Huzihiro Araki

Inequalities in Von Neumann Algebras

Les rencontres physiciens-mathématiciens de Strasbourg - RCP25, 1975, tome 22
« Exposés de : H. Araki, H.J. Borchers, J.P. Ferrier, P. Krée, J.F. Pommaret, D. Ruelle, R. Stora et A. Voros », , exp. n° 1, p. 1-25
<http://www.numdam.org/item?id=RCP25_1975__22__A1_0>

Inequalities in von Neumann algebras*

Huzihiro ARAKI

Research Institute for Mathematical Sciences
Kyoto University, Kyoto, JAPAN

Abstract Generalization of inequalities involving trace of matrices to von Neumann algebras not having traces in general is discussed.

§1. Introduction

There are some well-known useful inequalities involving the trace of matrices: Let $A^* = A$, $B^* = B$, $\rho \geqq 0$, $\sigma \geqq 0$ and x be finite matrices.

(i) Golden-Thompson inequality ([15], [22]):

$$\mathrm{tr}(e^A e^B) \geqq \mathrm{tr}\, e^{A+B}. \tag{1.1}$$

(ii) Peierls-Bogolubov inequality ([11], [18])

$$\mathrm{tr}\, e^{A+B} \geqq (\mathrm{tr}\, e^A)\exp\{\mathrm{tr}(e^A B)/\mathrm{tr}\, e^A\}. \tag{1.2}$$

(iii) Powers-Størmer inequality ([19]):

$$\| \rho - \sigma \|_{\mathrm{tr}} \geqq \| \rho^{1/2} - \sigma^{1/2} \|^2_{\mathrm{H.S.}} \tag{1.3}$$

---

* An expanded version of the talk given at Vingtieme Rencontre entre Physiciens Theoriciens et Mathematiciens at Strasbourg, May 22-24, 1975.

Here $\|x\|_{tr} \equiv tr\{(x^*x)^{1/2}\}$, $\|x\|_{H.S.} \equiv \{tr(x^*x)\}^{1/2}$.

(iv) Convexity of $\log tr\, e^A$ in A ([16]).

(v) Lieb concavity ([16]): $tr \exp(A+\log \rho)$ is convex in ρ.

(vi) Wigner-Yanase-Dyson-Lieb concavity ([16], [24]): Let $0 \leqq s$, $0 \leqq r$, $r+s \leqq 1$. Then $tr(x^*\sigma^s x \rho^r)$ is jointly concave in ρ and σ.

(vii) Properties of relative entropy ([17] , [23]): The relative entropy

$$S(\sigma/\rho) = tr\left(\rho \log \rho\right) - tr\left(\rho \log \sigma\right) \qquad (1.4)$$

satisfies the following properties (in addition to being lower semicontinuous in ρ and σ):

(α) Positivity: $S(\sigma/\rho) \geqq 0$ ($S(\sigma/\rho)=0$ only if $\sigma=\rho$) if $tr\,\sigma = tr\,\rho$.

(β) Convexity: $S(\sigma/\rho)$ is jointly convex in ρ and σ.

(γ) Monotonicity: Let E_N denote the conditional expectation of matrices to a $*$-subalgebra N relative to the trace. Then

$$S(E_N\sigma/E_N\rho) \leqq S(\sigma/\rho) \qquad (1.5)$$

In this review, we describe how to rewrite these inequalities without using "trace" so that the resulting expressions are meaningful for a general von Neumann algebra and inequalities remains true. We also sketch proofs for rewritten inequalities (ii), (v), (vi) and (vii). The proofs of (i), (ii) and (iv) are given for a general von Neumann algebra in [3] and (iii) in [4]. Also see [20]. The proof of (vi) and (viii) for a general von Neumann algebra will appear in a forth coming paper ([7]). The proof of (vi), (vii) (α) and (β) has already been given in [9].

Just to give an indication of what are our general idea, consider (i), (ii), (iv) and (v). Let M be a * algebra of matrices to which A,B and ρ belong. Any linear functional φ on M, which is positive in the sense that $\varphi(x^*x) \geqq 0$ for all $x \in M$ can be expressed in terms of a density matrix $\rho_\varphi \in M$ as

$$\varphi(x) = \mathrm{tr}(\rho_\varphi x) , \quad x \in M. \tag{1.6}$$

If we consider the case where $\rho_\varphi = e^A$, then

$$\mathrm{tr}\, e^A e^B = \varphi(e^B), \tag{1.7}$$

$$\mathrm{tr}\, e^A = \varphi(1), \tag{1.8}$$

$$\mathrm{tr}\, e^A B = \varphi(B). \tag{1.9}$$

Hence, if we somehow manage to define a positive linear functional φ^B on M from given φ with $\rho_\varphi = e^A$ and from $B=B^* \in M$, so that

$$\varphi^B(x) = \mathrm{tr}\,(e^{A+B}x), \tag{1.10}$$

then (i) and (ii) can be rewritten as

$$\varphi(e^B) \geqq \varphi^B(1) \geqq \varphi(1)\exp\{\varphi(B)/\varphi(1)\}. \tag{1.11}$$

(iv) is the convexity of $\log \varphi^B(1)$ in B and (v) is the concavity of $\varphi^{\log\rho}(1)$ in ρ.

