PUBLICATIONS MATHÉMATIQUES ET INFORMATIQUES DE RENNES

ARNAUD LE NY

Decimation on the Two Dimensionnal Ising Model : Non Gibbsianness at Low Temperature. Almost Gibbsianness or Weak Gibbsianness ?

Publications de l'Institut de recherche mathématiques de Rennes, 1998, fascicule 2
« Fascicule de probabilités », , p. 1-71
<http://www.numdam.org/item?id=PSMIR_1998___2_A6_0>

© Département de mathématiques et informatique, université de Rennes, 1998, tous droits réservés.

L'accès aux archives de la série « Publications mathématiques et informatiques de Rennes » implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Decimation on the two dimensionnal Ising model : non Gibbsianness at low temperature. Almost Gibbsianness or weak Gibbsianness ?

Arnaud Le Ny

Institut de Recherche Mathématique et CNRS (UMR 6625)
Université de Rennes 1
35042 Rennes, France
e-mail: aleny@maths.univ-rennes1.fr

Dublin Institute for Advanced Studies
10 Burlington Road, Dublin 4, Eire
e-mail: leny@stp.dias.ie

November 1998

Abstract

We show that the decimation on the $2\mathbb{Z}^2$-lattice for the two-dimensionnal Ising model leads to a non Gibbsian measure at low temperature. We provide a complete proof of a result sketched by van Enter et al.[5]. We also describe how we could restore this Gibbsianness, investigating, following the terminology of Maes et al.[11, 12, 13], the almost Gibbsianness and the weak Gibbsianness of this so-called decimated measure.

1 Introduction

It has been pointed out during the last decades that some pathologies may occur by using Renormalization Group Transformations (R.G.T) to various Gibbs measures [5, 12, 13, 1, 3, 7, 11]. A lot of examples are known and these transformations lead to new families of random fields. A main question is how to restore the Gibbsian formalism to these measures, fullfiling Dobrushin's claim :'*everything in the world is Gibbsian*'. After a quick, and non-exhaustive, description of the Gibbsian formalism, we study one of the simplest transformations leading to non-Gibbsianness: the decimation transformation on the two dimensionnal Ising model. We prove that, at low temperature, the image of *any* Gibbs measure for this model by this transformation *cannot* be Gibbsian. This proof was sketched in [5]. In the last section, we describe briefly what has been done to restore the Gibbsian formalism for those 'pathological' measures. We shall apply this to our example elsewhere.

2 Specifications and Gibbs measures

We introduce here the basic notions we need to define Gibbs measure, almost Gibbs measures and weak Gibbs measures. The reference for this section, and throughout this paper, are [9, 5].

2.1 Preliminaries

2.1.1 The configuration space

Let $(\Omega, \mathcal{F}, m)$ be a probability space defined as follows:

E is a finite set, and $\mathcal{E}$ a σ-algebra on it,

S is a countably infinite set,

And let m_0 be an a priori *finite* measure on $(E, \mathcal{E})$.

We define[1] then the product space $\Omega = E^S$ with its product σ-algebra $\mathcal{F} = \mathcal{E}^{\otimes S}$ and the usual product measure $m = m_0^{\otimes S}$.

The elements of Ω will be denoted by Greek letters ω, σ, τ and for each $\omega \in \Omega$, we denote ω_i the value of ω at the site $i \in S$. We often call these random variables *spins* (at site i).

In the main part of this paper, we will consider :

$$\Omega = \{-1,+1\}^{\mathbb{Z}^2} \ , \ \mathcal{E} = \mathcal{P}(\{-1,+1\}) \ , \ m_0 = \tfrac{1}{2}\delta_{-1} + \tfrac{1}{2}\delta_{+1}$$

in order to modelize the two-dimensionnal Ising model, which will more briefly be called 2d-Ising model.

We will always denote by $\mathcal{S}$ the set of all the finite subsets of S.

Moreover $\forall \Lambda \in \mathcal{S}$, we note $\Omega_\Lambda = E^\Lambda$ and ω_Λ the canonical projection of ω on Ω_Λ.

We also define, for all $\Lambda \subset S$, $\mathcal{F}_\Lambda$ to be the σ-algebra generated by the functions ($\omega \mapsto \omega_i$ for $i \in \Lambda$).

2.1.2 Topology and locality on Ω

The space Ω is called the configuration space and will also be equipped with the usual[2] product topology (with respect to the discrete topology on E). As the so called single-spin set E is finite, a typical neighbourhood of $\omega \in \Omega$ is given by

$$\mathcal{N}_\Lambda(\omega) = \{\sigma : \sigma_\Lambda = \omega_\Lambda, \sigma \text{ arbitrary outside } \Lambda\} \text{ with } \Lambda \in \mathcal{S}.$$

For the 2d-Ising model a basis of neighbourhoods for ω is given by the sets of the form $(\mathcal{N}_R)_{R>0}$ where

$$\mathcal{N}_R(\omega) = \{\sigma : \sigma = \omega \text{ on } \Lambda_R \text{ and } \sigma \text{ is arbitrary outside } \Lambda_R\}$$

[1]For more details about integration and measure theory, one could consult [6, 15].
[2]For any topological informations, see [4, 10].

and Λ_R is a square in $\mathbb{Z}^2$ of lentgh $2R$ centered at the origin, R being any integer strictly positive.

Definition 2.1 [locality and quasilocality] A function $f: \Omega \to \mathbb{R}$ is said to be *local* if $\exists \Lambda \in \mathcal{S}$ such that f is $\mathcal{F}_\Lambda$ -measurable, i.e f depends only on a finite number of spins.
A function $f: \Omega \to \mathbb{R}$ is said to be *quasilocal* if it is a uniform limit of some sequence of local functions f_n, i.e :

$$\lim_{n \to \infty} \sup_{\omega \in \Omega} | f_n(\omega) - f(\omega) | = 0$$

◇

The lemmata (2.1) and (2.2) provide a caracterisation of quasilocal functions which could be usefull. The proofs come from [9].

Lemma 2.1 *:*
A measurable function f *on* Ω *is* quasilocal *if and only if*

$$\lim_{\Lambda \uparrow \mathcal{S}} \sup_{\omega, \sigma \in \Omega, \omega_\Lambda = \sigma_\Lambda} | f(\omega) - f(\sigma) | = 0 \tag{1}$$

◇

Here the convergence[3], called " convergence along a net of finite subsets of S directed by inclusion ", should be taken as:

$$\lim_{\Lambda \uparrow \mathcal{S}} f(\Lambda) = a$$

iff

$$\forall \epsilon > 0, \exists\ K_\epsilon \in \mathcal{S} \text{ s.t } \mathcal{S} \ni \Lambda \supset K_\epsilon \implies | f(\Lambda) - a | \leq \epsilon$$

As $\mathcal{S}$ is directed by inclusion, one can consider the convergence along the following particulars nets.

[3]See [10, 5]. There is no need to add any further requirement on K_ϵ: it is not a convergence in the Van Hove sense.

Definition 2.2 [Cofinal sets] A subset $\mathcal{S}_0$ of $\mathcal{S}$ is called *cofinal* if each $\Lambda \in \mathcal{S}$ is contained in some $\Delta \in \mathcal{S}_0$. $\diamond$

For example, is $S = \mathbb{Z}^d$ for $d \geq 1$, then the set

$$\mathcal{S}_0 = \{[-n,n]^d \cap S \ , \ n \geq 1\}$$

of all centered cubes is cofinal.
As S is countable, one could show that there always exists a cofinal set $\mathcal{S}_0$ and moreover, it is always possible[4] to chose $\mathcal{S}_0 = (\Lambda_n)_{n\in\mathbb{N}}$ with $\Lambda_n \in \Lambda_{n+1}$ and $\lim_{n\to\infty} \Lambda_n \stackrel{\text{Def}}{=} \cup_{n\in\mathbb{N}}\Lambda_n = S$.

Proof of the lemma 2.1:

Let us first prove that each quasilocal function has the property (1). Let f be quasilocal. By definition, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of local functions such that

$$\lim_{n\to\infty} \sup_{\omega\in\Omega} \mid f_n(\omega) - f(\omega) \mid = 0 \tag{2}$$

$\forall \omega, \sigma \in \Omega$,

$$\mid f(\omega) - f(\sigma) \mid \leq \mid f_n(\omega) - f(\omega) \mid + \mid f_n(\sigma) - f(\sigma) \mid + \mid f_n(\omega) - f_n(\sigma) \mid \tag{3}$$

Let $\Lambda \in \mathcal{S}$ and let $\omega, \sigma \in \Omega$ such that $\omega_\Lambda = \sigma_\Lambda$. The above inequality (3) is obviously also true, hence we obtain

$$\sup_{\omega,\sigma\in\Omega,\omega_\Lambda=\sigma_\Lambda} \mid f(\omega) - f(\sigma) \mid \leq$$
$$2\sup_{\omega\in\Omega} \mid f_n(\omega) - f(\omega) \mid + \sup_{\omega,\sigma\in\Omega,\omega_\Lambda=\sigma_\Lambda} \mid f_n(\omega) - f_n(\sigma) \mid \tag{4}$$

Let $\epsilon > 0$. The equation (2) yields:

$$\exists N > 0, \ \forall n \geq N, \ 2\sup_{\omega\in\Omega} \mid f_n(\omega) - f(\omega) \mid < \epsilon$$

Let $n \geq N$.
f_n is a local function: there exists $\Lambda_n \in \mathcal{S}$ such that f_n is $\mathcal{F}_{\Lambda_n}$-measurable. If $\Lambda \supset \Lambda_n$, then for ω, $\sigma \in \Omega$ such that $\omega_\Lambda = \sigma_\Lambda$, we have $\mid f_n(\omega) - f_n(\sigma) \mid = 0$. Using equation (4) we obtain:

$$\forall \epsilon > 0, \ \exists K_\epsilon = \Lambda_n \text{ s.t } \mathcal{S} \ni \Lambda \supset K_\epsilon \Longrightarrow \sup_{\omega,\sigma\in\Omega,\omega_\Lambda=\sigma_\Lambda} \mid f(\omega) - f(\sigma) \mid < \epsilon$$

[4]The family of all subsets of S is not countable, but $\mathcal{S}$, family of all the *finite* subsets of S, is countable.

and the first statement of the lemma is proved.
Let f be any measurable function with the property (1) and let $\mathcal{S}_0 = (\Lambda_n)_{n\in\mathbb{N}}$ be a cofinal sequence of $\mathcal{S}$ with $\Lambda_n \subset \Lambda_{n+1}$ and $\lim_{n\to\infty}\Lambda_n = \cup_{n\in\mathbb{N}}\Lambda_n = \mathcal{S}$. Let $\sigma \in \Omega$ be any arbitrary configuration and define a sequence $(f_n)_{n\in\mathbb{N}}$ of local function by:

$$f_n(\omega) = f(\omega_{\Lambda_n}\sigma_{\mathcal{S}\setminus\Lambda_n})$$

$\forall n \in \mathbb{N}$, f_n is $\mathcal{F}_{\Lambda_n}$-measurable, then it is a sequence of local functions.
Let us recall the equation (1):

$$\lim_{\Lambda\uparrow\mathcal{S}} \sup_{\omega,\sigma\in\Omega,\omega_\Lambda=\sigma_\Lambda} \mid f(\omega) - f(\sigma) \mid = 0$$

With Λ_n and $\mathcal{S}_0$ defined above, we obtain:

$$\lim_{n\uparrow\infty} \sup_{\omega,\sigma\in\Omega,\omega_{\Lambda_n}=\sigma_{\Lambda_n}} \mid f(\omega) - f(\sigma) \mid = 0$$

Here we have:

$$\mid f_n(\omega) - f(\omega) \mid = \mid f(\omega_{\Lambda_n}\sigma_{\mathcal{S}\setminus\Lambda_n}) - f(\omega) \mid$$

and then

$$\begin{aligned}\sup_{\omega\in\Omega} \mid f_n(\omega) - f(\omega) \mid &= \sup_{\omega\in\Omega} \mid f(\omega_{\Lambda_n}\sigma_{\mathcal{S}\setminus\Lambda_n}) - f(\omega) \mid \\ &\leq \sup_{\omega,\sigma\in\Omega,\omega_{\Lambda_n}=\sigma_{\Lambda_n}} \mid f(\omega) - f(\sigma) \mid\end{aligned}$$

hence

$$\lim_{n\to\infty} \sup_{\omega\in\Omega} \mid f_n(\omega) - f(\omega) \mid = 0$$

and the lemma is proved.

◇

Thus[5], as pointed out in [9], a non constant tail-measurable function can never be quasilocal.

[5]This result could be used at the end of the 3^d part to prove the non-quasilocality of the decimated measure.

Example A function is said to be tail-measurable if it is measurable with respect to the so-called σ-algebra at infinity $\mathcal{T}_\infty \stackrel{\text{def}}{=} \cap_{\Lambda \in \mathcal{S}} \mathcal{F}_{\Lambda^c}$. For example, one could consider the event

$$A = \{\omega : \lim_{n \to \infty} \frac{1}{\mid \Lambda_n \mid} \sum_{i \in \Lambda_N} \omega_i = 0\}$$

where Λ_n is any cofinal sequence and $\mid \Lambda_n \mid = \text{card}(\Lambda_n)$. The indicator function of this event, defined by $f(\omega) = \mathbf{1}_A(\omega)$ is tail-measurable, non constant and non quasilocal (it is obviously non continuous, then non quasilocal by lemma (2.2) below.).

$\diamond$

Moreover, in our particular case, we have the the following

Lemma 2.2 *:*
When the single-spin space E is finite, $f : \Omega \longrightarrow \mathbb{R}$ is quasilocal ***iff*** *f is continuous:* $\diamond$

Proof:
Assume first that E is a separable metric space and endow Ω with the product topology of the product metric d. Let $f : \Omega \longrightarrow \mathbb{R}$ be uniformly continuous

$$\forall \epsilon > 0,\ \exists \eta > 0,\ \forall \omega, \sigma \in \Omega,\ d(\omega, \sigma) < \eta \implies \mid f(\omega) - f(\sigma) \mid < \epsilon$$

Using the previous lemma, we want to prove that f is quasilocal by proving:

$$\lim_{\Lambda \uparrow \mathcal{S}} \sup_{\omega, \sigma \in \Omega, \omega_\Lambda = \sigma_\Lambda} \mid f(\omega) - f(\sigma) \mid = 0$$

The definition of the product topology insures that there exists $K_\epsilon \in \mathcal{S}$ such that $\forall \Lambda \in \mathcal{S},\ \Lambda \supset K_\epsilon,\ \omega_\Lambda = \sigma_\Lambda \implies d(\omega, \sigma) < \eta$ which implies, by uniform continuity, $\mid f(\omega) - f(\sigma) \mid < \epsilon$. Hence f is quasilocal. If E is finite, it is a compact metric space and so is Ω. Every uniformly continuous function on Ω is continuous and then every continuous function is quasilocal.
Let us prove now that, when E is finite, every quasilocal function is continuous. The product space Ω is equipped with the product topology of the discrete topology on E and one could prove that

$$\delta(\omega, \sigma) = \sum_{i \in S} 2^{-n(i)} \mathbf{1}_{\{\omega_i \neq \sigma_i\}}$$

where $n: S \longrightarrow \mathbb{N}$ is any bijection, is a metric for the product topology on Ω.

Let f be a quasilocal function on Ω. Lemma (2.1) yields:

$$\lim_{\Lambda \uparrow S} \sup_{\omega,\sigma \in \Omega, \omega_\Lambda = \sigma_\Lambda} \mid f(\omega) - f(\sigma) \mid = 0$$

and then

$$\forall \epsilon > 0, \ \exists K_\epsilon \in \mathcal{S} \text{ s.t } \mathcal{S} \ni \Lambda \supset K_\epsilon \Longrightarrow \sup_{\omega,\sigma \in \Omega, \omega_\Lambda = \sigma_\Lambda} \mid f(\omega) - f(\sigma) \mid < \epsilon$$

Let us fix $\omega \in \Omega$. The previous statement yields

$$\forall \epsilon > 0, \ \exists K_\epsilon \in \mathcal{S} \text{ s.t } \Lambda \supset K_\epsilon$$

$$\Longrightarrow \forall \sigma \in \Omega \text{ s.t } \sigma_\Lambda = \omega_\Lambda, \mid f(\omega) - f(\sigma) \mid < \epsilon$$

and the expression of the product metric δ yields to

$$\exists \eta > 0 \text{ s.t } \delta(\omega, \sigma) < \eta \Longrightarrow \mid f(\omega) - f(\sigma) \mid < \epsilon$$

which proves that f is continuous.
$\diamond$

2.1.3 Interactions and Hamiltonians

Definition 2.3 [potential (or interaction)] A potential Φ is a family $\Phi = (\Phi_A)_{A \in \mathcal{S}}$ of functions indexed by the finite subsets of S

$$\begin{aligned} \Phi_A \ : \ & \Omega \longrightarrow \mathbb{R} \\ & \omega \longmapsto \Phi_A(\omega) \end{aligned}$$

such that $\forall A \in \mathcal{S}$, Φ_A is $\mathcal{F}_A$-measurable.$\diamond$

Definition 2.4 [Hamiltonian with free boundary condition] $\forall \Lambda \in \mathcal{S}$, the map

$$\begin{aligned} \mathbf{H}^\Phi_{\Lambda,f} \ : \ & \Omega \longrightarrow \mathbb{R} \\ & \omega \longmapsto \mathbf{H}^\Phi_{\Lambda,f}(\omega) = \sum_{A \in \mathcal{S}, A \subset \Lambda} \Phi_A(\omega) \end{aligned}$$

is called Hamiltonian at volume Λ with free boundary condition (for the interaction Φ).$\diamond$

Remark 2.1

As the sum involved in the definition of this Hamiltonian has a finite number of finite terms, these objects are always well defined.

