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HARMONIC ANALYSIS IN SCATTERING THEORY

by

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Abstract: The main part of this lecture is an exposition of the results of a recent paper entitled: "Volterra Algebra and Bethe--Salpeter Equation", written in collaboration with J.Faraut[(1)]. We do not give here the proofs of the theorems; but we shall try to illustrate the physical meaning of the results and to describe the framework of the theory where these results acquire meaning.

I - Introduction

N.N. Bogoliubov and D.V. Shirkov in the conclusions of the first edition of their book[(2)] "Introduction to the Theory of Quantized Field" (1959) wrote:

"Whereas in quantum electrodynamics, in view of the smallness of the square of the charge e^2 perturbation theory ceases to be applicable only at energies unattainable in practice, in pseudoscalar meson theory the method of weak coupling is not applicable at all, since the square of the corresponding charge $g^2/4 \quad = 15$ and one may consider expansions in power of this quantity only when one is in a state of utter desperation".

At that time prevailed the opinion that perturbation methods could not be used in strong-interaction physics. Therefore it was proposed an approach to the physics of hadrons based on those principles which are sufficient to provide a basis for the construction of the S-matrix (this approach was sometimes called S-matrix approach to strong interactions physics). These basic principles are:

1) Conservation of Probability (Unitarity of the S-matrix);

2) Causality;

3) Crossing.

In particular the crossing property (Bros Epstein and Glaser 1965[(3)]) guarantees that it is possible to continue analytically the mass-shell particle-particle amplitude to the particle-antiparticle amplitude. Of course it remains to see the practical ways for performing this analytic continuation.

But in 1968 the SLAC experiment[(4)] drastically changed the scenario of hadron physics. This experiment, which is usually referred as "deep inelastic scattering of electrons on hadrons"[5] can be schematically viewed as shown in fig.1

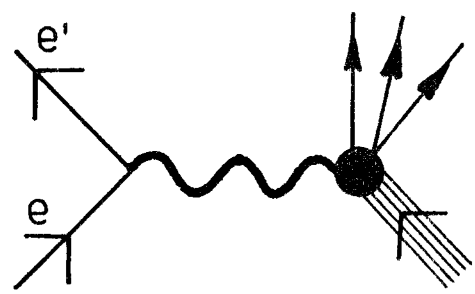

fig. 1

When a sufficiently high momentum transfer reaction takes place, the projectile, let it be an electron, sees the target as made up of nearly free constituents. Moreover the scattering from individual constituents is incoherent. The experiment indicates that:

The hadrons possess a "granular" structure and these "granules" behave as point-like almost free objects. This is the basic idea of Feymann "parton-model"[5] (1969).

Then these "constituents" may be identified with the "quarks". The quarks are fractionally charged particles, previously predicted by Gell-Mann on the basis of unitary symmetry model of elementary particles.

Besides the SLAC experiment one should also take into account another fact, which may be regarded as an experimental result (even if it is, in a certain sense, negative): in spite of many efforts it has never been possible to have a direct evidence of a free quark (a quark outside the hadron).

This fact might be explained by saying that the "quarks" are "confined" within the hadrons. Or, more precisely, that they are strongly bounded at a distance of the order of magnitude of the radius of the hadron.

These two experimental results give the following indications:

i) At large momentum transfer (small distances among the partons) the perturbative expansions are worth trying; indeed the quarks are weakly bounded. Therefore at large angles (backward scattering) the perturbative methods proper of a Lagrangian field theory (like Quantum-Cromo-Dynamics) may work satisfactorily(*).

ii) The fact that the "quarks" have never been observed as "free-particles" imposes restrictions to perturbation methods. Indeed at distances of the order of magnitude of the radius of the hadron they interact strongly and perturbative expansions do not work. Therefore at small momentum transfer (forward scattering) we must still use methods which are not based on perturbative expansions, like, for instance, a S-matrix approach.

In conclusion we can say that:

a) Lagrangian field theory is appropriate to describe backward scattering;

b) S-matrix non-perturbative methods are appropriate for describing forward scattering.

In this sense these two methods should be regarded as complementary.

(*) We see that the statement of Bogoliubov and Shirkov is unacceptable to-day, since the perturbative expansions may have a great role even in hadron physics.

II - Kinematics: Harmonic Analysis of Scattering Amplitude.

After this background let us come to scattering theory. At the beginning we focus our attention on the s-channel (direct-channel) and write a Fourier-Legendre (or partial-waves expansion) for the amplitude A(s,t) (s,t being the usual Mandelstam variables). In the case of equal mass spinless particles, we have:

$$A(s,t)=\sum_{\ell=0}^{\infty} a_\ell(s)\, P_\ell(\cos\vartheta)\ ; \qquad \cos\vartheta = 1+\frac{2t}{s-4m^2} \qquad (1)$$

ℓ being the relative angular momentum of the two particles.

The "Crossing Property" means that if we write a partial-waves expansion of the scattering amplitude in the different channels, we may regard these expansions as local representations of the same function. The usual way practised for connecting the direct channel with the crossed one, consists in trying to sum the series (1) with the Watson-Sommerfeld transform obtaining the so-called Regge representation.

$$A(s,t)=\frac{i}{2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \frac{a(s,\lambda)}{\sin\pi\lambda}\, P_\lambda\left(-1-\frac{2t}{s-4m^2}\right) d\lambda\, +$$

$$+\sum_{n=1}^{N} \frac{\gamma_n(s)}{\sin\pi\lambda_n(s)}\, P_{\lambda_n}\left(-1-\frac{2t}{s-4m^2}\right) \qquad (2)$$

This representation splits the amplitude in two terms: the first is the so-called background integral, the second is a sum over a finite number of poles of the scattering amplitude in the complex angular momentum plane. Now recalling the asymptotic behaviour of the Legendre functions one obtains:

$$A(s,t) \propto f(s)\, t^{\lambda_0(s)} \qquad (t \to \infty) \tag{3}$$

where λ_0 stands for the pole with the largest Re λ_n. Going in the crossed channel and exchanging s with t one could predict an asymptotic bahaviour for large value of s (high-energy) of the following type:

$$A(s,t) \propto f(t)\, s^{\lambda_0(t)} \qquad (s \to \infty) \tag{4}$$

Furthermore one could hope to connect the asymptotic behaviour in energy of the crossed reaction with the poles of the scattering amplitude which moving, as functions of s, in the complex angular momentum plane, are supposed to correlate various bound-states and resonances of the direct channel.