For general van Neumann algebra M, φ is taken to be normal faithful positive linear functional. Here "normal" refers to a continuity of $\varphi(x)$ in $x \in M$ relative to the σ-weak (or σ-strong) topology in M. Faithfulness refers to the property that $\varphi(x^*x) = 0$ occurs only if $x=0$. This property is equivalent to $\rho_\varphi > 0$ for the case of (1.6) and is automatically satisfied for $\rho_\varphi = e^A$. The only part which requires more sophiscated tool is the definition of φ^B —— a perturbed functional. The theory of modular operators [21] is used in an essential manner for this purpose.

§2. Modular operators

Let Ψ and Φ be cyclic and separating vector of a von Neumann algebra M on a Hilbert space $\mathfrak{h}$. (Ψ cyclic if $M\Psi$ is dense in $\mathfrak{h}$; separating if $x \in M$ and $x\Psi=0$ imply $x=0$ or equivalently $M'\Psi$ is dense.) Let $S_{\Phi,\Psi}$ be an antilinear operator defined on $M\Psi$ by

$$S_{\Phi,\Psi}\, x\Psi, = x^*\Phi, \quad x \in M. \tag{2.1}$$

Then $S_{\Phi,\Psi}$ has a closure $\bar{S}_{\Phi,\Psi}$, whose absolute square defines the relative modular operator:

$$\Delta_{\Phi,\Psi} = (\bar{S}_{\Phi,\Psi})^* \bar{S}_{\Phi,\Psi} . \tag{2.2}$$

The special case $\Delta_{\Psi,\Psi}$ is denoted by Δ_Ψ and called the modular operator. For given Ψ, $\Delta_{\Phi,\Psi}$ depends only on the normal faithful positive linear functional

$$\varphi(x) = (\Phi, x\Phi), \quad x \in M \tag{2.3}$$

and not on its representative vector Φ.

One of the main ingredients of Tomita-Takesaki theory ([21], also see [12]) is that $x \in M$ implies

$$\sigma_t^{\varphi}(x) \equiv (\Delta_{\Phi,\Psi})^{it} x (\Delta_{\Phi,\Psi})^{-it} \in M \tag{2.4}$$

for all real t. σ_t^{φ} is a continuous one-parameter group of automorphisms of M, called modular automorphisms. σ_t^{φ} depends only on φ and not on Ψ nor on the choice of the representative vector Φ of φ.

The polar decomposition

$$S_{\Psi,\Psi} = J_\Psi (\Delta_\Psi)^{1/2} \tag{2.5}$$

defines an antiunitary involution J_Ψ. (Namely $(J_\Psi f, J_\Psi g) = (g,\Psi)$, $(J_\Psi)^2 = 1$.) The other main ingredient of Tomita-Takesaki theory is that $x \in M$ implies

$$j_\Psi(x) \equiv J_\Psi x J_\Psi \in M'. \tag{2.6}$$

The closure of the set of vectors $(\Delta_\Psi)^{1/4} x\Psi$ where x runs over all positive elements of M is called natural positive cone and denoted by V_Ψ ([4], [8], [13]). It is a pointed closed convex cone, which is selfdual (i.e. $(f,g) \geq 0$ for all $g \in V_\Psi$ if and only if $f \in V_\Psi$). For any $\Phi \in V_\Psi$ and $x \in M$, $x j_\Psi(x)\Phi \in V_\Psi$ and the set of $x j_\Psi(x)\Psi$ for all $x \in M$ is dense in V_Ψ. Any vector $\Phi \in V_\Psi$ is cyclic if and only if it is separating. For such Φ in V_Ψ, $J_\Phi = J_\Psi$ and $V_\Phi = V_\Psi$ (the universality). For a general cyclic and separating Φ, there exists a unitary u' in

M' such that $V_\Phi = u'V_\Psi$, $J_\Phi = u'J_\Psi(u')^*$ and

$$S_{\Phi,\Psi} = u'J_\Psi(\Delta_{\Phi,\Psi})^{1/2}. \tag{2.7}$$

In our disscussion, we can use a fixed natural positive cone and hence we drop the suffix Ψ from J_Ψ, V_Ψ and j_Ψ in the following.

Any normal positive linear functional φ of M has a unique representative vector $\xi(\varphi)$ in V:

$$\varphi(x) = (\xi(\varphi), x\xi(\varphi)). \tag{2.8}$$

The mapping ξ is a concave monotone increasing (relative to the positive cones M^+ and V) homeomorphism, homogeneous of degree 1/2, satisfying

$$\|\xi(\varphi_1) + \xi(\varphi_2)\| \, \|\xi(\varphi_1) - \xi(\varphi_2)\| \geqq \|\varphi_1 - \varphi_2\| \geqq \|\xi(\varphi_1) - \xi(\varphi_2)\|^2. \tag{2.9}$$

For faithful φ of (2.3), $\xi(\varphi)$ is given by

$$\xi(\varphi) = (\Delta_{\Phi,\Psi})^{1/2}\Psi. \tag{2.10}$$

(For general φ with a support projection e, $\xi(\varphi)$ is obtained by the same formula in the subspace $ej(e)\mathfrak{H}$ with Ψ replaced by $ej(e)\Psi$ and with Δ defined relative to eMe.)

To understand all formulas above, we go back to the simple case of M being a matrix algebra and see what newly defined quantities look like.

