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Reduction in the Number of Independent Parameters for Models of Quantum Field Theory

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Reduction in the Number of Independent Parameters for Models of Quantum Field Theory

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1. Introduction

This talk concerns models of quantum field theory with several coupling parameters denoted by λ_o, $\lambda_1,\ldots,\lambda_n$. The model should be described by a renormalizable Lagrangian

$$\mathcal{L} = \mathcal{L}(\phi_1(x),\ldots,\phi_r(x),\ \lambda_o,\lambda_1,\ldots,\ \lambda_n,K^2,\ldots)$$

depending on fields $\phi_1,\ldots,\phi_r$, coupling parameters, a normalization mass $K^2 < 0$ and possibly particle masses. Primarily we consider models without intrinsic particle masses which, however, still involve a normalization mass $K^2 < 0$ needed for setting up perturbation theory in a consistent manner. A fundamental property of such models is the invariance under the renormalization group.[1] As a consequence of this invariance the Green's function

$$\tau = \langle T\phi_{i_1}(x_1)\ldots\phi_{i_N}(x_N)\rangle$$

(vacuum expectation values of time ordered products of field operators) satisfy a system of partial differential equations, the renormalization group equations[2],

$$\left(K^2\frac{\partial}{\partial K^2} + \sum_{i=1}^{n}\beta_i\frac{\partial}{\partial\lambda_i} + \sum_{\ell=1}^{N}\gamma_{i_\ell}\right)\tau = 0 \qquad (1)$$

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which express the possibility of changing the normalization of fields and parameters without modifying the physical contents of the theory. The normalization mass K which is not observable can be changed to a new value K' by rescaling the field operators and redefining the coupling parameters

$$\phi_j(x) \to \phi'_j(x) = \sqrt{Z_j}\ \phi_j(x), \quad Z_j > 0 , \tag{2}$$

$$K^2 \to K'^2, \quad \lambda_j \to \lambda'_j \quad .$$

The group of these transformations is called the renormalization group. The Callan-Symanzik functions β_i, γ_j depend only on the coupling parameters,

$$\beta_i = \beta_i(\lambda_o,\ldots,\lambda_n),\ \gamma_j = \gamma_j(\lambda_o,\ldots,\lambda_n) \ .$$

We may also include massive models in our discussion provided a formulation of the renormalization group is used where the β-functions do not depend on the mass ratios.[3]

2. Principle of Reduction

We ask ourselves the question if it is possible to reduce a system involving several couplings to a description in terms of a single coupling parameter only.[4] We select one of them, say λ_o, as the coupling on which all others should depend and call it the primary coupling. Of course, the functions $\lambda_i(\lambda_o)$ should not depend on the normalization mass K. It is natural to require, that in the limit of vanishing λ_o all other couplings are turned off too. Moreover, if the reduced model, should again resemble a renormalizable theory, all λ_j must be power series in λ_o. We thus impose the following three conditions

(i) $\lambda_j = \lambda_j(\lambda_o)$ independent of K (3)

(ii) $\lambda_j \to 0$ for $\lambda_o \to 0$ (4)

(iii) λ_i power series in λ_o (5)

and require renormalization group invariance for the original and the reduced model as well. Necessary and sufficient conditions for (1) follow by comparing the differential equations of the original and the reduced system.

$$\left(K^2 \frac{\partial}{\partial K^2} + \Sigma\, \beta_i \frac{\partial}{\partial \lambda_i} + \Sigma\, \gamma_\ell\right)\tau = 0$$

and

$$\left(K^2 \frac{\partial}{\partial K^2} + \beta_o' \frac{\partial}{\partial \lambda_o} + \Sigma\, \gamma_\ell'\right)\tau\Big|_{\lambda_i = \lambda_i(\lambda_o)} = 0$$

imply a system of ordinary differential equations for the functions $\lambda_j(\lambda_o)$:

$$\beta_o \frac{d\lambda_j}{d\lambda_o} = \beta_j \qquad j = 1,\ldots,n \tag{6}$$

Condition (i) can always be satisfied by solving (2) at a regular point near which a Lipschitz condition holds. Condition (ii) is restrictive. It cannot be satisfied for many models since $\lambda_i = \lambda_o = 0$ is a singular point where all β-functions vanish. (iii) only allows for a few solutions if any.