Definition 2.5 [Convergent interaction] A potential Φ is said to be *convergent* if the sum

$$\mathbf{H}_{\Lambda}^{\Phi}(\omega) \stackrel{\text{def}}{=} \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset} \Phi_A(\omega)$$

exists $\forall \omega \in \Omega$ and $\forall \Lambda \in \mathcal{S}$. $\diamond$

Remark 2.2

- By the existence of this sum, we mean the convergence of the net $(\sum_{A \in \mathcal{S}, A\cap\Lambda \neq \emptyset, A \subset \Delta} \Phi_A(\omega))_{\Delta \in \mathcal{S}}$ to a finite limit as $\Delta \uparrow \mathcal{S}$. Using cofinal sequences (definition 2.2), one could show that there is no need to fix a sequence of increasing volumes along which the limits have to be taken and it is enough to take Λ along an increasing sequence of a cofinal set. Moreover, those sums can be convergent without being absolutely convergent (see [13, 10]).

- If we do not precise the way this infinite sum is done, $\mathbf{H}_{\Lambda}^{\Phi}$ could be ill-defined. Let us consider the potential Φ defined on $\{-1, +1\}^{\mathbb{Z}}$ by

$$\begin{aligned} \forall \omega \in \Omega,\ \Phi_A(\omega) &= \frac{\omega_i \omega_j}{\mid i - j \mid} \text{ if } A = \{i, j\} \\ &= 0 \text{ otherwise} \end{aligned}$$

Let $\Lambda \in \mathcal{S}$ and define $B = \{i \in \mathbb{Z},\ \omega_i = +1\}$. One could write

$$\sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset} \Phi_A(\omega) = \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset} \Phi_{A'}(\omega) + \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset} \Phi_{A''}(\omega)$$

where $A' = A \cap B$ and $A'' = A \cap B^c$. Here,

$$\sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset} \Phi_{A'}(\omega) = \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset} \Phi_{A''}(\omega) = \sum_{i,j \in \mathbb{Z}} \frac{1}{\mid i - j \mid}$$

and the series are not convergent, whereas $\mathbf{H}_{\Lambda}^{\Phi}$ is well-defined if we use nets as above.

- When the interaction has finite range, as we will define it later, those series are always finite and we do not have to precise the way those infinite sums are taken.

Definition 2.6 [Hamiltonian at volume Λ with boundary condition τ] Given a convergent potential Φ, we can define, $\forall \tau \in \Omega, \forall \Lambda \in \mathcal{S}$, the following Hamiltonian at volume Λ with boundary condition τ,

$$\begin{aligned} \mathbf{H}^{\Phi}_{\Lambda,\tau} \ &: \ \Omega \longrightarrow \mathbb{R} \\ &\omega \longmapsto \mathbf{H}^{\Phi}_{\Lambda,\tau}(\omega) = \mathbf{H}^{\Phi}_{\Lambda}(\omega_{\Lambda}\tau_{\Lambda^c}) \stackrel{\text{def}}{=} \mathbf{H}^{\Phi}_{\Lambda}(\omega \mid \tau) \end{aligned}$$

where $\omega_{\Lambda}\tau_{\Lambda^c}$ is the configuration which agrees with ω in Λ and with τ in Λ^c. $\diamond$

Remark 2.3

A convergent potential is regular enough to define this Hamiltonian at finite volume with boundary condition, but it won't be enough to define the so-called Gibbs specifications. Thus, we need the

Definition 2.7 [Absolutely convergent potential] A potential Φ is said to be absolutely convergent if and only if

$$\forall i \in S, \ \|\Phi\|_i \stackrel{\text{def}}{=} \sum_{A \in \mathcal{S}, A \ni i} \sup_{\omega \in \Omega} \mid \Phi_A(\omega) \mid < +\infty \tag{5}$$

which is equivalent to the condition:

$$\forall \Lambda \in \mathcal{S}, \ \|\Phi\|_{\Lambda} \stackrel{\text{def}}{=} \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset} \sup_{\omega \in \Omega} \mid \Phi_A(\omega) \mid < \infty$$

$\diamond$

Example : finite range potentials

We say that a potential Φ has a finite range if there exists $R \in \mathbb{R}$ such that $\Phi_A \neq 0 \implies \text{diam}A \leq R$ where $\text{diam}A = \sup_{i,j \in S} \|i - j\|$. One could easily check that a finite range potential is absolutely convergent if and only if Φ_A is bounded for all $A \in \mathcal{S}$. A typical example is provided by the 2d-Ising model studied in the third part.

Example : A convergent potential non absolutely convergent

This example comes from one of Sullivan [18]. Let us deal with a slight modification of the one dimensional Ising model: take $\Omega = \{-1, 1\}^{\mathbb{Z}}$ and define a potential Φ such that $\forall A \in \mathcal{S}, \forall \omega \in \Omega$

$$\begin{aligned} \Phi_A(\omega) &= \frac{(-1)^n}{n^2} \text{ iff } \omega_i = +1 \, \forall i \in A = \{k, \cdots, k+n-1\}, k \in \mathbb{Z} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Thus Φ is non null only for the finite sets of adjacent sequences in $\mathbb{Z}$ on which the spins are all $+1$.

We prove now that Φ is a convergent potential, and even more, we prove that it is uniformly convergent in ω. We have to prove that for all $\Lambda \in \mathcal{S}$, the series $\mathbf{H}^{\Phi}_{\Lambda}(\omega) = \sum_{A \cap \Lambda \neq \emptyset, A \in \mathcal{S}} \Phi_A(\omega)$ are uniformly convergent in ω. One could show that it is enough to prove it for $\Lambda = \{0\}$ (see [5]). Thus, following the remark (2.2), we only have to prove that the sequence $(\sum_{A \ni 0, A \subset \Delta_N} \Phi_A(\omega))_{N \in \mathbb{N}}$, with $\Delta_N = [-N, N]$, converges uniformly in ω. Equivalently, we can prove that the sequence $(U_N)_{N \in \mathbb{N}}$ defined by

$$U_N = \sup_{\omega \in \Omega} \mid \sum_{A \ni 0, A \cap \Lambda_N^c \neq \emptyset} \Phi_A(\omega) \mid$$

converges to zero when N goes to infinity. but

$$\sum_{A \ni 0, A \cap \Lambda_N^c \neq \emptyset} \Phi_A(\omega) = \sum_{k > N} \sum_{A \ni 0, |A| = k} \frac{(-1)^k}{k^2} \prod_{i \in A} \mathbf{1}_{\{\omega_i = 1\}}(\omega)$$

where $|A| = \text{card}(A)$. Let $\omega \in \Omega$. We can distinguish three different behaviours.

ω is the "+"-configuration: $\forall i \in \mathbb{Z}$, $\omega_i = +1$.

We have:

$$\sum_{A \ni 0, A \cap \Lambda_N^c \neq \emptyset} \Phi_A(\omega) = \sum_{k > N} (k+1) \frac{(-1)^k}{k^2}$$

At least one sequence of "+" is semi-infinite around the origin:

We shall do it in one direction only, say left:

$$\exists n_0 > 0 \text{ s.t}: \omega_{n_0} = -1 \text{ and } \omega_i = +1 \; \forall i \in \mathbb{Z}, \; i \leq n_0$$

Here,

$$\sum_{A \ni 0, A\cap\Lambda_N^c \neq \emptyset} \Phi_A(\omega) = \sum_{k>N} \min(k+1, n_0)\frac{(-1)^k}{k^2} \leq \sum_{k>N}(k+1)\frac{(-1)^k}{k^2}$$

no sequence of "+" is infinite around the origin: In such a case,

$$\exists n_1 \in \mathbb{Z}^+ \text{ s.t } \omega_{n_1} = -1 \text{ and } \omega_i = +1 \; \forall i \in \mathbb{Z}^+, \; i \leq n_1$$
$$\exists n_2 \in \mathbb{Z}^- \text{ s.t } \omega_{n_2} = -1 \text{ and } \omega_i = +1 \; \forall i \in \mathbb{Z}^-, \; i \geq n_2$$

and then,

$$\forall N \geq \max(n_1, -n_2), \quad \sum_{A \ni 0, A\cap\Lambda_N^c \neq \emptyset} \Phi_A(\omega) = 0$$

If we consider now $N \geq \max(n_1, -n_2)$, we obtain

$$0 \; \leq U_N = \sup_{\omega\in\Omega} \mid \sum_{A \ni 0, A\cap\Lambda_N^c \neq \emptyset} \Phi_A(\omega) \mid \leq \mid \sum_{k>N}(k+1)\frac{(-1)^k}{k^2} \mid$$

and the term on the right is the tail of a well known alternated convergent serie. Thus, this potential is uniformly convergent.

But it is not absolutely convergent:

$$\begin{aligned} \sum_{A \ni 0, A\subset\Delta_N} \sup_{\omega\in\Omega} \mid \Phi_A(\omega) \mid &= \sum_{n=0}^{2N}(n+1) \mid \frac{(-1)^n}{n^2} \mid \\ &= \sum_{n=0}^{2N} \frac{n+1}{n^2} \end{aligned}$$

and this is well known as a non-convergent serie. Thus, this potential is not absolutely convergent[6].

[6]The same kind of examples is, for the same reasons, provided if instead of $\frac{1}{n^2}$ in the definition of the potential, one take any sequence $(c_n)_{n\in\mathbf{N}}$ such that the serie $\sum nc_n$ is divergent and the sequence $(nc_n)_{n\in\mathbf{N}}$ is a decreasing sequence which converges to zero.

Remark 2.4

- In the literature, absolutely convergent is sometimes replaced by absolutely summable ([7, 11, 9, 5]) or uniformly absolutely summable ([1, 13]).
- Requiring for a potential to be absolutely convergent will be enough to define a Gibbsian specification associated with this potential, and then to provide a 'reasonnable' modelization of the physical properties of the system. This actually seems to be too strong, and this strenghening causes the troubles we have in the 3^d part.

2.2 Specifications, quasilocality and Gibbs measures

The main references for this section are again [9, 5].

2.2.1 Specifications

Definition 2.8 [Specification on $(\Omega, \mathcal{F})$] A specification is a family $\gamma = \{\gamma_\Lambda, \Lambda \in \mathcal{S}\}$ of maps

$$\begin{aligned} \gamma_\Lambda \;:\; & \Omega \times \mathcal{F} \longrightarrow [0,1] \\ & (\omega, A) \longmapsto \gamma_\Lambda(\omega, A) \stackrel{\text{def}}{=} \gamma_\Lambda(A|\omega) \end{aligned}$$

which satisfy :

1. $\forall A \in \mathcal{F}, \gamma_\Lambda(A|\cdot)$ is $\mathcal{F}_{\Lambda^c}$-measurable.
2. $\forall \omega \in \Omega, \gamma_\Lambda(\cdot|\omega)$ is a probability measure on $(\Omega, \mathcal{F})$.
3. $\forall B \in \mathcal{F}_{\Lambda^c}\ ,\ \gamma_\Lambda(B|\omega) = \mathbf{1}_B(\omega)$
4. If $\Lambda \subset \Lambda'$ are finite sets, then $\gamma_{\Lambda'}\gamma_\Lambda = \gamma_{\Lambda'}$ where $\gamma_{\Lambda'}\gamma_\Lambda$ is a map on $\Omega \times \mathcal{F}$ defined by

$$\gamma_{\Lambda'}\gamma_\Lambda(\omega, A) = \gamma_{\Lambda'}\gamma_\Lambda(A|\omega) = \int_\Omega \gamma_\Lambda(A|\omega')\gamma_{\Lambda'}(d\omega'|\omega)$$

$\diamond$

Remark 2.5

- In a probabilistic point of vue, it is more natural to use the notation $\gamma_\Lambda(\omega, A)$, considering γ_Λ as a probability kernel, whereas in statistical mechanics, we use $\gamma_\Lambda(A|\omega)$ in order to consider it as a (regular) conditional probability.

- In other words a specification on $(\Omega, \mathcal{F})$ acts as a family of probability kernels from $\mathcal{F}_{\Lambda^c}$ to $\mathcal{F}$, which have the consistency property (4). We underline that because of properness (property 3), a specification γ also satisfies the converse consistency relation

$$\Lambda \subset \Lambda' \implies \gamma_\Lambda \gamma_{\Lambda'} = \gamma_{\Lambda'}$$

Let $\Lambda, \Lambda' \in \mathcal{S}$, $A \in \mathcal{F}$, $\omega \in \Omega$. Property (1) of a specification tells that $f(\cdot) = \gamma_{\Lambda'}(A|\cdot)$ is $\mathcal{F}_{\Lambda'^c}$-measurable, so it is $\mathcal{F}_{\Lambda^c}$-measurable because $\mathcal{F}_{\Lambda'^c} \subset \mathcal{F}_{\Lambda^c}$. It is also positive, then $\forall \omega \in \Omega$, $f(\omega) = \lim_{n\to\infty} \mathbf{1}_{A_n}(\omega)$, with, $\forall n \in \mathbb{N}$, $A_n \subset A_{n+1}$ and $A_n \in \mathcal{F}_{\Lambda^c}$. Using the Beppo-Levy property in the integration of any positive measurable function, we obtain

$$\begin{aligned}
\gamma_\Lambda \gamma_{\Lambda'}(A|\omega) &= \lim_{n\to\infty} \int_\Omega \mathbf{1}_{A_n}(\omega')\gamma_\Lambda(d\omega'|\omega) \\
&= \lim_{n\to\infty} \gamma_\Lambda(A_n|\omega) \\
&= \lim_{n\to\infty} \mathbf{1}_{A_n}(\omega) \quad \text{by properness} \\
&= f(\omega) = \gamma_{\Lambda'}(A|\omega)
\end{aligned}$$

- As described below, these objects are defined in order to specify some versions of conditionnal probabilities of a probability measure μ.
Let $\mu \in \mathcal{M}_1^+(\Omega, \mathcal{F})$, the set of all probability measures on $(\Omega, \mathcal{F})$, and assume[7] it is possible to define regular versions of conditional probabilities, with respect to the σ-algebra $\mathcal{F}_{\Lambda^c}$ for Λ finite subset of S.
$\mu[\cdot \mid \mathcal{F}_{\Lambda^c}]$ is defined μ-a.s[8] on $(\Omega, \mathcal{F})$ by the equalities :

$$\forall A \in \mathcal{F},\ \mu[A \mid \mathcal{F}_{\Lambda^c}](\cdot) = \mathbb{E}_\mu[\mathbf{1}_A \mid \mathcal{F}_{\Lambda^c}](\cdot)\ \mu - a.s$$

The well-known properties of the conditionnal expectation with respect to σ-algebras allow us to give the following for $\mu[\cdot \mid \mathcal{F}_{\Lambda^c}]$:

[7] Because our spaces are nice, this is always possible (see [17]).

[8] μ-a.s means that the claim is true on a set of μ-measure 1. My references in probability theory and integration are [15, 6].

1. $\forall A \in \mathcal{F}_{\Lambda^c}$, $\mu[A|\mathcal{F}_{\Lambda^c}](\omega)$ is $\mathcal{F}_{\Lambda^c}$-measurable for μ-almost every (ω).
2. μ-a.s(ω), $\mu[\cdot|\mathcal{F}_{\Lambda^c}](\omega)$ is a probability measure on $(\Omega, \mathcal{F})$ [9].
3. μ-a.s(ω),

$$\forall B \in \mathcal{F}_{\Lambda^c},\ \mu[B|\mathcal{F}_{\Lambda^c}](\omega) = \mathbb{E}_\mu[\mathbf{1}_B|\mathcal{F}_{\Lambda^c}](\omega) = \mathbf{1}_B(\omega)$$

   because $\mathbf{1}_B$ is $\mathcal{F}_{\Lambda^c}$-measurable.
4. if $\Lambda' \subset \Lambda$, then $\mathcal{F}_{\Lambda^c} \subset \mathcal{F}_{\Lambda'^c}$ and μ-a.s(ω),

$$\forall A \in \mathcal{F}, \mu[\mu[A|\mathcal{F}_{\Lambda^c}]|\mathcal{F}_{\Lambda'^c}](\omega) = \mu[A|\mathcal{F}_{\Lambda'^c}](\omega)$$

It appears that specifications are good objects to describe conditionnal probabilities, with the important objection that they are defined everywhere on Ω, for the convenient reason that we want to specify μ everywhere and not μ-a.s.

- $\forall \sigma, \omega \in \Omega, \forall \Lambda \in \mathcal{S}, \gamma_\Lambda(\sigma|\omega)$ depends only on σ_Λ and ω_{Λ^c}.

