VICTOR GUILLEMIN

On the Moment Mapping

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ON THE MOMENT MAPPING

Victor GUILLEMIN

§1 This talk will be concerned with the action of Lie groups on symplectic manifolds. I will begin by describing two pieces of symplectic machinery which are extremely important in this subject. Most of this talk will be devoted to some conventional and not-so-conventional applications of this machinery.

Let $\{X,\omega\}$ be a symplectic manifold and let P be the ring of smooth, real-valued functions on X. Equipped with the usual Poisson bracket, P is an (infinte-dimensional) Lie algebra. Let Σ be the Lie algebra of symplectic vector fields. There is a natural morphism of Lie algebras

$$\theta: P \to \Sigma$$

defined by

(1.1) $$\theta(f) = \xi^{\#} \Longleftrightarrow \iota(\xi^{\#})\omega = df.$$

Now let G be a compact, connected Lie group and let $\underline{g}$ be its Lie algebra. Given a symplectic action

$$\tau : G \times X \to X$$

one can differentiate τ to get a morphism of Lie algebras

(1.2) $$\tau^{\#}: \underline{g} \to \Sigma$$

One says that τ is a Hamiltonian action of G on X if there exists a morphism of Lie algebras,

(1.3) $$\phi: \underline{g} \to P$$

such that

(1.4) $$\theta \circ \phi = \tau^{\#} .$$

Suppose such a ϕ exists. If one chooses a basis, $\xi_1,\ldots,\xi_n$ of $\underline{g}$ then with respect to this basis one gets a mapping

(1.5) $$\Phi : X \longrightarrow \mathbb{R}^n$$

the i-th coordinate of Φ being the function $\phi(\xi_i)$. The mapping (1.5) is called (after Souriau) the <u>moment mapping</u>.

Example. Let X = classical phase space = $T^*\mathbb{R}^3$, and let q_i, $1 \leq i \leq 3$, and P_i, $1 \leq i \leq 3$, be the usual position and momentum coordinates. Let

$$G = E(3) = \mathbb{R}^3 \rtimes SO(3).$$

For the standard basis of $e(3)$ the associated moment mapping is:

$$\phi_i(P,q) = P_i,\ i = 1,2,3 .$$
$$\phi_4(P,q) = q_2P_3 - q_3P_2$$
$$\text{etc.}$$

i.e. the components of Φ are the classical linear and angular momenta.

It is useful to define the moment mapping without reference to a basis of $\underline{g}$. For each $x \in X$ and $\xi \in \underline{g}$ set

(1.6) $$\langle\Phi(x),\xi\rangle = \phi(\xi)(x).$$

The left hand side of (1.6) defines $\Phi(x)$ as an element of $\underline{g}^*$; so one can think of the moment mapping as being intrinsically a map

(1.7) $$\Phi: X \to \underline{g}^*$$

The virtue of this intrinsic definition is that it is easy to show that Φ is <u>equivariant</u> with respect to the given action of G on X and the co-adjoint action on $\underline{g}^*$.

The moment map is one of the two pieces of symplectic machinery about which this talk will be concerned. The other, which we will now describe, is the notion of <u>reduction</u>.

Consider the sequence of maps

(1.8) $$\underline{g} \xrightarrow{\tau^{\#}} \Sigma \xrightarrow{ev_x} T_x$$

The symplectic form on T_x gives one a canonical identification of T_x with T_x^*, i.e. a canonical bijection

(1.9) $$T_x \to T_x^*$$

Composing (1.8) with (1.9) one gets a map

(1.10) $$\underline{g} \to T_x^*$$

<u>Lemma</u> The map (1.10) is the transpose of

$$d\Phi_x: T_x \to \underline{g}^*$$

<u>Proof</u>: Just differentiate (1.6) keeping (1.1) in mind.

<u>Corollary 1</u> A point, $x \in X$, is a regular point of Φ if and only if the action of G is locally free in a neighborhood of x.

