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FOURIER INTEGRALS II.

J.J. DUISTERMAAT

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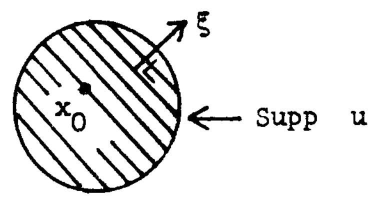
I would like to give some impression about the theory of Fourier integral operators of Hörmander [2]. Perhaps the best way to show how this theory can be used is to apply it to the well-known Cauchy problem for hyperbolic equations. As an illustration we shall also give an interpretation of the W K B-type results for the Schrödinger equation due to Maslov [5]. For regularity and existence theorems for equations $Pu = f$ and the construction of parametrices for large classes of operators P, see [1]. We start with a brief review of the calculus.

1. REVIEW OF THE CALCULUS.

If A is a distribution in $\mathbb{R}^n$ then A is C^∞ in a neighborhood of $x_0 \in \mathbb{R}^n$ if and only if $A.u \in C_0^\infty(\mathbb{R}^n)$ for some $u \in C_0^\infty(\mathbb{R}^n)$ with $u(x_0) \neq 0$. Because of the Paley-Wiener theorem this in turn is equivalent to the condition that the Fourier transform of $A.u$ is rapidly decreasing at infinity. In formula :

$$(1.1.) \qquad < A, ue^{-it <.,\xi>} > = O(t^{-k}) \quad \text{for } t \to \infty \text{ , any } k \text{ .}$$

It is assumed that the estimates in (1.1.) hold uniformly in $|\xi| = 1$. So we can find the singularities of A by testing with the rapidly oscillating test function $u(x)e^{-it <x,\xi>}$ with small support, and looking at the asymptotic behaviour as the frequency t approaches $+\infty$.



The test function u is used in order to localize with respect to the x-variables. Localizing also with respect to ξ (normal to the wave front $<x,\xi>$ = constant) this leads to the definition of the wave front set $W F(A)$:

If $(x_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^n\setminus\{0\}$ then we say that $(x_0,\xi_0) \notin W\,F(A)$

(1.2.) if and only if (1.1.) holds for some $u \in C_0^\infty$ with $u(x_0) \neq 0$ and uniformly for all ξ in some fixed neighborhood of ξ_0.

On a manifold X we no longer have invariantly defined linear phase functions $x \to \langle x,\xi\rangle$. In this case we obtain an invariant definition of $W\,F(A)$ by saying that for any $(x_0,\xi_0) \in T^*(X)\backslash 0$ (that is $x_0 \in X$, $\xi_0 \in T_{x_0}(X)^*$, $\xi_0 \neq 0$) we have $(x_0,\xi_0) \notin W\,F(A)$ if and only if

(1.3.) $$< A, ue^{-it\Psi} > = O(t^{-k}) \quad \text{for } t \to \infty, \text{ any } k.$$

Here $u \in C_0^\infty(X)$, $u(x_0) \neq 0$, $\Psi \in C^\infty(X)$, Ψ is real valued, $d\Psi \neq 0$ on supp u, $\xi_0 = d\Psi_{(x_0)}$. The phase function Ψ may depend on additional parameters and it is assumed that (1.3.) holds locally uniformly with respect to these parameters.

A special class of distributions are the <u>Fourier integrals</u> A defined by

(1.4.) $$< A,u > = \iint e^{i\varphi(x,\theta)}\, a(x,\theta)u(x)dxd\theta \;,\; u \in C_0^\infty(X).$$

Here $\theta = (\theta_1,\dots,\theta_N)$ are auxiliary variables called <u>frequency variables</u>. The <u>phase function</u> φ is a real valued homogeneous C^∞ function of degree 1 on $X \times \mathbb{R}^n\setminus\{0\}$ without stationnary points (that is $d_{(x,\theta)}\varphi \neq 0$ everywhere). For the <u>amplitude</u> $a(x,\theta)$ we may think of a C^∞ function on $X \times \mathbb{R}^n$, $a = 0$ in a neighborhood of $v = 0$ (where φ is singular) and $a = 0$ for x outside some compact subset of X, and finally $a(x,\theta) \sim \sum_{j=0}^{\infty} a_j(x,\theta)$ for $|\theta| \to \infty$ where $a_j(x,\theta)$ is homogeneous of degree $\mu - j$. The space of such amplitude functions will be denoted by $S^\mu(X \times \mathbb{R}^n)$ = space of symbols of growth order μ. A function $f(x,\theta)$ is called homogeneous of degree d (with respect to θ) if $f(x,t\theta) = t^d f(x,\theta)$ for all $t > 0$.