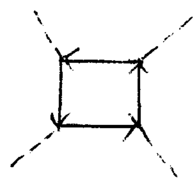
Reductions satisfying the conditions (i) - (iii) may be obtained by imposing a symmetry which is strong enough to relate all couplings such that only one remains independent. Then the Green's functions become power series in the remaining parameter provided there are no anomalies and the symmetry can be implemented in all orders of perturbation theory. But many reductions do not seem to be related to any symmetry. The principle of reduction may thus be viewed as a generalization of this particular aspect of symmetry.

3. Pseudoscalar Yukawa Interaction

The pseudoscalar Yukawa interaction

$$\mathcal{L}_I = ig_o\bar{\Psi}_o\gamma_5 A_o\psi_o$$

between a spinor field ψ_o and a pseudoscalar field A_o is not renormalizable in its original form. Divergent diagrams like



cannot be compensated by the parameters of the original Lagrangian. A new independent direct meson-meson interaction is needed for setting up a consistent renormalization scheme. The interaction Lagrangian then is

$$\mathcal{L}_I = ig_o\bar{\psi}_o\gamma_5 A_o\psi_o - \frac{\lambda_o}{4!} A_o^4$$

with the subscript 0 denoting unrenormalized quantities. The Green's functions then are power series in two independent (renormalized) parameters g and λ. Suppose a formulation is preferred involving power series in the original coupling g only. There is no obvious way of achieving this. Setting the renormalized coupling $\lambda = 0$, for instance, does not make sense. If this is done in one renormalization scheme, it would not be true in another. But the method of reduction provides a natural solution to this problem. In order to obtain a renormalizable description in terms of the original coupling g alone the system is reduced by requiring[4]

(i) $\lambda = \lambda(g^2)$

(ii) $\lambda \to +0$ for $g \to 0$

(iii) $\lambda = \rho_o g^2 + \rho_1 g^4 + \ldots$

combined with renormalization group invariance for the original and the reduced system as well. Necessary and sufficient for (i) is the ordinary differential equation (see eqn. (6))

$$\beta_{g^2} \frac{d\lambda}{dg^2} = \beta_\lambda \tag{7}$$

with

$$\beta_{g^2} = \frac{1}{16\pi^2} 5g^4 + \ldots$$

$$\beta_\lambda = \frac{1}{16\pi^2} (\frac{3}{2} \lambda^2 + 4\lambda g^2 - 24g^4) + \ldots$$

There is only one power series solution with $\lambda \geqq 0$, namely

$$\lambda = \frac{1}{3} (1 + \sqrt{145})\; g^2 + \rho_1 g^4 + \ldots \tag{8}$$

with all coefficients ρ_j uniquely determined.[4]

If the power series condition (iii) is dropped the general solution with $\lambda \geqq 0$ is given by the asymptotic expansion

$$\begin{aligned} \frac{\lambda}{g^2} &= \frac{1}{3} (1 + \sqrt{145}) + \rho_1 g^2 + \rho_2 g^4 + \ldots \\ &\quad + dg^{2\xi} + d_1 g^{2\xi+2} + \ldots \\ \xi &= \frac{1}{5} \sqrt{145} \end{aligned} \tag{9}$$

d is an arbitrary constant of integration. All other coefficients are uniquely determined.