But this procedure is not free from defects. Let us mention some of them:

a) The use of Watson-Sommerfeld transform requires certain properties of the partial waves $a_l(s)$, continued in the complex angular momentum plane. These properties can be proved only in non-relativistic potential scattering for certain classes of potentials like Yukawian potentials. Therefore all the theory is borrowed from a non-relativistic framework.

b) The idea of correlating resonances, which occur for different partial waves by means of poles moving in the complex angular momentum plane as function of s, does not have a sound fundation. Indeed numerical calculations of trajectories for Yukawa - type potentials (which are connected with the non-relativistic limit of quantum field theory for strongly interacting particles[6]), show that these trajectories do always leave the real axis very rapidly, and in practice a recurrence of resonances belonging to the same trajectory is never realized[7].

c) Last but not least, if we want to perform a phenomenological analysis of experimental data (necessarily affected by noise), connecting direct and crossed channel, uniqueness of analytic continuation is not sufficient to guarantee stability. Let us remind, in fact, that analytic continuation is a classical example of improperly posed problems in the sense of Hadamard.

All these difficulties suggest of reconsidering the problem of the asymptotic behaviour of the scattering amplitude for large values of s by means of different techniques. In this sense harmonic analysis could be a proper mathematical tool.

To this purpose let us return to the Fourier-Legendre expansion (1) and regard this series from the point of view of harmonic analysis. First of all one could observe that formula (1) describes at fixed s the angular distribution of the collision process; then it factorizes the amplitude into dynamics and symmetry: the coefficients $a_l(s)$ contain the dynamics and the Legendre polynomials reflect the symmetry. These latter functions, indeed, are the spherical functions on SO(3)/SO(2). Now SO(3) is precisely the group which leaves unchanged the vector $(\sqrt{s}, 0, 0, 0,)$; SO(3), in fact, is the group of spatial rotation and does not operate on the time-component. In conclusion this expansion is obtained coupling two incoming (or two outgoing) particles; the kinematics is properly described in the center of mass system. Finally a rigorous foundation of this expansion is given by a reduction of the product representations of the Poincaré group correspoding to the two-particles incoming or outgoing states into irreducible components. This reduction can be realized through the technique of the Clesbch-Gordan coefficients (see Joos[8]).

Then let us consider a coupling between an incoming with an outgoing particle. Now is the squared momentum transfer t which is fixed and we shall look for a group which leaves unchanged the vector $(0, 0, 0, \sqrt{-t})$. The group we are looking for is $SO_o(1,2)$ and $SO_o(1,3)$ in the case $t = 0$ (we shall consider the case $t \neq 0$ only). Observe that we still remain in the direct channel (s-channel), but now $t < 0$ is fixed (while s is distributed); and t is precisely the squared momentum carried by the exchanged particle.

Now we may formulate the question: is it possible to perform a decomposition of the scattering amplitude according to the quantum numbers of the "exchanged-object"?

This decomposition should be similar to the usual partial waves decomposition which is performed according to the quantum numbers of the "state" formed by the interacting particles. Therefore it may be called "crossed partial-waves decomposition"(9). The problem is still to reduce the product representation corresponding to an incoming and to an outgoing particles, into irreducible components. This reduction can be achieved by the use of the Clebsch-Gordan coefficients which have been evaluated by Moussa and Stora and used precisely in this context by Joos(10). This reduction, in the case of spinless particles, gives rise to a Fourier transform of the scattering amplitude $A(s,t)$ on the symmetric space $SO_o(1,2)/SO(2)$(11).

Now let us focus our attention on this Fourier transform. $SO_o(1,2)/SO(2)$ is the upper sheet of the two-sheeted hyperboloid; its Riemannian metric in terms of polar coordinates (r, ϑ) reads as follows:

$$ds^2 = dr^2 + (\sinh r)^2 \, d\vartheta^2 \tag{5}$$

The Laplace-Beltrami operator is:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\cosh r}{\sinh r}\frac{\partial}{\partial r} + \frac{1}{(\sinh r)^2}\frac{\partial^2}{\partial \vartheta^2} \tag{6}$$

In many istances (and especially for gaining a geometrical and intuitive insight in this Fourier transform) it is useful to consider models of hyperbolic geometry in $\mathbb{R}^2$. A model can be realized in the open unit disc D in $\mathbb{R}^2$, equipped with the following metric:

$$ds^2 = \frac{4\left[dx^2 + dy^2\right]}{\left[1 - x^2 - y^2\right]^2} \tag{7}$$

The transformation between the coordinates (r, ϑ) and (x,y) is given by:

$$\begin{cases} x = \tanh\left(\frac{r}{2}\right)\sin\vartheta \\ y = \tanh\left(\frac{r}{2}\right)\cos\vartheta \end{cases}$$

This non-euclidean disc can be identified with the symmetric space SU(1,1)/SO(2). In fact the group G = SU(1,1) $\left\{\text{group of matrices of the form, } \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, |a|^2 - |b|^2 = 1;\ a,b \in \mathbb{C}\right\}$ acts as a group of isometries of D by means of the maps:

$$g : z \longrightarrow \frac{az + b}{\bar{b}z + \bar{a}};\ z \in D,\ g \in G \tag{8}$$