Definition 2.9 [measure consistent with a specification] Let γ be a specification on $(\Omega, \mathcal{F})$. The set

$$\mathcal{G}(\gamma) \stackrel{\text{def}}{=} \{\mu \in \mathcal{M}_1^+(\Omega, \mathcal{F}) : \forall A \in \mathcal{F}, \forall \Lambda \in \mathcal{S}, \mu[A \mid \mathcal{F}_{\Lambda^c}](\cdot) = \gamma_\Lambda(A \mid \cdot)\ \mu - a.s\} \tag{6}$$

is the set of the probability measures *specified* by γ, or *consistent* with γ. $\diamond$

Remark 2.6

This definition reminds the Kolmogorov compatibility condition for the existence of a probability measure on an infinite product polish space. Instead of dealing with the family of marginals of this measure, we deal with its system of conditional probabilities. The Kolmogorov compatibility yields to the existence and unicity of such a measure, whereas $\mathcal{G}(\gamma)$ could have a lot of different structure. We provide here few examples:

A specification for which $\mathcal{G}(\gamma) = \{\mu\}$: This is the Gibbsian description of a reversible Markov chain on the integers. For more details about

[9] Because of the remark in the footnote 7.

this description, one could consult [9]. Let $\Omega = \{-1, +1\}^{\mathbb{Z}}$ and consider a stochastic matrix

$$M = \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}$$

with $p > 0,\ q > 0$ such that M is irreducible and aperiodic. Thus it is an ergodic Markov chain [10]. Hence, we have the following properties:

$$\exists\,!\ \nu \in \mathcal{M}_1^+(E, \mathcal{E}) \text{ s.t } \nu M = M$$

and

$$\exists\,!\ \mu \in \mathcal{M}_1^+(\Omega, \mathcal{F}) \text{ s.t } \forall \omega \in \Omega,\ \forall k \in \mathbb{N},\ \forall i_1, \cdots, i_k\ \in \mathbb{N}$$

$$\mu[(\omega_{i_1}, \cdots, \omega_{i_k})] = \nu(\omega_{i_1}) M^{i_2 - i_1}(\omega_{i_1}, \omega_{i_2}). \cdots . M^{i_k - i_{k-1}}(\omega_{i_{k-1}}, \omega_{i_k})$$

We also have the property

$$\forall j, k \in E,\ \lim_{n \to \infty} M^n(j,k) = \nu(k) > 0$$

Let $X = (X_n)_{n \in \mathbb{N}}$ be a sequence of random variables on $(\{-1, +1\}^{\mathbb{N}}, \mathcal{E}^{\otimes \mathbb{N}})$ of law μ:

$$\forall \omega \in \Omega,\ \forall k \in \mathbb{N},\ \forall i_1, \cdots, i_k\ \in \mathbb{N}$$

$$\begin{aligned} m[X_{i_1}(\omega) = \omega_{i_1}, \cdots, X_{i_k}(\omega) = \omega_{i_k}] &= \mu[(\omega_{i_1}, \cdots, \omega_{i_k})] \\ &= \nu(\omega_{i_1}) M^{i_2 - i_1}(\omega_{i_1}, \omega_{i_2}). \cdots . M^{i_k - i_{k-1}}(\omega_{i_{k-1}}, \omega_{i_k}) \end{aligned}$$

This sequence has the Markov property: $\forall k \in \mathbb{N},\ \forall i, j, \epsilon_{k-1}, \cdots, \epsilon_0\ \in E$

$$\begin{aligned} m[X_{k+1} = i | X_k = j, \cdots, X_0 = \epsilon_0] &= \mu[(\omega_{i_{k+1}} = i | \omega_{i_k} = j, \cdots, \omega_0 = \epsilon_0] \\ &= \mu[(\omega_{i_{k+1}} = i | \omega_{i_k} = j] = M(j, i) \end{aligned}$$

but, because it is ergodic, this Markov chain is reversible and $\forall k \in \mathbb{N},\ \forall l \in \mathbb{N},\ \forall i, j, \epsilon_{k+21}, \cdots, \epsilon_{K+l}\ \in E$

$$\begin{aligned} & m[X_k = i | X_{k+1} = j, \cdots, X_{k+l} = \epsilon_{k+l}] \\ = \ & \mu[(\omega_{i_k} = i | \omega_{i_{k+1}} = j, \cdots, \omega_{k+l} = \epsilon_{k+l}] \end{aligned}$$

[10] For more details about Markov chains, see [6].

$$
\begin{aligned}
&= \frac{\mu[(\omega_{i_k} = i, \omega_{i_{k+1}} = j, \cdots, \omega_{k+l} = \epsilon_{k+l}]}{\mu[\omega_{i_{k+1}} = j, \cdots, \omega_{k+l} = \epsilon_{k+l}]} \\
&= \frac{\nu(i) M(i,j). \cdots .M(\epsilon_{k+l-1}, \epsilon_{k+l})}{\nu(j) M(j, \epsilon_{k+2}). \cdots .M(\epsilon_{k+l-1}, \epsilon_{k+l})} \\
&= \frac{\nu(i) M(i,j)}{\nu(j)} = N(j,i)
\end{aligned}
$$

and N is a stochastic matrix associated to the reverse chain. Hence, we can extend this chain on $\Omega = \{-1,+1\}^{\mathbb{Z}}$ and it is still ergodic. We shall now define a specification γ such that $\mu \in \mathcal{G}(\gamma)$. We have to compute $\mu[\sigma|\sigma_{\Lambda^c} = \omega_{\Lambda^c}]$, $\forall \sigma \in \Omega$, $\forall \omega \in \Omega$, $\forall \Lambda \in \mathcal{S}$ but it is enough to construct it for all the finite subsets of $\mathcal{S}$ in the cofinal sequence of cubes, i.e $\forall n \in \mathbb{N}$ and $\Lambda_n = [-n, n]$. Let $n \in \mathbb{N}$ and $\sigma \in \Omega$

$$
\begin{aligned}
& \mu[\sigma_{\Lambda_n}|\sigma_{\Lambda_n^c} = \omega_{\Lambda_n^c}] \\
&= \frac{\mu[\omega_{]-\infty,-n-1]}\sigma_{\Lambda_n}\omega_{[n+1,\infty[}]}{\mu[\omega_{]-\infty,-n-1]}\omega_{[n+1,\infty[}]} \\
&= \frac{\mu[\omega_{-n-1}\sigma_{\Lambda_n}\omega_{n+1}]}{\mu[\omega_{-n-1}\omega_{n+1}]} \\
&= \frac{\nu(\omega_{-n-1}) M(\omega_{-n-1}, \sigma_{-n}) \bullet \cdots \bullet M(\sigma_n, \omega_{n+1})}{\nu(\omega_{-n-1}) M^{2n+2}(\omega_{-n-1}, \omega_{n+1})}
\end{aligned}
$$

If we define then

$$
\mathbf{Z}_{\Lambda_n}(\omega) = M^{2n+2}(\omega_{-n-1}, \omega_{n+1})
$$

and

$$
\gamma_{\Lambda_n}(\sigma|\omega) = \frac{1}{\mathbf{Z}_{\Lambda_n}(\omega)} M(\omega_{-n-1}, \sigma_{-n}) \bullet \cdots \bullet M(\sigma_n, \omega_{n+1})
$$

we define a specification γ such that $\mu \in \mathcal{G}(\gamma)$.
Now, $\mathcal{G}(\gamma)$ is not empty and take any $\mu' \in \mathcal{G}(\gamma)$ and prove that $\mu' = \mu$. Let us remark that μ' is Markovian by construction. The ergodicty of this chain prove that μ is the only probability measure on $(\Omega, \mathcal{F})$ such that

$$
\forall x, y \in \mathbb{Z}, \ \lim_{n \to \infty} M^n(x,y) \ = \ \mu(y)
$$

As they are probability measures on an infinite product space, it is enough to prove $\forall\omega \in \Omega$, $\forall k \in \mathbb{N}$, $\forall i_1, \cdots, i_k \in \mathbb{N}$,

$$\mu[(\omega_{i_1}, \cdots, \omega_{i_k})] = \mu'[(\omega_{i_1}, \cdots, \omega_{i_k})]$$

Let us prove it first for the one dimensionnal cylinder, i.e for $x \in E$, let us prove that $\mu'[\sigma_0 = x] = \mu[\sigma_0 = x]$. Because μ' is Markovian, we have $\forall n \in \mathbb{N}$

$$\mu'[\sigma_0 = x] = \sum_{a\in E, b\in E} \mu'[\sigma_0 = x|\sigma_{-n-1} = a, \sigma_{n+1} = b]\mu'[\sigma_{-n-1} = a, \sigma_{n+1} = b]$$

but $\forall a \in E$, $\forall b \in E$,

$$\mu'[\sigma_0 = x|\sigma_{-n-1} = a, \sigma_{n+1} = b] = \sum_{a\in E, b\in E} \frac{M^{n+1}(a,x)M^{n+1}(x,b)}{M^{2n+2}(a,b)}$$

and using ergodicity, one obtain $\forall a \in E$, $\forall a \in E$,

$$\lim_{n\to\infty} \mu'[\sigma_0 = x|\sigma_{-n-1} = a, \sigma_{n+1} = b] = \frac{\nu(x)\nu(b)}{\nu(b)}$$

and then $\mu'[\sigma_0 = x] = \nu(x) = \mu[\sigma_0 = x]$.
We obtain the equality of those measures on the other cylinders in the same way and $\mu' = \mu$.

◇

An example where $\mathcal{G}(\gamma)$ is empty: It has been provided by Spitzer and deals with random walks on $\mathbb{Z}$. This description comes from [2].
Let Y be a random walk on $(\Omega, \mathcal{F})$ and define a specification γ by its definition on cofinal sequence: $\forall n \in \mathbb{N}, \forall \sigma, \omega \in \Omega$

$$\gamma_{\Lambda_n}(\sigma_{\Lambda_n}|\omega_{\Lambda_n^c}) = m_{\Lambda_n}[Y_{\Lambda_n} = \sigma_{\Lambda_n}|Y_{-n-1} = \omega_{-n-1}, Y_{n+1} = \omega_{n+1}]$$

Assume now that there exists $\mu \in \mathcal{G}(\gamma)$ and define $\forall n \in \mathbb{N}$, $S_n = Y_0 - Y_{-n}$. Then S_n follows a law $\mathcal{B}(n, \frac{1}{2})$ and

$$\forall \epsilon > 0, \forall k \in \mathbb{Z}, \ \exists n \text{ s.t } m_{\Lambda_n}[S_n = k] \leq \epsilon$$

thus

$$\forall k, l \in \mathbb{Z}, \ \mu[Y_0 = k, Y_{-n} = l] \leq \epsilon$$

and then we must have

$$\mu[Y_0 = k] \leq \epsilon$$

Thus, μ can not be a probability measure and $\mathcal{G}(\gamma)$ is empty.

$\diamond$

A specification for which $\mathcal{G}(\gamma)$ is not empty, neither a singleton: It will be provided in the next part, where we will study the two-dimensionnal Ising-model at low temperature.

2.2.2 Quasilocality and continuity for specifications

There is a canonical action of the elements of a specification on functions and measures.
Let γ be a specification on $(\Omega, \mathcal{F})$

Definition 2.10 [Action on functions] Let f be any measurable function on $(\Omega, \mathcal{F})$

$$f \ : \ \Omega \longrightarrow \mathbf{R}$$
$$\sigma \longmapsto f(\sigma)$$

$\forall \Lambda \in \mathcal{S}$, we define

$$\gamma_\Lambda f \ : \ \Omega \longrightarrow \mathbf{R}$$
$$\omega \longmapsto \gamma_\Lambda f(\omega)$$

with

$$\gamma_\Lambda f(\omega) = \int_\Omega f(\sigma)\gamma_\Lambda(d\sigma|\omega)$$

$\diamond$

Definition 2.11 [Action on measures] Let μ be any measure on $(\Omega, \mathcal{F})$
$\forall \Lambda \in \mathcal{S}$, $\mu\gamma_\Lambda$ is a measure on $(\Omega, \mathcal{F})$ defined by :

$$\forall A \in \mathcal{F},\ \mu\gamma_\Lambda(A) = \int_\Omega \gamma_\Lambda(A|\omega)\mu(d\omega)$$

$\diamond$

Remark 2.7

- If μ is a *probability* measure, so is $\mu\gamma_\Lambda$ because γ_Λ is a *probability* kernel.

- Thus, the product of two elements (i.e two probability kernels) of a specifications $\gamma_{\Lambda'}\gamma_\Lambda$ is, for $\omega \in \Omega$, the action of γ_Λ on the measure $\gamma_{\Lambda'}(d\omega'|\omega)$ and could be seen as well as the action of $\gamma_{\Lambda'}$ on the measurable map $\gamma_\Lambda(A|\cdot)$

we can formulate the

Lemma 2.3 *:*
Let γ be a specification on $(\Omega, \mathcal{F})$ and μ a measure on $(\Omega, \mathcal{F})$. The following characterizations of consistency holds:

1.

$$\mu \in \mathcal{G}(\gamma) \iff \forall \Lambda \in \mathcal{S},\ \mu = \mu\gamma_\Lambda$$

2. $\mu \in \mathcal{G}(\gamma) \iff$ *There is a cofinal subset $\mathcal{S}_0$ of $\mathcal{S}$ s.t*

$$\forall \Lambda \in \mathcal{S}_0,\quad \mu = \mu\gamma_\Lambda$$

$\diamond$

Proof:

1. Let μ be consistent with γ. Definition (2.9) yields

$$\forall A \in \mathcal{F}, \forall \Lambda \in \mathcal{S}, \mu[A|\mathcal{F}_{\Lambda^c}](\cdot) = \gamma_\Lambda(A|\cdot)\ \mu - a.s$$

Let us compute what it means, using the definition of conditionnal probabilities.

$$\begin{aligned}
\forall B \in \mathcal{F}_{\Lambda^c},\ \int_B \mu[A|\mathcal{F}_{\Lambda^c}](\omega)\mu(d\omega) &= \int_B \mathbb{E}_\mu[\mathbf{1}_A|\mathcal{F}_{\Lambda^c}](\omega)\mu(d\omega) \\
&= \int_B \mathbf{1}_A(\omega)\mu(d\omega) \\
&= \int_\Omega \mathbf{1}_A(\omega)\mathbf{1}_B(\omega)\mu(d\omega) \\
&= \mu(A \cap B)
\end{aligned}$$

By consistency, this is also equal to

$$\int_B \gamma_\Lambda(A|\omega)\mu(d\omega) = \int_\Omega \mathbf{1}_B(\omega)\gamma_\Lambda(A|\omega)\mu(d\omega)$$

Now let us prove

$$\forall\omega \in \Omega,\ \forall A \in \mathcal{F},\ \forall B \in \mathcal{F}_{\Lambda^c},\ \mathbf{1}_B(\omega)\gamma_\Lambda(A|\omega) = \gamma_\Lambda(A \cap B|\omega) \tag{7}$$

Let $\omega \in \Omega$. By definition of a specification, $\gamma(\cdot|\omega)$ is a probability measure and then

$$\gamma_\Lambda(A \cap B|\omega) \leq \inf(\gamma_\Lambda(A|\omega), \gamma_\Lambda(B|\omega)) = \gamma_\Lambda(A|\omega)\mathbf{1}_B(\omega) \tag{8}$$

by property 3 of a specification (properness). Similarily,

$$\gamma_\Lambda(A \cap B^c|\omega) \leq \gamma_\Lambda(A|\omega)\mathbf{1}_{B^c}(\omega) \tag{9}$$

Now, the equality

$$\gamma_\Lambda(A \cap B|\omega) + \gamma_\Lambda(A \cap B^c|\omega) = \gamma_\Lambda(A|\omega)\mathbf{1}_B(\omega) + \gamma_\Lambda(A|\omega)\mathbf{1}_{B^c}(\omega)$$

proves that the inequalities (8) and (9) are equalities, thus (7) is proved and we obtain, $\forall A \in \mathcal{F}$, $\forall B \in \mathcal{F}_{\Lambda^c}$

$$\mu(A \cap B) = \int_\Omega \gamma_\Lambda(A \cap B|\omega)\mu(d\omega) = \mu\gamma_\Lambda(A \cap B)$$

Taking $B = \Omega \in \mathcal{F}_{\Lambda^c}$, we obtain $\mu = \mu\gamma_\Lambda$ as probability measures on $(\Omega, \mathcal{F})$.
The converse statement

$$\mu = \mu\gamma_\Lambda \Longrightarrow \mu \in \mathcal{G}(\gamma)$$

follows in the same way.

2. We only have to prove that if there exists a cofinal set $\mathcal{S}_0$ of $\mathcal{S}$ with $\mu = \mu\gamma_\Lambda$ $\forall\Lambda \in \mathcal{S}_0$, then $\mu = \mu\gamma_\Lambda$ holds $\forall\Lambda \in \mathcal{S}$. This follows directly from the definition of a cofinal set:

$$\forall\Lambda \in \mathcal{S},\ \exists\Delta \in \mathcal{S}_0 \text{ s.t } \Lambda \subset \Delta$$

and using the consistency property of any specification, one obtain:

$$\mu\gamma_\Lambda = \mu\gamma_\Delta\gamma_\Lambda = \mu\gamma_\Delta = \mu$$

The existence of a cofinal set is insured by the remark following the definition (2.2). ◇

Definition 2.12 [Quasilocal specification] A specification γ is said to be *quasilocal* if and only if, for all Λ in $\mathcal{S}$, for each (bounded) local function f, $\gamma_\Lambda f$ is (bounded) and quasilocal. $\diamond$

Lemma 2.4 *If γ is quasilocal, then for each (bounded) quasilocal function f, $\gamma_\Lambda f$ is (bounded) and quasilocal*[11].

Proof:
Let γ a quasilocal specification, $\Lambda \in \mathcal{S}$ and f be any quasilocal function on $(\Omega, \mathcal{F})$. f has the following property:

$$\forall \epsilon > 0, \ \exists g \text{ local s.t } \|f - g\|_\infty < \frac{\epsilon}{2}$$

As γ is a quasilocal specification and g a local function, $\gamma_\Lambda g$ is a quasilocal function. Hence,

$$\forall \epsilon > 0, \ \exists h \text{ local s.t } \|\gamma_\Lambda g - h\|_\infty < \frac{\epsilon}{2}$$

and then

$$\begin{aligned} \|\gamma_\Lambda f - h\|_\infty &\leq \|\gamma_\Lambda f - \gamma_\Lambda g\|_\infty + \|\gamma_\Lambda g - h\|_\infty \\ &< \|\gamma_\Lambda f - \gamma_\Lambda g\|_\infty + \frac{\epsilon}{2} \end{aligned}$$

But, $\forall \omega \in \Omega$

$$\begin{aligned} | \gamma_\Lambda f(\omega) - \gamma_\Lambda g(\omega) | &\leq \int_\Omega | f(\sigma) - g(\sigma) | \gamma_\Lambda(d\sigma|\omega) \\ &\leq \|f - g\|_\infty \int_\Omega \gamma_\Lambda(d\sigma|\omega) \\ &< \frac{\epsilon}{2} \end{aligned}$$

Thus, for all f quasilocal

$$\forall \epsilon > 0, \ \exists h \text{ local s.t } \|\gamma_\Lambda f - h\|_\infty < \epsilon$$

which proves that $\gamma_\Lambda f$ is quasilocal.
$\diamond$

[11] When E is discrete, all the local, quasilocal and continuous functions are bounded.

Remark 2.8

A typical example of quasilocal specification is given by Gibbsian specification, defined in the next section. The relationship between Gibbsian and quasilocal specifications is studied more precisely in the so-called Gibbs representation theorem (theorem (3.1)).