Proof: By definition, x is regular if and only if $d\Phi_x$ is surjective. By the lemma this is the case if and only if (1.8) is injective. But this is the case if and only if the stabilizer group of x is discrete. Q.E.D.

In the same way one proves

Corollary 2 The kernel of $d\Phi_x$ is the symplectic ortho-complement in T_x of the tangent space to the orbit of G through x.

Now suppose zero is a regular value of Φ. Let

$$Z = \{x \in X, \quad \Phi(x) = 0\}$$

This set is a G-invariant submanifold of X and, by corollary 1, the action of G on X is locally free. To simplify let's assume this locally free action is free. Let

$$X_0 = Z/G .$$

and let $\pi: Z \to X_0$ be the canonical projection. The triple (Z,X_0,π) can be viewed as a principal G-bundle. Let $\iota: Z \to X$ be the inclusion map, and consider the closed two-form $\iota^*\omega$. Since ω is G-invariant, so is $\iota^*\omega$. Moreover, by corollary 2, $\iota^*\omega$ is annihilated by vectors tangent to the fibers of $\pi: Z \to X_0$. This shows that $\iota^*\omega$ is "base-like": there is a closed two-form, ω_0, on X_0 such that

$$\iota^*\omega = \pi^*\omega_0 . \tag{1.11}$$

It is easy to check that ω_0 is symplectic. (This is another consequence of corollary 2.) So (X_0,ω_0) is a symplectic manifold. It is called the Marsden-Weinstein reduced space (See [12]) associated with the action of G on X.

Remark: If the action of G on Z is locally free, but not free, (X_0, ω_0) can have singularities; nevertheless it is still a symplectic "V-manifold."

§2. In this section we will describe some applications of the machinery described above.

Application 1 (Convexity theorems) There are a number of well-known theorems in linear algebra having to do with convexity properties of sets of matrices (e.g. Rayleigh [15], Shur [16], Horn [8]). We will be concerned with a generalization of these theorems due to Kostant: Let K be a compact semi-simple Lie group and G its Cartan subgroup, and let $\underline{k}$ and $\underline{g}$ be the Lie algebras of K and G. From the inclusion map $\iota: \underline{g} \to \underline{k}$ one gets a map the other way

$$\iota^*: \underline{k}^* \to \underline{g}^* .$$

Let X be a co-adjoint orbit of K in k^* and let

$$\Phi: X \to \underline{g}^*$$

be the restriction of ι^* to X. Since X is a K-space it is a fortiori a G-space.

Theorem (Kostant) a) The set of fixed points, S, of G in X is a finite set.

b) The image of Φ is the convex hull of the image of S.

See [11]. There is now a simple proof of this theorem, due to Heckman, [7], based on Morse-theoretic ideas. Heckman has also discovered an elegant generalization of this theorem in which G is replaced by an arbitrary closed subgroup of K. An even more striking generalization was recently discovered by V. Kac and D. Peterson. In their version, K is the "compact form" of an infinite dimensional semi-simple Lie group!

A couple years ago Sternberg and I [4] and Atiyah [1] discovered that the Kostant theorem has a generalization in another direction. Namely let X be a compact connected symplectic manifold and G a commutative compact connected Lie group. Let $\tau: G \times X \to X$ be a Hamiltonian action of G on X and let $\Phi: X \to \underline{g}^*$ be the associated moment mapping.

Theorem a) Let S be the fixed point set of the action of G on X. Then $\Phi(S)$ is a finite subset of $\underline{g}^*$.

b) The image of Φ is the convex hull of $\Phi(S)$.

For instance if K is a compact semi-simple Lie group containing G as its Cartan subgroup and X is a co-adjoint orbit of K, one can give X a canonical symplectic structure, following Kirillov and Kostant, and then Kostant's theorem becomes a special case of the result above.

Remark: There are examples where S itself isn't finite as in as in Kostant's case.