This action is transitive; the subgroup fixing the centre of the unit disc is SO(2) and we have the identification: D = SU(1,1)/SO(2). Here the Laplace-Beltrami operator is given by:

$$\Delta = \frac{1}{4}\left(1 - x^2 - y^2\right)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{9}$$

In this model a pencil of parallel straight-lines is given by arcs of circles orthogonal to the unit circle, lying in its interior, and intersecting the boundary B of the unit disc D at a common point b. The trajectories, orthogonal to this pencil of parallel geodesics, are the circles tangent from within to the boundary B at the point

b. These circles are the euclidean images of the horocycles (see fig.2)

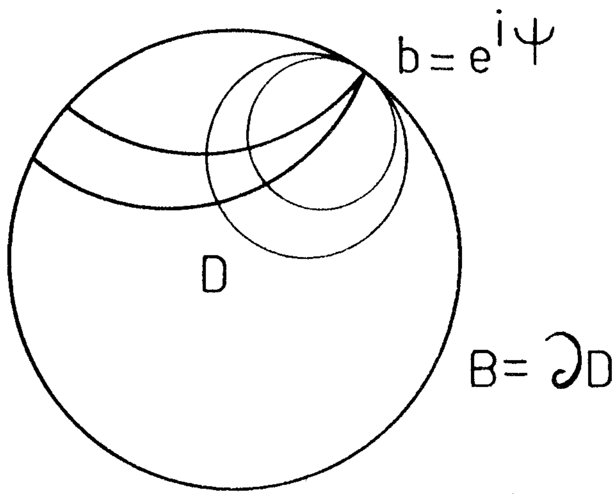

fig. 2

Now the classical Poisson kernel

$$P(z,b) = \frac{1-|z|^2}{1+|z|^2 - 2|z|\cos(\vartheta-\psi)} \qquad (10)$$

(where $z = |z|e^{i\vartheta}$, $b = e^{i\psi}$) has the following properties:

a) its level lines are the circles tangent to the unit circle at the point $b = e^{i\psi}$: the euclidean images of horocycles.

b) $[P(z,b)]^{\mu}$, $\mu \in \mathbb{C}$ is an eigenfunction of the Laplace-Beltrami operator.

c) P(z,b) is invariant with respect to any transformation that preserves the unit disc.

Therefore one can say (see S. Helgason[12]) that:

$$e^{\mu\langle z,b\rangle} = \left[\frac{1-|z|^2}{1+|z|^2 - 2|z|\cos(\vartheta-\psi)}\right]^{\mu}, \quad \mu \in \mathbb{C} \qquad (11)$$

may be regarded as the non-euclidean analog of the plane-wave. In fact P(z,b) is constant on each horocycle with normal b and passing through z. Indeed on has:

$$\langle z, b\rangle = d(0,z) = \log \frac{1+|z|}{1-|z|} \qquad (12)$$

These facts suggest to write a non-euclidean Fourier transform, in strict analogy with Fourier transform on $\mathbb{R}^2$ (Fourier-Bessel transform) as follows:

$$\hat{f}(\lambda, b) = \int_D e^{(-i\lambda+\frac{1}{2})\langle z, b\rangle} f(z)\,dz\,;\ \lambda \in \mathbb{R},\ b \in B \qquad (13)$$

where dz is the surface element on D. Then one can formulate the following theorem which is due to Helgason.

<u>Theorem</u> (Helgason[13]): For $f \in C_c^\infty(D)$ let $\hat{f}$ denote the Fourier transform:

$$\hat{f}(\lambda, b) = \int_D e^{(-i\lambda+\frac{1}{2})\langle z, b\rangle} f(z)\,dz\,;\ \lambda \in \mathbb{R},\ b \in B \qquad (14)$$

Then:

$$f(z) = \int_{\mathbb{R}}\int_B e^{(i\lambda+\frac{1}{2})\langle z, b\rangle} \hat{f}(\lambda, b)\,d\mu(\lambda, b) \qquad (15)$$

where

$$d\mu(\lambda, b) = (2\pi)^{-2}\lambda \tanh(\pi\lambda)\,d\lambda\,db$$

$d\lambda$ being the euclidean measure on $\mathbb{R}$, db the angular measure on B. Moreover if $\mathbb{R}^+$ denotes the set of positive reals, the mapping $f \longrightarrow \hat{f}$ extends to an isometry of $L^2(D, dz)$ to $L^2(\mathbb{R}^+ \times B, 2d\mu)$.

Now returning to the Poisson kernel and using the polar coordinates (i.e. writing: $z = \sqrt{x^2+y^2}\, e^{i\vartheta} = \tanh(r/2)e^{i\vartheta}$), we obtain:

$$e^{\mu\langle z,b\rangle} = \left[\frac{1-|z|^2}{1+|z|^2-2|z|\cos(\vartheta-\psi)}\right]^{\mu} =$$

$$= \left(\frac{1}{\cosh r - \sinh r \cos(\vartheta-\psi)}\right)^{\mu};\quad \mu\in\mathbb{C} \qquad (16)$$

Then the integral

$$\frac{1}{2\pi}\int_B e^{\mu\langle z,b\rangle}\, db = \frac{1}{2\pi}\int_0^{2\pi}(\cosh r - \sinh r\cos\varphi)^{-\mu}\, d\varphi =$$
$$= \mathcal{P}_{-\mu}(\cosh r) \qquad (17)$$

where $\mathcal{P}_{-\mu}(\cosh r)$ is the Legendre function of the first kind. These functions are spherical functions on G/K (G=SU(1,1), K=SO(2)). Indeed (let us denote these functions by Φ) we have:

i) They are bi-invariant under K

$$\Phi(k x k') = \Phi(x) \quad ; \quad x\in G\ ;\ k, k'\in K \qquad (18)$$

ii) They are eigenfunctions of the Laplace-Beltrami operator.

