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III

PLURISUBHARMONIC FUNCTIONS AND ENTIRE FUNCTIONS OF EXPONENTIAL TYPE (*)

par

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1 . Introduction and notations. Some new results were obtained recently on entire functions of exponential type of n complex variables. We give here a survey of these results which will appear in forthcoming papers of P. Lelong. [4, e , f], A. Martineau [5 , b] and C. O. Kiselman [3]. A first problem is to characterize the regularization of the two indicators (the circular indicator and the radial indicator) of entire functions of exponential type. This problem has recently received a complete solution for the two indicators in terms of plurisubharmonic functions. On the other hand the problems concerning the indicators themselves reduce to the study of certain "small" sets, i. e., sets of points where the indicator is smaller than its regularization. We give here properties of the class of "negligible" sets, a notion which probably can be used also for other -- classical or non-classical -- problems in the theory of functions of n complex variables.

We denote by $j : C^n \to R^{2n}$ the injection $(x, y) = j(z)$, $z \in C^n$, $(x, y) \in R^{2n}$, of the complex space C^n in the euclidean space R^{2n} ; the norm $\|z\|$ is the distance of z from the origin 0. If $f(z)$ is a real valued function

$$f^*(z) = \text{Reg. } f(z) = \limsup_{z' \to z} f(z') \ , \ z' \in C^n$$

is the smallest upper semicontinuous majorant of f and is called the regularization of f . An entire function $F(z) = F(z_1 , \dots , z_n)$ is of exponential type γ if

$$\limsup \ \|z\|^{-1} \log | F(z) | = \gamma \ , \ \|z\| \to +\infty \ , \ z \in C^n .$$

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E_n denotes the ring of the entire functions of finite exponential type in C^n. We shall be concerned first with the two following indicators, the circular indicator and the radial indicator of $F \in E_n$.

Definition 1.1. Given $F \in E_n$, we define the circular indicator of center $\zeta \in C^n$, of variable $z \in C^n$, as follows :

$$L_c(\zeta, z) = \limsup_{|u| \to +\infty} |u|^{-1} \log |F(\zeta + uz)|, \quad u \text{ complex}$$

If $n = 1$, L_c is the constant γ.

Definition 1.2. Given $F \in E_n$, the radial indicator of center $\zeta \in C^n$, of variable $z \in C^n$ is defined by

$$L_r(\zeta, z) = \limsup_{t \to +\infty} t^{-1} \log | F(\zeta + tz) |, \quad t \text{ real}, \quad t > 0 .$$

It is well known that for $n=1$, $L_r(\zeta, z)$ is a convex (and therefore continuous) function of $z \in C^1$ for each fixed ζ. Such simplicity is no longer true for $n > 1$. Instead of the convex functions, we must consider plurisubharmonic functions and negligible sets.

2 . Plurisubharmonic functions. $P(\Omega)$ denotes the class of the plurisubharmonic functions on a complex analytic manifold Ω. Perhaps it is useful to state the definition (although plurisubharmonic functions are now in textbooks on analytic functions).

Definition 2.1. A function f is a plurisubharmonic function in Ω if

(i) f is real valued, $-\infty \le f < +\infty$; $f \not\equiv -\infty$,

(ii) f is locally upper bounded (i.e., $\sup_{z \in K} f(z) = M(K)$ is finite for each compact $K \subset \Omega$),

(iii) given a linear mapping $m : z_k = z_k^o + a_k u$, $u \to z = (z_k) \in C^n$, $u \in C^1$, of C^1 in C^n, $f \circ m$ is locally subharmonic in u or is the constant $-\infty$.

The above definition was the original definition I gave in 1942 ([4, a] and [4, b]). The condition (ii) can be replaced by (ii)' : f is upper semicontinuous. A second (equivalent) definition is the following, using the injection j : $f \in P(\Omega)$, $\Omega \subset C^n$, if and only if

(i) $f \circ j^{-1}$ is a subharmonic function in Ω (considered as a domain in R^{2n})

(ii) for each complex linear mapping α of C^n: $z_k = z_k^{\,o} + \Sigma\, a_k^{\,j}\, z'_j$, $z = \alpha(z')$, $\|a_k^j\| \neq 0$, the function $f \circ \alpha \circ j^{-1}$ is subharmonic in the image (in R^{2n}) $j \circ \alpha^{-1}(\Omega)$.