If μ is too large, the integral (1.4.) will be defined as the limit of similar integrals with the amplitude replaced by a sequence $a^{(k)} \in S^{-\infty}(X \times \mathbb{R}^N) = \bigcap_\mu S^\mu(X \times \mathbb{R}^N)$ approaching a in a suitable manner as $k \to \infty$. An equivalent interpertation can be given using partial integrations.

In order to find $W F(A)$ we write

$$< A, ue^{-it\Psi} > = \iint e^{i[\varphi(x,\theta) - t\Psi(x)]} a(x,\theta)u(x)dx\, d\theta =$$

$$= t^N \iint e^{it[\varphi(x,\theta) - \Psi(x)]} a(x,t\theta)u(x)dx\, d\theta .$$

From the method of stationnary phase it follows that this integral is rapidly decreasing as $t \to \infty$ unless $d_{(x,\theta)}[\varphi(x,\theta) - \Psi(x)] = 0$, that is

(1.5.) $$d_x\varphi(x,\theta) = d\Psi(x) , \; d_\theta\varphi(x,\theta) = 0 .$$

Consequently $W F(A) \subset \Lambda_\varphi$, where

(1.6.) $$\Lambda_\varphi = \{(x, d_x\varphi(x,\theta)) \in T^*(X)\backslash 0 \; ; \; d_\theta\varphi(x,\theta) = 0\} .$$

A complete asymptotic development for $< A, ue^{it\Psi} >$ can be given if the stationnary points of $\varphi - \Psi$ are non-degenerate, that is if $Q = d^2(\varphi - \Psi)$ is non singular whenever $d(\varphi - \Psi) = 0$. In this case

(1.7.) $$< A, ue^{-it\Psi} > = t^N (\frac{2\pi}{t})^{(n+N)/2} e^{-it\Psi(X)} . |\det Q|^{-\frac{1}{2}} . e^{\frac{\pi i}{4} \operatorname{sgn} Q} .$$

$$. a(x,t\theta).u(x) + \text{terms of lower order as } t \to \infty .$$

Here Q is taken at the isolated stationnary point (x,θ) of $\varphi - \Psi$.

The non-singularity of Q implies that

(1.8.) $$d_{(x,\theta)}\, d_\theta\varphi \text{ has maximal rank } N$$

which in turn means that the set

(1.9) $$C_\varphi = \{(x,\theta) \in X \times \mathbb{R}^N\backslash\{0\} \; ; \; d_\theta\varphi(x,\theta) = 0 \}$$

is a C^∞ submanifold of $X \times \mathbb{R}^N\backslash\{0\}$ of dimension $(n + N) - N = n$. Moreover the non-singularity of Q implies that the mapping

(1.10.) $$C_\varphi \ni (x,\theta) \mapsto (x, d_x\varphi(x,\theta)) \in \Lambda_\varphi \subset T^*(X)\backslash 0$$

is an immersion of C_φ into $T^*(X)\backslash 0$ yielding Λ_φ as an n-dimensional C^∞ submanifold of $T^*(X)\backslash 0$. Finally, if (1.8.) holds whenever $d_\theta\varphi(x,\theta) = 0$ (in this case the phase function φ is called <u>non-degenerate</u>) then the condition that Q is non-singular is equivalent to the condition that the graph

$$\{(x, d\Psi(x)) \in T^*(X) \; ; \; x \in X\}$$

of the function $d\Psi$ intersects Λ_φ transversally. Note that (1.5.) just means that $d\Psi$ and Λ_φ intersect at (x,ξ) , $\xi = d\Psi(x) = d_x\varphi(x,\theta)$.

Because of the homogeneity of φ , Λ_φ is <u>conic</u> in $T^*(X)\backslash 0$, that is $(x,\xi) \in \Lambda_\varphi \Rightarrow (x,t\xi) \in \Lambda_\varphi$ for all $t > 0$. Secondly it turns out that Λ_φ is <u>Lagrangean</u>, that is the canonical 2-form σ of $T^*(X)$ vanishes on Λ_φ . (On local coordinates σ is given by $\sigma = \Sigma\, dx_j \wedge d\xi_j$). Conversely every conic Lagrangean submanifold Λ of $T^*(X)\backslash 0$ is locally equal to Λ_φ for some non-degenerate phase function φ .