Let the Hilbert space $\mathfrak{H}$ be M itself with inner product

$$\langle\eta(x),\ \eta(y)\rangle = \mathrm{tr}\ x^*y \tag{2.11}$$

where we have used the notation $\eta(x)$ for an element in $\mathfrak{h}$ to distinguish it from the operator $x \in M$, which is faithfully represented by the left multiplication:

$$\pi(x)\eta(y) \equiv \eta(xy). \tag{2.12}$$

The left multiplication

$$\pi'(x)\eta(y) \equiv \eta(yx) \tag{2.13}$$

defines operators $\pi'(x)$ which generates $\pi(M)'$. $\pi(M)$ which is isomorphic to M will take place of M in our general discussion.

Let ρ_ψ and ρ_φ be density matrices defined in (1.6). Let Ψ be $\eta(\rho_\psi{}^{1/2})$. Then for $x \in M$

$$\Delta_{\Phi,\Psi}\eta(x) = \eta(\rho_\varphi x \rho_\psi{}^{-1}), \tag{2.14}$$

$$J\eta(x) = \eta(x^*), \tag{2.15}$$

$$V = \eta(M^+), \tag{2.16}$$

$$\xi(\varphi) = \eta(\rho_\varphi{}^{1/2}), \tag{2.17}$$

$$\sigma_t^\varphi(\pi(x)) = \pi(\rho_\varphi x \rho_\varphi{}^{-1}). \tag{2.18}$$

It is now possible to rewrite inequalities (iii), (vi) and (vii) as follows. First note that

$$\|\xi(\varphi_1) - \xi(\varphi_2)\|^2 = \|\rho_{\varphi_1}^{1/2} - \rho_{\varphi_2}^{1/2}\|_{H.S.}^2 ,$$

$$\|\varphi_1 - \varphi_2\| = \sup_{\|x\| \leq 1} |\varphi_1(x) - \varphi_2(x)|$$

$$= \sup_{\|x\| \leq 1} |tr(\rho_{\varphi_1} - \rho_{\varphi_2})x| = \|\rho_{\varphi_1} - \rho_{\varphi_2}\|_{tr}.$$

Hence the second inequality of (2.9) is the generalization of the Powers-Størmer inequality (iii).

Next note that

$$(\Delta_{\Phi,\Psi})^{s/2} x\Psi = \eta(\rho_{\varphi}^{s/2} x \rho_{\psi}^{(1-s)/2})$$

which implies

$$\|(\Delta_{\Phi,\Psi})^{s/2} x\Psi\|^2 = tr(x^* \rho_{\varphi}^{s} x \rho_{\psi}^{1-s}). \qquad (2.19)$$

Hence the concavity of (2.19) generalizes the concavity in (vi) for $r + s = 1$. (The case $r + s \leq 1$ in (vi) follows from the case $r + s = 1$ and the operator concavity of $\rho \to \rho^p$ for $0 \leq p \leq 1$.)

Finally

$$S(\varphi/\psi) = -(\Psi, (\log\Delta_{\Phi,\Psi})\Psi) \qquad (2.20)$$

coincides with (1.4) with $\sigma = \rho_{\varphi}$ and $\rho = \rho_{\psi}$. Hence the positivity for $\varphi(1) = \psi(1)$, convexity and monotonicity of (2.20) generalize (vii), where the conditional expectation E_N in (1.5) is to be replaced by the restriction of a functional to von Neumann subalgebra N of M, because of the following circumstances: $E_N(\rho)$ is defined as the unique element in N satisfying

$$\operatorname{tr} \rho x = \operatorname{tr} E_N(\rho) x$$

for all $x \in N$. For $\rho = \rho_\varphi$, it coincides with the definition of the density matrix for the functional

$$\varphi^N(x) = \operatorname{tr} \rho x = \varphi(x), \qquad x \in N,$$

whcih is the restriction of φ to N.

We note that the concavity and monotonicity of ξ correspond to the operator concavity and monotonicity of $\rho \to \rho^{1/2}$.

§3. Perturbation of functionals.

To generalize the perturbed functional φ^B given by (1.10) to a general von Neumann algebra M, we define a vector $\Phi(h) \in V$ for given $\Phi \in V$ and $h = h^* \in M$ so that

$$\varphi^h(x) = (\Phi(h), x\Phi(h)), \qquad x \in M \tag{3.1}$$

is the desired perturbed functional. The formula (2.14) and (1.10) suggest

$$\log \Delta_{\Phi(h),\Phi} - \log \Delta_\Phi = h \tag{3.2}$$

which implies, due to (2.10),

$$\Phi(h) = \exp\{(\log \Delta_\Phi + h)/2\}\Phi. \tag{3.3}$$

An alternative expression can be found by using the expansion

$$e^{(A+B)t}e^{-tA} = \sum_{n=0}^{\infty} \int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n \sigma^{\varphi}_{-it_n}(B) \cdots \sigma^{\varphi}_{-it_1}(B),$$

$$\sigma^{\varphi}_t(B) = e^{itA}Be^{-itA},$$

to the representative vector $(e^{(A+B)/2}e^{-A/2})e^{A/2}$, where $\varphi(x) = \mathrm{tr}(e^A x)$. The resulting expression, written in terms of the modular operator Δ_Φ of $\Phi = e^{A/2}$ is

$$\Phi(h) = \sum_{N=0}^{\infty} \int_0^{1/2} dt_1 \cdots \int_0^{t_{n-1}} dt_n \Delta_\Phi^{t_n} h \Delta_\Phi^{t_{n-1}-t_n} h \cdots \Delta_\Phi^{t_1-t_2} h\Phi. \tag{3.4}$$