For other (equivalent) definitions see [4, b] and [4, d] . The definition 2.1 remains valid if Ω is a domain in a complex linear topological space, the condition (iii) being formulated : $f(x) \leq \pi^{-1} \int_D f(x+uy)\, d\lambda\,(u)$; $x + Dy \in \Omega$, where D is the disk $|u| \leq 1$, $u \in C^1$, and $d\lambda$ is the Lebesgue measure in $C^1 = R^2$.

Given a family $f_s \in P(\Omega)$, $s \in J$ locally upper bounded on Ω , we consider

(2.1) $\qquad W_1 = \sup_s f_s \;, \quad s \in J$

(2.2) $\qquad W_2 = \limsup_s \; f_s$ in the case J is a directed set ;

W_1 and W_2 are not in $P(\Omega)$, but the regularizations

$$W_1^* = \text{Reg. } W_1 \quad \text{and} \quad W_2^* = \text{Reg. } W_2$$

are in $P(\Omega)$. For the proof, see [4, d]. The property of W_1^* remains valid if Ω is an open set in a complex linear topological space with a countable base of the neighborhoods at the origin, for example a Banach space (see [1] and [3]).

3 . <u>Polar sets and negligible sets</u>. In the case n = 1 , the sets defined by $[z\,;\, W_1(z) < W_1^*(z)]$ or $[z\,;\, W_2(z) < W_2^*(z)]$ are sets of capacity zero. For $n > 1$, by the injection $j : C^n \to R^{2n}$, it is obvious that such sets are

of R^{2n} -capacity zero. But it is not a sufficiently precise property.

Definition 3.1. Given a complex manifold M^n ; E is called a polar set on M^n if there exists $f \in P(M^n)$ and

$$E \subset [z\ ;\ f(z) = -\infty\ ].$$

Theorem 3.2. If M^n is a countable union of compact sets $K_m \subset M^n$, such that a given compact $K \subset M^n$ belongs to K_m for sufficiently large m, then a countable union of polar sets on M^n is a polar set on M^n.
For the proof see [4, e, f].

Definition 3.3. $E \subset M^n$ is a negligible set on M^n if there exists an increasing sequence $f_q \in P(M^n)$, locally upper bounded and such that if $\lim f_q = W$, then $E \subset [z\ ;\ W(z) < W^*(z)\ ]$, where $W^* = \mathrm{Reg}\,.\ W$ on M^n. For the countable union of such sets in a domain of C^n we have :

Theorem 3.4. A countable union of negligible sets in a domain of C^n is a negligible set.

For the proof see [4, f]. Now we return to classes of functions and consider classes $M(\Omega)$, $M_o(\Omega)$ in a domain Ω of C^n ($\Omega = C^n$ is not excluded) defined as follows.

Definition 3.5. The class $M(\Omega)$ is a countable union of the following classes $C_q(\Omega)$:

$$C_o(\Omega) = P(\Omega) \text{ (plurisubharmonic functions)}$$

$C_1(\Omega)$ is defined by : $f \in C_{1a}(\Omega)$ if $f = \sup f_s$, $f_s \in P(\Omega)$ and the family f_s, $s \in J$, is locally upper bounded ; $f \in C_1(\Omega)$ if $f = \lim f_q$, $f_q \in C_{1a}(\Omega)$, $f \not\equiv -\infty$ and f_q is a decreasing sequence.

If $C_{q-1}(\Omega)$ is defined, we define $C_{qa}(\Omega)$ and $C_q(\Omega)$ by the same process : $f \in C_{qa}(\Omega)$ if $f = \sup f_s$, $f_s \in C_{q-1}(\Omega)$, $s \in J$, and the family f_s is locally upper bounded ; $f \in C_q(\Omega)$ if f is the limit of a decreasing sequence $f_q \in C_{qa}(\Omega)$, and $f \not\equiv -\infty$.

$M(\Omega)$ is the union of the classes $C_q(\Omega)$; it is also the limit of the increasing sequence of the $C_q(\Omega)$, $q \to +\infty$.

If we consider only countable families (i. e., J is a countable set), we obtain a subclass $M_o(\Omega) \subset M(\Omega)$ which contains only Baire functions.

<u>Theorem 3. 6.</u> (i) Given $W \in M(\Omega)$ there exists $W' \in M_o(\Omega)$ such that

$$W' \le W \text{ and Reg. } W' = \text{Reg. } W$$

(ii) $W^* = \text{Reg. } W$ is a plurisubharmonic function

(iii) $E = [z ; W(z) < W^*(z)]$ is a negligible set.