The following lemma establishes the equivalence between continuity and quasilocality for a finite single-spin set E, and is proved after the lemmata (2.2) and (2.4).

Lemma 2.5 *γ is quasilocal if and only if*

$$\forall f \, quasilocal, \forall \Lambda \in \mathcal{S}, \lim_{\Lambda' \in \mathcal{S}} \sup_{\omega,\sigma \in \Omega, \omega_{\Lambda'}=\sigma_{\Lambda'}} \mid (\gamma_\Lambda f)(\omega) - (\gamma_\Lambda f)(\sigma) \mid = 0$$

$\diamond$

2.2.3 Gibbs specification - Gibbs states

Definition 2.13 [partition function] Let Φ be a potential, ω a configuration in Ω, and let $\beta > 0$. If it exists, we call *partition function* at inverse temperature β, at volume Λ, with potential Φ and boundary condition ω, and we note it $\mathbf{Z}_\Lambda^{\beta\Phi}(\omega)$, the integral

$$\mathbf{Z}_\Lambda^{\beta\Phi}(\omega) = \int_\Omega \exp(-\beta \mathbf{H}_\Lambda^\Phi(\sigma)) m_\Lambda \otimes \delta_{\omega_{\Lambda^c}}^{\otimes \Lambda^c}(d\sigma) = \int_\Omega \exp(-\beta \mathbf{H}_\Lambda^\Phi(\sigma)) \kappa_\Lambda(d\sigma)$$

where we note[12] κ_Λ the product measure $m_\Lambda \otimes \delta_{\omega_{\Lambda^c}}^{\otimes \Lambda^c}$ on $(\Omega, \mathcal{F})$, where δ_x is the Dirac measure on $x \in E$ and $m_\Lambda = m_0^{\otimes \Lambda}$. $\diamond$

Remark 2.9

In order to highlight the role of the boundary condition ω, one could use the following expression for the partition function:

$$\mathbf{Z}_\Lambda^{\beta\Phi}(\omega) = \int_{\Omega_\Lambda} \exp(-\beta \mathbf{H}_\Lambda^\Phi(\sigma|\omega)) m_\Lambda(d\sigma_\Lambda)$$

[12] We underline here that κ_Λ is a measure on $(\Omega, \mathcal{F})$ which depends of ω_{Λ^c}.

Lemma 2.6 *If* Φ *is an* absolutely convergent *potential, then* $\forall \sigma \in \Omega, \forall \omega \in \Omega, \forall \Lambda \in \mathcal{S}$, $\mathbf{H}_\Lambda^\Phi(\sigma|\omega)$ *is bounded.* ◇

Proof:
By definition (2.6),

$$\mathbf{H}_\Lambda^\Phi(\sigma|\omega) = \sum_{A \in \mathcal{S}, A\cap\Lambda \neq \emptyset} \Phi_A(\sigma_\Lambda \omega_{\Lambda^c})$$

Thus $\forall \sigma \in \Omega, \forall \omega \in \Omega, \forall \Lambda \in \mathcal{S}$

$$\begin{aligned} \mid \mathbf{H}_\Lambda^\Phi(\sigma|\omega) \mid &\leq \sum_{A \in \mathcal{S}, A\cap\Lambda \neq \emptyset} \sup_{\omega\in\Omega} \mid \Phi_A(\omega) \mid \\ &\leq \sup_{\Lambda \in \mathcal{S}} \|\Phi\|_\Lambda \; < \; \infty \end{aligned}$$

and the lemma is proved.

◇

As Λ is finite, so is Ω_Λ. Then, as $\mathbf{H}_\Lambda^\Phi(\sigma|\omega)$ is finite for all $\sigma \in \Omega$, the partition function is always finite for an absolutely convergent potential and we can always give the following

Definition 2.14 [Gibbs distribution at volume Λ] Assume that Φ is an absolutely convergent potential. For $\Lambda \in \mathcal{S}$ and $\omega \in \Omega$, we call *Gibbs distribution at volume* Λ, with potential Φ, inverse temperature β and boundary condition ω, the probability measure $\gamma_\Lambda^{\beta\Phi}(\cdot|\omega)$ on $(\Omega, \mathcal{F})$ defined by:

$$\forall A \in \mathcal{F}, \gamma_\Lambda^{\beta\Phi}(A|\omega) = \frac{1}{\mathbf{Z}_\Lambda^{\beta\Phi}(\omega)} \int_\Omega \mathbf{1}_A(\sigma) \exp(-\beta \mathbf{H}_\Lambda^\Phi(\sigma)) \kappa_\Lambda(d\sigma) \tag{10}$$

where κ_Λ still denotes the product measure $m_\Lambda \otimes \delta_{\omega_{\Lambda^c}}^{\otimes \Lambda^c}$ on $(\Omega, \mathcal{F})$. ◇

Remark 2.10

As pointed out in the previous remark, one could write, in order to underline the boundary condition,

$$\gamma_\Lambda^{\beta\Phi}(A|\omega) = \frac{1}{\mathbf{Z}_\Lambda^{\beta\Phi}(\omega)} \int_{\Omega_\Lambda} \mathbf{1}_A(\sigma_\Lambda \omega_{\Lambda^c}) \exp(-\beta \mathbf{H}_\Lambda^\Phi(\sigma|\omega)) m_\Lambda(d\sigma_\Lambda)$$

We have the following lemma. The proof comes from [16]

Lemma 2.7 *Assume again that Φ is an absolutely convergent potential.*

1. *$\gamma^{\beta\Phi} = (\gamma_\Lambda^{\beta\Phi})_{\Lambda\in\mathcal{S}}$ is a specification on $(\Omega, \mathcal{F})$, called a* Gibbs specification.

2. *It is a* quasilocal specification.

◇

Proof:

1. Let Φ be any absolutely convergent potential. We know by the lemma (2.6) that the partition function at finite volume Λ exists and is finite. Let us prove first that the maps $\gamma_\Lambda^{\beta\Phi}$ are well defined $\forall\beta > 0$ and $\forall\Lambda \in \mathcal{S}$. Let $A \in \mathcal{F}$ and $\omega \in \Omega$ and define a measurable function f_Λ on $(\Omega, \mathcal{F})$ by

$$f_\Lambda(\sigma) = \frac{1}{\mathbf{Z}_\Lambda^{\beta\Phi}(\sigma)} \exp(-\beta \mathbf{H}_\Lambda^\Phi(\sigma))$$

and we have

$$\begin{aligned} \gamma_\Lambda^{\beta\Phi}(A|\omega) &= \int_\Omega \mathbf{1}_A(\sigma) f_\Lambda(\sigma) \kappa_\Lambda(d\sigma) \\ &= \int_{\Omega_\Lambda} \mathbf{1}_A(\sigma_\Lambda \omega_{\Lambda^c}) f_\Lambda(\sigma_\Lambda \omega_{\Lambda^c}) m_\Lambda(d\sigma_\Lambda) \end{aligned}$$

Using the proof of the lemma (2.6), we obtain

$$\forall A \in \mathcal{F},\ \forall\omega \in \Omega,\ 0 < \gamma_\Lambda^{\beta\Phi}(A|\omega) \leq 1$$

Hence, the maps $\gamma_\Lambda^{\beta\Phi}$ are well defined $\forall\beta > 0$ and $\forall\Lambda \in \mathcal{S}$. Let $\omega \in \Omega$. Let us study the map

$$\begin{aligned} \gamma_\Lambda(\cdot|\omega) \ :\ & \mathcal{F} \longrightarrow [0,1] \\ & A \longmapsto \gamma_\Lambda(A|\omega) \end{aligned}$$

$\forall A \in \mathcal{F}$, we can write it

$$\gamma_\Lambda(A|\omega) = \int_A f_\Lambda(\sigma) \kappa_\Lambda(d\sigma)$$

The function f_Λ is a positive measurable function on $(\Omega, \mathcal{F})$. The well-known properties of the integration of any positive measurable function with respect to a positive measure insure us that the map $\gamma_\Lambda^{\beta\Phi}(\cdot|\omega)$ is a positive measure on $(\Omega, \mathcal{F})$, and the normalisation by the partition function yields to a probability measure. It also insure[13] that $\forall A \in \mathcal{F}$ the map $\gamma_\Lambda^{\beta\Phi}(A|\cdot)$ is a measurable function on $(\Omega, \mathcal{F})$. Thus, the items (1) and (2) in the definition of a specification are checked.
Let $B \in \mathcal{F}_{\Lambda^c}$. $\forall \sigma, \omega \in \Omega$, $\mathbf{1}_B(\sigma_\Lambda \omega_{\Lambda^c})$ is independant of σ and

$$\mathbf{1}_B(\sigma_\Lambda \omega_{\Lambda^c}) = \mathbf{1}_B(\omega_\Lambda \omega_{\Lambda^c}) = \mathbf{1}_B(\omega)$$

and then, $\forall \omega \in \Omega$

$$\begin{aligned}
\gamma_\Lambda^{\beta\Phi}(B|\omega) &= \frac{1}{\mathbf{Z}_\Lambda^{\beta\Phi}(\omega)} \int_{\Omega_\Lambda} \mathbf{1}_B(\sigma_\Lambda \omega_{\Lambda^c}) \exp(-\beta \mathbf{H}_\Lambda^\Phi(\sigma|\omega)) m_\Lambda(d\sigma_\Lambda) \\
&= \frac{1}{\mathbf{Z}_\Lambda^{\beta\Phi}(\omega)} \int_{\Omega_\Lambda} \mathbf{1}_B(\omega_\Lambda \omega_{\Lambda^c}) \exp(-\beta \mathbf{H}_\Lambda^\Phi(\sigma|\omega)) m_\Lambda(d\sigma_\Lambda) \\
&= \frac{\mathbf{1}_B(\omega}{\mathbf{Z}_\Lambda^{\beta\Phi}(\omega)} \int_{\Omega_\Lambda} \exp(-\beta \mathbf{H}_\Lambda^\Phi(\sigma|\omega)) m_\Lambda(d\sigma_\Lambda) \\
&= \mathbf{1}_B(\omega)
\end{aligned}$$

and item (3) is also true.
Let us check item (4). We will now assume, without any loss of generality, that $\beta = 1$. Let $\Lambda, \Lambda' \in \mathcal{S}$ such that $\Lambda \subset \Lambda'$ and let $A \in \mathcal{F}$ and $\omega \in \Omega$. We want to prove

$$\gamma_{\Lambda'}(A|\omega) = \gamma_{\Lambda'}\gamma_\Lambda(A|\omega)$$

where

$$\gamma_{\Lambda'}(A|\omega) = \int_{\Omega_{\Lambda'}} \mathbf{1}_A(\tau_{\Lambda'} \omega_{\Lambda'^c}) f_{\Lambda'}(\tau_{\Lambda'} \omega_{\Lambda'^c}) m_{\Lambda'}(d\tau_{\Lambda'})$$

and

$$\gamma_{\Lambda'}\gamma_\Lambda(A|\omega) = \int_\Omega \gamma_\Lambda(A|\tau) \gamma_{\Lambda'}(d\tau|\omega)$$

$$= \int_{\Omega_{\Lambda'}} (\int_{\Omega_\Lambda} \mathbf{1}_A(\sigma_\Lambda \tau_{\Lambda' \setminus \Lambda} \omega_{\Lambda'^c}) f_\Lambda(\sigma_\Lambda \tau_{\Lambda' \setminus \Lambda} \omega_{\Lambda'^c}) d\sigma_\Lambda) f_{\Lambda'}(\tau_{\Lambda'} \omega_{\Lambda'^c}) d\tau_{\Lambda'}$$

[13] For more details on integration and measure theory, one could consult [15, 6].

where we have written $d\sigma_\Lambda$ instead of $m_\Lambda(d\sigma_\Lambda)$. Let us compute what $\gamma_{\Lambda'}\gamma_\Lambda(A|\omega)$ is.

$$\gamma_{\Lambda'}\gamma_\Lambda(A|\omega) = \int_{\Omega_{\Lambda'\setminus\Lambda}} g_{\Lambda,\Lambda'}(\tau_{\Lambda'\setminus\Lambda})d\tau_{\Lambda'\setminus\Lambda}$$

where

$$g_{\Lambda,\Lambda'}(\tau_{\Lambda'\setminus\Lambda})$$

$$\begin{aligned}
&= \int_{\Omega_\Lambda} f_{\Lambda'}(\tau_\Lambda\tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c})\left(\int_{\Omega_\Lambda} \mathbf{1}_A(\sigma_\Lambda\tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c})f_\Lambda(\sigma_\Lambda\tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c})d\sigma_\Lambda\right)d\tau_\Lambda \\
&= \int_{\Omega_\Lambda\times\Omega_\Lambda} \mathbf{1}_A(\sigma_\Lambda\tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c})f_\Lambda(\sigma_\Lambda\tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c})f_{\Lambda'}(\tau_\Lambda\tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c})d\sigma_\Lambda d\tau_\Lambda \\
&= \int_{\Omega_\Lambda\times\Omega_\Lambda} \mathbf{1}_A(\tau_\Lambda\tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c})f_\Lambda(\tau_\Lambda\tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c})f_{\Lambda'}(\sigma_\Lambda\tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c})d\sigma_\Lambda d\tau_\Lambda \qquad (11)
\end{aligned}$$

using the trivial change of variable $\phi(\sigma_\Lambda,\tau_\Lambda) = (\tau_\Lambda,\sigma_\Lambda)$. Now, let us recall that

$$f_\Lambda(\sigma) = \frac{1}{\mathbf{Z}_\Lambda^\Phi(\sigma)}\exp(-\mathbf{H}_\Lambda^\Phi(\sigma))$$

and let us prove the following

Lemma 2.8 *Let f_Λ be defined $\forall\Lambda\in\mathcal{S}$ as above. The following statements are equivalent:*

(a) $\forall\Lambda\subset\Lambda'\in\mathcal{S}$, $\forall\xi$ *and* $\xi'\in\Omega$ *s.t* $\xi_{\Lambda^c} = \xi'_{\Lambda^c}$,

$$f_{\Lambda'}(\xi)f_\Lambda(\xi') = f_\Lambda(\xi)f_{\Lambda'}(\xi') \qquad (12)$$

(b) $\forall\Lambda\subset\Lambda'\in\mathcal{S}$, $\forall\xi\in\Omega$,

$$f_{\Lambda'}(\xi) = f_\Lambda(\xi)\int_{\Omega_\Lambda} f_{\Lambda'}(\sigma_\Lambda\xi_{\Lambda^c})d\sigma_\Lambda \qquad (13)$$

◇

Proof:

(a) Let Λ, $\Lambda' \in \mathcal{S}$ such that $\Lambda \subset \Lambda'$ and let ξ and ξ' s.t $\xi_{\Lambda^c} = \xi'_{\Lambda^c}$. We have

$$\frac{\exp(-\sum_{A\cap\Lambda'\neq\emptyset} \Phi_A(\xi'))}{\exp(-\sum_{A\cap\Lambda'\neq\emptyset} \Phi_A(\xi))} = \exp(-(\sum_{A\cap\Lambda'\neq\emptyset} (\Phi_A(\xi') - \Phi_A(\xi)))$$
$$= \exp(-(\sum_{A\cap\Lambda\neq\emptyset} \Phi_A(\xi') - \Phi_A(\xi)))$$

because $\xi'_{\Lambda^c} = \xi_{\Lambda^c}$ and $\forall A \in \mathcal{S}$, Φ_A is $\mathcal{F}_A$-measurable. One could prove in the same way:

$$\frac{\mathbf{Z}_{\Lambda'}(\xi')}{\mathbf{Z}_{\Lambda'}(\xi)} = \frac{\mathbf{Z}_{\Lambda}(\xi')}{\mathbf{Z}_{\Lambda}(\xi)}$$

and (12) holds

(b) Let us prove that (a) $\Longrightarrow$ (b). Assume (a) holds for Λ, $\Lambda' \in \mathcal{S}$ such that $\Lambda \subset \Lambda'$ and let $\xi \in \Omega$.

$$\begin{aligned} f_\Lambda(\xi)\int_{\Omega_\Lambda} f_{\Lambda'}(\sigma_\Lambda\xi_{\Lambda^c})d\sigma_\Lambda &= \int_{\Omega_\Lambda} f_\Lambda(\xi)f_{\Lambda'}(\sigma_\Lambda\xi_{\Lambda^c})d\sigma_\Lambda \\ &= \int_{\Omega_\Lambda} f_{\Lambda'}(\xi)f_\Lambda(\sigma_\Lambda\xi_{\Lambda^c})d\sigma_\Lambda \\ &= f_{\Lambda'}(\xi)\int_{\Omega_\Lambda} f_\Lambda(\sigma_\Lambda\xi_{\Lambda^c})d\sigma_\Lambda \\ &= f_{\Lambda'}(\xi) \end{aligned}$$

because $\int_{\Omega_\Lambda} f_\Lambda(\sigma_\Lambda\xi_{\Lambda^c})d\sigma_\Lambda = 1$.

Let us prove that (b) $\Longrightarrow$ (a) and consider $\xi, \xi', \Lambda, \Lambda'$ as above, with $\xi_{\Lambda^c} = \xi'_{\Lambda^c}$. Using (12),

$$f_{\Lambda'}(\xi) = f_\Lambda(\xi)\int_{\Omega_\Lambda} f_{\Lambda'}(\sigma_\Lambda\xi_{\Lambda^c})d\sigma_\Lambda$$

and

$$f_{\Lambda'}(\xi') = f_\Lambda(\xi')\int_{\Omega_\Lambda} f_{\Lambda'}(\sigma_\Lambda\xi'_{\Lambda^c})d\sigma_\Lambda$$

but $\xi_{\Lambda^c} = \xi'_{\Lambda^c}$ yields to

$$\int_{\Omega_\Lambda} f_{\Lambda'}(\sigma_\Lambda\xi_{\Lambda^c})d\sigma_\Lambda = \int_{\Omega_\Lambda} f_{\Lambda'}(\sigma_\Lambda\xi'_{\Lambda^c})d\sigma_\Lambda$$

and then

$$f_{\Lambda'}(\xi')f_\Lambda(\xi)\int_{\Omega_\Lambda} f_{\Lambda'}(\sigma_\Lambda\xi_{\Lambda^c})d\sigma_\Lambda = f_\Lambda(\xi')(\int_{\Omega_\Lambda} f_{\Lambda'}(\sigma_\Lambda\xi_{\Lambda^c})d\sigma_\Lambda)\ f_{\Lambda'}(\xi)$$

If $\int_{\Omega_\Lambda} f_{\Lambda'}(\sigma_\Lambda\xi_{\Lambda^c})d\sigma_\Lambda = 0$ then[14] $f_{\Lambda'}(\xi) = f_{\Lambda'}(\xi')$ and (13) holds. Otherwise, it is strictly positive and (13) holds as well.