Sternberg and I have also proved a symplectic version of Heckman's theorem (for G non-commutative); however, as yet we don't know what the appropriate symplectic setting should be for the Kac-Peterson result (other than that it should involve the action of a compact Lie group on an infinite-dimensional symplectic manifold.)

Application 2 (Geometric invariant theory.)

Let X be a non-singular complex projective variety, G^C a complex algebraic group and $\tau: G^C \times X \to X$ an algebraic action of G^C on X. One of the goals of geometric invariant theory is to show that the quotient

$$X/G^C$$

is a "reasonable" space (e.g. an algebraic variety.) Unfortunately this quotient may not even be Hausdorff. However, Mumford showed in [13] that one can find an invariant Zariski-open set, Xst, in X (the stable points of X) such that

$$X_{st}/G^C \qquad (2.1)$$

is a projective variety. Because of this result, the quesiton of whether a point of X is stable or not is of considerable importance. In [9] Kempf and Ness gave a very manageable criterion for settling this question; and Mumford has recently discovered that their criterion is essentially a statement about moment mappings: To be specific, let G be the maximal compact subgroup of G^C. One can find a Kaehler form on X which is G-invariant and for which the action of G is Hamiltonian. (Take an arbitrary Kaehler form and average with respect to G.) Let $\Phi: X \to g^*$ be the resulting moment map.

Theorem A point, $p \in X$, is stable iff its G^C-orbit intersects the zero level set of Φ .

A corollary of this is that the Mumford quotient space, (2.1), is a reduced space in the sense of §1! Recently Frances Kirwan and Linda Ness have used the moment mapping to explore the structure of the non-stable points of X. For details of their work see [10] and [14].

Application 3 (Collective motion.) Let X be a Hamiltonian G-space and let $\Phi: X \to g^*$ be the moment mapping. Given a smooth real-valued function, f , on X one can pull it back to X by means of Φ to obtain a function

$$f \circ \Phi \tag{2.2}$$

on X. Functions of the form (2.2) are called collective functions, and the dynamical systems associated with them collective Hamiltonian systems. It has been known for a long time that a number of dynamical systems of considerable interest in physics are of this type, for instance the so-called "liquid-drop" model. Within the last decade it has been discovered that many other systems besides these are collective. Some spectacular examples are: i) motion of a particle on S^n with respect to a quadratic potential, ii) the n-dimensional Lagrange top, iii) geodesic flow on n-dimensional ellipsoids and iv) the periodic Toda lattice. These systems turn out to be collective because of the phenomenon of "hidden symmetries." The group of symmetries, G, acts on the phase space of the system, but there is no corresponding action on the configuration space.

The systems mentioned above are all completely integrable. For a systematic discussion of complete integrability from the collective point of view, see [5] or the talk by Sternberg in the proceedings of this conference.

Application 4 (Classical field theory.)

We will consider a very simple (and very artificial) problem: electro-dynamics on compact two-manifolds. Let M be a compact two-manifold and $B \overset{\pi}{\to} M$ a principal $U(1)$-bundle (i.e. circle bundle) over M. Let X be the space of connections on B. X is an (infinite dimensional) manifold and can be made into a symplectic manifold as follows: For $\alpha \in X$ the tangent space to X at α is the space of smooth one-forms on M. Given two such forms, α_1 and α_2, define the symplectic pairing of α_1 with α_2 by the formula

$$\Omega(\alpha_1,\alpha_2) = \int_M \alpha_1 \wedge \alpha_2$$

It is not hard to show that Ω is a symplectic form on X. Now let G be the group of bundle automorphisms of B. There is an obvious indentification:

$$G = \text{maps of } M \text{ into } S^1 . \tag{2.3}$$

It is not hard to show that the action of G on X is Hamiltonian. In fact the easiest way to show this is to exhibit explicitly the moment mapping: By (2.3) the Lie algebra, $\underline{g}$, of G can be identified with the space of smooth real-valued functions on X. Therefore the dual, $\underline{g}^*$, is the space of currents of degree 2, and contains, as a subspace, the space of smooth 2-forms.