iii) They satisfy the following equation

$$\int_K \Phi(x k y)\, dk = \Phi(x)\,\Phi(y);\ x, y\in G;\ k\in K \qquad (19)$$

which, returning to the classical notation, reads as follows:

$$\frac{1}{2\pi}\int_0^{2\pi}\mathcal{P}_\mu(\cosh r_1\cosh r_2 + \sinh r_1\sinh r_2\cos\varphi)\, d\varphi =$$

$$= \mathcal{P}_\mu(\cosh r_1)\,\mathcal{P}_\mu(\cosh r_2)\ ;\ \mu\in\mathbb{C}$$

which is the well-known product formula for the Legendre functions of the first kind.

Now let us return to the non-euclidean Fourier transform written before. Then if f is bi-invariant under K, then this Fourier transform reduce to classical Mehler transform; indeed we have:

$$\int_D e^{(-i\lambda+\frac{1}{2})\langle z,b\rangle} f(z)\,dz \longrightarrow$$

$$\longrightarrow 2\pi \int_0^{+\infty} \mathcal{P}_{i\lambda-\frac{1}{2}}(\cosh r)\, f(\cosh r)\,\sinh r\, dr \quad (20)$$

and

$$\int_{\mathbb{R}}\int_B e^{(i\lambda+\frac{1}{2})\langle z,b\rangle}\, \hat{f}(\lambda,b)\, d\mu(\lambda,b) \longrightarrow$$

$$\longrightarrow \frac{1}{2\pi}\int_0^{2\pi} \mathcal{P}_{i\lambda-\frac{1}{2}}(\cosh r)\,\hat{f}(\lambda)\,\tanh(\pi\lambda)\,\lambda\, d\lambda \quad (21)$$

the last one is a term similar to the background integral of the Regge representation continued in the crossed channel, if we observe that the kinematics of the crossed coupling (coupling between an incoming with an outgoing particle) gives precisely: $\cosh r = -1 + 2s/(4m^2-t)$ (in the case of equal mass particles).

<u>Remark</u>: Up to now we have obtained a term which corresponds to the background integral in the complex angular momentum representation; what about poles? We have seen that the asymptotic behaviour is dominated by the poles. Indeed, up to now we have considered amplitudes which are square-integrable on the group manifold; then these amplitudes approach zero faster than $s^{-1/2}$ as $s \to \infty$, well below the asymptotic value indicated by the experi-

ments.

A possible way for overcoming this difficulty is extend the domain of the Fourier integral operator to include realistic amplitudes which are polynomially bounded, recognizing that the range of the integral operator then contains generalized functions (distributions). This way has been followed by Rühl[14].

A second possibility involves to find a Laplace transform for functions defined on $SO_o(1,2)/SO(2)$. As far as we know no rigorous result has been obtained, up to now, in this direction (in spite of many conjectures dispersed in physical journals). I think that this is an open question.

III - Dynamics: Partial Diagonalization of Bethe-Salpeter Equation.

Bethe-Salpeter[15] in 1951 proposed an integral equation which permits the study of relativistic bound-states. As we have seen starting by the crossing property and the analytic continuation one could conjecture that the asymptotic behaviour of the scattering amplitude for high values of s is controlled by the bound-states and resonances in the crossed channel. Furthermore let us remind that the singularities of the scattering amplitude corresponding to bound-states are outside the convergence circle of perturbative expansions. Therefore a non-perturbative method apt for studying the high-energy cross-sections could be to consider the Bethe-Salpeter equation regarding the momentum tranfer as a fixed parameter.

This kind of Bethe-Salpeter equation in the case of elastic scattering of spinless particles reads as follows:

$$A(p_a, p_b, q) = B(p_a, p_b, q) + \int_{\mathbb{R}^4} K(p_a, p', q)\, A(p', p_b, q)\, d^4p' \qquad (22)$$