Now the asymptotic expansion (1.7.) leads to an invariant definition of the principal symbol a of A at $(x,\xi) \in \Lambda_\varphi$. Here "invariant" means independent of the testing phase function Ψ for which the graph of $d\Psi$ intersects Λ_φ transversally at (x,ξ) . Because of the factor $|\det Q|^{-\frac{1}{2}}$ the principal symbol is a density of order $\frac{1}{2}$ on Λ_φ , and because of the factor $e^{\frac{\pi i}{4}\operatorname{sgn} Q}$ it has its values in a complex line bundle L over Λ_φ with structive group $\mathbb{Z}$ mod. 4 . L is called the line bundle of Keller, Maslov and Arnold in [2] .

Now let Λ be any conic hagrangean submanifold of $T^*(X)\backslash 0$. <u>A global Fourier integral distribution</u> A <u>of order</u> m <u>corresponding to</u> Λ , <u>notation</u> $A \in I^m(X,\Lambda)$, is defined as a locally finite sum of Fourier integrals A_j defined by phase functions φ_j , amplitudes a_j , number of frequency variables N_j , such that:

(1.11.) The Λ_{φ_j} are a locally finite system of conic neighborhoods in Λ

$((x,\theta_j)$ only restricted to the conic support of $a_j)$, and

(1.12.) $$a_j \in S^{m+n/4-N_j/2}(X \times \mathbb{R}^{N_j}\setminus\{0\} .$$

Of course A also admits an asymptotic expansion as in (1.7.) , leading to the definition of the principal symbol of A as an element of $S^{m+n/4}(\Lambda\ ,\ \Omega_{\frac{1}{2}} \otimes L)$.

THEOREM 1.1. . - *If* Λ *is a closed conic Lagrangean submanifold of* $T^*(X)\setminus 0$ *then the mapping* :

(1.13.) $$I^m(X,\Lambda)/I^{m-1}(X,\Lambda) \to S^{m+n/4}(\Lambda,\Omega_{\frac{1}{2}} \otimes L)/S^{m+n/4-1}(\Lambda,\Omega_{\frac{1}{2}} \otimes L)$$

assigning to each $A \in I^m(X,\Lambda)$ *its principal symbol, is an isomorphism.*

This theorem is fundamental in all global constructions involving Fourier integrals since it says that for every $a \in S^{m+n/4}(\Lambda,\Omega_{\frac{1}{2}} \otimes L)$ there exists an $A \in I^m(X,\Lambda)$ with principal symbol equal to a , and secondly if A_1 , $A_2 \in I^m(X,\Lambda)$ have the same principal symbol modulo $S^{m+n/4-1}(\Lambda,\Omega_{\frac{1}{2}} \otimes L)$, then $A_1 - A_2 \in I^{m-1}(X,\Lambda)$.

If X and Y are C^∞ manifolds and K is a distribution in $X \times Y$, then $< Av,u > = < K,u \otimes v >$, $u \in C_0^\infty(X)$, $v \in C_0^\infty(Y)$, defines a continuous linear mapping $A : C_0^\infty(Y) \to \mathcal{D}'(X)$. If K is smooth then $(Av)(x) = \int K(x,y)v(y)dy$.

From the calculus of wave front sets it is known that if $WF(K)$ does not contain points of the form $(x,y,0,\eta)$ or $(x,y,\xi,0)$, then A can be extended to a continuous linear map : $\mathcal{E}'(Y) \to \mathcal{D}'(X)$ and

(1.14.) $$WF(Av) \subset WF'(A) \circ WF(v) .$$

Here

(1.15.) $$WF'(A) = \{((x,\xi),(y,-\eta)) \in T^*(X) \times T^*(Y)\ ;\ (x,y,\xi,\eta) \in WF(K)\} ,$$

and in (1.14.) we let $WF'(A)$ operate on $WF(v)$ as a relation :

(1.16.) $WF'(A) \circ WF(v) = \{(x,\xi) \in T^*(X) \; ; \; \exists (y,\eta) \in WF(v) : ((x,\xi),(y,\eta)) \in WF'(A)\}$.