4. General Class of Models with Two Couplings

For two variables the problem can be treated rigorously*. We assume that the β-functions in eqn. (7) are well defined in the neighborhood of the origin with asymptotic expansions of the form

$$\beta_{g^2} = bg^4 + \ldots, \quad \beta_\lambda = C_1\lambda^2 + C_2\lambda g^2 + C_3 g^4 + \ldots , \qquad (10)$$
$$b,\ C_1 \neq 0$$

The expansions (10) are presumably divergent for any non-trivial model in four dimensions. The special form of the leading term of β_{g^2} covers a large class of applications. The problem is to determine all solutions of (7) passing through the origin $\lambda = g^2 = 0$ with bounded ratio λ/g^2 and to establish their asymptotic form. A necessary condition is that the discriminant

$$\Delta = (C_2 - b)^2 - 4C_1C_3 \geqq 0 \qquad (11)$$

so that the quadratic equation

$$C_1\rho_\pm^2 + (C_2 - b)\rho_\pm + C_3 = 0 \qquad (12)$$

has real roots $\rho_+ \geqq \rho_-$. ρ_+ and ρ_- are the possible values of the limits

$$\lim_{g^2 \to 0} \frac{\lambda}{g^2} = \rho_\pm . \qquad (13)$$

We further introduce a characteristic exponent

$$\xi = -\frac{C_1}{b}(\rho_+ - \rho_-) \qquad (14)$$

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*For the mathematical literature see ref. [5]. The following discussion is based on ref. [7] where a rigorous treatment of the argument in ref. [4,6] is given. For the case b = 0 see ref. [8,9].

For non-integral ξ there are two formal power series solutions of (7)

$$\lambda = \rho_+ g^2 + \rho_{+1} g^4 + \ldots \tag{15}$$

$$\lambda = \rho_- g^2 + \rho_{-1} g^4 + \ldots \tag{16}$$

with uniquely determined coefficients. If the expansions (10) diverge the expansions (15, 16) are not expected to converge either. But the low order terms may be used to construct exact solutions with a similar asymptotic behavior for $g^2 \to 0$.

Let ξ, for example, be a positive non-integral number. Then all solutions of bounded λ/g^2 passing through $\lambda = g^2 = 0$ have the form

$$\lambda(g^2) = \rho_- g^2 + \ldots + \rho_{-n} g^{2n} + g^{2|\xi| + 2} \sigma(g^2) \tag{17}$$

$$(\xi > 0, \text{ non-integral})$$

n is the largest integer below $\xi + 1$. It can be shown that the differential equation in terms of σ satisfies a Lipschitz condition near the origin. Thus there is exactly one solution through a given point $\sigma = \sigma_o$, $g^2 = 0$ provided σ_o is sufficiently small. The case $\sigma_o = 0$ corresponds to the formal power series (16). The limit $|\sigma| \to \infty$ for $g^2 \to 0$ uniquely defines a solution with bounded λ/g^2 which corresponds to the other power series (17). Similar results hold for non-integral $\xi < 0$ with the subscripts + and - interchanged.

In the special case that the characteristic exponent ξ is an integer logarithmic terms may appear or there are infinitely many solutions with formal power series expansions. If $\xi = 0$ it is $\rho_+ = \rho_-$ and we define a function τ by

$$\lambda(g^2) = \rho_{\pm} g^2 - \frac{b g^2}{C_1(\tau(g^2) + \ell g(g^2))} \tag{18}$$

If $\xi = \pm 1, \pm 2, \ldots$ we define a function τ by

$$\lambda(g^2) = \rho_{\mp} g^2 + \sum_{j=1}^{|\xi|-1} \rho_{\mp j} \, g^{2j+2} \tag{19}$$

$$+ \, d_{\mp} g^{2|\xi|+2} \, \ell g \, g^2 + g^{2|\xi|+2} \, \tau(g^2)$$

In all cases the differential equation of τ satisfies a Lipschitz condition so that exactly one solution passes through any given point $\tau = \tau_o$, $g^2 = 0$ (sufficiently close to the origin). For $\lim\limits_{g^2 \to 0} |\tau| = \infty$ there is exactly one solution of bounded λ/g^2. This has a formal power series expansion

$$\begin{aligned} \lambda &= \rho_{\pm} g^2 + \sum_{i=1}^{\infty} \rho_i g^{2i+2} \qquad \text{if } \xi = 0 \\ \lambda &= \rho_{\mp} g^2 + \sum_{i=1}^{\infty} \rho_{\mp i} \, g^{2i+2} \qquad \text{if } \xi = \pm 1, \pm 2, \ldots \end{aligned} \tag{20}$$

Infinitely many power series occur if the (unique) coefficients d_- or d_+ in (19) vanish.

