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Local Cohomology and Its Structural Implications for Field Theory

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LOCAL COHOMOLOGY AND ITS STRUCTURAL IMPLICATIONS FOR FIELD THEORY *

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ABSTRACT

The concept of local cohomology is described and applications are made to the theory of superselection structure, solitons, spontaneously broken gauge symmetries and quantum electrodynamics.

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1. LOCAL COHOMOLOGY

Let me begin by recalling the standard notions[x)] of singular cohomology. If X is a topological space then an n-simplex of X is a continuous map of the standard n-simplex $\Delta^n = \{(t^0,t^1,\dots,t^n) \in \mathbb{R}^{n+1} : t^i \geq 0,\ \sum_{i=0}^{n} t^i = 1\}$ into X . Let $\Sigma_n(X)$ denote the set of n-simplexes of X . There are face maps $\partial_i : \Sigma_n(X) \to \Sigma_{n-1}(X)$, i = 0,1,...,n defined by

$$(\partial_i c)(t^0,t^1,\dots t^{n-1}) = c(t^0,t^1,\dots,t^{i-1},0,t^i,\dots,t^n).$$

An n-cochain of X with values in an Abelian group A is a mapping $f : \Sigma_n(X) \to A$. The set of n-cochains forms an Abelian group under addition denoted by $C^n(X,A)$. Using the face operators and the group structure of A one defines boundary operators $d : C^n(X,A) \to C^{n+1}(X,A)$ by

$$(df)(c) = \sum_{i=0}^{n+1} (-1)^i f(\partial_i c) .$$

One checks that $d^2 = 0$ and this gives rise to a cochain complex

$$C^0(X,A) \xrightarrow{d} C^1(X,A) \xrightarrow{d} C^2(X,A) \xrightarrow{d} \dots\dots$$

The n-cocycles $Z^n(X,A)$ and the n-coboundaries are the subgroups of $C^n(X,A)$ defined by $Z^n(X,A) = \ker d$, $B^n(X,A) = \operatorname{im} d$. Conventionally one sets $B^0(X,A) = 0$. The cohomology groups $H^n(X,A)$ are the quotient groups $Z^n(X,A)/B^n(X,A)$.
In fact, I am interested in the case that X is Minkowski space $\mathbb{R}^{s+1}$, s is the number of space dimensions, which has a trivial cohomology ; let me remind you why. One picks a fixed origin x_0 and if c is

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x For further details, the reader may consult any standard textbook, for example [1].

an n-simplex one lets $h(c)$ denote the n+1-simplex which is a cone with vertex x_0 and base c.

$$h(c)(t^0,t^1,\dots,t^{n+1}) = t^0 x_0 + (1-t^0)\,a(\tau t^1,\tau t^2,\dots,\tau t^n),\ t^0\neq 1$$

$$= x_0 \,, \qquad t^0 = 1 .$$

Here we have written $\tau = (1-t^0)^{-1}$. We have

$$\partial_0 h(c) = c \,, \qquad \partial_i h(c) = h(\partial_{i-1}c)\,, \quad i>0$$

except that if $c \in \Sigma_0(X)$ then $\partial_1 h(c) = x_0$. A mapping h with this property is called a contracting homotopy. Once we have a contracting homotopy we proceed as follows : if $z \in Z^n(X,A)$, $n \geq 1$, define $y \in C^{n-1}(X,A)$ by $y(b) = z(h(b))$, $b \in \Sigma_{n-1}(X)$. Then if $c \in \Sigma_n(X)$, $0 = (dz)(h(c)) = z(c) - \sum_{i=0}^{n} z(h\partial_i c)$ $= z(c) - dy(c)$. Hence $H^n(X,A) = 0$ for $n \geq 1$.

From now on we restrict ourselves to Minkowski space and omit the symbol $X = \mathbb{R}^{s+1}$, however the cohomology will not be trivial because I shall add a locality condition to the definition of cochain which takes into account the causal structure of Minkowski space. The causal structure of Minkowski space may be defined in terms of the quadratic form $(x,x) = x_0^2 - \sum_{i=1}^{s} x_i^2$. x and y are said to be timelike, lightlike and spacelike according as $(x-y)^2 > 0$, $(x-y)^2 = 0$, $(x-y)^2 < 0$.

Let $V_+ = \{x : x_0 \geq 0 \text{ and } (x,x) \geq 0\}$. If $x - y \in V_+$ and $x \neq y$ then $\mathcal{O} = (x - V_+) \cap (y + V_+)$ is said to be the double cone with vertices x and y. Let $\mathcal{K}$ denote the set of double cones ordered under inclusion and $\mathcal{K}_0$ the subset of double cones centred on the origin, i.e. with $y = -x$. Let $\mathcal{C}$ denote the set of compact subsets of Minkowski space ordered under inclusion. We now take as coefficients for local[x)] cohomology a <u>net</u> A of Abelian groups over $\mathcal{C}$, i.e. for each $F \in \mathcal{C}$, $A(F)$ is an Abelian group and if $F_1 \subset F_2$ then $A(F_1)$ is a subgroup of $A(F_2)$. Let A_∞ denote the inductive limit of the $A(F)$. A local n-cochain is an element of $C^n(A_\infty)$ such that there exists an $\mathcal{O} \in \mathcal{K}_0$ with

x The term local cohomology is appropriate within the context of algebraic field theory, in a wider context causal cohomology might be preferable.

$$f(c) \in A(\mathcal{O}+c) \ , \qquad c \in \Sigma_m \ .$$

Here we have written $\mathcal{O}+c$ for $\mathcal{O}+c(\Delta^m)$. Since $\mathcal{O}+\partial_i c \subset \mathcal{O}+c$, if f is local, df is also local.
Thus local cochains give rise to a subcomplex

$$C^0_\ell(A) \xrightarrow{d} C^1_\ell(A) \xrightarrow{d} C^2_\ell(A) \xrightarrow{d} \cdots$$

and we have the obvious definitions of local cocycles, local coboundaries and local cohomology groups. A contracting homotopy can now no longer be used to show that the resulting cohomology is trivial. One can only expect to find interesting results when A itself reflects the causal structure on Minkowski space.

For example one can construct a net $\mathcal{W}$ from the real C^∞-solutions of the wave equation by defining $\mathcal{W}(F)$ to consist of those solutions which vanish on F', the spacelike complement of F, $F' = \{ x : (x-y)^2 < 0, \ y \in F \}$. I present here a preview in tabular form of some results involving coefficients which are real C^∞-solutions of invariant partial differential equations with analogous support conditions.

	$\mathcal{W}$	$\dot{\mathcal{W}}$	$\mathcal{K}$	$\mathcal{L}$	$\mathcal{M}$
H^0_ℓ	0	0	0	0	0
H^1_ℓ, $s>1$	0	$\mathbb{R}$	0	0	0
H^2_ℓ, $s>2$	0	?	0	$\mathbb{R}$	$\mathbb{R}\times\mathbb{R}$

Here $\dot{\mathcal{W}}$ denotes the solutions ξ of the wave equation such that $\int \dot{\xi}(0,\underset{\sim}{x})\, d^s\underset{\sim}{x} = 0$; $\mathcal{L}$ denotes the vector wave equation with Lorentz condition $\Box \xi^\mu = 0$, $\partial_\mu \xi^\mu = 0$; $\mathcal{K}$ is the Klein-Gordon equation $(\Box + m^2)\xi = 0$ and $\mathcal{M}$ Maxwell's equations $\partial^{[\lambda} \xi^{\mu\nu]} = 0$, $\partial_\mu \xi^{\mu\nu} = 0$. The results for H^0_ℓ and H^1_ℓ

are trivial although the dimensionality restriction $s > 1$ is essential. If $s = 1$ one finds for example that $H^1_\ell(w)$ can be identified with the set of all real C^∞-solutions of the wave equation. By contrast the results for H^2_ℓ are not trivial and hinge on the fact that the sheaf of Cauchy data for the wave equation on a spacelike hyperplane is a soft sheaf. These results have some indirect physical interest ; $H^2_\ell(\mathcal{L})$ may be regarded as parametrized by an electric charge and $H^2_\ell(\mu)$ by an electric and magnetic charge, (see the discussion in section 4).