We adopt (3.4) as the definition of $\Phi(h)$ and (3.1) as the definition of φ^h for a general von Neumann algebra M. The absolute convergence of (3.4), uniform over $h \in (M)_k$ (the ball of radius k in M), follows from the following Lemma ([2], Theorem 3.1):

<u>Lemma 1</u> (1) A cyclic and separating vector Φ is in the domain of the operator

$$Q(z) \equiv \Delta_\Phi^{z_1} Q_1 \Delta_\Phi^{z_2} Q_2 \cdots \Delta_\Phi^{z_n} Q_n \tag{3.5}$$

for any integer n, any $Q_j \in M$ $(j=1,\ldots,n)$ and any complex number z_j $(j=1,\ldots,n)$ in the tube domain

$$\bar{I}_n^{1/2} \equiv \{z=(z_1,\ldots,z_n);\ \mathrm{Re}\ z_1 \geqq 0,\ldots,\mathrm{Re}\ z_n \geqq 0,\quad 1/2 \geqq \mathrm{Re}(z_1+\ldots z_n)\}. \tag{3.6}$$

(2) The vector-valued function $Q(z)\Phi$ of $z = (z_1,\ldots,z_n)$ is strongly continuous on $\bar{I}_n^{1/2}$, holomorphic in the interior $I_n^{1/2}$ of $\bar{I}_n^{1/2}$ and uniformly bounded by $\|\Phi\|\ \|Q_1\|\ldots\|Q_n\|$.

(3) Let $(M)_k^{*st}$ be the ball of radius k in M, equipped with *-strong operator topology. The vector $Q(z)\Phi$ is strongly continuous as a function of

$$(Q_1\ldots Q_n) \in (M)_k^{*st} \times \ldots \times (M)_k^{*st},$$

the continuity being uniform in $z_1\ldots z_n$ over any compact subset of the tube $\bar{I}_n^{1/2}$. ($k>0$ is arbitrary.)

(For the proof of (3), see Remark at the end of the section.)

The perturbed vector $\Phi(h)$ is automatically a cyclic and separating vector in the same natural cone as Φ and satisfies (3.2), (3.3) and the following properties ([2]):

$$\Phi(h_1) = \Phi(h_2) \text{ if and only if } h_1 = h_2. \tag{3.7}$$

$$[\Phi(h_1)](h_2) = \Phi(h_1+h_2). \tag{3.8}$$

$$[\Phi(h)](-h) = \Phi. \tag{3.9}$$

$$[\Phi(\lambda\mathbf{1})] = e^{\lambda/2}\Phi. \tag{3.10}$$

$$\log \Delta_{\Phi(h)} = \log \Delta_\Phi + h - j(h). \tag{3.11}$$

$$\sigma_t^{\varphi^h}(x) = u_t \sigma_t^{\varphi}(x) u_t^*, \tag{3.12}$$

$$u_t \equiv (\Delta_{\Phi(h),\Phi})^{it}\Delta_\Phi^{-it}$$
$$= \sum_{n=0}^{\infty}\int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n \sigma_{t_n}^{\varphi}(h)\ldots\sigma_{t_1}^{\varphi}(h). \quad (3.13)$$

$$(d/dt)\{\sigma_t^{\varphi^h}(x) - \sigma_t^{\varphi}(x)\}_{t=0} = i[h,x]. \quad (3.14)$$

$$(d/dt)u_t = u_t\sigma_t^{\varphi}(h). \quad (3.15)$$

From Lemma 1(3) and the uniform bound of Lemma 1(2), it follows that $\Phi(h)$ is strongly continuous as a function of $h \in (M)_k$.

For our application, it is important to find an analytic continuation in h. For example, the vector $\Phi(h)$ can be defined for arbitrary $h \in M$ by (3.4). It is then seen from the uniform bound of Lemma 1(2) that $\Phi(h(z))$ is holomorphic in z if $h(z)$ is holomorphic in z. The following Lemma ([2], Theorem 3.2) yields such result for $\varphi^h(1)$:

<u>Lemma 2</u> (1) For any $Q_j \in M$ $(j=1,\ldots,n+1)$, the following formula defines a single-valued function $f(z)$ for $z \in \bar{I}_n^1$ (defined by (3.6) in which 1/2 is replaced by 1):

$$f_{n+1}(z) = (\Delta_\Phi^{\bar{z}_{j2}} Q_{j+1}^* \Delta_\Phi^{\bar{z}_{j+1}} \ldots \Delta_\Phi^{\bar{z}_n} Q_{n+1}^* \Phi\ ,$$
$$\Delta_\Phi^{z_{j1}} Q_j \Delta_\Phi^{z_{j-1}} \ldots \Delta_\Phi^{z_1} Q_1 \Phi), \quad (3.16)$$

where

$$z = (z_1,\ldots,z_n) \in \bar{I}_n^1\ , \qquad z_j = z_{j1}+z_{j2},$$

$$\mathrm{Re}(z_1+\ldots+z_{j-1}+z_{j1}) \leqq 1/2,$$

$$\mathrm{Re}(z_{j2}+z_{j+1}+\ldots+z_n) \leqq 1/2.$$

(2) The function $f_{n+1}(z)$ so defined is continuous on $\bar{I}_n^1$, holomorphic in the interior I_n^1 of $\bar{I}_n^1$, and uniformly bounded on $\bar{I}_n^1$ by $\|\Phi\| \, \|Q_1\| \ldots \|Q_{n+1}\|$.