Now a <u>Fourier integral operator of order</u> m <u>defined by the relation</u> $C = \Lambda'$ <u>from</u> $T^*(Y)$ <u>to</u> $T^*(X)$ simply is defined as an operator A with kernel $K \in I^m(X \times Y \; ; \; \Lambda)$. The set of all such operators A will be denoted by $I^m(X,Y \; ; \; C)$. Note that $WF(Au) \subset C \circ WF(u)$ since $WF(K) \subset \Lambda$, if we assume that C does not contain points of the form $((x,\xi),(y,0))$ or $((x,0),(y,\eta))$.

The condition that Λ is Lagrangean in $T^*(X \times Y)$ means that $\sigma_{T^*(X)} - \sigma_{T^*(Y)}$ vanishes on $C = \Lambda'$. If C is the graph of a mapping $\Phi : T^*(Y)\backslash 0 \to T^*(X)\backslash 0$ this would mean that Φ preserves the canonical 2-forms, that is Φ is a canonical transformation . It is homogeneous of degree 1 because C is conic. In general C will not be the graph of a mapping (we will see some natural examples below) and C then is called a <u>homogeneous canonical relation from</u> $T^*(Y)$ <u>to</u> $T^*(X)$.

THEOREM 1.2. . - <u>Let</u> C_1 <u>and</u> C_2 <u>be homogeneous canonical relations from</u> $T^*(Y)$ <u>to</u> $T^*(X)$ <u>and from</u> $T^*(Z)$ <u>to</u> $T^*(Y)$ <u>respectively, such that</u>

(1.17.) $C_1 \times C_2$ <u>intersects the diagonal in</u> $T^*(X) \times T^*(Y) \times T^*(Y) \times T^*(Z)$ <u>transversally and not in points</u> $(x,0,y,\eta,y,\eta,z,0)$, <u>and the projection of the intersection to</u> $T^*(X) \times T^*(Z)$ <u>is a proper mapping</u>.

<u>Then the image</u> $C_1 \circ C_2$ <u>is a homogeneous canonical relation from</u> $T^*(Z)$ <u>to</u> $T^*(X)$.

<u>Secondly, if</u> $A_1 \in I^{m_1}(X,Y \; ; \; C_1)$, $A_2 \in I^{m_2}(Y,Z \; ; \; C_2)$ <u>and</u>

(1.18.) <u>the projection from the intersection of</u> $\mathrm{Supp}\, A_1 \times \mathrm{Supp}\, A_2$ <u>with the diagonal in</u> $X \times Y \times Y \times Z$ <u>to</u> $X \times Z$ <u>is proper</u>,

<u>then</u> $A_1 \circ A_2 \in I^{m_1+m_2}(X,Z \; ; \; C_1 \circ C_2)$ <u>and the principal symbol of</u> $A_1 \circ A_2$ <u>is equal to the product of the principal symbols of</u> A_1 <u>and</u> A_2 .

The last sentence should be read as follows. If $c_1 = (x,\xi,y,\eta) \in C_1$, $c_2 = (y,\eta,z,\zeta) \in C_2$ then there is a canonically defined bilinear mapping $(a_1,a_2) \to a_1.a_2$ from the fiber of $\Omega_{\frac{1}{2}} \otimes L$ over C_1 at c_1 and the fiber of $\Omega_{\frac{1}{2}} \otimes L$ over C_2 at c_2 to the fiber of $\Omega_{\frac{1}{2}} \otimes L$ over $C_1 \circ C_2$ at $c = (x,\xi,z,\zeta)$. If $a_j(c_j)$ denotes the principal symbol of A_j at $c_j \in C_j$, $j=1,2$ then the principal symbol of $A_1 \circ A_2$ at $(x,\xi,z,\zeta) \in C_1 \circ C_2$ is given by

$$(1.19.) \qquad \sum_{(y,\eta)} a_1(x,\xi,y,\eta).a_2(y,\eta,z,\zeta) .$$

the sum being extended over the finitely many (y,η) such that $(x,\xi,y,\eta) \in C_1$ and $(y,\eta,z,\zeta) \in C_2$.

If $X = Y$, C = graph of the identity : $T^*(X)\backslash 0 \to T^*(X)\backslash 0$, then $I^m(X,X ; I) = L^m(X)$ = space of pseudo-differential operators of order m on X . (If m is a positive integer then the partial differential operators of order m form a subclass of $L^m(X)$) . The principal symbol of $P \in L^m(X)$ can be identified with a homogeneous function of degree m on $T^*(X)\backslash 0$. Finally, if in Theorem 1.3. either A_1 or A_2 is a pseudo-differential operator then the multiplication of the principal symbols reduces to the usual scalar multiplication.