Unfortunately the local cohomology which is of direct interest for algebraic field theory involves another essential, but disjoint, complication in that the coefficients are not nets of Abelian groups. This compells me to say something about non-Abelian cohomology. Suppose one were to try and make a non-Abelian group G into the coefficient group for cohomology (locality is irrelevant here). There is no problem for $n = 0$, 1 . For example a 1-cocycle is defined by the identity

$$z(\partial_0 c)\, z(\partial_2 c) = z(\partial_1 c), \qquad c \in \Sigma_2 .$$

Two 1-cocycles z , z' are cohomologous if there is a $y: \Sigma_0 \to G$ such that

$$z(b)\, y(\partial_1 b) = y(\partial_0 b)\, z'(b), \qquad b \in \Sigma_1 .$$

z is a 1-coboundary if it is cohomologous to the trivial 1-cocycle, $z'(b) = e$, $b \in \Sigma_1$. Comparing these formulae with the corresponding formulae in Abelian cohomology, we see that the problem lies in ordering the terms. There is no ordering which works for $n > 1$ and in fact even for $n = 1$ something has been lost because the product of 1-cocycles cannot be defined. There is a way out which I can only hint at here : non-Abelian cohomology needs coefficient objects which have a richer algebraic structure. This structure becomes increasingly complicated with increasing n . A relevant example of such a coefficient object will be discussed in sections 3 and 4.

2. ALGEBRAIC FIELD THEORY

Algebraic field theory tries to describe the structural properties of elementary particle physics in terms of the "algebra of local observables" ; this is a net $\mathfrak{A}$ of von Neumann algebras [x]. Experience with quantum theory tells us that observables can be represented as self-adjoint operators on a Hilbert space and $\mathfrak{A}(\mathcal{O})$ is to be thought of as the von Neumann algebra generated by the observables one can measure within the space-time region $\mathcal{O} \in \mathcal{K}$. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated then the causality principle implies that no signal can connect $\mathcal{O}_1$ and $\mathcal{O}_2$ so that measurements in $\mathcal{O}_1$ do not interfer with those in $\mathcal{O}_2$. Again experience with quantum theory tell us that consequently $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$ commute. Thus $\mathfrak{A}$ is a local net :

$$A_1 A_2 = A_2 A_1 , \qquad A_1 \in \mathfrak{A}(\mathcal{O}_1), \; A_2 \in \mathfrak{A}(\mathcal{O}_2) , \; \mathcal{O}_1 \subset \mathcal{O}_2' .$$

The Poincaré invariance of the theory is expressed by saying that there is a continuous representation $L \rightarrow \alpha_L$ of the Poincaré group $\mathcal{P}$ by automorphisms of $\mathfrak{A}$ such that

$$\alpha_L (\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(L\mathcal{O}) , \qquad L \in \mathcal{P}, \; \mathcal{O} \in \mathcal{K} .$$

A state of the physical system is represented by a state of the C^*-algebra which is the C^*-inductive limit of the local algebras $\mathfrak{A}(\mathcal{O})$. By abuse of notation this algebra is again denoted by the symbol $\mathfrak{A}$. The most important state is the vacuum state ω_0 , a Poincaré invariant state describing the complete absence of particles,

$$\omega_0 \circ \alpha_L = \omega_0 , \qquad L \in \mathcal{P} .$$

Associated with the state ω_0 by the GNS construction is a representation π_0 of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}_0$ with a cyclic vector Ω

x For details on von Neumann algebras, the reader might first consult the review article by A. Connes in these proceedings and then a standard textbook, e.g. [2, 3].

and a continuous unitary representation $L \to U_0(L)$ of the Poincaré group satisfying

$$U_0(L)\,\pi_0(A)\,\Omega = \pi_0(\alpha_L(A))\,\Omega,\ L \in \mathcal{P},\ A \in \mathfrak{A}.$$

ω_0 is a state with minimal energy so that the spectrum of the representation $a \to U_0(a)$ of the subgroup of spacetime translations is contained in the forward light cone V_+ in momentum space.

In general, one can find several states fitting this description of the vacuum state although physical intuition suggests that the vacuum state is unique. I shall pick a pure state ω_0 as "the" vacuum state and the discussion that follows is relative to this one pure vacuum state.

Fortunately of the totality of all states of $\mathfrak{A}$ very few have any relevance to elementary particle physics and I shall make a preliminary reduction in the number of states by taking π_0 to be faithful and by treating only locally normal states. On the other hand it is one of the striking features of elementary particle physics as opposed to elementary quantum theory that one must take into account states which are not represented by vectors or density matrices in $\mathcal{H}_0$, i.e. states which are not normal states of π_0. The pure states represented by vectors of $\mathcal{H}_0$ constitute the <u>vacuum sector</u>. The appearance of other superselection sectors in elementary particle physics was first pointed out in [4] and interpreted in an algebraic setting in [5]. It seems that the states of $\mathfrak{A}$ relevant to elementary particle physics are those which are local finite-energy perturbations *) of the vacuum state ω_0. One of the important problems of algebraic field theory is to describe the structure of these states and their relation to the algebraic structure of the net $\mathfrak{A}$.

The first results in this direction may be found in [6] and the most systematic treatment to date in [7, 8]. The selection criterion used in [7] is to consider states which are normal states of representations π such that there exists $\mathcal{O} \in \mathcal{K}_0$ with

*) Similar problems have recently been investigated in the context of classical field theory. See the article by G. Velo in these proceedings.

$$\pi_0 \upharpoonright \mathfrak{A}(\mathcal{O}'+a) \cong \pi \upharpoonright \mathfrak{A}(\mathcal{O}'+a)\,, \quad a \in \mathbb{R}^{s+1}. \tag{S}$$

Here $\mathfrak{A}(\mathcal{O}')$ denotes the C^*-subalgebra of $\mathfrak{A}$ generated by the $\mathfrak{A}(\mathcal{O}_1)$ with $\mathcal{O}_1 \in \mathcal{K}$, and $\mathcal{O}_1 \subset \mathcal{O}'$, and the symbols $\upharpoonright$ and $\cong$ denote respectively restriction and unitary equivalence. In [8], in order to treat particle structure and construct scattering states, π is in addition supposed to be a covariant representation, so that there is a continuous unitary representation $L \to U_\pi(L)$ of the covering group $\tilde{\mathcal{P}}$ of the Poincaré group with

$$U_\pi(L)\,\pi(A) = \pi(\alpha_L(A))\,U_\pi(L), \quad L \in \tilde{\mathcal{P}}, \quad A \in \mathfrak{A}.$$

In this case it is sufficient of course to assume (S) for $a = 0$.

Unfortunately, some of the most interesting quantum field theory models do not fall within the scope of this analysis. In particular states with non-zero electric charge are not normal states of representations satisfying (S) because, by Gauss'law, a localized electric charge produces a constant flux of electric field strength through any enveloping sphere. Hence states with differing electric charge will never coincide on $\mathfrak{A}(\mathcal{O}')$ however large $\mathcal{O}$ itself is.