5. Two Scalar Fields

The general form of an interaction for two scalar fields A_1, A_2 with dimensionless couplings is

$$\mathcal{L}_I = \frac{1}{4!} \sum_{abcd} \lambda_{abcd} \, A_a A_b A_c A_d \tag{21}$$

Since the λ_{abcd} may be chosen symmetric there are five independent components

$$\lambda_o = \lambda_{1111}, \ \lambda_1 = \lambda_{1122}, \ \lambda_2 = \lambda_{1112}, \ \lambda_3 = \lambda_{12222}, \ \lambda_4 = \lambda_{2222} \quad .$$

Setting up and solving the reduction equations one finds that the O(2) -

symmetric case represents the only non-trivial reduction. Choosing λ_o as primary coupling one obtains $\lambda_1 = \frac{1}{3}\lambda_o + \rho_{11}\lambda_o^2 + \ldots$, $\lambda_2 = \lambda_3 = 0$, $\lambda_4 = \lambda_o + \rho_{41}\lambda_o^2 + \ldots$. All coefficients on the expansions are uniquely determined. The Lagrangian does not immediately appear in symmetric form, since the defining relations for the coupling parameters could not be assumed to be symmetric. But by a redefinition of the coupling parameters it is always possible to make the lowest order relations exact

$$\lambda_1' = \frac{1}{3}\lambda_o' \;,\; \lambda_4' = \lambda_o' \;,$$

so that the interaction Lagrangian becomes manifestly O(2)-invariant:

$$\mathcal{L}_I = \frac{1}{4!}\lambda_o'(A_1^2 + A_2^2)^2 \;. \tag{22}$$

There are more solutions of the reduction equations, but they all correspond to commuting subsystems, for further details see ref. [10].

6. Application to N = 2 Supersymmetric Yang-Mills Theory

The reduction method can be used to construct minimal Yang-Mills interactions involving the gauge coupling as the only coupling parameter.[11] Suppose a system of free massless matter fields is given, for instance, a spinor field ψ, a scalar field A and a pseudoscalar field B, all in the adjoint representation of SU(2). The aim is to couple the system minimally to a Yang-Mills field with gauge group SU(2). The classical minimal Lagrangian is

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}\,G^{\mu\nu}_a - \frac{1}{2}\,D_\mu A\,D^\mu A - \frac{1}{2}\,D_\mu B D^\mu B - \bar{\psi}\gamma_\mu D^\mu\psi \tag{23}$$

to which gauge fixing terms should be added for quantizing. $G^a_{\mu\nu}$ denotes the gauge field strength, D_μ the covariant derivative. In this form the Lagrangian cannot be quantized since quartic and Yukawa couplings are generated. Adding

$$-i\sqrt{\lambda_1}\ \varepsilon^{abc}\,\overline{\psi}^a(A^b + \gamma_5 B^b)\psi^c - \frac{1}{4}\lambda_2(\vec{A}^2 + \vec{B}^2)^2$$
$$+ \frac{1}{4}\lambda_3\ ((\vec{A}^2)^2 + (\vec{B}^2)^2 + 2(\vec{A}\vec{B})^2)$$

the most general form of a Lagrangian is obtained which has dimension four and is invariant under the symmetries of the original system. In this form the model can be renormalized, but it involves λ_1, λ_2 and λ_3 as additional independent parameters. We now impose that the λ_j be functions of the gauge coupling g demanding renormalization group invariance with the requirements (i) to (iii) for the reduced model. Then the formulation involves the gauge coupling g as the only coupling parameters with asymptotic power series expansions in g for the Green's functions. This comes closest to the original concept of a minimal Yang-Mills interaction. The solutions of (6) with $\lambda_o = g^2$ being the primary coupling which satisfy the requirements (i) to (iii) are as follows:

(a) Supersymmetric solution

$$\lambda_j = g^2 + \rho_{j1}\, g^4 + \ldots \qquad i = 1,2,3 \qquad (24)$$

with all coefficients uniquely determined. This reduction represents the supersymmetric N = 2 Yang - Mills theory. Its β-function is

$$\beta(g^2) = \beta_o(g^2,\ \lambda_1(g^2),\ \lambda_2(g^2),\ \lambda_3(g^2))\ .$$

Thus the β-function of the supersymmetric model is uniquely determined by the non-symmetric formulation with independent λ_1, λ_2, λ_3.[12] By a redefinition of the coupling parameters λ_j the lowest order can be made exact so that $\lambda_j = g^2$ for the supersymmetric case.

(b) One parametric family of solutions breaking supersymmetry

$$\lambda_1 = g^2 + \ldots, \ \lambda_2 = g^2 + ag^3 + \ldots, \ \lambda_3 = g^2 + 3ag^3 + \ldots \ .$$

Here the coefficient a is arbitrary, all higher order coefficients are unique.

(c) Isolated solution not known to be related to a symmetry

$$\lambda_1 = g^2 + \ldots, \ \lambda_2 = \frac{9}{\sqrt{105}} \ g^2 + \ldots, \quad \lambda_3 = \frac{7}{\sqrt{105}} g^2 + \ldots \ .$$

7. Asymptotic Solutions in General

We consider the case where the β-function of the primary coupling has the asymptotic form

$$\beta_o = b\lambda_o^2 + \ldots \qquad \text{with } b \neq 0 \ . \tag{25}$$

The leading terms of the other β-functions are assumed to be quadratic forms in $\lambda_o, \lambda_1, \ldots, \lambda_n$. It is possible to work out the detailed conditions for the existence of power series solutions of (6)[11]

$$\frac{\lambda_j}{\lambda_o} = \rho_{j0} + \rho_{j1} \lambda_o + \rho_{j2}\lambda_o^2 + \ldots \tag{26}$$

The lowest order values ρ_{i0} are real roots at a system of bilinear equations. A power series solution (26) of (6) is either isolated for $\lambda \to 0$ as embedded in a family of solutions in most cases involving non-integral powers

$$\frac{\lambda_j}{\lambda_o} = \rho_{j0} + \rho_{j1} \lambda_o + \ldots + \sum_{\alpha=1}^{\ell} a_{j\alpha}\lambda_o^{\xi_\alpha} + \ldots \tag{27}$$

The general expansion term is of the term

$$g^{2(p_1\xi_1 + \ldots + p_\ell\xi_\ell + q)} \tag{28}$$

where $p_1, \ldots, p_\ell$, q are non-negative integers. The exponents $\xi_1, \ldots, \xi_\ell$ are the positive eigenvalues of the nxn matrix[11]

$$\Xi = \left\| \frac{1}{b} \frac{\partial \beta_j^{(o)}}{\partial \rho_{ko}} - \delta_{jk} \right\| \quad , \quad \beta_j^{(o)} = \lim_{\lambda_o \to 0} \frac{\beta_j}{\lambda_o} \ . \tag{29}$$

In all applications so far the eigenvalues of Ξ are real. The modification of (27, 28) for a matrix Ξ with complex eigenvalues is straightforward. In some cases also logarithmic terms occur in the expansion (27) of the general solution.

8. Effective Couplings

In models of quantum field theory effective couplings $\bar{\lambda}_j$ are defined by certain Green's functions as functions depending on the square k^2 of a momentum variable k. At the normalization mass $k^2 = K^2$ the effective couplings coincide with the original coupling parameter,

$$\bar{\lambda}_j = \lambda_j \text{ at } k^2 = K^2 \ .$$

Effective couplings provide a qualitative measure for the coupling strength in dependence on energy. They satisfy a system of ordinary differential equations,

$$u \frac{\partial \bar{\lambda}_j}{\partial u} = \bar{\beta}_j \ , \quad u = \frac{k^2}{K^2} \quad , \tag{30}$$