Theorem The moment map, $\Phi: X \to \underline{g}^*$, associated with the action of G on X is the map which to each $\alpha \in X$ associates $\operatorname{curv}(\alpha)$, the curvature form of α.

<u>Corollary</u> The reduced space associated with the action of G on X is the space of flat connections modulo equivalence.

There is a very simple topological description of this space. The exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 0$$

induces a long exact sequence on cohomology

$$0 \to H^1(M,\mathbb{Z}) \to H^1(M,\mathbb{R}) \to H^1(M,S^1) \overset{\delta}{\to} H^2(M,\mathbb{Z}) \tag{2.4}$$

One can identify the space of flat connections, modulo equivalence, with the set

$$\{\mu \in H^1(M,S^1), \quad \delta\mu = 0\}$$

By (2.4) this space is isomorphic with

$$H^1(M,\mathbb{R})/H^1(M,\mathbb{Z}) \ ,$$

a torus of dimension equal to twice the genus of M.

Atiyah and Bott have shown that the discussion above can be generalized to $U(N)$-bundles. In this case the reduced space is still finite-dimensional but a lot more complicated. We refer to [2] for details.

§3 I want to devote the concluding portion of this tale to a theorem about the local structure of the moment mapping. Let G be a compact Lie group, X a symplectic manifold on which G acts in a Hamiltonian fashion and $\Phi: X \to \underline{g}^*$ the moment mapping. Finally let x be an arbitrary point of X.

<u>Theorem</u> Φ is determined up to isomorphism in a G-invariant neighborhood of x by the following data.

a) The value, $\alpha = \Phi(x)$, of Φ at X.

b) The stabilizer group, G_x, of x.

c) The linear representation of G_x on the tangent space at x.

Before sketching the proof of this theorem, I will, for motivation, review an elementary theorem in differential topology: Let M be a compact manifold and let

(3.1) $$\iota_1: M \to X_1 \quad \text{and} \quad \iota_2: M \to X_2$$

be imbeddings. ι_1 and ι_2 are said to be <u>equivalent</u> if there exist neighborhoods, U_1 and U_2, containing $\iota_1(M)$ and $\iota_2(M)$, and a diffeomorphism

(3.2) $$f: U_1 \to U_2$$

such that $\iota_2 = f \circ \iota_1$. The following theorem can be regarded as a classification theorem for imbeddings, up to equivalence. Its proof is an easy consequence of the tubular neighborhood theorem.

<u>Theorem</u> The imbeddings, (3.1) are equivalent if and only if the normal bundle of M in X_1 is isomorphic (as a vector bundle) to

the normal bundle of M in X_2.

Suppose now that X is a symplectic manifold and $\iota: M \to X$ an imbedding. ι is said to be isotropic if $\iota^*\omega = 0$. This property can also be described as follows: Let T'_m be the image of T_m in T_x, and let $(T'_m)^\perp$ be its symplectic ortho-complement. Then if the imbedding is isotropic $(T'_m) \subset (T'_m)^\perp$ for all $m \in M$ and visa versa. Let

$$N_n = (T'_m)^\perp / T_m .$$

This space is called the symplectic normal space to M at m, and the vector bundle over M whose fiber at m is N_m is called the symplectic normal bundle of M in X.

Suppose that the manifolds, X_1 and X_2 in (3.1) are symplectic and the imbeddings, ι_1 and ι_2, are isotropic. We will say that ι_1 and ι_2 are equivalent as isotropic imbeddings if the map, f, in (3.2) is a symplectomorphism. In [19] Weinstein proved the following symplectic version of the theorem quoted above.

Theorem The isotropic imbeddings, (3.1), are equivalent if and only if their symplectic normal bundles are isomorphic (as symplectic vector bundles).