A grafical representation of eq.(22) is given in fig.3

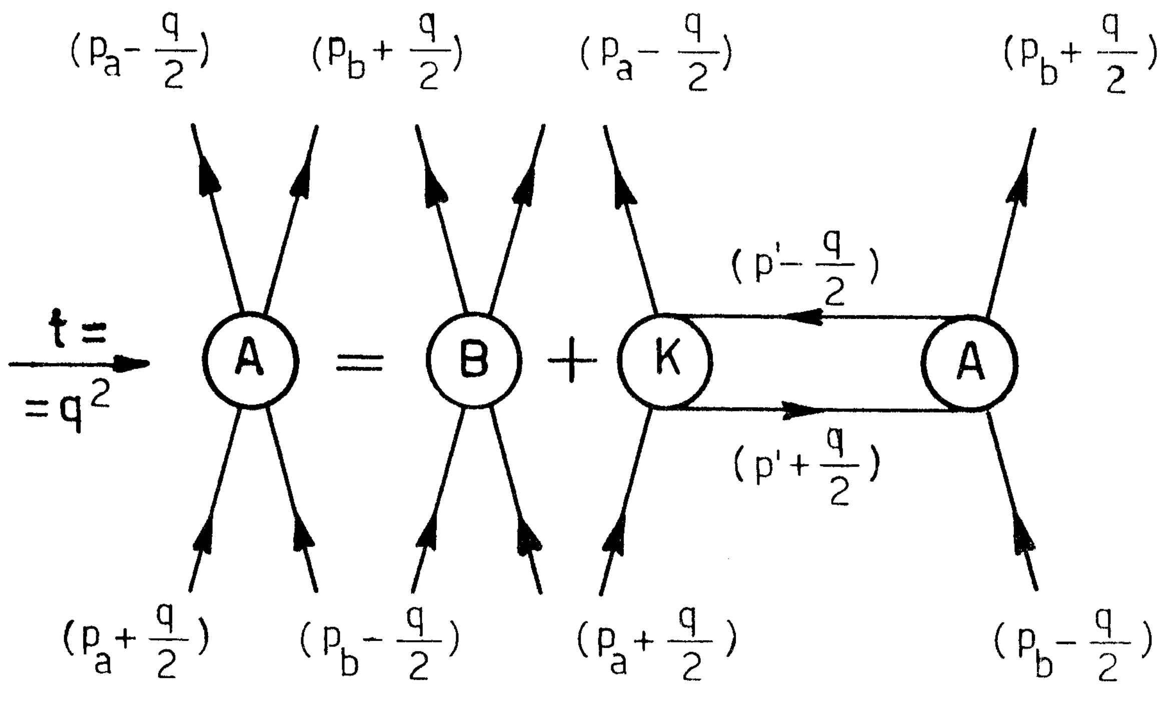

fig. 3

In this equation the amplitude A is an unknown function, while B and K are supposed to be known. The functions A, B and K are functions invariant under $SO_o(1,3)$, depending on three four-vectors (p_a, p_b, q). But for <u>fixed q (fixed momentum transfer)</u> the functions A, B and K are functions depending on two four-vectors, and are invariant under the subgroup G of the Lorentz transformation fixing the vector q:

a) if $q = 0$, the group G is $SO_o(1,3)$ itself;

b) In the case $t = q^2 < 0$; one may choose a coordinate system such that: $q = (0,0,0,\sqrt{-t})$ is fixed and the group G is then $SO_o(1,2)$.

One of the difficulties in treating eq.(22) is due to the fact that without suitable assumptions the integration domain in the last term of this equation is not bounded. But if on assumes that the amplitude A is not the whole amplitude but the absorptive part only, then one can use support conditions.

The support condition are the following:

i) for the amplitude A and the potential B the support is contained in the set

$$\left\{ p_a , p_b \;\middle|\; (p_a + p_b)^2 \geq 0 \quad ; \quad \left((p_a)_0 + (p_b)_0\right) \geq 0 \right\}$$

$(\;)_0$ is the time-component;

ii) for the kernel K the support is contained in the set

$$\left\{ p_a , p' \;\middle|\; (p_a - p')^2 \geqslant 0 \quad ; \quad \left((p_a)_0 - (p')_0\right) \geqslant 0 \right\}$$

If we make the further assumption that the external momenta p_a and p_b are spacelike, then also p' is necessarily spacelike.

At this point one can still make the position $p_a = x$, $p_b = -y$, $p' = z$; forget about q which is fixed and rewrite the Bethe-Salpeter equation in the following form:

$$A(x,y) = B(x,y) + \int_{D(y,x)} K(x,z)\, A(z,y)\, dz \tag{23}$$

which turns out to be a Volterra integral equation of the second kind.

Indeed let be $X = \mathbb{R}^n$ and Ω the forward light-cone in $\mathbb{R}^n$ defined by:

$$x_0^2 - x_1^2 - \cdots - x_{n-1}^2 > 0 \quad , \quad x_0 > 0$$

We consider on X the <u>ordering associated with Ω</u>: for x, y in X we note $x > y$ if $(x - y)$ belongs to Ω, and $x \geqslant y$ if $(x - y)$ belongs to the closure $\bar{\Omega}$ of Ω. For this ordering the set

$$D(y,x) = \left\{ z \in X \;\middle|\; y \leqslant z \leq x \right\}$$

is bounded. It is empty if $(x - y)$ does not belong to $\bar{\Omega}$.

A kernel $K(x,y)$ is said to be a Volterra kernel if $K(x,y)$ is continuous on $\Gamma = \left\{ (x,y) \,\middle|\, (x-y) \in \bar{\Omega} \right\}$, and vanishes out of Γ.

<u>Remark:</u> Observe that we are using the support conditions for the absorptive part and the assumptions that p_a and p_b are spacelike but we add also the hypothesis that the amplitude A, the potential B and the kernel K are continuous functions on the sets containing the support; this latter assumption is quite restrictive, but at this stage of our research we do not take care of specific and more realistic models for the kernel and the potential. However it is reasonable to think that it is not too difficult to remove this condition.

The product of two Volterra kernels K_1 and K_2 is given by:

$$K_1 \# K_2 (x,y) = \int_{D(y,x)} K_1 (x,z)\, K_2 (z,y)\, dz \tag{24}$$

where $K_1 \# K_2$ is again a Volterra kernel. Hence the space $V(X)$ of Volterra kernels is an algebra: the Volterra algebra of the ordered space X.

If K is a Volterra kernel we define $K^{\# k}$ by:

$$\begin{cases} K^{\# 1} = K & (25.a) \\ K^{\# k} = K^{\# (k-1)} \# K & (25.b) \end{cases}$$

<u>Problem:</u> For K and B given in $V(X)$ find a kernel A in $V(X)$ such that:

$$A(x,y) - \int_{D(y,x)} K(x,z)\, A(z,y)\, dz = B(x,y) \tag{26}$$

This is a Volterra integral equation of the second kind and it can also be written:

$$A - K \# A = B$$

We can prove a theorem on the existence and uniqueness of the solution of eq.(26).

<u>Theorem:</u> The equation (26) has a solution which is unique. It is given by:

$$A(x,y) = B(x,y) + \int_{D(y,x)} R(x,z)\, B(z,y)\, dz \tag{27}$$

where

$$R(x,y) = \sum_{k=1}^{\infty} K^{\# k}(x,y) \tag{28}$$

The series (28) converges uniformly on bounded sets and R is a Volterra kernel.