The part (iii) is a consequence of theorem 3.4. For the proof see [4 , f]. By (i) we have $E \subset E'$ where E' is negligible and a Baire set. In [4 , c] other properties are given : if E is a negligible set in $\Omega \subset C^n$, we have $E \subset \sum_1^n \eta_k$, where η_k is a Baire set whose sections by the $C^1(z_k)$ are sets of R^2-capacity zero. An important property is the following (see [4 , c]):

<u>Theorem 3.7.</u> The restriction of a negligible set on the real subspace R_x^n is a set of R^n-measure zero.

The subspace R_x^n is defined by putting $z_k = x_k + iy_k$, and $y_k = 0$, $1 \le k \le n$, The property given by theorem 3.7. remains valid if we use a holomorphic mapping ; for example , the restriction of a negligible set on the distinguished boundary of an n-cylinder in C^n is of Lebesgue measure zero.

<u>Theorem 3. 8.</u> If $W(\zeta, z) \in M(\Omega_\zeta \times \Omega_z)$, we have

$$W^*(\zeta, z) = \text{Reg.}_{\zeta \times z} W(\zeta, z) = \text{Reg.}_\zeta [\text{Reg}_z W(\zeta, z)] = \text{Reg.}_z [\text{Reg.}_\zeta W(\zeta, z)].$$

As a corollary, if $W(z_1, \dots, z_n) \in M(\Omega)$, then W^* can be obtained by using the regularizing process successively for the n variables.

<u>Comparison between the polar sets and the negligible sets.</u> A polar set E in Ω is negligible in Ω ; for if $E \subset [z ; V(z) = -\infty]$ where $V \in P(\Omega)$, we consider $W(z) = \limsup_{n \to +\infty} \frac{1}{n} V(z)$; $W^*(z) \equiv 0$, and we have $E \subset [z ; W < W^*]$. But is the converse true ? The following statement gives an answer only in a particular case :

<u>Theorem 3.9.</u> If $E \subset [z ; W(z) < W^*(z)]$ and $W = \lim \nearrow f_n$, $f_n \in P(\Omega)$, and if W^* is <u>pluriharmonic,</u> then E is a polar set in Ω.

4 . <u>Homogeneous plurisubharmonic functions.</u> $V(z)$ is positively homogeneous of order λ if

$$V(tz) = t^\lambda V(z) \qquad t > 0 ;$$

$V(z)$ is complex homogeneous of order λ if

$$V(uz) = |u|^\lambda V(z) , \quad u \text{ complex.}$$

In the following, homogeneous means homogeneous of order 1. It is convenient to introduce the following definition.

<u>Definition 4.1.</u> $V \in P(C^n)$ is of exponential type γ if

$$\limsup \|z\|^{-1} V(z) = \gamma .$$

Obviously if $V \in P(C^n - 0)$ is bounded in a neighborhood of the origin 0 , and positively homogeneous, then $V \in P(C^n)$ and V is of exponential type.

<u>Theorem 4.2.</u> (i) For $n=1$, if $V \in P(C^1)$ is positively homogeneous, then $V(z)$ reduces by j to a convex and positively homogeneous function $V(x, y)$ in R^2.

(ii) If $V \in P(C^n)$ is positively homogeneous, then $V(z) > -\infty$, $z \in C^n$.

(iii) If $V \in P(C^n)$ is complex homogeneous, then $V(z) \geq 0$, $z \in C^n$, and $U = \log V \in P(C^n)$. Furthermore there exists a sequence $P_m(z)$ of homogeneous polynomials, degree $P_m = m$, such that the sequence

$$V_m = \frac{1}{m} \log |P_m|$$

is locally upper bounded and $W^*(z) = U(z)$, where $W(z) = \limsup_{m \to +\infty} V_m(z)$.

The statement (i) is classical and can be proved by considering in an angle α, $0 < \alpha < \pi$, the positively homogeneous harmonic function $h = ax+by$ which takes the values of V on the boundary of the angle. The inequality $V \le h$ is equivalent to the convexity of V.

For (ii) and (iii) see [4, f] and [3]. The open set $D = [z ; V(z) < 1]$, if V is complex homogeneous, is a pseudo-convex domain. Let us put $z = u\lambda$, $\|\lambda\| = 1$; D is defined by $V(u\lambda) = |u| V(\lambda) < 1$, or $|u| < R(\lambda)$ with

$$\operatorname{Log} R(\lambda) = -\log V(\lambda)$$

and, by a classical property of $R(\lambda)$ for the pseudo convex domains, $U(z) = \log V(z)$ is a plurisubharmonic function.