(3) The values of $f_{n+1}(z)$ at distinguished boundaries of $\bar{I}_n^1$ are given by

$$f_{n+1}(it_1-it_2,\ldots,it_n-it_{n+1}) = \varphi(\sigma^{\varphi}_{t_{n+1}}(Q_{n+1})\ldots\sigma^{\varphi}_{t_1}(Q_1)), \quad (3.17)$$

$$f_{n+1}(it_1-it_2,\ldots,it_j-it_{j+1}+1,\ldots,it_n-it_{n+1})$$
$$= \varphi(\sigma^{\varphi}_{t_j}(Q_j)\ldots\sigma^{\varphi}_{t_1}(Q_1)\sigma^{\varphi}_{t_{n+1}}(Q_{n+1})\ldots\sigma^{\varphi}_{t_{j+1}}(Q_{j+1})), \quad (3.18)$$

where $t_1,\ldots,t_{n+1}$ are real and $j=1,\ldots,n$.

(4) $f_{n+1}(z)$ is a continuous function of

$$(Q_1,\ldots,Q_{n+1}) \in (M)_k^{st} \times\ldots\times (M)_k^{st},$$

the continuity being uniform in z over any compact subset of $\bar{I}_n^1$. ($k>0$ is arbitrary.) Here $(M)_k$ is equipped with strong operator topology. (For Bergman-Weil formula, see [1], Corollary 3.4 and Remark 3.5.)

<u>Remark</u> (1) Lemma 2(4) can be proved as follows: To make dependence on $Q = (Q_1,\ldots,Q_{n+1})$ explicit, we write

$$F(z;Q) = e^{(z_1^2+\ldots+z_n^2)} f_{n+1}(z) \qquad (3.19)$$

where the Gaussian factor is introduced to make F uniformly vanishing for infinite z in $\bar{I}^1_{n+1}$. It is enough to show that for any $\varepsilon>0$,

$$|F(z;Q') - F(z;Q)| < \varepsilon$$

for Q' in a suitable strong neighbourhood of Q within $(M)_k^{st} \times\ldots\times(M)_k^{st}$, the neighbourhood being independent of z as long as z is in any given compact subset of $\bar{I}^1_{n+1}$. Due to the analyticity in z and vanishing at infinite z, $|F(z;Q') - F(z;Q)|$ is bounded by the supremum of its values on distinguished boundaries, which consists of the following $n+1$ planes:

$$B_0 = \{z \; ; \; \mathrm{Re}\; z = 0\} \; , \qquad (3.20)$$

$$B_j = \{z \; ; \; \mathrm{Re}\; z_j = 1 \text{ and } \mathrm{Re}\; z_\ell = 0 \text{ for } \ell \neq j\} \; , \qquad (3.21)$$

where $j=1,\ldots,n$. Since $F(z;h)$ tends to 0 as $z\to\infty$ from within $\bar{I}^1_{n+1}$, uniformly in $h \in (M)_k^{st}\times\ldots\times(M)_k^{st}$, it is enough to see that the supremum of $|F(z;Q') - F(z;Q)|$ over z in some compact subset of a distinguished boundary is bounded by a given ε. For this it is enough to see that $F(z;Q)$ is a continuous function of $(z,Q)\in B_j\times(M)_k\times\ldots\times(M_k)$ for $j=0,\ldots,n$. The function $f(z;Q)$ is given by Lemma 2(3), which can be rewritten as the expectation value in Φ of a product of some of operators $Q_1,\ldots,Q_{n+1},\Delta_\Phi^{i(t_{n+1}-t_1)},\ldots,\Delta_\Phi^{i(t_n-t_{n+1})},\Delta_\Phi^{i(t_{n+1}-t_1)}$ in a certain order. Since a product of operators is simultaneously strongly continuous as long as operators are in a uniformly bounded set, and since Δ_Φ^{is} is strongly continuous in real variable s (with norm 1), we have the desired continuity of $f(z;Q)$ in (z,Q) with z on distinguished boundaries.

(2) Lemma 1 (3) can be proved as follows: Let

$$\Phi(z;Q) = e^{z_1^2+\ldots+z_n^2} Q(z)\Phi. \tag{3.22}$$

We have to show that

$$\|\Phi(z;Q') - \Phi(z;Q)\| = \sup_{\|\Psi\|=1} |(\Psi,\Phi(z;Q') - \Phi(z;Q))| < \varepsilon$$

for $Q' = (Q_1'\ldots Q_n')$ in a suitable strong neighbourhood of $Q = (Q_1\ldots Q_n)$ within $(M)_k^{*st}\times\ldots\times(M)_k^{*st}$, the neighbourhood being independent of z as long as z is in a given compact subset of $\bar{I}_{n+1}^1$. As above, the problem is reduced to the strong continuity of $\Phi(z;Q)$ in (z,Q) for z in the distinguished boundaries of $\bar{I}_n^{1/2}$ and Q in $(M)_k^{*st}\times\ldots\times(M)_k^{*st}$. This follows again from the strong continuity of product of operators in a uniformly bounded set applied to the following expressions for real $s = (s_1\ldots s_n)$:

$$\Phi(is_1\ldots is_n;Q) = \Delta_\Phi^{is_n}Q_n\ldots\Delta_\Phi^{is_1}Q_1\Phi,$$

$$\Phi(is_1\ldots is_j+1/2\ldots is_n;Q) = \Delta_\Phi^{is_n}Q_n\ldots\Delta_\Phi^{is_{j+1}}Q_{j+1}\Delta_\Phi^{i(s_1+\ldots+s_j)}$$
$$Q_1^*\Delta_\Phi^{-is_1}Q_2^*\Delta_\Phi^{-is_2}\ldots\Delta_\Phi^{-is_{j-1}}Q_j^*\Phi.$$

(3) In the proof of Theorem 3.2 of [2], a factor $e^{-(z_1^2+\ldots+z_n^2)}$ is missing from the definition of $F^\beta(z)$ on page 173. With this factor, it is enough to prove the simultaneous continuity of $F^\beta(x-i\lambda^{(j)})$ in Q's and x's for each j, which follows again from the strong continuity of product on bounded set.