◇

Then we obtain after (11):

$$g_{\Lambda,\Lambda'}(\tau) =$$

$$\begin{aligned} &\int_{\Omega_\Lambda} \mathbf{1}_A(\tau_\Lambda\tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c})f_\Lambda(\tau_{\Lambda'}\omega_{\Lambda'^c})(\int_{\Omega_\Lambda} f_{\Lambda'}(\sigma_\Lambda\tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c})d\sigma_\Lambda)d\tau_\Lambda \\ = &\int_{\Omega_\Lambda} \mathbf{1}_A(\tau_{\Lambda'}\omega_{\Lambda'^c})f_{\Lambda'}(\tau_{\Lambda'}\omega_{\Lambda'^c})d\tau_\Lambda \end{aligned}$$

and then

$$\gamma_{\Lambda'}\gamma_\Lambda(A|\omega) =$$

$$\begin{aligned} &\int_{\Omega_{\Lambda'\setminus\Lambda}}(\int_{\Omega_\Lambda} \mathbf{1}_A(\tau_{\Lambda'}\omega_{\Lambda'^c})f_{\Lambda'}(\tau_{\Lambda'}\omega_{\Lambda'^c})d\tau_\Lambda)d\tau_{\Lambda'\setminus\Lambda} \\ = &\int_{\Omega_{\Lambda'}} \mathbf{1}_A(\tau_{\Lambda'}\omega_{\Lambda'^c})f_{\Lambda'}(\tau_{\Lambda'}\omega_{\Lambda'^c})d\tau_{\Lambda'} \\ = &\ \gamma_{\Lambda'}(A|\omega) \end{aligned}$$

and item (4) is proved.

2. Let us prove it is a quasilocal specification and without any loss of generality, we assume again that $\beta = 1$.
   We first prove that, when Φ is an absolutely convergent potential, the

[14] Properties 1,2 and 3 of a specification hold for γ, and this requires f positive.

Hamiltonians $\mathbf{H}_\Lambda^\Phi$ are quasilocal functions for all $\Lambda \in \mathcal{S}$. Let $\Lambda \in \mathcal{S}$. We want to prove, using lemma (2.1):

$$\lim_{\Lambda' \uparrow \mathcal{S}} \sup_{\sigma, \omega \in \Omega, \sigma_{\Lambda'} = \omega_{\Lambda'}} \mid \mathbf{H}_\Lambda^\Phi(\omega) - \mathbf{H}_\Lambda^\Phi(\sigma) \mid = 0$$

Let $\mathcal{S} \ni \Lambda' \supset \Lambda$ and consider two configurations σ and ω such that $\sigma_{\Lambda'} = \omega_{\Lambda'}$.

$$\mathbf{H}_\Lambda^\Phi(\omega) - \mathbf{H}_\Lambda^\Phi(\sigma) = \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset} (\Phi_A(\omega) - \Phi_A(\sigma))$$

$$= \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset, A \subset \Lambda'} (\Phi_A(\omega) - \Phi_A(\sigma)) + \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset, A \cap \Lambda'^c \neq \emptyset} (\Phi_A(\omega) - \Phi_A(\sigma))$$

the definition of a potential proves that

$$\sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset, A \subset \Lambda'} (\Phi_A(\omega) - \Phi_A(\sigma)) = 0$$

because Φ_A is $\mathcal{F}_A$-measurable and $\sigma_{\Lambda'} = \omega_{\Lambda'}$. Hence

$$\sup_{\sigma, \omega \in \Omega, \sigma_{\Lambda'} = \omega_{\Lambda'}} \mid \mathbf{H}_\Lambda^\Phi(\omega) - \mathbf{H}_\Lambda^\Phi(\sigma) \mid \leq 2 \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset, A \cap \Lambda'^c \neq \emptyset} \sup_{\omega \in \Omega} \mid \Phi_A(\omega) \mid$$

If Φ is an absolutely convergent potential, we can write in $\mathbb{R}$:

$$\sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset, A \cap \Lambda'^c \neq \emptyset} \sup_{\omega \in \Omega} \mid \Phi_A(\omega) \mid$$

$$= \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset,} \sup_{\omega \in \Omega} \mid \Phi_A(\omega) \mid - \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset, A \subset \Lambda'} \sup_{\omega \in \Omega} \mid \Phi_A(\omega) \mid$$

The absolute convergence of Φ means that :

$$\lim_{\Lambda' \uparrow \infty} \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset, A \subset \Lambda'} \sup_{\omega \in \Omega} \mid \Phi_A(\omega) \mid = \sum_{A \in \mathcal{S}, A \cap \Lambda \neq \emptyset} \sup_{\omega \in \Omega} \mid \Phi_A(\omega) \mid < \infty$$

Hence

$$\lim_{\Lambda' \uparrow \mathcal{S}} \sup_{\sigma, \omega \in \Omega, \sigma_{\Lambda'} = \omega_{\Lambda'}} \mid \mathbf{H}_\Lambda^\Phi(\omega) - \mathbf{H}_\Lambda^\Phi(\sigma) \mid = 0$$

which proves that $\mathbf{H}_\Lambda^\Phi$ is quasilocal.

Remark 2.11

Requiring for a potential to be absolutely convergent is actually to strong to prove this quasilocality. Let us prove here that the uniform convergence is enough.

$$\sup_{\sigma,\omega\in\Omega,\sigma_{\Lambda'}=\omega_{\Lambda'}} | \mathbf{H}_{\Lambda}^{\Phi}(\omega) - \mathbf{H}_{\Lambda}^{\Phi}(\sigma) | \leq 2\sup_{\omega\in\Omega} | \sum_{A\in\mathcal{S},A\cap\Lambda\neq\emptyset,A\cap\Lambda'^c\neq\emptyset} \Phi_A(\omega) |$$

and

$$\lim_{\Lambda'\uparrow\mathcal{S}} \sup_{\sigma,\omega\in\Omega} | \sum_{A\in\mathcal{S},A\cap\Lambda\neq\emptyset,A\cap\Lambda'^c\neq\emptyset} \Phi_A(\omega) |= 0$$

is exactly the expression of the uniform convergence of this potential. $\diamond$

Let $\epsilon > 0$. The last result means that there exists h_Λ local on Ω such that

$$\sup_{\omega\in\Omega} | \mathbf{H}_{\Lambda}^{\Phi}(\omega) - h_\Lambda(\omega) | < \epsilon$$

Let f be any local function on Ω. We want to prove that there exists a local function k_Λ on Ω such that

$$\sup_{\omega\in\Omega} | \gamma_\Lambda f(\omega) - k_\Lambda(\omega) | < \epsilon$$

recall that

$$\gamma_\Lambda f(\omega) = \frac{1}{\mathbf{Z}_\Lambda(\omega)} \int_{\Omega_\Lambda} f(\sigma_\Lambda\omega_{\Lambda^c}) \exp(-\mathbf{H}_\Lambda(\sigma_\Lambda\omega_{\Lambda^c}))m_\Lambda(d\sigma_\Lambda)$$

and define

$$k_\Lambda(\omega) = \frac{1}{\mathbf{Z}_\Lambda(\omega)} \int_{\Omega_\Lambda} f(\sigma_\Lambda\omega_{\Lambda^c}) \exp(-h_\Lambda(\sigma_\Lambda\omega_{\Lambda^c}))m_\Lambda(d\sigma_\Lambda)$$

Then we have

$$| \gamma_\Lambda f(\omega) - k_\Lambda(\omega) | \leq$$

$$\frac{1}{\mathbf{Z}_\Lambda(\omega)} \int_{\Omega_\Lambda} | f(\sigma_\Lambda\omega_{\Lambda^c}) || \exp(-\mathbf{H}_\Lambda(\sigma_\Lambda\omega_{\Lambda^c}))-\exp(-h_\Lambda(\sigma_\Lambda\omega_{\Lambda^c})) | m_\Lambda(d\sigma_\Lambda)$$

As both $\mathbf{H}_\Lambda$ and h_Λ are bounded[15], there exists $K_\Lambda \in \mathbb{R}$ such that

$$\sup_{\sigma,\omega\in\Omega} | \exp(-\mathbf{H}_\Lambda(\sigma_\Lambda\omega_{\Lambda^c})) - \exp(-h_\Lambda(\sigma_\Lambda\omega_{\Lambda^c})) |$$

[15]They are quasilocal and local, and E is finite.

$$\leq K_\Lambda \sup_{\sigma,\omega\in\Omega} \mid \mathbf{H}_\Lambda(\sigma_\Lambda\omega_{\Lambda^c}) - k_\Lambda(\sigma_\Lambda\omega_{\Lambda^c}) \mid$$

leading to

$$\sup_{\omega\in\Omega} \mid \gamma_\Lambda f(\omega) - k_\Lambda(\omega) \mid \leq \sup_{\omega\in\Omega} \mid f(\omega) \mid K_\Lambda\, \epsilon$$

which proof the second statement of the lemma.

$\diamond$

Definition 2.15 [Gibbs states (or Gibbs measures)] A Gibbs state is a measure consistent with a Gibbs specification. We often say that it is consistent with an absolutely convergent potential. $\diamond$

Remark 2.12

- Let μ be a Gibbs state (for an absolutely convergent potential). Then by the definition of consistency (equation (6))

$$\forall A \in \mathcal{F},\ \mu[A|\mathcal{F}_{\Lambda^c}](\cdot) = \mathbb{E}_\mu[\mathbf{1}_A|\mathcal{F}_{\Lambda^c}](\cdot) = \gamma_\Lambda(A|\cdot)\ \mu - a.s \qquad (14)$$

  this caracterisation of Gibbs measures is called *D.L.R equation*, after Dobrushin, Lanford and Ruelle who introduced it first.
  As γ is a quasilocal specification, *no version of the conditional probabilities* of μ with respect to the σ-algebra generated by the cofinite[16] subsets of S *can be discontinuous* as function of the boundary condition ω. One often say that they are *essentially discontinuous.*

- A Gibbs specification is quasilocal but the converse is not true in general. However, *most* of the quasilocal specications are Gibbsian, and we precise this now.

Definition 2.16 [non nullness] A specification γ is said to be :

1. non null iff $\forall \Lambda \in \mathcal{S}, \forall A \in \mathcal{F}$ verifying $m(A) > 0$, we have :

$$(\gamma_\Lambda(A|\omega) > 0,\ \forall \omega \in \Omega\ )$$

2. uniformly non null iff $\forall \Lambda \in \mathcal{S}, \exists \alpha_\Lambda, \beta_\Lambda$ *with* $0 < \alpha_\Lambda \leq \beta_\Lambda < \infty$ such that

$$\alpha_\Lambda m(A) \leq \gamma_\Lambda(A|\omega) \leq \beta_\Lambda m(A)\ \ \forall \omega \in \Omega,\ \forall A \in \mathcal{F}.$$

[16] A cofinite set is the complementary of a finite set.

◇

We can now formulate the important

Theorem 2.1 (Gibbs representation theorem) *Let γ be a specification on $(\Omega, \mathcal{F})$. The following statements are equivalent:*

1. *There exists an absolutely convergent potential Φ such that γ is the Gibbsian specification for Φ (and the a priori measure m).*

2. *γ is quasilocal and uniformly non null (with respect to m).*

Moreover, if the single-spin set is finite, these are equivalent to

- *γ is quasilocal and non null with respect to m.*

◇

Proof:

1. Let γ be a specification consistent with an absolutely convergent potential Φ. We know by lemma (2.6) that it is a quasilocal specification. Moreover, it is non null by construction. As the distribution defined by γ is absolutely continuous with respect to the a priori measure m, this is equivalent to uniform non-nullness. Thus, any Gibbsian specification is uniformly non-null and quasilocal.

2. Let γ be any quasilocal specification uniformly non null with respect to the a priori measure m. $\forall\omega \in \Omega$, $\forall\Lambda \in \mathcal{S}$, $\gamma_\Lambda(\cdot|\omega)$ is absolutely continuous with respect to m and one could write:

$$\gamma_\Lambda(A|\omega) = \int_A g_\Lambda(\sigma) m(d\sigma)$$

where g_Λ is a non negative measurable function on $(\Omega, \mathcal{F})$. Moreover, the properties 1, 2 and 3 of any specification enable us to write it

$$\gamma_\Lambda(A|\omega) = \int_\Omega \mathbf{1}_A(\sigma_\Lambda\omega_{\Lambda^c}) f_\Lambda(\sigma_\Lambda\omega_{\Lambda^c}) \kappa_\Lambda(d\sigma)$$

with $f_\Lambda(\sigma) > 0$ for all $\sigma \in \Omega$ by non-nullness.

Let us use now the property (4) of a specification:

$$\forall \Lambda \subset \Lambda' \in \mathcal{S},\ \gamma_{\Lambda'}\gamma_\Lambda = \gamma_{\Lambda'}$$

The proof of lemma (2.7) yields to the statement: $\forall \omega \in \Omega,\ \forall A \in \mathcal{S}$

$$\begin{aligned}
&\int_{\Omega_{\Lambda'}} \mathbf{1}_A(\tau_{\Lambda'}\omega_{\Lambda'^c}) f_{\Lambda'}(\tau_{\Lambda'}\omega_{\Lambda'^c}) d\tau_{\Lambda'} \\
= &\int_{\Omega_{\Lambda'}} \mathbf{1}_A(\tau_{\Lambda'}\omega_{\Lambda'^c}) f_\Lambda(\tau_{\Lambda'}\omega_{\Lambda'^c})(\int_{\Omega_\Lambda} f_{\Lambda'}(\sigma_\Lambda \tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c} d\sigma_\Lambda) d\tau_{\Lambda'}
\end{aligned}$$

But, because $\forall \Lambda \in \mathcal{S}, f_\Lambda \geq 0$, this means that for $m_{\Lambda'}$-almost every $\tau_{\Lambda'} \in \Omega_{\Lambda'}$, i.e $\forall \tau \in \Omega$,

$$f_{\Lambda'}(\tau_{\Lambda'}\omega_{\Lambda'^c}) = f_\Lambda(\tau_{\Lambda'}\omega_{\Lambda'^c}) \int_{\Omega_\Lambda} f_{\Lambda'}(\sigma_\Lambda \tau_{\Lambda'\setminus\Lambda}\omega_{\Lambda'^c}) d\sigma_\Lambda$$

using lemma (2.8), one obtain $\forall \Lambda \subset \Lambda' \in \mathcal{S}, \forall \xi, \xi'$ s.t $\xi_{\Lambda^c} = \xi'_{\Lambda^c}$

$$f_\Lambda(\xi) f_{\Lambda'}(\xi') = f_\Lambda(\xi') f_{\Lambda'}(\xi) \tag{15}$$

Let a be any fixed configuration in Ω and define:

$$\Omega_a = \{\sigma \in \Omega :\ \exists \Lambda \in \mathcal{S},\ \sigma_{\Lambda^c} = a_{\Lambda^c}\}$$

Let $\omega \in \Omega_a$: $\exists \Lambda \in \mathcal{S}$ s.t $\sigma_{\Lambda^c} = a_{\Lambda^c}$ and define $\Lambda_0(\omega) = \cap_{\Lambda : \omega_{\Lambda^c} = a_{\Lambda^c}} \Lambda$ in order to obtain

$$\forall \Lambda \in \mathcal{S},\ \omega_{\Lambda^c} = a_{\Lambda^c} \implies \Lambda_0(\omega) \subset \Lambda$$

Now, let us prove that we can define a unique function $H^a : \Omega_a \longrightarrow \mathbb{R}$ such that:

- $H^a(a) = 0$
- $\forall \omega \in \Omega_a,\ \forall \Lambda \in \mathcal{S}$,

$$f_\Lambda(\omega) = \frac{\exp(H^a(\omega))}{\int_{\Omega_\Lambda} \exp(H^a(\sigma_\Lambda \omega_{\Lambda^c}) d\sigma_\Lambda} \tag{16}$$

We still consider $\omega \in \Omega_a$ and let us define

$$H^a(\omega) = \ln(f_{\Lambda_0(\omega)}(\omega)) - \ln(f_{\Lambda_0(\omega)}(a))$$

We have $H^a(a) = 0$ and let us first consider $\Lambda \supseteq \Lambda_0(\omega) = \Lambda_0$. Using the equation (15), we obtain

$$\begin{aligned} \exp(H^a(\omega)) &= \frac{f_{\Lambda_0}(\omega)}{f_{\Lambda_0}(a)} \\ &= \frac{f_{\Lambda_0}(\omega_{\Lambda_0} a_{\Lambda_0^c})}{f_{\Lambda_0}(a)} \\ &= \frac{f_\Lambda(\omega)}{f_\Lambda(a)} \end{aligned} \tag{17}$$