<u>Proof:</u> The proof runs essentially along the lines indicated by M. Riesz in his classical paper "L'intégrale de Riemann-Liouville et le problème de Cauchy"[16].

Now let us sketch a theory of <u>Volterra algebra on the hyperboloid with one sheet.</u>

Let X be the hyperboloid with one sheet in $\mathbb{R}^3$, defined by:

$$-x_0^2 + x_1^2 + x_2^2 = 1$$

The Lorentz group $G = SO_0(1,2)$ acts transitively on X. The pseudo-riemannian metric induced on X by the Minkowsky metric:

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2$$

is invariant under G. We shall denote by $d\sigma$ the corresponding surface element. The isotropy subgroup of the point $e_2 = (0,0,1)$ is

$H = SO_o(1,1)$; i.e. the one parameter subgroup of the following matrices:

$$h_\vartheta = \begin{pmatrix} \cosh\vartheta & \sinh\vartheta & 0 \\ \sinh\vartheta & \cosh\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let us also introduce the one parameter subgroup A of the following matrices:

$$a_\xi = \begin{pmatrix} \cosh\xi & 0 & \sinh\xi \\ 0 & 1 & 0 \\ \sinh\xi & 0 & \cosh\xi \end{pmatrix}$$

and we define:

$$A_+ = \left\{ a_\xi \in A \,\middle|\, \xi > 0 \right\}$$

For $x = (x_o, x_1, x_2)$ and $y = (y_o, y_1, y_2)$, we note $x > y$ if $(x-y)$ belongs to Ω (the forward light-cone in $\mathbb{R}^3$), and $x \geqslant y$ if $(x-y)$ belongs to the closure $\bar{\Omega}$ of Ω. For this ordering the set

$$D(y,x) = \left\{ z \in X \,\middle|\, y \leqslant z \leqslant x \right\} \tag{29}$$

is bounded. This ordering is invariant under G. Let G_+ be the set of g in G such that $ge_2 > e_2$, $e_2 = (0,0,1)$. The set G_+ is a semi-group and $A_+ = A \cap G_+$.

We can prove the following

Proposition: The semi-group G_+ has the following decomposition

$$G_+ = H\,A_+\,H \tag{30}$$

Remark: The numbers (ξ, ϑ) will be called polar coordinates of X; in terms of them the surface element is given by: $d\sigma = \sinh \xi \, d\xi \, d\vartheta$.

Now let Γ be the graph of the ordering restricted to the hyperboloid with one-sheet: $\Gamma = \{ (x,y) \in X \times X \mid y \leq x \}$. A function $K(x,y)$ on $X \times X$ is called a Volterra kernel if K is continuous on Γ, and vanishes out of Γ; if K_1 and K_2 are two Volterra kernels their product is given by:

$$K_1 \# K_2 (x,y) = \int_{D(y,x)} K_1(x,z)\, K_2(z,y)\, d\sigma(z) \tag{31}$$

where the set $D(y,x)$ is defined by equality (29). The kernel $K_1 \# K_2$ is again a Volterra kernel, hence the set of Volterra kernels is an algebra (Volterra algebra $V(X)$ of the ordered space X).

The kernel K is said to be invariant under G if, for any g in G

$$K(gx, gy) = K(x,y)$$

The set $V(X)^{\#}$ of the invariant Volterra kernels is a subalgebra of $V(X)$.

One can identify an invariant Volterra kernel K with a function f on G which is continuous on the closure $\overline{G}_+$ of G_+, vanishes out of $\overline{G}_+$, and it is bi-invariant under H. The identification is given by:

$$K\,(g e_2, e_2) = f(g)$$

Hence we shall consider the element of the algebra $V(X)^{\#}$ as functions on G as well. Now we have the following

<u>Theorem</u>: <u>The algebra $V(X)^{\#}$ of invariant Volterra kernels is commutative.</u>

A function belonging to $V(X)^{\#}$ depends only on one variable; for such a function f and for $\xi \geqslant 0$, we will use the following notation:

$$f\,(h_1\, a_\xi\, h_2) = f\,[\cosh \xi]\,;\; h_1, h_2 \in H\,,\; a_\xi \in A$$

Then we can provide an explicit formula for the composition product in $V(X)^{\#}$:

<u>Proposition:</u> For two functions f_1 and f_2 in $V(X)^{\#}$ we have:

$$f_1 \# f_2\,[\cosh \xi] =$$

$$= 2\int_0^{\xi} \left\{ \int_0^{\alpha(\xi,\tau)} f_1\left[\cosh \xi \cosh \tau - \sinh \xi \sinh \tau \cosh \vartheta\right] d\vartheta \right\} f_2\,[\cosh \tau] \sinh \tau \, d\tau \tag{32}$$

where $\alpha = \alpha(\xi, \tau)$ is the positive root of the equation:

$$\cosh \xi \cosh \tau - \sinh \xi \sinh \tau \cosh \alpha = 1$$

Next we introduce the <u>Poisson kernel</u> and the <u>spherical functions</u> on the one-sheet hyperboloid.

To this purpose we introduce the one-parameter subgroup N of G, consisting in the following matrices:

$$n(z) = \begin{pmatrix} 1+\frac{1}{2}z^2 & z & \frac{1}{2}z^2 \\ z & 1 & z \\ -\frac{1}{2}z^2 & -z & 1-\frac{1}{2}z^2 \end{pmatrix}$$

The map

$$N \times A \longrightarrow X$$

$$(n(z), a_\xi) \longrightarrow n(z)\, a_\xi\, e_2$$

is a diffeomorphism of N x A on the open set $\{x \in X \mid x_0+x_2 > 0\}$, which contains the set $\{x \in X \mid x > e_2\}$. If $x = n(z)a_\xi e_2$, we have:

$$\begin{cases} x_0 = \sinh\xi + \frac{1}{2}z^2 e^\xi \\ x_1 = z e^\xi \\ x_2 = \cosh\xi - \frac{1}{2}z^2 e^\xi \end{cases}$$

and $n(z)a_\xi$ belongs to G_+ (i.e. $g\, e_2 > e_2$), iff $\xi > 0$ and

$|z|<1-e^{-\xi}$. It follows that G_+ is contained in NA_+H.