The analysis in [7, 8] also relies on a structural assumption on $\mathfrak{A}$, known as <u>duality</u> :

$$\pi_0(\mathfrak{A}(\mathcal{O}'))' = \pi_0(\mathfrak{A}(\mathcal{O}))\,, \qquad \mathcal{O} \in \mathcal{K}.$$

In this connection it may be helpful to point out that if $\mathfrak{A}$ is any net of von Neumann algebras over $\mathcal{K}$ on a Hilbert space, one may define a dual net $\mathfrak{A}^d$ by setting

$$\mathfrak{A}^d(\mathcal{O}) = \mathfrak{A}(\mathcal{O}')'\,, \qquad \mathcal{O} \in \mathcal{K}.$$

Trivially $\mathfrak{A}_1 \subset \mathfrak{A}_2$ implies $\mathfrak{A}_1^d \supset \mathfrak{A}_2^d$ and $\mathfrak{A} \subset \mathfrak{A}^{dd}$. Thus the operation $\mathfrak{A} \to \mathfrak{A}^d$ has the same formal properties as building a commutant. A net is local if $\mathfrak{A} \subset \mathfrak{A}^d$. If $\mathfrak{A}$ is local, so is $\mathfrak{A}^{dd}$. A net satisfies duality if $\mathfrak{A} = \mathfrak{A}^d$ and this implies in particular that $\mathfrak{A}$ is a maximal local net. Let me introduce one further concept and say that a net satisfies <u>essential duality</u> if $\mathfrak{A}^d = \mathfrak{A}^{dd}$. The reason for this is that, as will be discussed further in section 3, duality for $\pi_0(\mathfrak{A})$ is too restrictive because it rules out spontaneously broken gauge symmetries. Instead I shall suppose that $\pi_0(\mathfrak{A})$ satisfies essential duality.

As a first application of local cohomology, I shall show how the condition (S) may be analysed with its help. To simplify notation in the remainder of this section, the symbol π_0 will usually be suppressed and $\mathfrak{A}$ will be treated as a net in $\mathcal{H}_0$. Consider $\mathfrak{A}^d$ as a net over $\mathcal{C}$ by defining

$$\mathfrak{A}^d(F) = \mathfrak{A}(F')' \equiv \bigcap \{ \mathfrak{A}(\mathcal{O})' : \mathcal{O} \in \mathcal{K}, \mathcal{O} \subset F' \} .$$

Let $\mathcal{U}(\mathfrak{A}^d)$ denote the unitary group of $\mathfrak{A}^d$ with the induced net structure.

<u>2.1. Proposition</u>. The unitary equivalence classes of representations satisfying (S) are in 1-1 correspondence with the cohomology classes $H^1_\ell(\mathcal{U}(\mathfrak{A}^d))$, provided $s > 1$.

I will not give a proof, but indicate how the correspondence is established. Given π and $a \in \Sigma_0$, pick a unitary operator $\psi_a : \mathcal{H}_\pi \to \mathcal{H}_0$ such that

$$\psi_a \, \pi(A) = A \, \psi_a , \qquad A \in \mathfrak{A}(\mathcal{O}' + a)$$

and define

$$z(b) = \psi_{\partial_0 b} \, \psi^*_{\partial_1 b} , \qquad b \in \Sigma_1$$

then $z \in Z^1_\ell(\mathcal{U}(\mathfrak{A}^d))$. Conversely, given $z \in Z^1_\ell(\mathcal{U}(\mathfrak{A}^d))$

and $a \in \Sigma_0$, one may define a representation π_a by requiring that

$$\pi_a(A) = z(b)\, A\, z(b)^*$$

whenever $b \in \Sigma_1$, $\partial_0 b = a$, $A \in \mathfrak{A}(\mathcal{O}_1)$ and $\mathcal{O}_1 \subset \mathcal{O}' + \partial_1 b$ (compare [9 ; Thm. 2.2]). Furthermore

$$z(b)\, \pi_{\partial_1 b}(A) = \pi_{\partial_0 b}(A)\, z(b), \quad b \in \Sigma_1,\ A \in \mathfrak{A},$$

$$\pi(A) = A, \qquad A \in \mathfrak{A}(\mathcal{O}' + a).$$

This implies that π_a satisfies (S) and also that $\pi_a(\mathfrak{A}) \subset \mathfrak{A}^d$.

In the above, the net structure on $\mathfrak{U}(\mathfrak{A}^d)$ would a priori be changed in one replaced $\mathfrak{A}$ by $\mathfrak{A}^{dd}$ in its definition. However a little argument shows that the notion of local 1-cocycle remains unchanged. Consequently when $s > 1$, the given representation π of $\mathfrak{A}$ may be extended in a canonical manner to a representation $\tilde{\pi}$ of $\mathfrak{A}^{dd}$ by setting

$$\tilde{\pi}(A) = \psi_a^* A \psi_a, \qquad A \in \mathfrak{A}^{dd}(\mathcal{O}' + a)$$

and $\tilde{\pi}$ satisfies (S) (with π_0 of course being replaced by $\tilde{\pi}_0$ the defining representation of $\mathfrak{A}^{dd}$).

Specializing this result to the case of a net satisfying essential duality, we have

<u>2.2. Theorem</u>. If $s > 1$ and the observable algebra satisfies essential duality, then every representation π satisfying (S) has a canonical extension to a representation $\tilde{\pi}$ of $\mathfrak{A}^d$ which also satisfies (S).

This shows that the analysis of superselection structure in [7 , 8] applies equally well to essentially dual observable algebras.

If $s = 1$, $\mathcal{O}'$ is no longer path-connected but decomposes into two components denoted by $\mathcal{O}^r$ and $\mathcal{O}^\ell$, the right and left spacelike complements of $\mathcal{O}$ respectively. One may define (compare [9 ; Prop. 3.2]).

$$\tilde{\pi}^\ell(A) = \psi_a^* A \psi_a , \quad A \in \mathfrak{A}(\mathcal{O}_1) , \quad \mathcal{O}+a \subset \mathcal{O}_1^\ell$$

$$\tilde{\pi}^r(A) = \psi_a^* A \psi_a , \quad A \in \mathfrak{A}(\mathcal{O}_1) , \quad \mathcal{O}+a \subset \mathcal{O}_1^r$$

In general $\tilde{\pi}^\ell \neq \tilde{\pi}^r$. On the left $\tilde{\pi}^r$ behaves as $\tilde{\pi}_0$, the vacuum representation of $\mathfrak{A}^d$, on the right it behaves as $\tilde{\pi}_0'$, which is in general a different extension of π_0 to $\mathfrak{A}^d$. Typically one expects ([9 ; Thm. 3.3] is applicable here) that $\tilde{\pi}_0'$ is the vacuum representation for a different choice of vacuum state $\tilde{\omega}_0' = \tilde{\omega}_0 \circ g$ of $\mathfrak{A}^d$ where g is a gauge automorphism (see Section 3). The interpretation here is that π describes a "soliton sector". It is a representation relevant to elementary particle physics because it represents a local finite-energy perturbation of the vacuum representation. The soliton aspect only becomes apparent when one tries to extend π to a larger algebra $\mathfrak{A}^d$ containing some "field quantities". However this extension is done it exhibits non-local features usually expressed in some way in terms of a homotopy invariant. This is analogous to the interpretation of the chemical potential in [10] where this too is a latent parameter which appears when a KMS state of the observable algebra is extended to a KMS state of the field algebra.

Examples of soliton behaviour which fit into this pattern may be found in [11, 12] . Of course the analysis presented here applies only to a 2-dimensional space-time. However the class of representations singled out by (S) are those usually described by gauge groups of the first kind. The usual argument for showing that gauge groups of the second kind are needed to produce soliton behaviour in more than two space-time dimensions is an energy argument [13] . It is remarkable how two apparently unrelated arguments lead to the same qualitative conclusions. For a review of soliton behaviour in higher dimensions illustrating the role of homotopy, the reader may profitably consult [14].