§4. Proof of Lieb convexity

We use the method of Epstein ([14]), for which we need an analytic continuation of $\varphi^h(1)$ in h, given by the following formula:

$$f(Q,\varphi) \equiv \varphi(1)+\varphi(Q)+\sum_{n=2}^{\infty}\int_0^1 dt_1\ldots\int_0^{t_{n-1}} dt_n f_n(t_1-t_2,\ldots,t_{n-1}-t_n). \tag{4.1}$$

By Lemma 2(2), the expression (4.1) is convergent and defines a holomorphic function of Q in the sense that $f(Q(z),\varphi)$ is holomorphic in z whenever Q(z) is holomorphic in z. It is also strongly continuous as long as Q is in a bounded set. If $Q = h = h^*$, then

$$f(h,\varphi) = \varphi^h(1), \tag{4.2}$$

which can be proved as follows.

It is enough to prove (4.2) for a dense set of h and hence we assume that $\sigma_t^\varphi(h)$ is an entire function of t. In this case the following formula holds for real z and $H = \log \Delta_\phi$:

$$e^{iz(H+h)}e^{-izH} = \sum_{n=0}^{\infty}(iz)^n\int_0^1 dt_1 \ldots \int_0^{t_{n-1}} dt_n \sigma^{\varphi}_{zt_n}(h)\ldots\sigma^{\varphi}_{zt_1}(h). \tag{4.3}$$

See, for example, [6] Theorem 14.) Due to $H\Phi = 0$, we have

$$e^{iz(H+h)}\Phi = \sum_{n=0}^{\infty}(iz)^n\int_0^1 dt_1 \ldots \int_0^{t_{n-1}} dt_n \sigma^{\varphi}_{zt_n}(h)\ldots\sigma^{\varphi}_{zt_1}(h)\Phi, \tag{4.4}$$

at first for real z. Since

$$(e^{-i\bar{z}(H+h)}\Psi, \Phi)$$

for any entire vector Ψ of $H+h$ (which is selfadjoint) and the inner product of Ψ with the right hand side of (4.4) are both an entire function of z and coincides for real t, they are equal. It follows that Φ is in the domain of $e^{iz(H+h)}$ and (4.4) holds for all z. For $z = -i/2$, (4.4) gives $\Phi(h)$ (the right handside gives (3.4) and the left hand side gives (3.3)). Hence

$$\varphi^h(1) = (\Phi, e^{H+h}\Phi)$$

$$= \varphi(1) + \varphi(h) + \sum_{n=z}^{\infty}\int_0^1 dt_1 \ldots \int_0^{t_{n-1}} dt_n (\Phi, \sigma^{\varphi}_{-it_n}(h)\ldots\sigma^{\varphi}_{-it_1}(h)\Phi). \tag{4.5}$$

The desired result (4.1) follows (4.5) due to the formula

$$(\Phi, \sigma^{\varphi}_{t_n}(h)\ldots\sigma^{\varphi}_{t_1}(h)\Phi) = f_n(it_1 - it_2, \ldots, it_n - it_{n-1}), \tag{4.6}$$

which obviously holds for real t and hence by analytic continuation for all t where f_n is defined. This concludes the proof of (4.2).

We now apply Lemma 3 of [14] to the function $\rho \to f(\log \rho, \varphi)$ defined on

$$D = \bigcup \{A;\ \mathrm{Re}\ e^{-i\theta} A \geqq \varepsilon\} \tag{4.7}$$

where the union is over real $\varepsilon > 0$ and $\theta \in [-\pi/2, \pi/2]$, and $\mathrm{Re}\ C$ denotes $(C+C^*)/2$. The convexity of $\Phi(\log \rho) = f(\log \rho, \ )$ in $\rho \in M^+$ follows from the following conditions to be satisfied by f:

(i) f is holomorphic in $\rho \in D$.

(ii) If $\mathrm{Im}\ \rho > 0$ and $\rho \in D$, then $\mathrm{Im}\ f(\log \rho, \varphi) \geqq 0$. If $\mathrm{Im}\ \rho < 0$ and $\rho \in D$, then $f(\log \rho, \varphi) \leqq 0$. Here $\mathrm{Im}\ \rho$ denotes $(\rho - \rho^*)/(2i)$.

(iii) For every real r and $\rho \in D$,

$$f(\log (r\rho), \varphi) = r^s f(\log \rho, \varphi) \tag{4.8}$$

where $0 < s \leqq 1$.

Since $\rho \to \log \rho$ is holomorphic in the domain (4.7) ([14]), (i) is satisfied. Since $\varphi^{h+c\mathbf{1}}(1) = e^c \varphi^h(1)$, the corresponding equation holds for its analytic continuation and hence (4.8) holds with $s = 1$.