5 . Properties of the indicators $L_c(\zeta, z)$, $L_r(\zeta, z)$. Simplifications arise from the following properties.

Theorem 5. 1.
$$\operatorname{Reg.}_\zeta L_c(\zeta, z) \le \operatorname{Reg.}_z L_c(\zeta, z)$$
$$\operatorname{Reg.}_\zeta L_r(\zeta, z) \le \operatorname{Reg.}_z L_r(\zeta, z)$$

These are consequences of the following geometric situation. Let us consider the mapping $\pi : C^n - \{0\} \to P^{n-1}$ where P^n is the projective map, and $z^o \neq 0$, u' is some bounded neighborhood of ζ^o in C^n, then there exists t_o such that

$$u' + z^o t \subset \zeta^o + \pi^{-1}(U)$$

for $t \ge t_o$.

Theorem 5.2. (i) $\mathrm{Reg.}_{\zeta\times z}\, L_r(\zeta, z) = \mathrm{Reg.}_z\, L_r(\zeta, z)$.

(ii) $\mathrm{Reg.}_z\, L_r(\zeta, z)$ is independent of ζ.

(iii) The same properties are true for $L_c(\zeta, z)$.

Theorem 5.2. leads to the following definitions.

Definitions 5.3. We put

$$L^*_c(z) = \mathrm{Reg.}_{\zeta\times z}\, L_c(\zeta, z) \qquad \text{(circular global indicator)}$$

$$L^*_r(z) = \mathrm{Reg.}_{\zeta\times z}\, L_r(\zeta, z) \qquad \text{(radial global indicator)}$$

6 . Circular global indicator. It is obvious that $L^*_c(uz) = |u|\, L^*_c(z)$: $L^*_c(z)$ is therefore a complex homogeneous plurisubharmonic function. The converse is true (see [4, e, f]).

Theorem 6.1. In order that $V(z)$ be a global circular indicator of an entire function $F \in E_n$, it is necessary and sufficient that $V \in P(C^n)$ be complex homogeneous of order one.

The proof makes use of the fact that the domain $[z ; V(z) < 1]$ which was considered in theorem 4.2. is a holomorphy domain (by the theorem of Oka-Norguet).

For fixed ζ , the set $[z ; L_r(\zeta, z) < L^*_r(z)]$ is a negligible set of complex lines L^1 . Example : if $n=2$, this set can be identified with a set of capacity zero in the projective space P^1.

7 . Radial global indicator. $L^*_r(z)$ is a positively homogeneous plurisubharmonic function. The converse is true and was proved recently and independently by A. Martineau [5, b] and C. O. Kiselman [3]. It is possible to associate to $L^*_r(z)$ an open set Δ of holomorphy (see P. Lelong [4, f] defined by $\Delta = \Delta_1 \cup \Delta_2$, where $\Delta_1 = [z ; L^*_r(z) < 1]$, $\Delta_2 = [z ; L^*_r(-z) \leq -1]$, but the proof of the theorem given by Kiselman uses the following construction of a

domain of holomorphy.

Lemma 7.1. Let F be a positively homogeneous and plurisubharmonic function in C^n. Then the open set (Re means the real part) :

$$\Omega_F = [z \in C^n \ ; \ \inf_t (F(tz) - \operatorname{Re} t) < 0 \ ] \ , \quad t \text{ complex},$$

is a pseudo-convex domain, and therefore a holomorphy domain.

Let us denote by π the mapping of $C^{n+1} - \{0\} \to P^n$ on the projective space. We consider in C^{n+1} the open set :

$$\omega_F = [z \in C^{n+1} \ ; \text{ for some } \ t \in C^1 \ , \ F(tz_1, \ldots, tz_n) < \operatorname{Re} tz_{n+1}]$$

ω_F is a connected Stein manifold ; $\omega'_F = \pi(\omega_F)$ determines F uniquely. If $F, G \in P(C^n)$, are positively homogeneous of order one, we have $F \leq G$ if and only if $\omega'_F \supset \omega'_G$. As a consequence, we have

Theorem 7.2. If $F \in P(C^n)$ is positively homogeneous of order one, then there exists an entire function of exponential type having F as global radial indicator.