Weinstein also proved that if G is a compact Lie group, the G-equivariant version of this theorem is true.

Let's come back to the moment mapping $\Phi: X \to g^*$ in the vicinity of the point, $x \in X$. We want to show that Φ is determined locally by the value, α , of Φ at x, the stabilizer group, G_x, and the linear representation of G_x on T_x. We will skip the first step in the proof, which is a reduction to the case, $\alpha = 0$. Assuming $\alpha = 0$ we claim:

Lemma The G-orbit through x is isotropic.

Proof: Since Φ is equivariant the set, $\Phi^{-1}(0)$, contains the G-orbit through x; so the tangent space to this orbit at x is contained in kernel $d\Phi_x$. On the other hand by corollary 2 of §1, kernel $d\Phi_x$ is the symplectic ortho-complement of the tangent space to the orbit.

Q.E.D.

The orbit of G through x is the homogeneous space, G/G_x, and its symplectic normal bundle is the homogeneous G-bundle associated with the linear isotropy representation of G_x on the symplectic normal space at x. Thus, in view of Weinstein's theorem, the local structure of X in the vicinity of the G-orbit through x is completely determined by this linear representation.

For further details, as well as some applications of this result, see [6].

Bibliography

1. M. Atiyah,"Convexity and commuting Hamiltonians," Bull. Lon. Math. Soc. 14(1982) 1-15.

2. M. Atiyah and R. Bott, "The Yang-Mills equations over Riemann surfaces" Compositio Math. (to appear).

3. V. Guillemin and S. Sternberg, "The moment mapping and collective motion" Annals of Physics 127(1980) 220-253.

4. V. Guillemin and S. Sternberg, "Convexity properties of the moment mapping" Inv. Math. 67(1982) 491-513.

5. V. Guillemin and S. Sternberg, "On collective complete integrability according to the method of Thimm," Ergodic Theory and Dyn. Systems (to appear).

6. V. Guillemin and S. Sternberg, "Multiplicity-free spaces" (to appear).

7. G. Heckman, "Projections of orbits and asymptotic behavior of multiplicites for compact connected Lie groups" Inventiones, 67(1982) 333-356.

8. A. Horn, "Doubly stochastic matrices and the diagonal of a rotation matrix," Amer. Jour. Math. 76(1954) 620-630.

9. G. Kempf and L. Ness, "The length of vectors in representation spaces," Algebraic Geometry, Proceedings, Copenhagen 1978 Lecture Notes in Math. Springer 732(1979).

10. F. Kirwan, "The cohomology of quotient spaces in algebraic geometry" Thesis Oxford (1982).

11. B. Kostant, "On convexity, the Weyl group and the Iwasawa decomposition" Ann. Sci. Ec. Norm. Sup. 6(1973) 413-455.

12. J. Marsden and A. Weinstein, "Reduction of symplectic manifolds with symmetry," Jour. Math. Phys. 5(1974) 121-130.

13. D. Mumford and J. Fogarty, "Geometric Invariant Theory, Second Enlarged Edition, Springer-Verlag, New York, 1982.

14. L. Ness, "A stratification of the null cone via the moment map" Amer. Jour. Math. (to appear).

15. J.W.S. Rayleigh, "Note on the finite vibrations of a system about a configuration of equilibrium" Phil Mag. 20(1910) 450-456.

16. I. Shur, "Über eine Klasse von Mittelbildungen mit Anwendungen auf der Determinanten theorie," Sitzungsberichte der Berliner Math. Gesellshaft 22(1923).

17. J.M. Souriau, Structure des systèmes dynamiques, Paris, Dunod (1969).

18. A. Weinstein, "Neighborhood classfication of isotropic imbeddings", Jour. Diff. Geom. 16(1981) 125-128.

M.I.T.
Cambridge, Mass. 02139 (U.S.A.)