problem, then presumably, inside the horizon, the curvature and its gradient are of one and the same order of magnitude. Therefore, in this case, the expansion above is not already going to produce quantitatively correct results.

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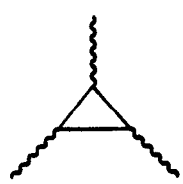
16) This can be seen quite generally. At the horizon, the curvature and its derivatives in the frame (u_-, v_-) co-moving with the source are finite, whereas

$$\left.\frac{du_-}{du_+}\right|_{\mathcal{H}} = 0 \, .$$

Therefore, whenever the derivative d/du_+ acts on the curvature, the result vanishes at the horizon.

17) The Hawking effect is in fact a vertex effect described by the one-loop triangular graph



with three external gravitons (or curvatures) and a given vacuum particle circulating in the loop. However, the propagators are not Feynman's. There are several such graphs with the retarded, advanced and Feynman propagators in the diagrammatic technique for expectation values (see, e.g., E.S. Fradkin and D.M. Gitman, *Fortschr. der Phys.* **29** (1981) 381). It is the sum of these graphs that gives rise to the retarded form factors in the effective equations (A.O. Barvinsky and G.A. Vilkovisky, *Nucl.Phys.* **B282** (1987) 163).

I want to conclude by making two remarks. First, the heat kernel is a very grateful object. The moment you learn how to calculate it better, it thanks you by giving back a lot of physical effects. Second, loops of field theory may be only a sign of the epoch: time will pass and they may perish. But the heat kernel will not perish; it is a cultural treasure that can be put in the British Museum. It is worth the effort.

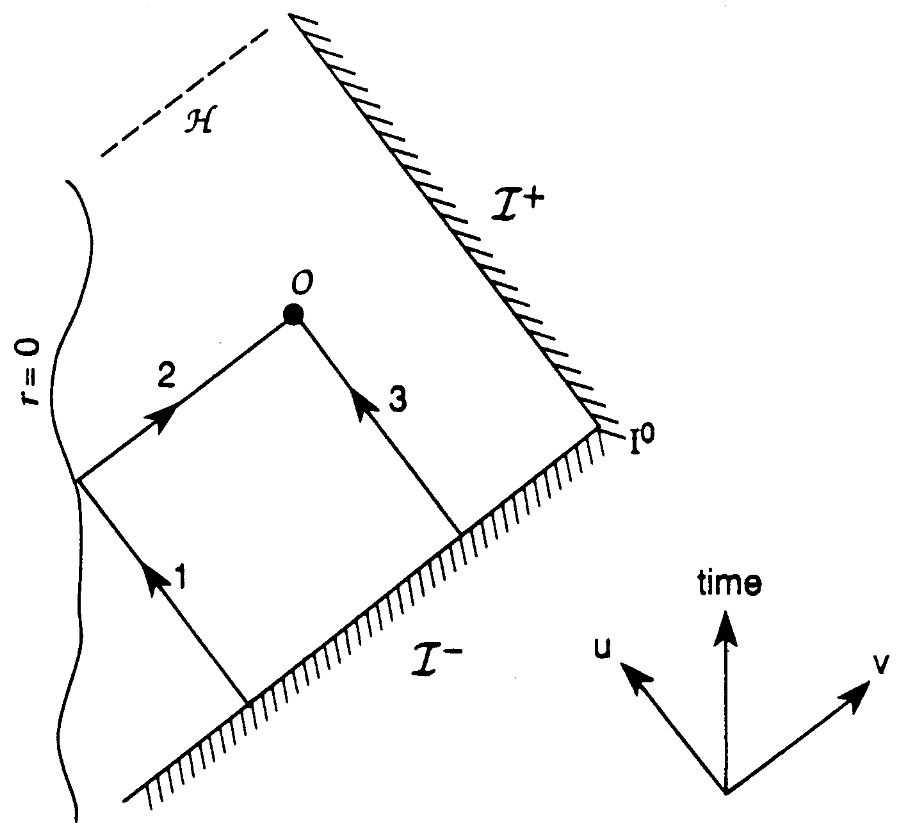


FIGURE CAPTION

Two – dimensional section of a four – dimensional spherically symmetric asympto– tically flat spacetime

Null co-ordinates u, v are employed. The timelike line is $r = 0$, and $\mathcal{I}^- \cup I^0 \cup \mathcal{I}^+$ is $r = \infty$. The broken line $\mathcal{H}$ is the event horizon (if there is one in the metric). The null paths $1 \cup 2$ and 3 are the world lines of the two radial light rays coming to an observation point $\mathcal{O}$.