It may happen that the principal symbol of $A_1 \circ A_2$ of order $m_1 + m_2$ vanishes identically although neither of the principal symbols of A_1 or A_2 vanish identically. An important case is treated below.

THEOREM 1.3. . - Suppose $P \in L^m(X)$ has a principal symbol p . Let C be a homogeneous canonical relation from $T^*(Y)$ to $T^*(X)$ such that p vanishes on the projection of C in $T^*(X)\backslash 0$. If $A \in I^\mu(X,Y ; C)$ and (1.18.) holds for $P = A_1$, $A = A_2$, then $PA \in I^{m+\mu-1}(X,Y ; C)$ with principal symbol of order $m + \mu - 1$ equal to

$$(1.20.) \qquad \frac{1}{i} H_{\tilde{p}} a + c.a .$$

Here a is the principal symbol of A , $\tilde{p}(x,\xi,y,\eta) = p(x,\xi)$, $H_{\tilde{p}}$

is the Hamilton field defined by $\tilde{p}$. Finally C is the subprincipal symbol of order $m - 1$ of the operator P .

Recall that the Hamilton field H_f of a function f on a cotangent bundle $T^*(X)$ is the vector field given by $H_f = \Sigma \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial x_j} - \Sigma \frac{\partial f}{\partial x_j} \frac{\partial}{\partial \xi_j}$ on local coordinates. Let us restrict for simplicity that f is real and we are at a zero of f . Because the vector H_f spans the orthogonal complement of the tangent space of $f = 0$ (orthogonal with respect to the canonical : 2-form σ) it follows that H_f is tangent to any Lagrangean submanifold of $f = 0$. So $H_{\tilde{p}}$ is tangent to C and (1.20.) makes sense.

2. INITIAL VALUE PROLEMS.

Let X_0 be a submanifold of X of codimension k . Then the restriction mapping $\rho : C^\infty(X) \to C^\infty(X_0)$ is a Fourier integral operator of class $I^{k/4}(X_0, X \,;\, R_0)$, with

(2.1.) $$R_0 = \{(x_0,\xi_0,x,\xi) \;;\; x = x_0 \in X_0 \;,\; \xi | T_{x_0}(X_0)\} \;.$$

Note that R_0 , regarded as a relation : $T^*(X)\backslash 0 \to T^*(X_0)$ is neither a mapping nor injective. In order to see that ρ is such a Fourier integral operator it suffices to consider the case that $X = \mathbb{R}^n$, $X_0 = \mathbb{R}^{n-k}$ and then it is seen from the formula

(2.2.) $$(\rho u)(x_0) = (2\pi)^{-n} \iint e^{i\langle x_0 - y, \eta\rangle} u(y)dy\, d\eta \;.$$

From the calculus of wave front sets it follows that ρ can be applied to any distribution u for which $WF(u)$ does not contain any (x,ξ) which by R_0 is related to an element of the form $(x_0, 0) \in T^*(X_0)$. In other words, ρ can be extended continuously to all $u \in \mathcal{D}'(X)$ such that $WF(u)$ does not meet the normal bundle $X_0^\perp$ in $T^*(X)$ of X_0 .

If P is a pseudo-differential operator $\in L^m(X)$ with principal symbol p , then $Pu \in C^\infty(X)$ implies that $p = 0$ on $WF(u)$. Writing $Pu \equiv 0$ if $Pu \in C^\infty(X)$ we thus obtain that ρ can be extended to all distribution solutions of $Pu \equiv 0$, if the characteristic set

$$N = \{(x,\xi) \in T^*(X)\backslash 0 \ ; \ p(x,\xi) = 0\} \tag{2.3.}$$

does not meet $X_0^\perp$. In this case X_0 is called <u>non-characteristic with respect to</u> P .

Now assume that p is real and that X_0 has codimension 1 . Let $Q_j \in L^{m_j}(X)$, $j = 1 ,\ldots, \mu$ be a number of pseudo-differential operators. We want to find operators E_j , $j = 1 ,\ldots, \mu$, such that

$$PE_j \equiv 0 \tag{2.4.}$$

$$\rho Q_j E_k \equiv \delta_{jk} . \text{identity on } X_0 . \tag{2.5.}$$

Here $A \equiv B$ for operators A , B means that $A - B$ is an integral operator with C^∞ kernel. The operators E_j are the solution operators (modulo C^∞) of the Cauchy problem $Pu \equiv 0$, $\rho Q_j u \equiv f_j$, since $u = \Sigma E_j f_j$ satisfies these equations.