$$\bar{\lambda}_j = \bar{\lambda}_j \ (u, \lambda_o, \ldots, \lambda_n), \quad \bar{\beta}_j = \beta_i(\bar{\lambda}_o, \ldots, \bar{\lambda}_n)$$

which are called the evolution equations. Elimination of the scale variable u in (30) leads to

$$\bar{\beta}_o \frac{d\bar{\lambda}_j}{d\bar{\lambda}_o} = \bar{\beta}_j \ , \ \bar{\lambda}_j = \bar{\lambda}_j \ (u(\bar{\lambda}_o), \lambda_o, \ldots, \lambda_n) \ . \tag{31}$$

These are the reduction equations (6) but with a different interpretation. The system (31) represents identities which always hold for effective couplings.[4,6]

Now let us suppose that the reduction equations (6) allow for solutions satisfying (4) with

$$\lim_{\lambda_o \to 0} \frac{\lambda_j}{\lambda_o} \text{ finite} \quad . \tag{32}$$

This condition can always be satisfied by an appropriate selection of the primary coupling λ_o. Then either[11]

$$\beta_o \equiv 0$$

or

$$\beta_o < 0 \quad \text{(asymptotic freedom}^{13}\text{)},$$

or

$$\beta_o > 0 \quad \text{(asymptotic freedom in the infrared region for massless models)}$$

holds for the β-function of the reduced theory. For the effective couplings this implies

$$\begin{aligned} \beta_o &\equiv 0 : && \text{all } \bar{\ell}_i \text{ constant ,} \\ \beta_o &< 0 : && \overline{\lambda_j} \to 0 \quad \text{for } k^2 \to -\infty \, , \\ \beta_o &> 0 : && \overline{\lambda_i} \to 0 \quad \text{for } k^2 \to 0 \\ & && \text{(in case of massless models).} \end{aligned} \tag{33}$$

This clarifies the connection between the reduction principle and the asymptotic behavior for large or small momenta.

Closely related is further the concept of ultraviolet stability. A system is called ultraviolet stable with the primary coupling λ_o if

$$\delta\bar{\lambda}_o \to 0 \quad , \quad \delta(\bar{\lambda}_j/\bar{\lambda}_o) \to 0 \tag{34}$$

uniformly for $|k^2| \to \infty$ provided the variations at a fixed momentum k_o are chosen small enough. This is stability in the sense of Lyapunov for $\bar{\lambda}_o$ and $\bar{\lambda}_j/\bar{\lambda}_o$

in the limit $|k^2| \to \infty$.[14] An application of Lyapunov's theory yields the following stability criterion.[11] Again we assume that the β-function of the primary coupling is of the asymptotic form (25) while the asymptotic expansions of the other β-functions start off with general quadratic forms. Then a system is ultraviolet stable if $b < 0$ and $\mathrm{Im}\kappa < 0$ for all eigenvalues κ of the stability matrix

$$\Gamma = \left\| \frac{\partial\beta_j^{(o)}}{\partial\rho_{ko}} - b\, d_{jk} \right\| , \quad \beta_j^{(o)} = \lim_{\lambda_o \to 0} \frac{\beta_j}{\lambda_o} \tag{35}$$

For $b < 0$ the system is ultraviolet unstable it $\mathrm{Im}\kappa > 0$ for at least one eigenvalue of Γ . The exponent matrix (29) and the stability matrix (35) are related by

$$\Xi = \frac{1}{b}\,\Gamma$$

The eigenvalues of Γ with $\mathrm{Im}\kappa < 0$ lead to characteristic exponents ξ of the general solution to (6) with the requirement (4). The number of eigenvalues of Γ with $\mathrm{Im}\kappa < 0$ equals the number of integration constants in the general solution.

9. Application to the Standard Model[15]

The gauge symmetry group of the standard model of the strong, electromagnetic and weak interactions is SU(3) x SU(2) x U(1) with the gauge coupling parameters g_s, g and g' respectively. The three gauge coupling parameters cannot be reduced due to opposite signs of the β-functions. Grand unification provides a more natural way of obtaining a single parameter description.