The numbers (z, ξ) are called horocyclic coordinates. In terms of them the surface element is given by $d\sigma = e^{\xi}\, dz\, d\xi$.

For a complex number λ we define the function P^{λ} by:

$$P^{\lambda}(x) = e^{-\lambda\xi} \qquad \text{if} \qquad x = n(z)\, a_{\xi}\, e_2$$

and the Poisson kernel $P^{\lambda}(x, \vartheta)$ by:

$$P^{\lambda}(x, \vartheta) = P^{\lambda}(h_{-\vartheta}\, x)$$

we have

$$P^{\lambda}(x) = (x_0 + x_2)^{-\lambda}$$

$$P^{\lambda}(a_{\xi} e_2, \vartheta) = (\cosh\xi + \sinh\xi \cosh\vartheta)^{-\lambda}$$

Then we have the following

Proposition:

a) The function P^{λ} satisfies the relationship:

$$P^{\lambda}(n(z)\, a_{\xi}\, x) = e^{-\lambda\xi}\, P^{\lambda}(x) \tag{33}$$

b) For $x > e_2$, $\operatorname{Re}\lambda > 0$

$$\int_H |P^{\lambda}(hx)|\, dh = \int_{-\infty}^{+\infty} |P^{\lambda}(x, \vartheta)|\, d\vartheta < \infty \tag{34}$$

c) Furthermore, if f belongs to G_+

$$\int_H P^{\lambda}(ghx)\, dh = P^{\lambda}(ge_2) \int_H P^{\lambda}(hx)\, dh \tag{35}$$

Next for $\mathrm{Re}\,\lambda > 0$ we define the spherical function Φ_λ on G_+ by:

$$\Phi_\lambda(g) = \int_H P^\lambda(h\, g\, e_2)\, dh \tag{36}$$

The function Φ_λ is bi-invariant under H and

$$\Phi_\lambda[\cosh\xi] = \int_{-\infty}^{+\infty} (\cosh\xi + \sinh\xi \cosh\vartheta)^{-\lambda}\, d\vartheta \tag{37}$$

This function coincides with the <u>Legendre function of the second kind.</u> With the classical notation we have:

$$\Phi_\lambda[\cosh\xi] = 2\, Q_{\lambda-1}(\cosh\xi)$$

Next we have the following

<u>Theorem:</u> The spherical function Φ_λ satisfies a product formula: for g_1 and g_2 in G_+

$$\int_H \Phi_\lambda(g_1 h\, g_2)\, dh = \Phi_\lambda(g_1)\, \Phi_\lambda(g_2)$$

This formula can also be written when $g_1 = a_\xi$, $g_2 = a_\tau$, as follows:

$$\int_{-\infty}^{+\infty} \Phi_\lambda[\cosh\xi \cosh\tau + \sinh\xi \sinh\tau \cosh\vartheta]\, d\vartheta =$$
$$= \Phi_\lambda[\cosh\xi]\, \Phi_\lambda[\cosh\tau] \tag{38}$$

As well as in the case of two-sheeted hyperboloid we obtain, by the use of the Poisson kernel, the product formula for the Legendre functions of the first kind; analogously in the case of one-sheet hyperboloid we obtain, through the Poisson kernel introduced above, the product formula for the Legendre function of the second kind.

Then we can define the spherical Laplace transform f of a function f belonging to the Volterra algebra $V(X)^{\#}$ by:

$$\tilde{f}(\lambda) = \int_X f(x)\; P^{\lambda}(x)\; d\sigma(x) \qquad (39)$$

whenever the integral converges. Integrating in polar coordinates we obtain:

$$\tilde{f}(\lambda) = \int_0^{+\infty} f[\cosh\xi]\; \Phi_\lambda[\cosh\xi]\; \sinh\xi\, d\xi \qquad (40)$$

Next we can prove the following

<u>Theorem:</u> Let $\alpha > 0$, and let $V(X)_\alpha^{\#}$ be the space of functions f in $V(X)^{\#}$ such that

$$\| f \|_\alpha = \int_X | f(x) |\; P^{\alpha}(x)\, dx < \infty$$

The space $V(X)_\alpha^{\#}$ is a subalgebra of $V(X)^{\#}$, and for two functions f_1 and f_2 in $V(X)_\alpha^{\#}$ we have:

$$\| f_1 \# f_2 \|_\alpha \leq \| f_1 \|_\alpha \cdot \| f_2 \|_\alpha$$

The spherical Laplace transform $\tilde{f}$ of a function f in $V(X)_\alpha^{\#}$ is defined for $\mathrm{Re}\,\lambda \geq \alpha$ analytic for $\mathrm{Re}\,\lambda > \alpha$ and for f_1 and f_2 in $V(X)_\alpha^{\#}$ we have:

$$\widetilde{f_1 \# f_2}(\lambda) = \tilde{f}_1(\lambda)\; \tilde{f}_2(\lambda) \qquad (41)$$

Remark: Since

$$\Phi_\lambda[\cosh\xi] \sim c(\lambda)\, e^{-\lambda\xi} \quad (\xi \to \infty) \tag{42}$$

with

$$c(\lambda) = 2^{(\lambda+1)} \int_0^{+\infty} (1+\cosh\vartheta)^{-\lambda}\, d\vartheta =$$

$$= 2\, \frac{\sqrt{\pi}\;\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})}$$

a function f of $V(X)^{\#}$ belongs to $V(X)^{\#}_{\alpha}$ iff

$$\int_0^{+\infty} \left| f[\cosh\xi] \right| e^{(1-\alpha)\xi}\, d\xi < \infty$$