The idea is to try $E_j \in I^{\nu_j}(X,X_0 \ ; C_0)$ for some orders ν_j and some canonical relation C_0 to be determined below. Because of (2.4.) we take C_0 such that $p(x,\xi)=0$ if $(x,\xi,x_0,\xi_0) \in C_0$. As remarked after Theorem 1.3 this implies that C_0 is invariant under $H_{\tilde{p}}$. Since the <u>bicharacteristic strips of</u> P are defined as the solution curves in N of the vector field H_p , this means that $(x,\xi,x_0,\xi_0) \in C_0$ if $(y,\eta,x_0,\xi_0) \in C_0$ and (x,ξ) is lying on the bicharacteristic strip passing through (y,η) . On the other hand (2.5.) leads to the condition $(x,\xi,x_0,\xi_0) \in C_0$, $x \in X_0 \Rightarrow x = x_0$ and $\xi|_{T_{x_0}(X_0)} = \xi_0$. Conversely this situation should occur for every $(x_0,\xi_0) \in T^*(X_0)\backslash 0$ in order to get $R_0 \circ C_0$ = graph of the identity : $T^*(X_0)\backslash 0 \to T^*(X_0)\backslash 0$. So we are lead almost automatically to the following definition of C_0 :

$$(x,\xi,x_0,\xi_0) \in C_0 \Leftrightarrow \text{There exists } (x_0,\eta) \text{ such that}$$

(2.6.) (i) $\eta|_{T_{x_0}(X_0)} = \xi_0$ and $p(x_0,\eta) = 0$,

(ii) (x,ξ) is on the bicharacteristic strip emanating from (x_0,η) .

Now the following theorem can be regarded as an interpretation of the results of Lax [3] and Ludwig [4] .

THEOREM 2.1. - Let X_0 be a connected submanifold of X $(n \geq 3)$ of codimension 1 , non-characteristic with respect to P . Assume that every bicharacteristic curve of P (= projection into X of a bicharacteristic strip of P) intersects X_0 at most once and transversally. Then the number μ of solutions $\eta = \eta_j(x_0,\xi_0)$, $j = 1,\dots,\mu$ of (2.6.) , (i) does not depend on $(x_0,\xi_0) \in T^*(X_0)\backslash 0$ and the η_j depend smoothly on (x_0,ξ_0) . The set C_0 defined in (2.6.) is a homogeneous canonical relation from $T^*(X_0)\backslash 0$ to $T^*(X)\backslash 0$, and in addition closed in $(T^*(X) \times T^*(X_0))\backslash 0$, if

(2.7.) No bicharacteristic curve emanating from X_0 is contained in a compact subset of X , and

(2.8.) For each compact subset K_0 of X_0 , K of X there exists a compact subset K' of X such that every interval on a bicharacteristic curve with one end point in K_0 and the other in K , is contained in K' .

Secondly, if $Q_j \in L^{m_j}(X)$ have principal symbols q_j and

(2.9.) The matrix $q_j(x_0,\eta_k(x_0,\xi_0))$, j , $k = 1,\dots,\mu$ is non-singular for every $(x_0,\xi_0) \in T^*(X_0)\backslash 0$,

then there exist $E_j \in I^{-\frac{1}{4} - m_j}(X,X_0 ; C_0)$ satisfiyng the equations (2.3.) and (2.4.) .

PROOF. - The transversality of the bicharacteristic curves means that

$d_\xi p(x_0,\eta) \notin T_{x_0}(X_0)$, that is $d_\xi p(x_0,\eta) \neq 0$ when restricted to $T_{x_0}(X_0)^\perp$ (if $p(x_0,\eta)=0$) . But this means that the solutions of (2.6.) , (i) are simple. In view of the condition that X_0 is non-characteristic this implies that their number is finite and locally constant, and that the solution depend smoothly on (x_0,ξ_0) . Because $T^*(X_0)\backslash 0$ is connected $(n \geq 3!)$ the number of solutions is constant on all of $T^*(X_0)\backslash 0$. For the proof that C_0 is a closed C^∞ submanifold of $(T^*(X) \times T^*(X_0))\backslash 0$, and Lagrangean for $\sigma_{T^*(X)} - \sigma_{T^*(X_0)}$, we refer to [1] , section 6.5.