3. FIELD ALGEBRAS

In practice one does not usually proceed by constructing the observable net $\mathfrak{A}$ in the vacuum representation and then looking for new representations of $\mathfrak{A}$. Instead one constructs a net of fields in some irreducible vacuum representation on a Hilbert space $\mathcal{H}$. $\mathfrak{F}(\mathcal{O})$ may be thought of as the von Neumann algebra generated by the bounded functions of the fields smeared with test functions with support in $\mathcal{O}$. $\mathfrak{A}$ is then considered as a subnet of $\mathfrak{F}$, $\mathfrak{A} \subset \mathfrak{F}$. In favourable cases, the relevant states of $\mathfrak{A}$ are the normal states of the defining representation on $\mathcal{H}$. Associated with $\mathfrak{F}$ one has a continuous representation $L \to \beta_L$ of $\tilde{\mathcal{P}}$ by automorphisms of $\mathfrak{F}$ inducing $L \to \alpha_L$ on $\mathfrak{A}$ and a vacuum state ϕ_0 on $\mathfrak{F}$ inducing ω_0 on $\mathfrak{A}$. Some condition is needed to ensure that the normal states of $\mathfrak{F}$ are physically relevant as states of $\mathfrak{A}$. One way of ensuring this is to suppose that $\mathfrak{F}$ is relatively local to $\mathfrak{A}$.

$$\mathfrak{A}(\mathcal{O}) \subset \mathfrak{F}(\mathcal{O}')', \qquad \mathcal{O} \in \mathcal{K}.$$

A gauge automorphism of $\mathfrak{F}$ is an automorphism g of $\mathfrak{F}$ such that

$$g(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(\mathcal{O}), \quad \mathcal{O} \in \mathcal{K} \qquad \text{and} \qquad g(A) = A, \quad A \in \mathfrak{A}.$$

Let $\mathcal{G}$ denote the group of gauge automorphisms of $\mathfrak{F}$. If $g \in \mathcal{G}$, then $_L g \equiv \beta_L g \beta_L^{-1} \in \mathcal{G}$. $\mathcal{G}$ is said to be a gauge group of the first kind if

$$_L g = g, \qquad g \in \mathcal{G}, \quad L \in \tilde{\mathcal{P}}.$$

Let $\mathcal{G}_0 = \{ g \in \mathcal{G} : \phi_0 \circ g = \phi_0 \}$ be the stability subgroup of the vacuum. The elements of $\mathcal{G}_0$ can be represented by unitary operators $V(g)$ in $\mathcal{H}$

$$g(F)\Omega = V(g)F\Omega, \qquad F \in \mathfrak{F}, \; g \in \mathcal{G}_0 .$$

If $\mathcal{G} \neq \mathcal{G}_0$, then one talks about spontaneously broken gauge symmetries. It will also prove convenient to introduce a net $\mathcal{B}$ intermediate between $\mathfrak{A}$ and $\mathfrak{F}$. Set

$$\mathcal{B}(\mathcal{O}) = \{ F \in \mathfrak{F}(\mathcal{O}) : g(F) = F, \ g \in \mathcal{G}_0 \}, \qquad \mathcal{O} \in \mathcal{K}.$$

The net $\mathfrak{F}$ may be usefully regarded as an extension of the net $\mathfrak{A}$. There has been as yet no systematic study of such extensions although incidentally many partial results are known. Let me give a long list of properties which seem to be valid for the most useful extensions in field theory at the same time giving some references to the partial results which go some way towards establishing these properties. The first property is usually used to define $\mathfrak{A}$ in terms of $\mathfrak{F}$ and $\mathcal{G}$ by a principle of gauge invariance.

1) $$\mathfrak{A}(\mathcal{O}) = \{ F \in \mathfrak{F}(\mathcal{O}) : g(F) = F, \ g \in \mathcal{G} \}.$$

2) $\mathcal{G}_0$ is compact, considered for example in the topology induced by strong operator convergence on $\mathcal{H}$ [15, 16].

3) $\mathcal{G}_0$ commutes with Poincaré transformations [15, 17, 18].

4) $\mathfrak{F}$ has "Bose-Fermi commutation relations", this is most conveniently described as follows : there is an involution $k \in \mathcal{G}$ such that if $\mathcal{O}_1 \subset \mathcal{O}_2'$ then

$$F_1 F_2 = F_2 F_1, \qquad F_2 \in \mathfrak{F}(\mathcal{O}_2), \quad F_1 \in \mathfrak{F}_+(\mathcal{O}_1)$$

$$F_1 F_2 = k(F_2) F_1, \qquad F_2 \in \mathfrak{F}(\mathcal{O}_2), \quad F_1 \in \mathfrak{F}_-(\mathcal{O}_1)$$

Here $\mathfrak{F}_+ = \{ F \in \mathfrak{F} : k(F) = F \}$, $\mathfrak{F}_- = \{ F \in \mathfrak{F} : k(F) = -F \}$
In fact $k \in \mathcal{G}_0$ [19].

5) $\mathfrak{F}$ satisfies twisted duality [16, 20].

$$\mathfrak{F}(\mathcal{O}')' = \mathfrak{F}^t(\mathcal{O}) \equiv \mathfrak{F}_+(\mathcal{O}) + V(k)\mathfrak{F}_-(\mathcal{O}), \quad \mathcal{O} \in \mathcal{K} .$$

6) $\mathfrak{B}^- = \mathfrak{A}^-$ and even $\mathfrak{A}(\mathcal{O}')^- = \mathfrak{B}(\mathcal{O}')^-$, $\mathcal{O} \in \mathcal{K}$ [19].

7) $\mathfrak{B}^- = V(\mathcal{G}_0)'$. Furthermore the irreducible subrepresentations of the defining representation of $\mathfrak{B}$ (and hence by 6) of $\mathfrak{A}$) on $\mathcal{H}$ are in 1-1 correspondence with the irreducible representations of $\mathcal{G}_0$ [16].

8) $\mathfrak{B}$ satisfies duality on $\mathcal{H}_0 = [\mathfrak{B}\Omega] = [\mathfrak{A}\Omega]$ and hence by 6) , $\mathfrak{A}$ satisfies essential duality on $\mathcal{H}_0$ [16] .

Of course, $\mathfrak{A}$ itself does not satisfy duality on $\mathcal{H}_0$ unless $\mathcal{G} = \mathcal{G}_0$ and the presence of spontaneously broken symmetries is the main reason for insisting on essential duality in place of duality. $\mathfrak{B}$ plays the role of the net $\mathfrak{A}^d$ of section 2 and is isomorphic to this net.

Before going on to list two further properties, I must describe some techniques introduced in [21] and developed in [22] . If $\mathcal{M}$ is a von Neumann algebra, then a Hilbert space in $\mathcal{M}$ is a norm-closed linear subspace H of $\mathcal{M}$ such that $\psi \in H$ implies $\psi^*\psi \in \mathbb{C}I$ and $A \in \mathcal{M}$ and $A\psi = 0$, $\psi \in H$ implies $A = 0$.

As the name implies, H is actually a Hilbert space with the norm induced by $\mathcal{M}$. If $\{\psi_j\}_{j\in J}$ is an orthonormal basis for H , then the $\psi_j\psi_j^*$ are mutually orthogonal projections and $\sum_{j\in J}\psi_j\psi_j^* = I$. Of course, unless $\mathcal{M}$ is properly infinite, all Hilbert spaces in $\mathcal{M}$ are one dimensional. Associated with H is a morphism ρ_H of $\mathcal{M}$, i.e. an identity preserving, normal $*$-homomorphism of $\mathcal{M}$ into itself, defined by

$$\rho_H(A)\psi = \psi A , \qquad \psi \in H, \quad A \in \mathcal{M} .$$

In fact, $\rho_H(A) = \sum_{j\in J}\psi_j A \psi_j^*$. If H_1 and H_2 are Hilbert spaces in $\mathcal{M}$, then the bounded linear operators from H_1 to H_2 can be identified with

$$(H_1, H_2) = \{ A \in \mathcal{M} : AH_1 \subset H_2 \}$$

If $A \in (H_1, H_2)$ then

$$A \rho_{H_1}(B) = \rho_{H_2}(B) A , \qquad B \in \mathcal{M} .$$

If α is an automorphism of $\mathcal{M}$, $\alpha(H)$ is again a Hilbert space in $\mathcal{M}$ and there is a unique unitary $U \in (H, \alpha(H))$ such that $U\psi = \alpha(\psi)$, $\psi \in H$.