Hence, $\forall \Lambda \supseteq \Lambda_0(\omega)$,

$$H^a(\omega) = \ln(f_\Lambda(\omega)) - \ln(f_\Lambda(a))$$

and equation (17) yields to

$$\begin{aligned} \int_{\Omega_\Lambda} \exp(H^a(\sigma_\Lambda \omega_{\Lambda^c})) d\sigma_\Lambda &= \int_{\Omega_\Lambda} \exp(H^a(\sigma_\Lambda a_{\Lambda^c})) d\sigma_\Lambda \\ &= \int_{\Omega_\Lambda} \frac{f_\Lambda(\sigma_\Lambda a_{\Lambda^c})}{f_\Lambda(a)} d\sigma_\Lambda \\ &= \frac{1}{f_\Lambda(a)} \end{aligned}$$

and (16) is true $\forall \omega \in \Omega_a$, $\forall \Lambda \supseteq \Lambda_0(\omega)$
Let us consider now $\Lambda \subset \Lambda_0(\omega)$. Using lemma (2.7), we write:

$$f_{\Lambda_0}(\omega) = f_\Lambda(\omega) \int_{\Omega_\Lambda} f_{\Lambda_0}(\sigma_\Lambda \omega_{\Lambda^c}) d\sigma_\Lambda$$

i.e

$$f_\Lambda(\omega) = \frac{f_{\Lambda_0}(\omega)}{\int_{\Omega_\Lambda} f_{\Lambda_0}(\sigma_\Lambda \omega_{\Lambda^c}) d\sigma_\Lambda}$$

and (16) yields to

$$f_\Lambda(\omega) = \frac{\exp(H^a(\omega))}{\int_{\Omega_{\Lambda_0}} \exp(H^a(\sigma_{\Lambda_0} \omega_{\Lambda_0^c})) d\sigma_{\Lambda_0}} \frac{1}{\int_{\Omega_\Lambda} f_{\Lambda_0}(\sigma_\Lambda \omega_{\Lambda^c}) d\sigma_\Lambda}$$

but

$$\begin{aligned} f_{\Lambda_0}(\sigma_\Lambda \omega_{\Lambda^c}) &= f_{\Lambda_0}(\sigma_\Lambda \omega_{\Lambda_0 \setminus \Lambda} \omega_{\Lambda_0^c}) \\ &= \frac{\exp(H^a(\sigma_\Lambda \omega_{\Lambda_0 \setminus \Lambda} \omega_{\Lambda_0^c}))}{\int_{\Omega_{\Lambda_0}} \exp(H^a(\tau_{\Lambda_0} \omega_{\Lambda_0^c})) d\tau_{\Lambda_0}} \end{aligned}$$

and then we obtain

$$\int_{\Omega_\Lambda} f_{\Lambda_0}(\sigma_\Lambda \omega_{\Lambda^c}) d\sigma_\Lambda = \frac{\int_{\Omega_\Lambda} \exp(H^a(\sigma_\Lambda \omega_{\Lambda^c})) d\sigma_\Lambda}{\int_{\Omega_{\Lambda_0}} \exp(H^a(\tau_{\Lambda_0} \omega_{\Lambda_0^c})) d\tau_{\Lambda_0}}$$

hence

$$f_\Lambda(\omega) = \frac{\exp(H^a(\omega))}{\int_{\Omega_\Lambda} \exp(H^a(\sigma_\Lambda \omega_{\Lambda^c})) d\sigma_\Lambda}$$

and (16) is true $\forall \Lambda \in \mathcal{S}$. Thus the function H^a is well defined by (16) on Ω_a and the value in a brings the unicity.
Now Define for all $A \in \mathcal{S}$

$$\begin{aligned} \Phi_A \ &: \ \Omega \longrightarrow \mathbb{R} \\ &\sigma \longmapsto \Phi_A(\sigma) = \sum_{B \subset A} (-1)^{|A \setminus B|} H^a(\sigma_B a_{B^c}) \end{aligned}$$

and use the convention $\Phi_\emptyset = 0$. these maps are well-defined on Ω because we only use $H^a(\sigma_B a_{B^c})$. We can also define, $\forall \Lambda \in \mathcal{S}$, $\forall \sigma \in \Omega$, a map H^a_Λ by the expression

$$H^a_\Lambda(\sigma) = H^a(\sigma_\Lambda a_{\Lambda^c})$$

such that, by Möbius'inversion formula ([8]), $\forall \Lambda \in \mathcal{S}$, $\forall \sigma \in \Omega$

$$H^a_\Lambda(\sigma) = \sum_{A \subset \Lambda} \Phi_A(\sigma)$$

and this yields to

$$H^a(\sigma) = \sum_{A \in \mathcal{S}} \Phi_A(\sigma)$$

Moreover, $\forall A \in \mathcal{S}$, Φ_A is $\mathcal{F}_A$-measurable, up to property (3) of a specification which proves that $H^a(\cdot \times a_{\Lambda^c})$ is $\mathcal{F}_\Lambda$-measurable.

Define now, provided these sums are well defined[17], for all $\Lambda \in \mathcal{S}$

$$\begin{aligned} g_\Lambda \ : \ & \Omega_a \longrightarrow \mathbb{R} \\ & \sigma \longmapsto g_\Lambda(\sigma) = \sum_{A \cap \Lambda \neq \emptyset, A \in \mathcal{S}} \Phi_A(\sigma) \end{aligned}$$

then

$$\begin{aligned} H^a(\sigma_\Lambda \omega_{\Lambda^c}) - g_\Lambda(\sigma_\Lambda \omega_{\Lambda^c}) &= \sum_{A \subseteq \Lambda^c, A \in \mathcal{S}} \Phi_A(\sigma_\Lambda \omega_{\Lambda^c}) \\ &= \sum_{A \subseteq \Lambda^c, A \in \mathcal{S}} \Phi_A(\omega_{\Lambda^c}) \end{aligned}$$

This is independant of ω_Λ and $\forall \xi \in \Omega$, we obtain $\forall \omega, \sigma \in \Omega, \forall \Lambda \in \mathcal{S}$,

$$g_\Lambda(\sigma_\Lambda \omega_{\Lambda^c}) = H^a(\sigma_\Lambda \omega_{\Lambda^c}) - H^a(\xi_\Lambda \omega_{\Lambda^c})$$

Now, we can compute

$$\begin{aligned} f_\Lambda(\sigma_\Lambda \omega_{\Lambda^c}) &= \frac{\exp(H^a(\sigma_\Lambda \omega_{\Lambda^c}))}{\int_{\Omega_\Lambda} \exp(H^a(\tau_\Lambda \omega_{\Lambda^c})) d\tau_\Lambda} \\ &= \frac{\exp(g_\Lambda(\sigma_\Lambda \omega_{\Lambda^c})). \exp(H^a(\xi_\Lambda \omega_{\Lambda^c}))}{\int_{\Omega_\Lambda} \exp(g_\Lambda(\tau_\Lambda \omega_{\Lambda^c})). \exp(H^a(\xi_\Lambda \omega_{\Lambda^c})) d\tau_\Lambda} \\ &= \frac{\exp(g_\Lambda(\sigma_\Lambda \omega_{\Lambda^c}))}{\int_{\Omega_\Lambda} \exp(g_\Lambda(\tau_\Lambda \omega_{\Lambda^c})) d\tau_\Lambda} \end{aligned}$$

Define now $\forall \sigma \in \Omega$, $\forall \Lambda \in \mathcal{S}$, a configuration σ^Λ by

$$\begin{aligned} \sigma_i^\Lambda &= \sigma_i \text{ if } i \in \Lambda \\ &= a_i \text{ otherwise} \end{aligned}$$

We can then define $\forall A \in \mathcal{S}$, $\forall \Lambda \in \mathcal{S}$, the maps V_A and G_Λ on Ω by

$$V_A(\sigma) = \Phi_A(\sigma^\Lambda)$$

and

$$G_\Lambda(\sigma) = \sum_{A \cap \Lambda \neq \emptyset, A \in \mathcal{S}} V_A(\sigma)$$

[17] Φ is a vaccum potential defined from γ with vaccum state a. It does not need to be convergent or consistent with γ. See [12, 9]

in order to obtain $\forall \sigma \in \Omega$, $\forall \omega \in \Omega$, $\forall \Lambda \in \mathcal{S}$

$$f_\Lambda(\sigma_\Lambda \omega_{\Lambda^c}) = \frac{\exp(G_\Lambda(\sigma_\Lambda \omega_{\Lambda^c}))}{\int_{\Omega_\Lambda} \exp(G_\Lambda(\tau_\Lambda \omega_{\Lambda^c})) d\tau_\Lambda}$$

Thus, $G_\Lambda(\sigma|\omega) = G_\Lambda(\sigma_\Lambda \omega_{\Lambda^c})$ is a good candidate to describe the Hamiltonian at volume Λ with boundary condition ω with interaction potential V. We shall prove now that it is an absolutly convergent potential, using the quasilocality of the specification.
First of all, one should prove, using indicator functions, that this imply $\forall \sigma \in \Omega$ the quasilocality of the function $\omega \longmapsto f_\Lambda(\sigma_\Lambda \omega_{\Lambda^c})$.
The absolute convergence of V means that

$$\sum_{A \in \mathcal{S}} \sup_{\omega \in \Omega} |V_A(\omega)| < +\infty$$

i.e

$$\lim_{\Lambda' \uparrow \mathcal{S}} \sum_{A \subset \Lambda'^c} \sup_{\omega \in \Omega} |V_A(\omega)| = 0$$

but, $\forall \omega \in \Omega$, $\forall A \in \Lambda'^c$

$$V_A(\omega) = V_A(\omega_{\Lambda'^c})$$

then

$$\sum_{A \subset \Lambda'^c} \sup_{\omega \in \Omega} |V_A(\omega)| \leq \sum_{A \subset \Lambda'^c} \sum_{B \subset A} \sup_{\omega_B = a_B, \omega \in \Omega} | \ln(\frac{f_B(\omega_B a_B^c)}{f_B(a)}) |$$

the quasilocality of f_B proves that this converges to zero when $\Lambda' \uparrow \mathcal{S}$.
Thus, every quasilocal and uniformly non null specification is Gibbsian.
When E is finite, we do not need uniform non-nullness to characterize the absolute continuity of measures and non-nullness is enough to write f_Λ in an exponential form. The proof follows from the general case.

2.2.4 Some results about $\mathcal{G}(\gamma)$

We only describe here what we'll need in the next part. Those sets are studied in much more details in [9].

Definition 2.17 [Feller property] A specification is said to be *Feller* if for each $\Lambda \in \mathcal{S}$,

$$f \in \mathcal{C}(\Omega) \implies \gamma_\Lambda f \in \mathcal{C}(\Omega)$$

◇

Here, $\mathcal{C}(\Omega)$ is the space of the continuous functions in Ω. In the field studied in this paper, this is equivalent to the quasilocality property mentionned above.
We still denote $(\Lambda_n)_{n\in\mathbb{N}}$ a cofinal sequence of S.

Lemma 2.9 *:*
Assume Ω is a compact metric space and let γ be a Feller specification. The following statements hold:

1. *$\mathcal{G}(\gamma)$ is not empty.*
2. *Moreover, μ is consistent with γ if and only if it is a limit point (in the weak topology[18]) of a sequence $(\nu_n\gamma_{\Lambda_n})_{n\in\mathbb{N}}$ for some arbitrary sequence $(\nu_n)_{n\in\mathbb{N}}$ of probability measures on $(\Omega, \mathcal{F})$.*

◇

Proof: Let us first prove the second statement.

- Let $\mu \in \mathcal{G}(\gamma)$. Then, by lemma 2.2,

$$\forall \Lambda \in \mathcal{S},\ \mu\gamma_\Lambda = \mu$$

  hence

$$\forall n \in \mathbb{N},\ \mu\gamma_{\Lambda_n} = \mu$$

  Define now $\nu_n = \mu$, $\forall n \in \mathbb{N}$, we obtain $\mu = \nu_n\gamma_{\Lambda_n}$, $\forall n \in \mathbb{N}$.

- Denote $\forall n \in \mathbb{N}$, $\mu_n = \nu_n\gamma_{\Lambda_n}$. The compactness of Ω yields to this of $\mathcal{M}_1^+(\Omega)$, and this means that there exist a convergent subsequence for the weak topology, still denoted $(\mu_n)_{n\in\mathbb{N}}$. We note μ its limit and let us prove that μ is consistent with γ by checking

$$\forall \Lambda \in \mathcal{S},\ \mu\gamma_\Lambda = \mu$$

[18]a sequence (μ_n) of probability measures on $(\Omega, \mathcal{F})$ converges to a probability measure on $(\Omega, \mathcal{F})$ μ in the weak topology iff $\lim_{n\to\infty}(\int_\Omega f d\mu_n) = \int_\Omega f d\mu$ for each continuous function f.

We prove first that we can write

$$\mu\gamma_\Lambda = \lim_{n\to\infty} \nu_n\gamma_{\Lambda_n}\gamma_\Lambda$$

because, if f is any continuous function on $(\Omega, \mathcal{F})$, Then

$$\nu_n\gamma_{\Lambda_n}\gamma_\Lambda[f] = \int_\Omega \gamma_\Lambda[f|\omega]\nu_n\gamma_{\Lambda_n}[d\omega]$$

the Feller property of γ proves that $\gamma_\Lambda[f|\omega]$ is a continuous function of ω and then

$$\begin{aligned}\lim_{n\to\infty} \nu_n\gamma_{\Lambda_n}\gamma_\Lambda[f] &= \int_\Omega \gamma_\Lambda[f|\omega]\mu[d\omega] \\ &= \mu\gamma_\Lambda[f]\end{aligned}$$

But we also know that $\forall\Lambda \in \mathcal{S}, \exists n_0 \in \mathbb{N}$, s.t $n > n_0 \implies \Lambda_n \supset \Lambda$. Thus, $\forall n > n_0$, $\gamma_{\Lambda_n}\gamma_{\Lambda_{n_0}} = \gamma_{\Lambda_n}$ which proves that

$$\begin{aligned}\mu\gamma_\Lambda[f] &= \lim_{n\to\infty} \nu_n\gamma_{\Lambda_n}\gamma_\Lambda[f] \\ &= \lim_{n\to\infty} \nu_n\gamma_{\Lambda_n}[f] \\ &= \mu[f]\end{aligned}$$

and $\mu \in \mathcal{G}(\gamma)$.

The first statement comes directly from this. let $a \in \Omega$ and define $\forall n \in \mathbb{N}$, $\nu_n = \delta_a$, the Dirac measure in a. Then, we easily check that $\mu^a = \nu_n\gamma_{\Lambda_n} = \gamma_{\Lambda_n}(\cdot|a)$ exists and is a probability measure consistent with γ.
◇

We shall now use these results for the Ising model on $\mathbb{Z}^2$, called 2d-Ising model.

3 An example where non-Gibbsianness arises: decimation on the 2d-Ising model

3.1 Introduction to the 2d-Ising model

3.1.1 Configuration space and interaction

Let us consider now the particular case :

$$\Omega = \{-1,+1\}^{\mathbb{Z}^2}\ ,\ \mathcal{E} = \mathcal{P}(\{-1,+1\})\ ,\ m_0 = \tfrac{1}{2}\delta_{-1} + \tfrac{1}{2}\delta_{+1}$$

Take also a constant $J > 0$ and denote by $\langle ij \rangle$ a pair of nearest neighbourgs[19] in $\mathbb{Z}^2$.

$\forall \Lambda \in \mathcal{S},\ \forall \sigma, \omega \in \Omega,\ \forall \beta > 0,$

$$\mathbf{H}^{\beta}_{\Lambda,\omega}(\sigma) = \mathbf{H}^{\beta}_{\Lambda}(\sigma|\omega) \stackrel{\text{def}}{=} -\sum_{\langle ij \rangle \subset \Lambda} \beta J \sigma_i \sigma_j \ - \sum_{\langle ij \rangle, i \in \Lambda, j \in \Lambda^c} \beta J \sigma_i \omega_j \qquad (18)$$

is the Hamiltonian at volume Λ and boundary condition ω for the Ising model at temperature β with coupling J. For more tractability, we shall assume that $\beta = 1$ and we will write J instead of βJ. As the range R of this nearest neighbourgs interaction is finite ($R = 1$), it is an absolutely convergent potential and the lemma (2.7) yields that the Gibbsian specification γ which arises is *quasilocal.*

Let's write what γ is:
$\forall \Lambda \in \mathcal{S},\ \forall A \in \mathcal{F},\ \forall \omega \in \Omega,$

$$\gamma_\Lambda(A|\omega) = \frac{1}{\mathbf{Z}_\Lambda(\omega)} \int_\Omega \mathbf{1}_A(\sigma) \exp\Big(\sum_{\langle ij \rangle \subset \Lambda} J \sigma_i \sigma_j \ + \sum_{\langle ij \rangle, i \in \Lambda, j \in \Lambda^c} J \sigma_i \omega_j \Big)\ \kappa_\Lambda(d\sigma) \qquad (19)$$

where we still use the notation $\kappa_\Lambda(d\sigma) = m_\Lambda \otimes \delta^{\otimes \Lambda^c}_{\omega_{\Lambda^c}}(d\sigma)$ and where the partition function is given as usual by

$$0 \ < \ \mathbf{Z}_\Lambda(\omega) = \int_\Omega \exp\Big(\sum_{<ij> \subset \Lambda} J \sigma_i \sigma_j \ + \sum_{<ij>, i \in \Lambda, j \in \Lambda^c} J \sigma_i \omega_j \Big)\ \kappa_\Lambda(d\sigma) \ < \ \infty$$

[19] i,j $\in \mathbb{Z}^2$ are nearest neighbourgs iff $||i-j|| = 1$ where $||.||$ is the usual euclidean norm in $\mathbb{R}^2$.