If e_j is the principal symbol of E_j then (2.4.) , (2.5.) lead to the equations

(2.10.) $$\frac{1}{i} H_{\tilde{p}}\, e_j + c.e_j = 0 \text{ , and}$$

(2.11.) $$\sum_{\ell} r(x_0,\xi_0,x_0,\eta_\ell(x_0,\xi_0)).q_j(x_0,\eta_\ell(x_0,\xi_0)).e_k(x_0,\eta_\ell(x_0,\xi_0)\ ,\ x_0,\xi_0)) = \delta_{jk}$$

according to Theorems 1.3. , 1.2. respectively. Here r is the principal symbol of ρ . Because of (2.9.) the equations (2.11.) have unique solutions $e_k(x_0,\eta_\ell(x_0,\xi_0)\ ,\ x_0,\xi_0)$, k , $\ell = 1 ,\dots, \mu$, which can be regarded as the initial values for the solutions e_j of first order differential equation (2.10.) along the bicharacteristic strips.

So (2.10.) , (2.11.) have a unique solution e_j .

Taking $E_j^{(0)} \in I^{-\frac{1}{4} - m_j}(X,X_0 ; C_0)$ arbitrarily with these principal symbols, we obtain that

$$PE_j^{(0)} \in I^{m - \frac{1}{4} - m_j - 2}(X,X_0 ; C_0) ,$$

$$\rho Q_j E_k^{(0)} - \text{id.} \in L^{-1}(X_0) .$$

Solving similar equations for the principal symbols of operators $E_j^{(r)} \in I^{-\frac{1}{4} - m_j - r}(X,X_0 ; C_0)$ we obtain recurrently

$$P(E_j^{(0)} + \dots + E_j^{(r)}) \in I^{m - \frac{1}{4} - m_j - r - 2}(X,X_0 ; C_0) ,$$

$$\rho Q_j(E_k^{(0)} + \dots + E_k^{(r)}) - \text{id.} \in L^{-r-1}(X_0) .$$

By taking asymptotic sums of the amplitudes we obtain operators $E_j \in I^{-\frac{1}{4} - m_j}(X, X_0 ; C_0)$ such that $E_j - (E_j^{(0)} + \dots + E_j^{(r)}) \in I^{-\frac{1}{4} - m_j - r - 1}(X, X_0 ; C_0)$ for all r, and we see that these operators solve (2.4.) and (2.5.) .

Note that (2.9.) is satisfied if $Q_j = (\frac{\partial}{\partial n})^{j-1}$, n = transversal vectorfield to X_0 , by looking at a Vandermonde determinant. This leads to the classical Cauchy problem. The conditions of Theorem 2.1. are fulfilled if P is a strictly hyperbolic differential operator in the usual sense, that is if $\mu = m$ and

(2.12.) $X = X_0 \times \mathbb{R}$, $X \times (t)$ is non-characteristic for P and the bicharacteristic curves intersect $X_0 \times (t)$ transversally for all $t \in \mathbb{R}$.

Note that for general pseudo-differential operators there is no natural relation between μ and m since m may be any real number. Finally we remark that if $\rho(t)$ is the restriction operator : $C^\infty(X) \to C^\infty(X_{(t)})$ then the matrix of operators

$$\rho(t) Q_j E_k \in I^{m_j - m_k}(X_{(t)} , X_{(0)} ; R_{(t)} \circ C_0) \tag{2.13.}$$

transforms the Cauchy data at $t = 0$ to the Cauchy data at time t . Here $X_{(t)} = X_0 \times (t)$ and $R_{(t)}$ denotes the canonical relation of the restriction operator $\rho(t)$.