Without symmetry breaking the standard model does not allow for mass terms neither for the W-meson mediating the weak interactions nor for the leptons and quarks. In order to provide for particle masses a Higgs coupling λ

is introduced with spontaneous symmetry breaking providing a mass for the W-meson. Further Yukawa couplings between the fermion fields and the Higgs field are added which allow for arbitrary quark and lepton masses.

While reduction is not possible for the gauge couplings it seems reasonable to apply the reduction principle to the Higgs coupling and the Yukawa couplings of the quarks. One starts by neglecting the electroweak couplings

$$g = g' = 0 .$$

We further neglect family mixing assuming that the mass matrix implied by the Yukawa couplings is diagonal. The remaining coupling parameters of the model are

(a) strong gauge coupling g_s,

(b) Higgs coupling λ,

(c) 6 Yukawa couplings G^u_i, G^d_j, $j = 1, 2, 3$.

The case of three fermion families is assumed with n denoting up quarks and d denoting down quarks. The couplings λ, G^u_j, G^d_j determine the corresponding masses of the Higgs particle and the six quarks.

We apply the reduction principle to these couplings by setting up the reduction equations (6) with

$$x = \frac{g_s^2}{4\pi} \tag{36}$$

as the primary coupling. At first power series solutions satisfying (4) are determined. One possibility is the trivial solution

$$G_q \equiv 0 , \quad \lambda \equiv 0 \tag{37}$$

which represents pure quantum chromodynamics without Higgs and Yukawa couplings. Apart from this trivial reduction there are several non-trivial reductions. Realistic is only

$$G_q \equiv 0 \quad \text{except for the top quark ,}$$

$$\frac{G_{top}^2}{4\pi} = \frac{2}{9}\,x + \sum_{j=2}^{\infty} a_j x^j \tag{38}$$

for the top quark (the up quark of the third family),

$$\frac{\lambda}{4\pi} = \frac{\sqrt{689} - 25}{18}\,x + \sum_{j=2}^{\infty} b_j\, x^j \ .$$

The coefficients a_j and b_j are uniquely determined. With $x = 0.1$ (at the normalization mass M_w) the corresponding masses would be

$$\begin{aligned} &m_{top} \simeq 90 \text{ GeV} \qquad m_{Higgs} \simeq 50 \text{ GeV} \\ &m_q = 0 \qquad\qquad q \neq \text{top} \end{aligned} \tag{39}$$

neglecting terms of order x^2. Corrections to these values are due to higher orders in x (expected to be small), to other quark masses and to weak/electromagnetic interactions.

A. Corrections from Other Quark Masses

The power series expansions (38) imply $G_q = 0$ to all orders for $q \neq$ top unless unexpected anomalies occur. In order to allow for non-perturbative effects like mass generation we drop the power series requirement demanding only $G_q^2 \to +0$, $\lambda \to +0$ for $g_s^2 \to +0$ with the ratios of the non-trivial coupling (38) in the limit. The corresponding general solution has the asymptotic expansion

$$\frac{G_q^2/4\pi}{x} = a_q\, x^{\frac{1}{21}} + \Sigma\ a_{qn}\, x^{\frac{n}{21}} \tag{40}$$

for quarks q of the first two families,

$$\frac{G_{bottom}^2/4\pi}{x} = a_b x^{\frac{2}{21}} + \Sigma\ a_{bn} x^{\frac{n}{21}} \tag{41}$$

for the bottom quark (the down quark of the third family),

$$\frac{G^2_{top/4\pi}}{x} = \frac{2}{g} + \Sigma\ a_{tn}\ x^{\frac{n}{21}} \tag{42}$$

for the top quark (the up quark of the third family),

$$\frac{\lambda/4\pi}{x} = \frac{\sqrt{689} - 25}{18} + \Sigma\ a_{Hn}\ x^{\frac{n}{21}} . \tag{43}$$