Remark 3.1

Here, the σ-algebras $\mathcal{F}_\Lambda$ are atomic and the Gibbs specification can be defined atom per atom:
$\forall \Lambda \in \mathcal{S},\ \forall \sigma \in \Omega,\ \forall \omega \in \Omega,$

$$\gamma_\Lambda(\sigma|\omega) = \frac{1}{\mathbf{Z}_\Lambda(\omega)} \exp\Big(\sum_{\langle ij\rangle \subset \Lambda} J\sigma_i\sigma_j \ + \sum_{\langle ij\rangle, i\in\Lambda, j\in\Lambda^c} J\sigma_i\omega_j \Big)$$

As the specification is quasilocal and the single-spin set is finite, lemma (2.2) yields that it is a Feller specification and by the lemma (2.9), we know that $\mathcal{G}(\gamma)$ is not empty. Moreover we have the following

Theorem 3.1 *There exists $J_c > 0$ such that:*

1. *for $J < J_c$, there is a unique Gibbs measure μ.*
2. *for $J > J_c$, $|\ \mathcal{G}(\gamma)\ | > 1$.*
   Moreover, $\mathcal{G}(\gamma)$ is a convex set whose extreme points are the measures μ_+ and μ_-, which can be selected respectively by the '+' and the '−' boundary condition[20]*. We also have $\mathbf{M}_o(J) \stackrel{\text{def}}{=} \mu_+[\sigma_0] = -\mu_-[\sigma_0] > 0$.*

$\diamond$

Remark 3.2

In the second statement of this theorem, when $\mathcal{G}(\gamma)$ isn't a singleton, we say that there is a *phase transition.* In this case with the lattice $\mathbb{Z}^2$, a well known proof of the arising of a phase transition is based on the so-called *Peierls'argument*([9, 14]).

3.1.2 The decimation transformation

Definition 3.1 [Decimation] The decimation transformation on $\mathbb{Z}^2$ with spacing 2 to is the transformation

$$T \ :\ \Omega \longrightarrow T(\Omega) = \Omega' = \Omega$$
$$\omega \longmapsto \omega'$$

defined by $\forall i \in \mathbb{Z}^2,\ \omega'_i = \omega_{2i}$ $\diamond$

[20]The '+' (resp. the '−') b.c is the configuration $\omega \in \Omega$ for which $\omega_i = +1$ (resp -1), $\forall i \in \mathbb{Z}^2$.

Remark 3.3

$(\Omega, \mathcal{F}, \lambda) = (\Omega', \mathcal{F}', \lambda')$. Then why the primes ? We shall use the notation with a prime ' for all the objects studied *after* the decimation transformation, and without any prime ' when they are considered *before* it. This is just a trick to know which kind of object we are studying.

Weshall now, and during all the study of the decimation, fix μ to be a[21] *Gibbs measure for this 2d Ising model.*

T acts on measures: we define[22] the *decimated measure* ν to be the image of μ under the decimation transformation T:

$$\nu = T\mu$$

and we can describe this in two ways:

1. $\forall A' \in \mathcal{F}'$
$$\nu(A') = \mu(T^{-1}(A')) = \mu(A)$$
with the notation $A = T^{-1}(A') \in \mathcal{F}$.
2. $\forall f$ measurable and bounded on Ω
$$\int_{\Omega'} f(\sigma')d\nu(\sigma') = \int_{\Omega} f(\sigma)d\mu(\sigma)$$

T acts on subsets of S: we can define a canonical decimation action T on the 'even' sites of $\mathbb{Z}^2$, i.e $2\mathbb{Z}^2$ defined by:

$$T : x = 2x' \longmapsto x'$$

and this gives rise to an action on subsets:

$$\forall \Lambda \subset 2\mathbb{Z}^2, \Lambda' = T(\Lambda) = \{x \in \mathbb{Z}^2,\ 2x \in \Lambda\} \subset \mathbb{Z}^2.$$

we have to underline here that it maps the finite subsets of $2\mathbb{Z}^2$ on the finite subsets of $\mathbb{Z}^2$, but its inverse transformation T^{-1} *does not* map

[21] In case of phase transition, we do not precise which it is.

[22] For the measures and for functions, we do not use any prime !

the *co*finite subsets of $\mathbb{Z}^2$ on the *co*finite subsets of $2\mathbb{Z}^2$. Let us deal with the case we will study untill the end, when $\Lambda' = \{0\}$ consists of the origin of $\mathbb{Z}^2$. Then

$$\Lambda'^c = \mathbb{Z}^2 \backslash \{0\}$$

and

$$\begin{aligned} T^{-1}(\Lambda'^c) &= \{x = 2x' \text{ s.t } x' \in \Lambda'^c\} \\ &= \{x = 2x',\ x' \in \mathbb{Z}^2,\ x' \neq 0\} \\ &= 2\mathbb{Z}^2 \backslash \{0\} \\ &= [(\mathbb{Z}^2 \backslash 2\mathbb{Z}^2) \cup \{0\}]^c \\ &= \lambda^c \end{aligned}$$

where

$$\lambda = (\mathbb{Z}^2 \backslash 2\mathbb{Z}^2) \cup \{0\}$$

isn't a finite subset of $\mathbb{Z}^2$. This will bring some troubles for the computation of the conditional probabilities of ν, and it will be detailed more precisely in the next section (see fig 1).

3.2 The decimated measure ν

3.2.1 Introduction

We claim here that for suitable coupling[23] J the decimated measure ν is *not* gibbsian for *any* gibbs measure μ of the 2d-Ising model.
Assume here that ν is Gibbsian. From the previous part, we know that it should be consistent with a Gibbs specification. Then, there exists a quasilocal specification γ consistent with ν and verifying:

$$\forall \Lambda' \in \mathcal{S}',\ \forall A' \in \mathcal{F}',\ \nu[A'|\mathcal{F}_{\Lambda'^c}](\cdot) \ = \ \gamma_{\Lambda'}(A'|\cdot)\ \nu - a.s$$

In order to prove that ν is not Gibbsian, we shall prove that there exists a Λ' finite and a f local on Ω' such that no version of $\nu[f|\mathcal{F}_{\Lambda'^c}](\cdot)$ is quasilocal[24]. Equivalently, we want to find ω' in Ω' for which there exists a f local with

[23] We recall that by J we mean βJ: hence suitable J means here suitable temperature.

[24] For any probability measure ν, we note $\nu(f)$ or $\nu[f]$ the expectation of f under ν when it exists.

$\nu[f|\mathcal{F}_{\Lambda'^c}](\omega')$ *essentially discontinuous*[25].
Let Λ' be any finite subset of $\mathbb{Z}^2$. We have to compute the conditional probabilities $\nu[\cdot|\mathcal{F}_{\Lambda'^c}]$. They are defined ν-a.s by:

$$\forall A' \in \mathcal{F}',\ \nu[A'|\mathcal{F}_{\Lambda'^c}] = \mathbb{E}_\nu[\mathbf{1}_{A'}|\mathcal{F}_{\Lambda'^c}]$$

And more precisely, using the definition of the conditional expectation with respect to a σ-algebra, $\nu(A'|\mathcal{F}_{\Lambda'^c})$ is defined as an equivalence class of random variables Y on $(\Omega, \mathcal{F}, \nu)$ equals ν almost surely and verifying:

1. Y is $\mathcal{F}_{\Lambda'^c}$-measurable.
2. $Y \in \mathcal{L}^1(\Omega, \mathcal{F}, \nu)$.
3. $\forall B' \in \mathcal{F}_{\Lambda'^c},\ \int_{B'} \nu[A'|\mathcal{F}_{\Lambda'^c}](\omega')\nu(d\omega') \ = \ \nu(A' \cap B')$.

If we use now the definition of ν, we obtain:

$$\nu(A' \cap B') = \mu(T^{-1}(A' \cap B')) = \mu(A \cap B)$$

with $A = T^{-1}(A')$ and $B = T^{-1}(B')$. When B' describes $\mathcal{F}_{\Lambda'^c}$, B describes $\mathcal{F}_{\lambda^c}$ with $\lambda^c = T^{-1}(\Lambda'^c)$. Using again the properties of the conditional expectations, we obtain

$$\nu(A' \cap B') = \int_B \mu[A|\mathcal{F}_{\lambda^c}](\omega)\mu(d\omega)$$

and then

$$\nu[A'|\mathcal{F}'_{\lambda'^c}](\omega') = \mu[A|\mathcal{F}_{\lambda^c}](T^{-1}\omega')\ \nu - a - s(\omega')$$

so we have to compute the conditional probabilities $\mu[A|\mathcal{F}_{\lambda^c}]$ for λ non finite.

Remark 3.4

- We wrote $\lambda^c = T^{-1}(\Lambda'^c)$. This is not mistyping. We do not use Λ^c because $T^{-1}(\Lambda'^c)$ *is not* a cofinite set (and we usually use this notation for cofinite sets) as shown on the figure 1 below. A short computation leads to

$$\lambda^c \ = \mathbb{Z}^2 \backslash (2\Lambda') \ = \ T^{-1}(\Lambda') \cup (\mathbb{Z}^2 \backslash 2\mathbb{Z}^2).$$

[25] see remark 2.12.

- We should also emphasize that $\lambda^c = T^{-1}(\Lambda'^c)$ *does not* imply $\lambda = T^{-1}(\Lambda')$.

Let us consider, untill the end of this paper, the simple case $\Lambda' = \{0\}$. In this case, λ^c consists of all the spins of $2\mathbb{Z}^2$ except the origin: If we 'knew' everything except the origin on the decimated system Ω, we 'know' the spins on $2\mathbb{Z}^2$ except at the origin. Then λ is the origin *plus* the sites which are not in $\mathbb{Z}^2$ (see figure 1 and figure 2 next page : the letters denote the value of spins on the underlying sites and the ? indicate that the spin over the underlying site is unknown).
Thus in order to compute the conditional probability $\nu[\cdot|\mathcal{F}_{\Lambda'^c}]$, we have to compute the conditional probability for $\mu[\cdot|\mathcal{F}_{\lambda^c}]$ with λ *non finite.* We are in trouble here because, as μ is a Gibbs measure for the 2d-Ising model, the D.L.R equations give these probabilities *only* for the finite sets.

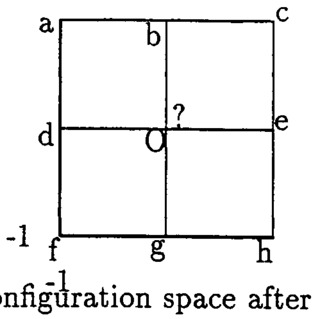

fig 1 : The configuration space after decimation, Ω'

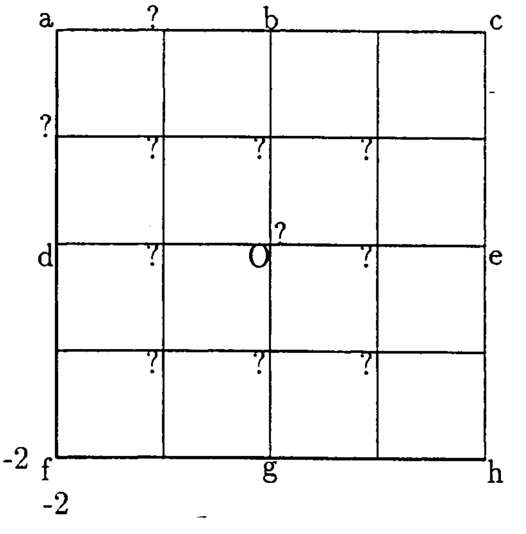

fig 1b : the configuration space before Ω

3.2.2 Study of the conditional probabilities for μ

We want to compute $\mu[\cdot|\mathcal{F}_{\lambda^c}](\omega)$ when $\omega \in T^{-1}(\omega')$, with $\omega' \in \Omega'$. We know that μ is a Gibbs measure for the $2d$-Ising model. Then there exists Ω_o with $\mu(\Omega_o) = 1$ such that:

$$\forall \omega \in \Omega_o,\ \forall \sigma \in \omega,\ \forall \Lambda \in \mathcal{S}$$

$$\mu[\sigma|\sigma_{\Lambda^c} = \omega_{\Lambda^c}] = \frac{1}{\mathbf{Z}_\Lambda(\omega)} \exp\Big(\sum_{\langle ij\rangle \subset \Lambda} J\sigma_i\sigma_j + \sum_{\langle ij\rangle, i\in\Lambda, j\in\Lambda^c} J\sigma_i\omega_j \Big) \tag{20}$$

but we want to study $\mu[\cdot|\mathcal{F}_{\lambda^c}]$ with λ *non finite*. One could then prove the following

Lemma 3.1 :
Let $\omega' \in \Omega'$ and let λ a infinite subset of $\mathbb{Z}^2$. Then the restriction[26] of $\mu[\cdot|\mathcal{F}_{\lambda^c}](\omega')$ to $(\Omega_\lambda, \mathcal{F}_\lambda)$ is a Gibbs measure for a potential (absolutely convergent) $\Phi = \Phi(\lambda, \omega')$ which depends on λ and ω'. $\diamond$

We shall not prove this lemma in the general case, because we do not need it. We will prove it in the next section for particular choices of Ω'. We could do it in the general case exactly in the same way. We will choose a configuration ω' in order to obtain a failure of quasilocality for all the conditional probabilities $\mu[\cdot|\mathcal{F}_{\lambda^c}](\omega)$, $\forall \omega \in T^{-1}(\omega')$.

3.3 Non-Gibbsianness of the decimated measure

3.3.1 Study of a particular boundary condition

Let ω' be the *alternative* configuration, defined by

$$\forall x = (x_1, x_2) \in \mathbb{Z}^2,\ \omega'_x, = (-1)^{x_1+x_2}$$

Define $\mu^{\omega',\lambda}$ as the restriction of $\mu[\cdot|\mathcal{F}_{\lambda^c}](\omega')$ to $(\Omega_\lambda, \mathcal{F}_\lambda)$. As λ is fixed (we always take now $\Lambda' = \{0\}$), we will forget it and note $\mu^{\omega',\lambda} = \mu^{\omega'}$. We want to prove that it is a Gibbs measure on $(\Omega_\lambda, \mathcal{F}_\lambda)$.
In order to do it, let $\Delta \subset \lambda$ finite and let $\tau \in \Omega_\lambda$, which yields the following picture:

[26] We define the restriction μ_λ of a probability measure μ from $(\Omega, \mathcal{F})$ to $(\Omega_\lambda, \mathcal{F}_\lambda)$ by: $\forall A \in \mathcal{F}_\lambda,\ \mu_\lambda(A) = \mu(A \times \Omega_{\lambda^c})$.

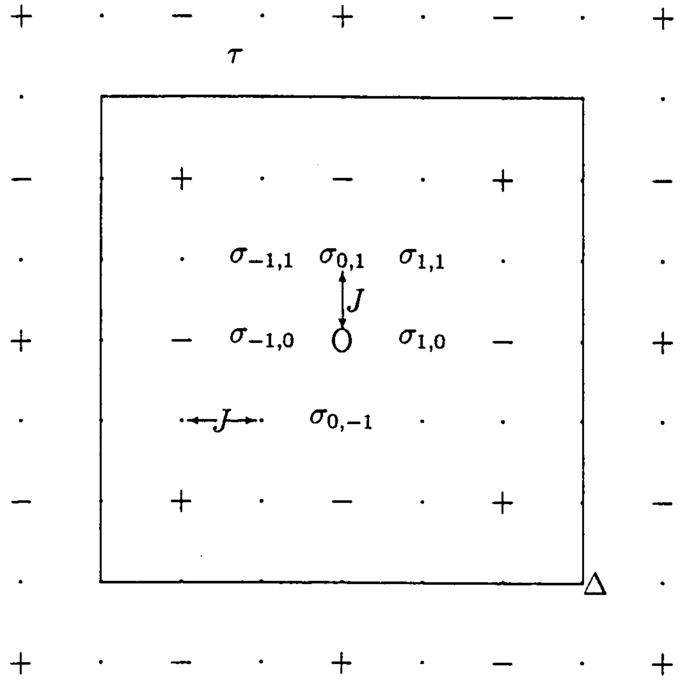

fig 2 : Configuration space Ω_λ with the alternative configuration in λ^c.

We want to check the D.L.R equations for a suitable interaction and to compute for $\mu^{\omega'}$-almost τ and $\forall\, \sigma_\lambda \in \Omega_\lambda$

$$\mu^{\omega'}[\sigma_\lambda | \sigma_{\lambda\setminus\Delta} = \tau_{\lambda\setminus\Delta}] = \sum_{\sigma_{\lambda^c}\in\Omega_{\lambda^c}} \mu[\sigma | \sigma_{\lambda\setminus\Delta} = \tau_{\lambda\setminus\Delta},\ \sigma_{\lambda^c} = \omega_{\lambda^c}]$$

by definition of the restriction of a probability measure on a subspace.
Here, we took the obvious notation: σ is the configuration which agrees with σ_λ in λ and with σ_{λ^c} in λ^c. ω is still in $T^{-1}(\omega')$.
Only one term in the previous sum is not zero: when $\sigma_{\lambda^c} = \omega_{\lambda^c}$, which is the alternative configuration on λ^c
then we have:

$$\mu^{\omega'}[\sigma_\lambda | \sigma_{\lambda\setminus\Delta} = \tau_{\lambda\setminus\Delta}] = \mu[\sigma | \mathcal{F}_{\Delta^c \cup \lambda^c}](\tau_\lambda \omega_{\lambda^c})$$

But $\Delta^c \cup \lambda^c = (\Delta \cap \lambda)^c$, and $\Delta \cap \lambda = \Delta$ is a finite subset of $\mathbb{Z}^2$. We can now use[27] the D.L.R equations (14) to obtain $\mu - a.s(\tau_\lambda \omega_{\lambda^c})$:

$$\mu[\sigma_\lambda | \sigma_{\Delta^c} = \tau_{\Delta_c}] = \frac{1}{\mathbf{Z}_\Delta^{\omega'}(\tau)} \exp(J(\sum_{\langle ij \rangle \subset \Delta} \sigma_i \sigma_j + \sum_{\langle ij \rangle, i \in \Delta, j \in \lambda^c} \sigma_i \omega_j + \sum_{\langle ij \rangle, i \in \Delta, j \in \lambda \cap \Delta^c} \sigma_i \tau_j)) \tag{21}$$

where as usual

$$\mathbf{Z}_\Delta^{\omega'}(\tau) = \sum_{\sigma_\lambda \in \Omega_\lambda} \exp(\sum_{\langle ij \rangle \subset \Delta} J\sigma_i\sigma_j + \sum_{\langle ij \rangle, i \in \Delta, j \in \lambda^c} J\sigma_i\omega_j + \sum_{\langle ij \rangle, i \in \Delta, j \in \lambda \cap \Delta^c} J\sigma_i\tau_j)$$

and in the sum $\sum_{\langle ij \rangle, i \in \Delta, j \in \lambda^c} J\sigma_i\omega_j$, the j are 'even', i.e $j = 2k$ with $k \in \mathbb{Z}^2$ such that $\omega_j = \omega'_k$ is fixed in the alternative configuration.
Assume[28] that we can find an $\omega \in T^{-1}(\omega')$ such that $\omega \in \Omega_o$, the set on which the D.L.R equations occur for μ. Then, we obtain the validity of (21) for $\mu^{\omega'}$-almost $\tau \in \Omega_\lambda$.
Thus we have proved the :

Lemma 3.2 *Let ω' be the alternative configuration defined above and assume there exists $\omega \in T^{-1}(\omega')$ for which the D.L.R for μ are valid. Then $\mu^{\omega'}$, the restriction of $\mu[\cdot|\mathcal{F}_{\lambda^c}](\omega)$ on $(\Omega_\lambda, \mathcal{F}_\lambda)$ is a Gibbs measure for some absolutely convergent potential.* $\diamond$

We shall give the expression of the potential later, during the computation of a quantity we will define later, the so-called the magnetisation. We shall then observe (fig 3) that the coupling which comes from the 'even' sites cancels and we obtain a Gibbs measure for an Ising model on $(\Omega_\lambda, \mathcal{F}_\lambda)$, with the same definition of the nearest neighbourgs than in $\mathbb{Z}^2$, with an external magnetic field $h = -J$.
We shall explicit this later: we do not need it now. We just need to know that there is *some* Gibbs measure for the interaction of the previous equation.
In case of phase transition, we *do not* know which it could be, and we shall prove that local variations in ω' could change drastically the selected phase[29]. This will yield to a non-Gibbsianness of the decimated measure. In order to do it, we shall compute the so-called magnetisation, defined below, for ω' in a neighbourhood of the alternative configuration.