In order to obtain the inversion of the Laplace transform, it is convenient to compute the spherical Laplace transform of a function f, using horicyclic coordinates:

$$\tilde{f}(\lambda) = \int_0^{+\infty} \left[\int_{|z| \leq 1-e^{-\xi}} f(n(z)\, a_\xi)\, dz \right] e^{-(\lambda-1)\xi}\, d\xi \tag{43}$$

Next we define the Abel transform of f as:

$$\mathcal{A}f(\xi) = e^{\xi/2} \int_{|z| \leq 1-e^{-\xi}} f(n(z)\, a_\xi)\, dz \tag{44}$$

so that the spherical Laplace transform is the composition of the Abel transform and the usual Laplace transform:

$$\tilde{f}(\lambda) = \int_0^{+\infty} \mathcal{A}f(\xi)\, e^{-(\lambda-\frac{1}{2})\xi}\, d\xi \tag{45}$$

Therefore we can arrive at the following

Proposition: Let f be a function in $V(X)^{\#}_{\alpha}$ such that, for $\sigma \geq \alpha$

$$\int_{-\infty}^{+\infty} \left| \tilde{f}\,(\sigma + i\nu) \right| |\nu| \, d\nu < \infty$$

then for $\xi > 0$

$$f\left[\cosh \xi\right] = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \vartheta\left(\xi, \sigma + i\nu\right) \tilde{f}\left(\sigma + i\nu\right) d\nu \qquad (46)$$

with

$$\vartheta\left(\xi, \lambda\right) = \left(\lambda - \frac{1}{2}\right) P_{(\lambda-1)}\left(\cosh \xi\right)$$

where $P_{(\lambda-1)}(\cosh\xi)$ is the Legendre function of the first kind.

Analogous treatment can be done on the hyperboloid with one sheet in $\mathbb{R}^4$; this latter analysis is relevant in the case t = 0 (forward scattering), where the little group is $SO_o(1,3)$.

Finally we return to the Bethe-Salpeter equation in the case $t < 0$. We write $(x = x', x_3)$, with $x' = (x_o, x_1, x_2)$; $y = (y', y_3)$, with $y' = (y_o, y_1, y_2)$. The vectors x' and y' are the components of x and y orthogonal to the fixed momentum transfer q. We will further assume that x' and y' are spacelike. Then we can write:

$$\begin{cases} x' = \rho u \;, \quad \rho > 0 \;; \qquad -u_o^2 + u_1^2 + u_2^2 = 1 \\ y' = r v \;, \quad r > 0 \;; \qquad -v_o^2 + v_1^2 + v_2^2 = 1 \end{cases}$$

Introducing polar coordinates we can rewrite the Bethe-Salpeter convolutive kernel as follows:

$$K \# A\left[\rho, r ; \cosh\xi ; x_3, y_3\right] =$$

$$= 2\int_{\beta_1}^{\beta_2} dz_3 \int_{\sqrt{r\rho}\, e^{-\xi/2}}^{\sqrt{r\rho}\, e^{\xi/2}} \rho'^2\, d\rho' \int_0^{\xi} d\tau \sinh\tau\, A\left[\rho', r; \cosh\tau ; x_3, y_3\right] \cdot$$

$$\cdot \left\{ \int_0^{\alpha(\xi,\tau)} K\left[\rho, \rho'; \cosh\xi\cosh\tau - \sinh\xi \sinh\tau \cosh\vartheta ; z_3, y_3\right] d\vartheta \right\} \qquad (47)$$

β_1 and β_2 being the lower and upper limits for z_3 given by:

$$\frac{x_3 + y_3}{2} - \frac{1}{2}\rho \sinh\xi < z_3 < \frac{x_3 + y_3}{2} + \frac{1}{2}\rho \sinh\xi$$

Taking the spherical Laplace transform we have:

$$\tilde{A}\left(\rho, r ; \lambda ; x_3, y_3\right) =$$

$$= \int_0^{+\infty} A\left[\rho, r; \cosh\xi ; x_3, y_3\right] \Phi_\lambda\left[\cosh\xi\right] \sinh\xi\, d\xi \qquad (48)$$

with

$$\Phi_\lambda\left[\cosh\xi\right] = 2\, Q_{\lambda-1}(\cosh\xi)$$

Since the Laplace transform carries convolutive product into ordinary product, at the end we obtain the following equation:

$$\tilde{A}\left(\rho, r; \lambda; x_3, y_3\right) = \tilde{B}\left(\rho, r ; \lambda ; x_3, y_3\right) +$$

$$+ \int_{-\infty}^{+\infty} dz_3 \int_0^{+\infty} \rho'^2\, d\rho'\, \tilde{K}\left(\rho, \rho' ; \lambda; x_3, z_3\right) \tilde{A}\left(\rho', r ; \lambda ; x_3, y_3\right). \qquad (49)$$

Analogously in the case t = 0 (working with the little group $SO_o(1,3)$) we reduce to the following equation:

$$\tilde{A}(\rho, r; \lambda) = \tilde{B}(\rho, r; \lambda) + \int_0^{+\infty} \tilde{K}(\rho, \rho'; \lambda)\ \tilde{A}(\rho', r; \lambda)\, \rho'^3 d\rho' \quad (50)$$

These equations can be further diagonalized, if the equations remain invariant under dilatation. In this case one can perform a Mellin transform involving the radial variables: ρ , ρ', and r. Let us recall that this dilatation invariance holds true in the limit of zero mass for the internal particles.

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