Returning now to the field net $\mathfrak{F}$, let $\mathcal{H}(\mathcal{O})$ denote the set of Hilbert spaces in $\mathfrak{F}(\mathcal{O})$ such that $g(H) = H$, $g \in \mathcal{G}$. Thus each element of $\mathcal{H}(\mathcal{O})$ carries a unitary representation of $\mathcal{G}$. If $H_1, H_2 \in \mathcal{H}(\mathcal{O})$ then $S \in (H_1, H_2)$ is in $\mathfrak{A}$ if and only if S intertwines the corresponding representations of $\mathcal{G}$. If $H \in \mathcal{H}(\mathcal{O})$ then ρ_H may be considered as a morphism of the net $\mathfrak{F}$ and $\rho_H(\mathfrak{A}(\mathcal{O})) \subset \mathfrak{A}(\mathcal{O})$.

The two further properties of field nets are

9) $\mathfrak{F}(\mathcal{O})$ is generated as a von Neumann algebra by $\mathcal{H}(\mathcal{O})$ [21].

10) If H is a Hilbert space in $\mathfrak{F}(\mathcal{O})$ and $\rho_H(\mathfrak{A}(\mathcal{O}_1)) \subset \mathfrak{A}(\mathcal{O}_1)$ for $\mathcal{O}_1 \supset \mathcal{O}$ then $H \in \mathcal{H}(\mathcal{O})$.

If 10) is used to define $\mathcal{H}(\mathcal{O})$ then 9) provides what is perhaps a useful notion of extension of nets analogous to the use of normal subgroups in the theory of group extensions. The gauge group $\mathcal{G}$ appears as the Galois group of the extension.

We have seen in Proposition 2.1 how the local 1-cohomology of $\mathcal{U}(\mathfrak{B})$ gives information on the sectors of $\mathfrak{A}$ and hence on the representation theory of $\mathcal{G}_0$. We shall now show how the local 1-cohomology of $\mathcal{U}(\mathfrak{A})$ gives one information on the field net and the representation theory of $\mathcal{G}$, provided $\mathcal{G}$ is a gauge group of the first kind.

We may regard $\mathfrak{A}$ as a net over $\mathcal{K}$ by defining

$$\mathfrak{A}(F) = \mathfrak{A} \cap \mathfrak{A}(F')', \qquad F \in \mathcal{E}.$$

This is consistent with the net structure over $\mathcal{K}$. We give $\mathcal{U}(\mathfrak{A})$ the induced net structure over $\mathcal{E}$.

Given $\mathcal{O} \in \mathcal{K}_0$ and $H \in \mathcal{H}(\mathcal{O})$, $\alpha_a(H) \in \mathcal{H}(\mathcal{O}+a)$ when $\mathcal{G}$ is a gauge group of the first kind. Let $\{\psi_j\}_{j\in J}$ be a basis for H ; we set

$$z(b) = \sum_{j\in J} \alpha_{\partial_0 b}(\psi_j)\, \alpha_{\partial_1 b}(\psi_j)^*, \qquad b \in \Sigma_1 .$$

<u>3.1. Proposition</u>. $z \in Z^1_\ell(\mathcal{U}(\mathfrak{A}))$ and z is a local 1-coboundary if and only if H is a 1-dimensional Hilbert space in $\mathfrak{A}(\mathcal{O})$.

It is easily checked that $z \in Z^1_\ell(\mathcal{U}(\mathfrak{A}))$ and the remainder of the Proposition can be deduced from [9 ; Thm. 2.2] .

One defect of this local 1-cohomology is evident : there will always be Hilbert spaces in $\mathfrak{A}(\mathcal{O})$ of dimension greater than one. The corresponding local 1-cocycles are of no importance but this information is not contained in the local 1-cohomology. There is also an analogous defect in the local 1-cohomology used in Proposition 2.1. The reason is that the appropriate coefficient object is not $\mathcal{U}(\mathfrak{A})$ but the algebraic system of morphisms and intertwiners, which we shall now describe.

If ρ and ρ' are morphisms of $\mathfrak{A}$, then an intertwiner [7, 23] from ρ to ρ' is a triple $(\rho' | S | \rho)$ with $S \in \mathfrak{A}$ such that

$$\rho'(A)\, S = S\, \rho(A), \qquad A \in \mathfrak{A}.$$

The set of intertwiners from ρ to ρ' forms a linear space. An adjoint is defined by setting

$$(\rho' | S | \rho)^* = (\rho | S^* | \rho') .$$

There is a composition of intertwiners defined if the adjoining morphisms coincide

$$(\rho'' | S' | \rho') \circ (\rho' | S | \rho) = (\rho'' | S'S | \rho)$$

and an associative product

$$(\rho_1' | S_1 | \rho_1) \times (\rho_2' | S_2 | \rho_2) = (\rho_1' \rho_2' | S_1 \rho_1(S_2) | \rho_1 \rho_2) .$$

At some risk of confusion, the triple $(\rho' | S | \rho)$ will be denoted simply by S . One has

$$(S_1 \times S_2)^* = S_1^* \times S_2^*$$

$$(S_1' \circ S_1) \times (S_2' \circ S_2) = (S_1' \times S_2') \circ (S_1 \times S_2) ,$$

the interchange law being valid whenever the left hand side is defined. This structure was also discussed in [22] under the name monoidal W^*-category $^{*)}$. Here we have a monoidal C^*-category or, if one takes the local structure into account, a net of monoidal C^*-categories which will be denoted by End $\mathfrak{A}$.

End $\mathfrak{A}$ plays an important role in describing extensions : the field net $\mathfrak{F}$ determines a full monoidal subcategory of End $\mathfrak{A}$ whose objects are $\{ \rho_H \restriction \mathfrak{A} : H \in \mathcal{H}(\mathcal{O}), \mathcal{O} \in \mathcal{K} \}$. This subcategory plays the role of a "quotient object of $\mathfrak{F}$ by $\mathfrak{A}$" and is represented in $\mathfrak{F}$ as a category of representations of $\mathcal{G}$ (Compare [21 , Thm. 3.6]).

A (unitary) 1-cocycle in End $\mathfrak{A}$ is a pair (y, z) where for each $a \in \Sigma_0$, $y(a)$ is a morphism of $\mathfrak{A}$ and for each $b \in \Sigma_1$, $z(b)$ is a unitary intertwiner from $y(\partial_1 b)$ to $y(\partial_0 b)$

$$z(b)\, y(\partial_1 b)(A) = y(\partial_0 b)(A)\, z(b), \quad A \in \mathfrak{A},\ b \in \Sigma_1$$

such that

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* For the more standard notions of category theory used in the sequel, the reader may consult [24] for example.

$$z(\partial_0 c) \circ z(\partial_2 c) = z(\partial_1 c), \qquad c \in \Sigma_2 .$$

The right way to describe the non-Abelian cohomology is not to stress the equivalence relation "cohomologous" but to regard the 1-cocycles as the objects of a category $Z^1(\mathrm{End}\,\mathfrak{A})$. In this category, $t : (y,z) \to (y',z')$ if, for each $a \in \Sigma_0$, $t(a)$ is an intertwiner

$$t(a)\, y(a)(A) = y'(a)(A)\, t(a) , \qquad A \in \mathfrak{A}$$

and

$$z'(b) \circ t(\partial_1 b) = t(\partial_0 b) \circ z(b) , \qquad b \in \Sigma_1$$

If t^* is defined by $t^*(a) = t(a)^*$, $a \in \Sigma_0$, then $t^* : (y',z') \to (y,z)$. (y,z) and (y',z') are cohomologous if there is a $t : (y,z) \to (y',z')$ with $t(a)$ unitary for each $a \in \Sigma_0$.