Consequently, the regularization of the circular and of the radial indicators are completely determined by domains of holomorphy in C^n ; a classical result of the theory $(n = 1)$ is so extended to the general case $n \geq 1$.

8 . It is important to develop our knowledge of the entire functions of exponential type of n variables not as a pure generalization of the classical theory $(n = 1)$ but in close connection with the main problems of functional analysis. The ring E_n contains the Fourier transforms of most of the usual linear operators (convolution is the image of the multiplication of the ring E_n). On the other hand, many problems of the theory of analytic functionals reduce to problems on E_n . Sometimes the properties of regularizations of the indicators are not

sufficient, and properties of negligible sets are to be used. A negligible set is of Lebesgue measure zero on the real subspace R^n_x. As a consequence of this property we have given in [4, c] a theorem of Hartogs, for the real subspace R^n_x, and applications to entire functions of exponential type. This method gives a generalization of a result of Martineau [5, a]. The "Hartogs theorem for the reals" is the following [4, c] :

Theorem 8.1. Let us consider a domain $\Omega \subset C^n$, such that $\Omega \cap R^n_x = d$, where d is a domain in the real subspace R^n_x of C^n. Let $V_s(z)$, $s \in J$, where J is a directed set be a family, $V_s \in P(\Omega)$, locally upper bounded in Ω. We denote by $z = x + iy$ the decomposition $C^n = R^n_x \times R^n_y$. If $f(x)$ is a continuous function on d and $\overline{f}(z)$ a continuous continuation of f to Ω [for example, $\overline{f}(x+iy) = f(x)$], and if we have

$$\limsup_s V_s(x) \le f(x), \quad x \in d, \quad x \text{ real}$$

then the following property is true : given K compact in $d \subset R^n_x$, and $\varepsilon > 0$, there exists s_0 and a neighborhood $U(K)$ of K for the C^n-topology, such that

$$V_s(z) \le \overline{f}(z) + \varepsilon$$

if $s > s_0$, and $z \in U(K)$.

An equivalent property is the following : if $W(z) \in M(\Omega)$, we have $W^*(z) = \limsup_{x \to 0} W(z+x)$, x real. For each function $W \in M(\Omega)$ the regularization can be made using only the restriction of the function to the real values. An easy consequence (see [4, f]) is the following :

Theorem 8.2. Let $\psi(x)$ be a continuous function of the real $x \in R^n_x$, and $\psi(x)$ be positive and positively homogeneous of order one on R^n_x (i. e., $\psi(tx) = t\,\psi(x)$, $t > 0$). Then, to given $\varepsilon > 0$, there corresponds a finite number $C(\varepsilon)$

such that, if V is a plurisubharmonic function of exponential type γ, then the condition

$$L_r(0,x) \le \psi(x) \quad \text{for } x \text{ real}$$

gives

$$L^*_r(z) \le \psi(x) + \epsilon \, \|x\| + C(\epsilon) \, \|y\|$$

and $C(\epsilon)$ depends only upon γ and ϵ.

As a corollary we have : with the same hypothesis on $\psi(x)$, the majorization $\limsup_{t\to+\infty} t^{-1} \log |F(tx)| \le \psi(x)$, x real, where F is an entire function of exponential type, gives.

$$V(z) = \log |F(z)| \le \psi(x) + \epsilon \, \|x\| + C(\epsilon) \, \|y\| + C'(\epsilon)$$

where $C(\epsilon)$ and $C'(\epsilon)$ are two finite constants. A particular situation for entire functions or plurisubharmonic functions of exponential type is obtained if $\psi(x) \equiv 0$. Then $L_r(0,x) \le 0$ gives $L^*_r(x) = 0$, x real, and we have

$$L^*_r(iy) = \sup_x L^*_r(x+iy) = h(y)$$

and $h(y)$ is a convex function in R^n_y. For fixed y, the set $[\zeta \,;\, L_r(\zeta, iy) < h(y)]$ is a polar set E_y. On the other hand, if $\zeta \notin E_y$ we have

$$\limsup_{t\to+\infty} t^{-1} V(\zeta_k + te^{i\theta} y_k) = h(y_k) \sin\theta \quad , \quad 0 < \theta < \pi$$

(see [4, f] and [2]). Such results have applications in functional analysis (see for example [2]) and are generalizations of classical results given for entire functions of one complex variable or for subharmonic functions in a half-plane.

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