3. THE SCHRÖDINGER EQUATION FOR $h \to 0$.

We consider solutions $u(t,x,h)$ of the Schrödinger equation

$$h \frac{i}{c} \frac{\partial u}{\partial t} = h^2 \Delta_x u + V(t,x).u \tag{3.1.}$$

depending on $h > 0$. We want to study their asymptotic behaviour as $h \to 0$ ("classical limit"). Dividing by h^2 and writing $\sigma = 1/h$ we obtain

$$\sigma . \frac{i}{c} \frac{\partial u}{\partial t} = \Delta_x u + \sigma^2 . V(t,x).u \tag{3.2.}$$

and we are interested in the asymptotic behaviour for $\sigma \to \infty$. Now write

$$u(t,x,\sigma) = \int e^{-i\sigma s} v(t,x,s)ds \tag{3.3.}$$

= Fourier transform at σ with respect to a new variable s . Then we recognize the asymptotics for $\sigma \to \infty$ as a wave front investigation of the singularities of v . For v we obtain the equation

$$Pv = \frac{1}{c}\frac{\partial^2 v}{\partial s \partial t} - \Delta_x v - V(t,x).\frac{\partial^2 v}{\partial s^2} = 0 , \tag{3.4.}$$

where P is a real operator, and all terms are of the same order 2 . The only drawback of P is that the initial manifold $t = 0$ in (t,x,s) - space is characteristic for the operator P . For this reason we prefer to work here with the relativistic Schrödinger equation

$$h^2 \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = h^2 \Delta_x u + V(t,x).u \tag{3.5.}$$

Applying the same trick we are led to the equation

$$Pv = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} - \Delta_x v + V(t,x).\frac{\partial^2 v}{\partial s^2} = 0 , \tag{3.6.}$$

where the operator P is strictly hyperbolic if $V(t,x)<0$ for all t,x . ($V(t,x) = -m^2c^2$ for a free particle. (For the Dirac equation we obtain a hyperbolic system).

Now assume that we have highly oscillatory initial values

$$\begin{aligned} u(0,x,h) &= e^{-i\frac{1}{h}\Psi_0(x)} a_0(x,\tfrac{1}{h}) \\ \frac{\partial u}{\partial t}(0,x,h) &= e^{-i\frac{1}{h}\Psi_1(x)} a_1(x,\tfrac{1}{h}) , \end{aligned} \tag{3.7.}$$

with $a_j(x,\sigma) \sim \sum_{k=0}^{\infty} a_j^{(k)}(x).\sigma^{\mu_j - k}$ for $\sigma \to \infty$, $j = 0,1$.

This implies

$$v_0(x,s) = v(0,x,s) = (2\pi)^{-1} \int e^{i\sigma s}.e^{-i\sigma \Psi_0(x)} a_0(x,\sigma)\, d\sigma \tag{3.8}$$

which is a Fourier integral with phase function $\sigma(s - \Psi_0(x))$.

Similarly $v_1(x,s) = \frac{\partial v}{\partial t}(0,x,s)$ is a Fourier integral with phase function $\sigma(x - \Psi_1(x))$. (σ is the frequency variable). So these distributions belong to the class $I^{\mu_j - \frac{1}{2}}(X_0,\Lambda_j)$, where Λ_j = normal bundle of the manifold $s = \Psi_j(x)$.

It follows that $E_j v_j \in I^{\mu_j - 3/4 - j}(X_0, C_0 \circ \Lambda_j)$, where $C_0 \circ \Lambda_j$ is the Lagrangean manifold in $T^*(X)\backslash 0$ obtained from Λ_j by applying the relation C_0 defined in (2.6.) . In the general points $C_0 \circ \Lambda_j$ will be the normal bundle of a manifold $s = \Psi_j(t,x)$ which in analogy with (3.8.) leads to an asymptotic expansion of the form

$$(3.9.) \qquad u(t,x,\sigma) \sim e^{-i\sigma \Psi_0(t,x)} \sum_{k=0}^{\infty} a_0^{(k)}(t,x).\sigma^{\mu_0 - k} + e^{-i\sigma \Psi_1(t,x)} \sum_{k=0}^{\infty} a_1^{(k)}(t,x).\sigma^{\mu_1 - 1 - k} .$$

The points where $C_0 \circ \Lambda_j$ is not locally equal to the normal bundle of a manifold $s = \Psi_j(t,x)$ are called caustics in analogy with the terminology of geometrical optics. Of course (3.9.) are just the asymptotic expansions of the WKB-method. However we have given a proof,which is globally valid,that the solutions satisfy such expansions if the initial data do. Note that the asymptotic expansions are exact modulo terms of order $-\infty$ because the calculus of Fourier integral operators is exact modulo C^∞ . The WKB-method only gives results up to the caustics, whereas we also obtain asymptotic expansions at points lying beyond the caustics. Moreover, since we have an integral representation of the solutions,a more refined stationnary phase analysis also leads to certain asymptotic expansions at the caustics, at least in special cases.

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