There is a monoidal $*$-functor [22] $G : \mathrm{End}\,\mathfrak{A} \to Z^1(\mathrm{End}\,\mathfrak{A})$ defined as follows : let $S = (\rho' | S | \rho)$ then $G(\rho) = (y, z)$ where $y(a) = \rho$, $z(b) = 1_\rho \equiv (\rho | I | \rho)$, $G(S)(a) = S$, $a \in \Sigma_0$, $b \in \Sigma_1$. The image of G consists of the trivial 1-cocycles. There are also monoidal $*$-functors $F_a : Z^1(\mathrm{End}\,\mathfrak{A}) \to \mathrm{End}\,\mathfrak{A}$ defined by evaluation at $a \in \Sigma_0$:

$$F_a(y,z) = y(a) , \qquad F_a(t) = t(a) .$$

F_a is a left inverse of G. A contracting homotopy h based on $a \in \Sigma_0$ gives a natural transformation $\eta(h) : \mathrm{id}_{Z^1(\mathrm{End}\,\mathfrak{A})} \to GF_a$ defined by

$$\eta(h)_{(y,z)}(a') = z(ha') , \qquad a' \in \Sigma_0 .$$

This shows that $Z^1(\mathrm{End}\,\mathfrak{A})$ is monoidally equivalent to $\mathrm{End}\,\mathfrak{A}$ and expresses the triviality of the cohomology.

To define the local 1-cohomology, one looks at $\mathrm{End}\,\mathfrak{A}$ as a net over $\mathcal{K}$. If $F \in \mathcal{K}$, we define a full monoidal subcategory $\mathrm{End}\,\mathfrak{A}(F)$ of $\mathrm{End}\,\mathfrak{A}$ by requiring that $(\rho' | S | \rho)$ is an arrow of End $\mathfrak{A}(F)$ if $\rho(A) = \rho'(A) = A$ for $A \in \mathfrak{A}(\mathcal{O})$, $\mathcal{O} \subset F'$. This automatically implies that $S \in \mathfrak{A}(F)$. We now define a 1-cocycle (y, z) to be local if there exists an $\mathcal{O} \in \mathcal{K}_0$ such that $y(a)$ is an object of End $\mathfrak{A}(\mathcal{O}+a)$, $a \in \Sigma_0$. This automatically implies that $z(b)$ is an arrow of End $\mathfrak{A}(\mathcal{O}+b)$, $b \in \Sigma_1$. An arrow $t: (y, z) \to (y', z')$ of local 1-cocycles is automatically local in the sense that there exists an $\mathcal{O} \in \mathcal{K}_0$ such that $t(a)$ is an arrow of End $\mathfrak{A}(\mathcal{O}+a)$, $a \in \Sigma_0$. The resulting C^*-monoidal category will be denoted by $Z^1_\ell(\mathrm{End}\,\mathfrak{A})$.

A 1-cocycle of the form $G(\rho)$ is local if and only if $\rho = \iota$, the identity automorphism of $\mathfrak{A}$. A Hilbert space $H \in \mathcal{H}(\mathcal{O})$ still generates a local 1-cocycle (y, z) as in Proposition 3.2, if we use the previous definition for $z(b)$ and set $y(a) = \rho_{\alpha_a}(H) \upharpoonright \mathfrak{A}$. If H is a Hilbert space in $\mathfrak{A}(\mathcal{O})$ and $\psi \in H$, there is an arrow $t_\psi : G(\iota) \to (y, z)$ defined by $t_\psi(a) = \alpha_a(\psi)$, $a \in \Sigma_0$.

The image of $Z^1_\ell(\mathrm{End}\,\mathfrak{A})$ under the monoidal $*$-functors F_a is independent of a and is denoted by $\mathcal{J}(\mathfrak{A})$. A morphism is an object of $\mathcal{J}(\mathfrak{A})$ if and only if it is a transportable localized morphism, i.e. if there are unitary intertwiners $(\rho_a | u_a | \rho)$ with $\rho_0 = \rho$ and an $\mathcal{O} \in \mathcal{K}$ such that

$$\rho_a(A) = A, \quad A \in \mathfrak{A}(\mathcal{O}' + a), \quad a \in \Sigma_0 .$$

This is the analogue for morphisms of the condition (S) for representations studied in Section 2. $F_a : Z^1_\ell(\mathrm{End}\,\mathfrak{A}) \to \mathcal{J}(\mathfrak{A})$ is an equivalence of categories. For this reason we shall talk about $\mathcal{J}(\mathfrak{A})$ rather than Z^1_ℓ (End $\mathfrak{A}$) in the remainder of this section. Statements about $\mathcal{J}(\mathfrak{A})$ have pointwise extensions to statements about $Z^1_\ell(\mathrm{End}\,\mathfrak{A})$.

The results on superselection structure in [7, 8] rested on an analysis of $\mathcal{J}(\mathfrak{A})$ where it was assumed that $\mathfrak{A}$ satisfies duality. We have already seen in Section 2, however, how these results can be extended to the case that $\mathfrak{A}$ satisfies essential duality. In fact, the same is true of the analysis of $\mathcal{J}(\mathfrak{A})$ and this provides an abstract way of studying spontaneously broken gauge symmetries of the first kind without explicitly introducing field nets of gauge groups. The key result which is just a variation on Theorem 2.2 is

3.2. <u>Theorem</u>. If $s > 1$ and $\mathfrak{A}$ satisfies essential duality then every localized transportable morphism ρ of $\mathfrak{A}$ can be extended to a unique localized transportable morphism $\tilde{\rho}$ of $\mathfrak{A}^d$. Furthermore $\rho \mapsto \tilde{\rho}$ and $t \mapsto t$ is a monoidal $*$-functor from $\mathcal{J}(\mathfrak{A})$ to $\mathcal{J}(\mathfrak{A}^d)$.

This functor from $\mathcal{J}(\mathfrak{A})$ to $\mathcal{J}(\mathfrak{A}^d)$ is the abstract version of the operation of restricting representations from $\mathcal{G}$ to $\mathcal{G}_o$.

We may construct, as in [7, 23], the permutation symmetry ε for $\mathcal{J}(\mathfrak{A})$. Furthermore, if we suppose that

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{A} \cap \mathfrak{A}^d(\mathcal{O}), \qquad \mathcal{O} \in \mathcal{K}$$

which is the case if $\mathfrak{A}$ is derived from $\mathfrak{F}$ as envisaged at the beginning of this section, the construction of conjugate morphisms [8, 23] may be carried out within $\mathcal{J}(\mathfrak{A})$.

4. QUANTUM ELECTRODYNAMICS

The best way to approach the second cohomology is first to consider where one might expect to find a non-trivial local 2-cocycle. Let me recall the relationship between the cohomology of differential forms and singular differential cohomology. If A^μ is a vector field [x)] satisfying $\partial^\mu A^\nu - \partial^\nu A^\mu = 0$ and σ is a differentiable 1-simplex, one may define a 1-cocycle A by

$$A(\sigma) = \int A^\mu(x)\, d\sigma_\mu(x) = \int_0^1 A^\mu(\sigma(s)) \frac{d\sigma_\mu}{ds}\, ds$$

since, by Stokes' Theorem, if c is a differentiable 2-simplex, $A(\partial c) = 0$. A is a 1-coboundary if there is a scalar field ϕ with $A^\mu = \partial^\mu \phi$. Similarly if $F^{\mu\nu}$ is an antisymmetric tensor with $\partial^{[\lambda} F^{\mu\nu]} = 0$ then one gets a 2-cocycle F by setting

$$F(c) = \int F^{\mu\nu}(x)\, dc_{\mu\nu}(x) .$$

F is a 2-coboundary if $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$.