To prove (ii), we introduce

$$h_\beta \equiv \int \sigma_t^{\varphi}(\log \rho) e^{-t^2/\beta} dt/(2\pi\beta)^{1/2}. \tag{4.9}$$

We can verify (ii) if we show that $\mathrm{Im}\, f(h_\beta, \varphi) \geqq 0$ if $\mathrm{Im}\,\rho > 0$, $\rho \in D$ and $f(h_\beta, \varphi) \leqq 0$ if $\mathrm{Im}\,\rho > 0$, $\rho \in D$, because $\lim_{\beta \to +0} h_\beta = \log \rho$ and $f(Q, \varphi)$ is continuous in Q.

Let E_λ for $\lambda \in [0,1]$ be the spectral projection of Δ_Φ for the spectral set $[\lambda, 1/\lambda]$. Then $E_\lambda H$ is bounded and $\lim_{\lambda \to 0} E_\lambda = 1$. By Remark 4 of [14], $0 < \mathrm{Im} \log \rho < \pi$ if $\mathrm{Im}\,\rho > 0$. This implies $0 < \mathrm{Im}\, h_\beta < \pi$ if $\mathrm{Im}\,\rho > 0$. By Remark 2 of [14], $0 < \mathrm{Im}\ \mathrm{Sp}\, h_\beta < \pi$ where Sp denotes the spectrum. Hence $\mathrm{Im}\,\mathrm{Sp}(e^{HE_\lambda + h_\beta}) \geqq 0$ and

$$\mathrm{Im}\,(\Phi,\ e^{HE_\lambda + h_\beta}\Phi) \geqq 0$$

whenever $\mathrm{Im}\,\rho > 0$. We now prove

$$\lim_{\lambda \to 0} (\Phi,\ e^{HE_\lambda + h_\beta}\Phi) = f(\log \rho, \varphi), \tag{4.10}$$

which will complete the proof of Lieb convexity for a general von Neumann algebra.

By the formula (4.3) with H replaced by HE_λ and iz by 1, we obtain by using $e^{-HE_\lambda}\Phi = \Phi$

$$(\Phi,\ e^{HE_\lambda + h_\beta}\Phi) = \sum_{n=0}^{\infty} \int_0^1 dt_1 \ldots \int_0^{t_{n-1}} dt_n\, g(t_1 \ldots t_n), \tag{4.11}$$

$$g(t_1 \ldots t_n) = (\Phi,\ h_\beta e^{(t_{n-1} - t_n)HE_\lambda} \ldots e^{(t_1 - t_2)HE_\lambda} h_\beta \Phi). \tag{4.12}$$

We replace each exponential in (4.12) by the formula

$$e^{sHE_\lambda} = \{\Delta_\Phi^s E_\lambda + (1 - E_\lambda)\}$$

and obtain 2^{n-1} terms of the following type

$$(\Phi, h_\beta e_{n-1}\sigma^{\varphi}_{-is_{n-1}}(h_\beta)\dots e_1\sigma^{\varphi}_{-is_1}(h_\beta)\Phi), \tag{4.13}$$

where

$$e_j = \varepsilon_j E_\lambda + (1-\varepsilon_j)(1-E_\lambda),$$

$$s_j = \sum_{\ell=j}^{n-1} \varepsilon_\ell (t_\ell - t_{\ell+1}),$$

and ε_j is either 0 or 1. By the continuity of the product of uniformly bounded operators, (4.13) is continuous in $(\lambda, s_1, \dots, s_{n-1})$ and hence tends to zero as $\lambda \to 0$, except that the term with all $\varepsilon_j = 1$ tends to

$$(\Phi, h_\beta\sigma^{\varphi}_{-i(t_{n-1}-t_n)}(h_\beta)\dots\sigma^{\varphi}_{-i(t_1-t_n)}(h_\beta)\Phi)$$

$$= (\Phi, \sigma^{\varphi}_{-it_n}(h_\beta)\dots\sigma^{\varphi}_{-it_1}(h_\beta)\Phi)$$

where all convergence is uniform in $(t_1 \dots t_n)$ within the compact region of integration in (4.11). (4.13) is also bounded by

$$2^{n-1}\{\sup_{0\leqq s\leqq 1} \|\sigma^{\varphi}_{-is}(h_\beta)\|\}^n \|\Phi\|^2$$

independent of $(\lambda, t_1, \dots, t_n)$. Hence the series (4.11) is absolutely convergent uniformly in λ and we obtain (4.10) from the convergence of (4.13).

§5. Relative Entropy

Let E_λ be the spectral projection of $\Delta_{\Phi,\Psi}$. Then the definition (2.20) is

$$S(\varphi/\psi) = -\int_0^\infty \log\lambda \, d(\Psi, E_\lambda \Psi). \tag{5.1}$$

By a numerical inequality

$$\log\lambda \leqq \lambda - 1, \tag{5.2}$$

we have

$$\begin{aligned} S(\varphi/\psi) &\geqq \int_0^\infty (1-\lambda) d(\Psi, E_\lambda \Psi) \\ &= \|\Psi\|^2 - \|(\Delta_{\Phi,\Psi})^{1/2}\Psi\|^2 \\ &= \psi(1) - \varphi(1). \end{aligned} \tag{5.3}$$

Hence we have the positivity

$$S(\varphi/\psi) \geqq 0 \tag{5.4}$$

if $\varphi(1) = \psi(1)$. Since the equality in (5.2) holds only if $\lambda = 1$, the eqality in the inequality of (5.3) holds if the measure $d(\Psi, E_\lambda \Psi)$ is concentrated at $\lambda = 1$, i.e.

$$\Phi = (\Delta_{\Phi,\Psi})^{1/2}\Psi = \Psi.$$

Hence if $\varphi(1) = \psi(1)$, then

$$S(\varphi/\psi) = 0$$

holds if and only if $\varphi = \psi$. (Strict positivity.)