a_q and a_b are five arbitrary coefficients. All other coefficients are uniquely determined. Hence there are five free parameters which may be used to adjust for the five known quark masses. The top quark and the Higgs mass (both yet unknown) become functions of the five lower quark masses. No arbitrary parameters occur in the expansions of the top Yukawa and the Higgs coupling since the corresponding eigenvalues of the exponent matrix (29) are negative. The characteristic exponents for the five lower quark Yukawa couplings are given by the positive eigenvalues of (29), namely

$$\xi_a = \frac{1}{21} \quad \text{for all quarks of families 1 and 2,}$$

$$\xi_a = \frac{2}{21} \quad \text{for the bottom quark}$$

Neglecting contributions from quarks of the first and second family the leading terms for the bottom and top Yukawa couplings are (the Higgs coupling does not contribute in this order)

$$\frac{G^2_{bottom}/4\pi}{x} = d_b\ x^{\frac{2}{21}} + \dots ,$$

$$\frac{G^2_{top}/4\pi}{x} = \frac{2}{9} - \frac{1}{5}\, d_b\ x^{\frac{2}{21}} + \dots$$

$$= \frac{2}{9} - \frac{1}{5}\ \frac{G^2_{bottom/4\pi}}{x} + \dots . \tag{44}$$

Using the experimental value $m_{bottom} \simeq 5$ GeV we obtain

$$\frac{G^2_{bottom}/4\pi}{x} \simeq 7.10^{-4}$$

Hence the contribution from the bottom quark is negligible as compared to the leading term 2/9. A similar result is obtained for the Higgs mass. Corrections from quarks of the first and second family are even smaller.

B. Weak and Electromagnetic Corrections

If the reduction equations are solved with the dependence on g and g^1 included one finds the corrected values

$$m_{top} = 81 \text{ GeV} \quad m_{Higgs} = 61 \text{ GeV} \tag{45}$$

for the non-trivial reduction.[14] An expansion with respect to powers of

$$\frac{g^2}{g_s^2} = 0.37 \qquad \frac{g'^2}{g_s^2} = 0.1$$

was used. The error of the predictions (45) due to uncertainties of the value of $\sin^2\theta_W$ and α_s is estimated to be of the order 10 - 15%.

C. Trivial Deduction

Finally we discuss the trivial reduction (37). Since the corresponding Higgs and quark masses all vanish this does not look promising as a starting point for generating heavy masses. However, the surrounding general solution with

$$\frac{G_q^2}{g_s^2} \to +0 \, , \qquad \frac{\lambda}{g_s^2} \to +0$$

involves six free parameters. The generation of large quark masses seems unlikely, but formally the adjustment of all six quark masses is possible. The Higgs m_H is then a function of the quark masses.

Except for the mass of the top quark all other quark masses may be neglected. The values (45) of the non-trivial reduction are upper bounds for the allowed values of the trivial reduction. Following are some values of the Higgs mass for given top mass

m_{top}	40	60	80
m_{Higgs}	44	49	60

(electroweak corrections are included).

The asymptotic expansions embedding the trivial or non-trivial reduction are the only realistic solutions of the reduction equations compatible with asymptotic freedom for the strong interactions (including Higgs and quark Yukawa couplings).

10. Summary

The principle of reduction generalizes symmetry constraints which relate the couplings among each other so that only one of them remains independent. Since all such symmetries are covered by the reduction method (provided they can be implemented in all orders) new symmetries could be discovered in this way. While no new symmetry has been found so far, the reduction principle allows for many other possibilities which do not seem to be related to any symmetry. An example is the application to the standard model where for one solution the top quark and the Higgs mass are determined (non-trivial reduction). Another - less convincing - solution of the reduction equations (trivial reduction) determines the Higgs mass as a function of the top mass with the values of the non-trivial reduction as upper bounds. The numerical values obtained should be taken with some caution since the standard model is only on effective theory approximating a more fundamental interaction. Therefore, its β-functions are also only approximate and changes in their lowest order coefficients may have large effects on the reduction solutions.

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