Now one knows that in quantum electrodynamics, the electromagnetic field $F^{\mu\nu}$ is a local field satisfying $\partial^{[\lambda} F^{\mu\nu]} = 0$ but that there is no local electromagnetic potential A^μ such that $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, unless one is prepared to sell one's soul, abandon Maxwell's equations and adopt the Gupta-Bleuler formalism.

In the case of free quantum electrodynamics, the ideas of local cohomology provide a new, and physically more transparent, proof of this well-known result [25] . Let $F^{\mu\nu}$ be the free electromagnetic field and f a C^∞-function of compact support in $\mathbb{R}^3$. Set

$$F_f^{\mu\nu}(x) = \int F^{\mu\nu}(x + \vec{y})\, f(\vec{y})\, d^3\vec{y} ,$$

so that we still have $\partial^{[\lambda} F_f^{\mu\nu]} = 0$. Now let

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[x] We adopt the physicist's convention and talk of a covariant vector field in place of a 1-form. In this section, we take $s = 3$.

$$F_f(c) = \int F_f^{\mu\nu}(x)\, dc_{\mu\nu}(x)$$

and consider the associated inner automorphism

$$z(c)(A) = e^{iF_f(c)} A\, e^{-iF_f(c)}, \quad A \in \mathfrak{A},$$

where $\mathfrak{A}$ is the net of von Neumann algebras generated by the free electromagnetic field. z is a 2-cocycle in the sense that for each differentiable 3-simplex d,

$$z(\partial_2 d)\, z(\partial_0 d) = z(\partial_1 d)\, z(\partial_3 d).$$

There is no ordering problem here because these automorphisms commute. More precisely, let A denote the subgroup of all inner automorphisms of $\mathfrak{A}$ induced by the Weyl operators, i.e. the subgroup of 1-particle automorphisms, then z is a local 2-cocycle with values in the Abelian group A . This is the origin of the examples illustrating the concept of local cohomology in Section 1, because the C^∞-solutions of Maxwell's equations corresponding to Cauchy data with compact support can be regarded as a subgroup of A and z even takes values in this subgroup.

4.1. <u>Theorem</u>. If $\int f(\vec{y})\, d^3\vec{y} \neq 0$, then z is a non-trivial local 2-cocycle of A . In fact there is no function y on differentiable 1-simplexes with values in morphisms of $\mathfrak{A}$, such that there exists an $\mathcal{O} \in \mathcal{K}_0$ with

$$y(b)(A) = A, \quad A \in \mathfrak{A}(\mathcal{O}_1), \quad \mathcal{O} + b \subset \mathcal{O}_1', \quad b \in \Sigma_1,$$

$$z(c)\, y(\partial_0 c)\, y(\partial_2 c) = y(\partial_1 c).$$

To prove this result; suppose y exists and $\mathcal{O} + \partial b \subset \mathcal{O}_1'$, then we may find a differentiable 2-simplex c with $\partial_1 c = b$ and $(\mathcal{O} + \partial_0 c) \cup (\mathcal{O} + \partial_2 c) \subset \mathcal{O}_1'$. If $A \in \mathfrak{A}(\mathcal{O}_1)$, $y(\partial_0 c)(A) = y(\partial_2 c)(A) = A$ so

$$y(b)(A) = y(\partial_1 c)(A) = z(c)(A), \quad A \in \mathfrak{A}(\mathcal{O}_1). \tag{x}$$

We show that (x) leads to inconsistencies. Before proceeding to a formal proof, it may be useful to sketch the ideas involved. $y(b)$ if it existed, would create a charge q at $\partial_0 b$ and $-q$ at $\partial_1 b$ contradicting $0 = j^\mu = \partial_\nu F^{\mu\nu}$. This suggests what to do : take for A the surface integral of the electric field over part of the surface of a small sphere surrounding $\partial_0 b$ such that b intersects this surface.-A is equally the surface integral of the electric field over the complementary part of the surface of the sphere. In the latter case (x) would give $y(b)(A) = A$ and in the former $y(b)(A) = A + q$.

Now let $F^{*\mu\nu}$ denote the dual tensor of $F^{\mu\nu}$ and g be a C^∞-function with compact support in $\mathbb{R}^3$. We can define another local 2-cocycle z' with values in A by setting

$$z'(c)\,(A) = e^{iF^*_g(c)}\,A\,e^{-iF^*_g(c)}\,, \quad A \in \mathfrak{A}$$

Now define $\langle z'(c'), z(c)\rangle$ [x] by

$$\langle z'(c'), z(c)\rangle\, I = e^{-iF^*_g(c')}\,e^{iF_f(c)}\,e^{iF^*_g(c')}\,e^{-iF_f(c)}\,.$$

$\langle z'(c'), z(c)\rangle$ takes values in the circle and is a 2-cocycle in c and c' separately. As such its values depend only on ∂c and $\partial c'$. The interesting configurations are those for which $(\partial c + \mathrm{supp} f) \subset (\partial c' + \mathrm{supp}\, g)'$. In fact if ∂c_1 and ∂c_2 are homologous as cycles of $(\partial c' + \mathrm{supp}\, g - \mathrm{supp} f)'$ then $\langle z'(c'), z(c_1)\rangle = \langle z'(c'), z(c_2)\rangle$. This suggests evaluating $\langle z'(c'), z(c)\rangle$ when ∂c has a non-trivial homology class in $(\partial c' + \mathrm{supp}\, g - \mathrm{supp} f)'$. Just such a situation is involved in calculating $y(b)(A)$ using (x) if A is the surface integral of the electric field as discussed above. Taking ∂c to be $x_0 = 0$, $x_1^2 + x_2^2 = a^2$, $x_3 = 0$ and $\partial c'$ to be $x_0 = 0$, $x_2 = 0$, $(x_1 - a)^2 + x_3^2 = a^2$, an explicit computation gives $\langle z'(c'), z(c)\rangle = e^{ik\int g(\vec{x})d^3\vec{x}\int f(\vec{x})d^3\vec{x}}$ for a sufficiently large, where $|k| \neq 0$ depends only on the normalization of the basic field $F^{\mu\nu}$ and the sign of k depends on the relative orientations of ∂c and $\partial c'$. In particular choosing

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x A similar coupling of local cocycles takes place at the level of the 1-cohomology for the massless "scalar field" with s=1 discussed in [26]. This is really a vector field A^μ satisfying the self-dual equations $\partial_\mu A^\mu = 0$ and $\partial^\mu A^\nu - \partial^\nu A^\mu = 0$.

$\int g(\vec{x})\, d^3\vec{x}$ suitably, $\langle z'(c'), z(c) \rangle \neq 1$, whereas if there were to be a y with the properties described in Theorem 4.1, one would need $\langle z'(c'), z(c) \rangle = 1$ for a sufficiently large.
Of course, the situation is symmetric as regards z and z'. Thus z' is a non-trivial local 2-cocycle because any local y' would have to be such that $y'(b)$ creates a magnetic charge q at $\partial_0 b$ and $-q$ at $\partial_1 b$. This is the reason for regarding $H^2_\ell(\mathcal{M})$ in Section 1 as being parametrized by an electric and a magnetic charge.