We now consider perturbed functional φ^{h-c1} where $h = h^* \in M$ and the number c is chosen to be

$$c = \log(\varphi^h(1)/\varphi(1)) \tag{5.5}$$

so that $\varphi^{h-c1}(1) = \varphi(1)$. By (3.2) and $\Delta_\Phi \Phi = \Phi$, we have

$$\begin{aligned} S(\varphi^{h-c1}/\varphi) &= -\varphi(h-c1) \\ &= \varphi(1)c - \varphi(h). \end{aligned} \tag{5.6}$$

The positivity and (5.5) imply

$$\varphi(h) \leqq \varphi(1) \log(\varphi^h(1)/\varphi(1)), \tag{5.7}$$

which is the Peierls-Bogolubov inequality (the second inequality of (1.11)).

The WYDL concavity has been generalized ([7],[9]) to the joint concavity of $\|(\Delta_{\Phi,\Psi})^{p/2}x\Psi\|^2$ in faithful normal positive functionals φ and ψ for $0 \leqq p \leqq 1$. This implies the concavity of

$$\begin{aligned} S_p(\varphi/\psi) &\equiv \int_0^\infty \lambda^p d(\Psi, E_\lambda \Psi) \\ &= \|(\Delta_{\Phi,\Psi})^{p/2}\Psi\|^2 \end{aligned} \tag{5.8}$$

and hence the convexity of

$$S(\varphi/\psi) = \lim_{p\to 0} p^{-1}\{\psi(1)-s_p(\varphi/\psi)\} \tag{5.9}$$

jointly in φ and ψ.

This convexity can by used to prove the monotonicity

$$S(\varphi/\psi) \geqq S(E_N\varphi/E_N\psi) \tag{5.10}$$

where E_N denotes the restriction of functionals to N and the proof has been found so far ([7]) for a general M and for a von Neumann subalgebra N of M belonging to one of the following cases:

(1) $M = N \otimes N_1$ for $N_1 = M \cap N'$.

(2) $N = A' \cap M$ for a finite dimensional abelian von Neumann subalgebra A of M.

(3) N is an approximate finite von Neumann algebra. This includes any finite dimensional N, which is the case needed in applications ([5], [10]).

References

[1] H. Araki: Publ. RIMS Kyoto Univ. 4A, 361-371 (1968).

[2] ________: Publ. RIMS Kyoto Univ. 9, 165-209 (1973).

[3] ________: Commun. Math. Phys. 34, 167-178 (1973).

[4] ________: Pacific J. Math. 50, 309-354 (1974).

[5] ________: Commun. Math. Phys. 38, 1-10 (1974); On uniqueness of KMS states of one-dimensional quantum lattice systems. To appear in Commun. Math. Phys.

[6] ________: Ann. Sci. École Norm. Sup. 6, 67-84 (1973).

[7] ________: Relative entropy of states of von Neumann algebras. In preparation.

[8] ________: Positive cone, Radon-Nikodym theorems, relative Hamiltonian and the Gibbs condition in statistical mechanics. To appear in Lecture Notes of Varenna Summer School, 1973.

[9] ________: Recent developments in the theory of operator algebras and their significance in theoretical physics. To appear in Proceedings of convegno sulle algebre C* e loro applicazioni in fisica teorica, Rome 1975.

[10] ________: Relative entropy and its applications. To appear in Proceedings of International colloquium on mathematical methods of quantum field theory, Marseille, 1975.

[11] N.N.Bogolubov: Dubna report, 1962 (unpublished) [German transl.: Phys. Abh. SU 6, 1 (1962); 6, 113 (1962); 6, 229 (1962)]

[12] A.Connes: Ann. Scient. Ecole Norm. Sup. 4ème Série 6, 133-252 (1973)

[13] A.Connes, Caractérisation des algèbres de von Neumann comme espaces vectoriels ordonnés.

[14] H.Epstein, Commun. Math. Phys. 31, 317-325 (1973).

[15] S.Golden, Phys. Rev. 137, B1127-1128 (1965).

[16] E.H.Lieb, Advances in Math. 11, 267-288 (1973).

[17] G.Lindblad, Commun. Math. Phys. 39, 111-119 (1974).

[18] R.Peierls, Proc. Camb. Phil. Soc. 32, 477-481 (1936).

[19] R.T.Powers and E.Størmer, Commun. Math. Phys. 16, 1-33 (1970).

[20] M.B.Ruskai, Commun. Math. Phys. 26, 280-289 (1972).

[21] M.Takesaki, Tomita's Theory of Modular Hilbert Algebras and its Applications, Springer Verlag, 1970.

[22] C.Thompson, J. Math. Phys. 6, 1812-1813 (1965).

[23] H.Umegaki, Kōdai Math. Sem. Rep. 14, 59-85 (1962).

[24] E.P.Wigner and M.M.Yanase, Proc. Nat. Acad. Sci. U.S.A. 49, 910-918 (1963); Canad. J. Math. 16, 397-406 (1964).