The same argument, replacing group-theoretical commutators by commutators, will show that, if $\int f(\vec{x})\, d^3\vec{x} \neq 0$, there is no function $G_b(P)$ with the properties expected of $\iint (\Omega, [A^\mu(x+\vec{y}), P]\Omega) f(\vec{y})\, d^3\vec{y}\, db_\mu(x)$ for a local electromagnetic potential, where P is a polynomial in the fields.

In interacting quantum electrodynamics the dearth of rigorous results forces me to rely on a speculative chain of reasoning. In the first place, everyone believes that here too there is no local electromagnetic potential A^μ. However, as there are electrically charged particles in the interacting theory, the reason is not just that $e^{iA_f(b)}$ tries to create charges at the vertices of b, but probably that such charges would, as in the free theory, not be quantized.
The next step is to realize that there is a formal expression with just the right properties for being an electromagnetic "potential" of a quantized charge for the interacting theory :

$$\psi(\partial_0 b)^* \, e^{ie \int A^\mu(x)\, db_\mu(x)} \, \psi(\partial_1 b) .$$

This expression is gauge invariant in the conventional sense and should therefore generate local quantities in the vacuum Hilbert space of quantum electrodynamics, which cannot be expressed as local functions of the electromagnetic field. In other words, one would expect that if $\mathfrak{A}$ is the net generated by the vacuum representation of the electromagnetic field, then $\mathfrak{A}$ does not satisfy duality although it should still satisfy essential duality in the light of the general arguments in [20].
The "true electromagnetic potential" might be thought of as a path-dependent

field taking values in the circle. More precisely, I suppose that, in place of the formal expression above, for some (and presumably any) $\mathcal{O} \in \mathcal{K}_0$ there is a unitary $u(b) \in \mathfrak{A}^d(\mathcal{O}+b)$, $b \in \Sigma_1$ such that

$$z(c) \equiv u(\partial_1 c)\, u(\partial_2 c)^{-1}\, u(\partial_0 c)^{-1} \in \mathfrak{A}(\mathcal{O}+c)\,, \quad c \in \Sigma_2 .$$

In Section 3, gauge automorphisms of field nets were defined and one may ask whether there are any gauge automorphisms of $\mathfrak{A}^d$ considered as a field net. The interpretation of $u(b)$ as a potential leads one to suppose that there are such gauge automorphisms and that if $\mathcal{G}$ denotes the gauge group of $\mathfrak{A}^d$, presumably Abelian, then

$$g(u(b)) = \chi_b(g)\; u(b) \;, \qquad g \in \mathcal{G}$$

where χ_b is character of $\mathcal{G}$. This is not a logical contradiction to the gauge invariance of the formal expression above but at most a linguistic one. If $X(F)$, for $F \in \mathcal{C}$, denotes the set of characters χ of $\mathcal{G}$ such that there exists a unitary u in $\mathfrak{A}^d(F)$ with $g(u) = \chi(g)\,u$, $g \in \mathcal{G}$ then χ_b is a local 1-cocycle with values in the net X . A natural conjecture is $H^1_{\ell}(X) = \mathbb{Z}$, where the parameter is interpreted as an electric charge.

However, returning to the 2-cohomology, the algebraic structure of End $\mathfrak{A}$ also allows it to be used as the coefficient object for a 2-cohomology. I will not attempt to describe the general version which would be necessary for non-Abelian gauge theories of the second kind but will just present the simplified version relevant to the Abelian case. A (unitary) 2-cocycle with values in End $\mathfrak{A}$ is a pair (y,z) where for each $b \in \Sigma_1$, $y(b)$ is a morphism of $\mathfrak{A}$ and for each $c \in \Sigma_2$, $z(c)$ is a unitary intertwiner

$$z(c)\, y(\partial_0 c)\, y(\partial_2 c)\, (A) = y(\partial_1 c)(A)\, z(c), \quad A \in \mathfrak{A}, \quad c \in \Sigma_2$$

such that

$$z(\partial_2 d) \circ (z(\partial_0 d) \times 1_{y(\partial_2 \partial_3 d)}) = z(\partial_1 d) \circ (1_{y(\partial_0 \partial_1 d)} \times z(\partial_3 d)), \quad d \in \Sigma_3 .$$

As in the case of the 1-cohomology, the 2-cocycles can be regarded as the objects of a category $Z^2(\mathrm{End}\, \mathfrak{A})$. In this category, $(v,w) : (y,z) \to (y',z')$ if, for each $a \in \Sigma_0$, $v(a)$ is a morphism of $\mathfrak{A}$ and $w(b)$ is an intertwiner

$$w(b)\, v(\partial_0 b)\, y(b)(A) = y'(b)\, v(\partial_1 b)(A)\, w(b), \quad b \in \Sigma_1$$

such that for $c \in \Sigma_2$

$$(z'(c) \times 1_{v(\partial_1 \partial_2 c)}) \circ (1_{y'(\partial_0 c)} \times w(\partial_2 c)) \circ (w(\partial_0 c) \times 1_{y(\partial_2 c)}) = w(\partial_1 c) \circ (1_{v(\partial_0 \partial_1 c)} \times z(c)).$$

The composition law is given by $(v', w') \times (v,w) = (\hat{v}, \hat{w})$ where

$$\hat{v}(a) = v'(a)\, v(a), \quad a \in \Sigma_0$$

$$\hat{w}(b) = (w'(b) \times 1_{v(\partial_1 b)}) \circ (1_{v'(\partial_0 b)} \times w(b)), \quad b \in \Sigma_1 .$$

In fact, there is much more algebraic structure ; in particular, $Z^2(\mathrm{End}\, \mathfrak{A})$ may be considered as a 2-category but I refrain from giving further details. End $\mathfrak{A}$, being a monoidal category, is also a 2-category in a natural way and it is this aspect which allows it to be used as the coefficient object for 2-cohomology. Note, however, that this 2-cohomology does not allow a product of 2-cocycles and this is a defect in the sense that, in the application to quantum electrodynamics, the "product" of 2-cocycles corresponds to the addition of electric charge.

Adding a locality condition to define $Z^2_\ell(\mathrm{End}\,\mathfrak{A})$ poses no problems as one may follow the pattern established in defining $Z^1_\ell(\mathrm{End}\,\mathfrak{A})$ in Section 3.

On the basis of the conjectures made above for quantum electrodynamics, one may define a non-trivial local 2-cocycle (y,z) by setting

$$y(b)(A) = u(b)\,A\,u(b)^{-1}, \qquad A \in \mathfrak{A},\ b \in \Sigma_1$$
$$z(c) = u(\partial_1 c)\,u(\partial_2 c)^{-1}\,u(\partial_0 c)^{-1}, \qquad c \in \Sigma_2 .$$

Now $u(b)$ is associated with a local 1-cocycle χ_b with values in X and a tedious but trivial computation shows that cohomologous local 1-cocycles give rise to cohomologous local 2-cocycles. In other words one has a "connecting map" $H^1_\ell(X) \to H^2_\ell(\mathrm{End}\,\mathfrak{A})$.

In general terms, it should be expected that, just as theories involving fields and a principle of gauge invariance of the first kind can be described intrinsically by saying that $Z^1_\ell(\mathrm{End}\,\mathfrak{A})$ is non-trivial, so theories involving fields and a principle of gauge invariance of the second kind will be able to be described intrinsically by saying that $Z^2_\ell(\mathrm{End}\,\mathfrak{A})$ is non-trivial.

Theorem 4.1 expresses the fact that in free quantum electrodynamics the local 2-cocycle z with values in A cannot be derived from a local 2-cocycle (y,z') with values in End $\mathfrak{A}$ by setting $z(c)(A) = z'(c)\,A\,z'(c)^{-1}$, $c \in \Sigma_2$. It may perhaps be considered as illustrating the dictum that the principle of gauge invariance of the second kind forces an interaction on the theory.

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