[27] Assuming the D.L.R equations for μ are valids for ω.

[28] We shall not check this assumption now because we shall only be intersted by the study for ω' in a neighbourhood of the alternative configuration. We shall check that it is then always possible such configurations.

[29] We sometimes call phase a Gibbs measure.

3.3.2 Computation of the magnetisation

We want to prove a non quasilocality of ν at sufficiently low temperature. We have then to consider the action of the conditional probabilities on the local functions. The local function we choose should be characteristic of the phase transition mentioned above. Namely, it should be an *order parameter of the phase transition*[30]. We shall consider here the so-called *magnetisation* which can be defined in our model as the mean spin at the origin. Hence, we consider the local function

$$\begin{aligned} f \; : \; \Omega' &\longrightarrow \mathbf{R} \\ \sigma' &\longmapsto f(\sigma') = \sigma'_{0,0} \end{aligned}$$

and we want to study $\nu[\sigma'_{0,0}|\mathcal{F}_{\Lambda'^c}](\omega')$ for differents values of ω'. Let us consider first that ω' is in the alternative configuration. Then

$$\nu[\sigma'_{0,0}|\mathcal{F}_{\Lambda'^c}](\omega') = \mu^{\omega'}[\sigma_{0,0}]$$

as described in the previous section.
We know that this is a Gibbs measure for some interaction, then by the lemma (2.9) their exists a sequence $(\nu_R\gamma_{\Lambda_R})_{R\in\mathbb{N}}$ whose weak limit is $\mu^{\omega'}$.
Let R be any positive integer.
Let Λ_R be the intersection between λ and the square centered at the origin of length $2R$. We then know that their exists a sequence ν_R such that:

$$\langle\sigma_{0,0}\rangle^{\omega'} \stackrel{\text{def}}{=} \mu^{\omega'}[\sigma_{0,0}] = \lim_{R\to\infty}\langle\sigma_{0,0}\rangle^{\omega',\nu_R}$$

where

$$\langle\sigma_{0,0}\rangle^{\omega',\nu_R} \stackrel{\text{def}}{=} \int_{\Omega}\mu^{\omega'}[\sigma_{0,0}|\mathcal{F}_{\Lambda_R^c}](\tau_R)d\nu_R[\tau_R]$$

is the expectation of the spin at the origin when the boundary conditions which selects $\mu^{\omega'}$ have the law ν_R. Let first fix one boundary condition τ_R and note $\langle\cdot\rangle^{\omega',\tau_R}$ the expectation under the measure $\mu^{\omega'}[\cdot|\mathcal{F}_{\Lambda_R^c}](\tau_R)$. We know that $\mu^{\omega'}$ is a Gibbs measure on $(\Omega_\lambda,\mathcal{F}_\lambda)$. The lattice on which it is defined is composed by all the non-even spins plus the origin. In order to study this

[30]In statistical mechanics, an order parameter of an absolutely convergent potential which admit a family $\{\mu_j, j\in J\}$ of distinct Gibbs measures is a finite system $\{f_1,\ldots,f_n\}$ of local functions which discriminated these Gibbs measures by meaned of the associated expectation values $\{\mu_j(f_1),\ldots,\mu_j(f_n)\}$. See[9].

measure on a more conventionnal lattice, let us try to fix the spin at the origin.
Define $L_R = \{i \in \Lambda_R \text{ s.t } i_1 \textit{ and } i_2 \text{ are both odds}\}$ and $H_R = \Lambda_R \backslash L_R$.
we have, using the notation[31] $\kappa_R^\lambda(d\sigma_\lambda) = m_{\Lambda_R} \otimes \delta_{\tau_{\lambda\backslash\Lambda_R}}^{\otimes\lambda\backslash\Lambda_R}(d\sigma_\lambda)$

$$\langle\sigma_{0,0}\rangle^{\omega',\tau_R} =$$

$$\frac{1}{\mathbf{Z}^{\omega',\tau_R}}\int_{\Omega_\lambda} \sigma_{0,0}e^{J(\sigma_{0,0}-1)(\sum_{\langle i0\rangle}\sigma_i)}e^{\sum_{\langle ij\rangle, i\in\Lambda_R, j\in\lambda\backslash\Lambda_R} J\sigma_i\tau_j}\prod_{a\in L_R}(e^{\sum_{\langle ia\rangle\subset\Lambda_R}J\sigma_a\sigma_i})\kappa_R^\lambda(d\sigma_\lambda) \tag{22}$$

and

$$\mathbf{Z}^{\omega',\tau_R} = \int_{\Omega_\lambda} e^{J(\sigma_{0,0}-1)(\sum_{\langle i0\rangle}\sigma_i)}e^{\sum_{\langle ij\rangle, i\in\Lambda_R, j\in\lambda\backslash\Lambda_R} J\sigma_i\tau_j}\prod_{a\in L_R}(e^{\sum_{\langle ia\rangle\subset\Lambda_R}J\sigma_a\sigma_i})\kappa_R^\lambda(d\sigma_\lambda)$$

Those integrals are finite and positive and we can integrate out with respect to the origin first. We obtain, with $\lambda^\star = \lambda\backslash\{0\}$ where 0 denotes the origin of the lattice and $\kappa_R^{\lambda^\star}(d\sigma_{\lambda_\star}) = m_{\Lambda_R^\star} \otimes \delta_{\tau_{\Lambda_R^{\star c}}}^{\otimes\Lambda_R^{\star c}}(d\sigma_{\lambda^\star})$ with $\Lambda_R^\star = \Lambda_R\backslash\{0\}$:

$$\langle\sigma_{0,0}\rangle^{\omega',\tau_R} =$$

$$\frac{1}{\mathbf{Z}^{\omega',\tau_R}}(1-\int_{\Omega_{\lambda^\star}} e^{-2J(\sum_{\langle i0\rangle}\sigma_i)}e^{\sum_{\langle ij\rangle, i\in\Lambda_R, j\in\lambda^\star\backslash\Lambda_R} J\sigma_i\tau_j}\prod_{a\in L_R}(e^{\sum_{\langle ia\rangle\subset\Lambda_R}J\sigma_a\sigma_i})\kappa_R^{\lambda^\star}(d\sigma_{\lambda^\star})) \tag{23}$$

and for the partition function

$$\mathbf{Z}^{\omega',\tau_R} = 1+\int_{\Omega_{\lambda^\star}} e^{-2J(\sum_{\langle i0\rangle}\sigma_i)}e^{\sum_{\langle ij\rangle, i\in\Lambda_R, j\in\lambda\backslash\Lambda_R} J\sigma_i\tau_j}\prod_{a\in L_R}(e^{\sum_{\langle ia\rangle\subset\Lambda_R}J\sigma_a\sigma_i})\kappa_R^{\lambda^\star}(d\sigma_{\lambda^\star}) \tag{24}$$

Where $\sum_{\langle i0\rangle}$ means that the sum is taken over all the spins attached to the origin.
Hence, we only have to compute the expectation of $e^{-2J(\sum_{\langle i0\rangle}\sigma_i)}$ with respect to the Gibbs distribution with boundary condition τ_R for an Ising model on $(\Omega_{\lambda^\star}, \mathcal{F}_{\lambda^\star})$ when the spin is fixed to be '+' at the origin. We obtain this model because of the very particular interaction we get with the alternative configuration: The contributions of the 'even sites', which are on the alternative configuration, annihilate themselves.

[31] κ_R^λ is the restriction of κ_{Λ_R} to $(\Omega_\lambda, \mathcal{F}_\lambda)$.

We have then the alternative configuration everywhere on $2\mathbb{Z}^2$ and an Ising distribution on the so-called *decorated lattice* λ^*.
We note μ^{+,ω',τ_R} this measure and $\langle \cdot \rangle^{+,\omega',\tau_R}$ the expectation with respect to it. Hence, (23) yields to:

$$\langle \sigma_{0,0} \rangle^{\omega',\tau_R} = \frac{1 - \langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{-1,0}+\sigma_{0,-1})} \rangle^{+,\omega',\tau_R}}{1 + \langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{-1,0}+\sigma_{0,-1})} \rangle^{+,\omega',\tau_R}} \tag{25}$$

In order to study this model, and because we will have to compute it, we shall study $\langle \sigma_{0,1} \rangle^{+,\omega',\tau_R}$, the expectation of one spin attached to the origin. We have the :

Lemma 3.3

$$\begin{aligned} \langle \sigma_{0,1} \rangle^{+,\omega',\tau_R} &= \langle \tanh(J(\sigma_{1,1} + \sigma_{-1,1})) \rangle^{+,\omega',\tau_R} \\ &= \langle (\frac{1}{2} \tanh(2J))(\sigma_{1,1} + \sigma_{-1,1}) \rangle^{+,\omega',\tau_R} \end{aligned}$$

where $\sigma_{1,1}$ and $\sigma_{-1,1}$ are the spins attached to $\sigma_{0,1}$.

Proof :

Let us establish the first equality. The Bayes'formula yields:

$$\mu^{+,\omega',\tau_R}(\sigma_{0,1} = +) =$$

$$\sum_{\xi \in \Omega_{\lambda^*}} \mu^{+,\omega',\tau_R}[\sigma_{0,1} = + | \sigma_x = \xi_x, x \neq (0,1)] \mu^{+,\omega',\tau_R}[\sigma_x = \xi_x, x \neq (0,1)]$$

but $\mu^{+,\omega',\tau_R}(\sigma_{0,1} = + | \sigma_x = \xi_x, x \neq (0,1))$

$$= \frac{\mu^{+,\omega',\tau_R}(\sigma_{0,1} = +, \sigma_x = \xi_x, x \neq (0,1))}{\mu^{+,\omega',\tau_R}(\sigma_x = \xi_x, x \neq (0,1))}$$

$$= \frac{\exp(J \sum_{\langle x,(0,1) \rangle} \xi_x \ + J \sum_{\langle i,j \rangle \subset \Lambda_R} \xi_i \xi_j)}{\exp(J \sum_{\langle x,(0,1) \rangle} \xi_x \xi_0 \ + J \sum_{\langle i,j \rangle \subset \Lambda_R} \xi_i \xi_j)}$$

The terms coming from bonds which do not touch $(0,1)$ cancel and, if we note $x \sim (0,1)$ when x is attached to $(0,1)$, we obtain:

$$\mu^{+,\omega',\tau_R}(\sigma_{0,1} = + | \sigma_x = \xi_x, x \neq (0,1)) =$$

$$\mu^{+,\omega',\tau_R}(\sigma_{0,1} = + | \sigma_x = \xi_x, x \sim (0,1))$$

Hence

$$\begin{aligned}\langle\sigma_{0,1}\rangle^{+,\omega',\tau_R} &= \mu^{+,\omega',\tau_R}(\sigma_{0,1}=+) - \mu^{+,\omega',\tau_R}(\sigma_{0,1}=-)\\ &= \langle\tanh(J\sum_{x\sim(0,1)}\sigma_x)\rangle^{+,\omega',\tau_R}\\ &= \langle\tanh(J(\sigma_{1,1}+\sigma-1,1))\rangle^{+,\omega',\tau_R}\end{aligned}$$

and the first equality is proved.
The second equality is just a trick using the fact that the spins take values in $\{-1,+1\}$, then $\sigma_{1,1}+\sigma_{-1,1}\in\{-2,0,+2\}$.

$$\langle\sigma_{1,1}+\sigma_{-1,1}\rangle^{\cdot} = 2\mu^{\cdot}[\sigma_{1,1}+\sigma_{-1,1}=2] - 2\mu^{\cdot}[\sigma_{1,1}+\sigma_{-1,1}=-2]$$

and, using $\tanh 0 = 0$ and $\tanh(-x) = -\tanh x$

$$\langle\tanh(J(\sigma_{1,1}+\sigma_{-1,1}))\rangle^{\cdot} =$$

$$\tanh(2J)\mu^{\cdot}[\sigma_{1,1}+\sigma_{-1,1}=2] - \tanh(2J)\mu^{\cdot}[\sigma_{1,1}+\sigma_{-1,1}=-2]$$

◇

Thus, we shall only have to study the distribution of the spins in L_R, the lattice of spins whose coordinates are both odd.

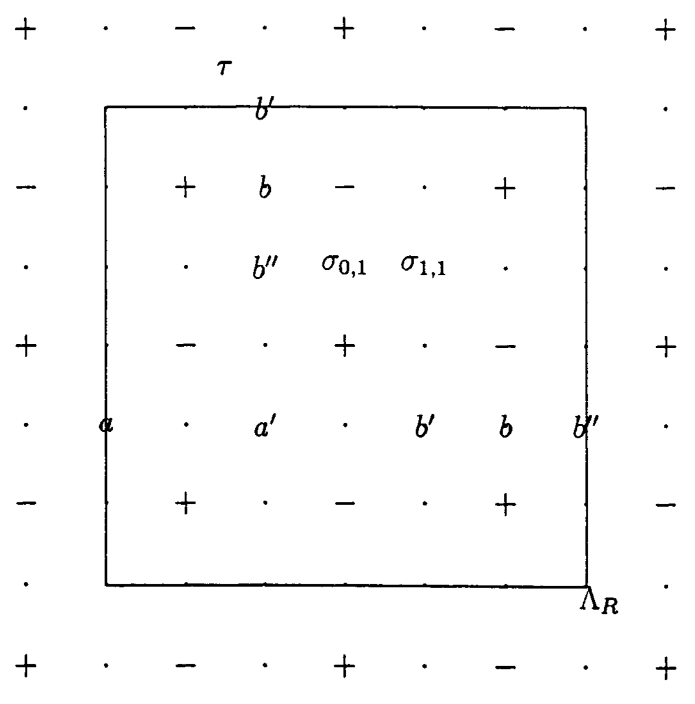

fig 3 : Ising model on the decorated lattice λ^*

We then have to compute $\langle\sigma_{1,1}\rangle^{+,\omega'\tau_R}$. As claimed before[32], we can integrate out as we want and we will begin by integrating out with respect to the spins in H_R, the sites of the decorated lattice which have exactly two neighbours (see fig 3).
We call $H_R^o = H_R \backslash \Gamma_R$ where $\Gamma_R = \Lambda_R \backslash \Lambda_{R-1}$ is the boundary of Λ_R. The sites in H_R^o are those which have two neighbours *in* Λ_R.
We also call $H_R^1 = H_R \cap \Gamma_R$: it is the set of the sites which have two neighbourgs in the lattice λ^*, one in Λ_R and the other, 'filled' by the boundary condition τ, outside Λ_R. Let's compute:

$$\langle\sigma_{1,1}\rangle^{+,\omega'\tau_R} =$$

$$\frac{1}{\mathbf{Z}^{+,\omega',\tau_R}} \int_{\Omega_{\lambda^*}} \sigma_{1,1} A_R^0(\sigma, d\sigma) A_R^1(\sigma, d\sigma) A_R(\sigma, d\sigma)$$

where

$$A_R^0(\sigma, d\sigma) = \prod_{b \in H_R^o} (e^{J\sigma_b(\sigma_{b'}+\sigma_{b''})} m_o(d\sigma_b))$$

$$A_R^1(\sigma, d\sigma) = \prod_{b \in H_R^1} (e^{J\sigma_b(\sigma_{b'}+\tau_{b''})} m_o(d\sigma_b))$$

$$A_R(\sigma, d\sigma) = \prod_{a \in L_R} m_o[d\sigma_a] \otimes \delta_{\tau_{\lambda\backslash\Lambda_R}}^{\lambda\backslash\Lambda_R}(d\sigma_{\lambda\backslash\Lambda_R})$$

where for each $b \in H_R$, we call b' and b'' its neighbourgs who are in L_R or filled by the boundary condition τ in Λ_{R+1}.
If we compute the integral above, we obtain

$$\int_{\Omega_{L_R}} \sigma_{1,1}(\prod_{b \in H_R} \int_E e^{J\sigma_b(\sigma_{b'}+\sigma_{b''})} m_o[d\sigma_b]) \prod_{a \in L_R} m_o[d\sigma_a] \otimes \delta_{\tau_{\lambda\backslash\Lambda_R}}^{\otimes\lambda\backslash\Lambda_R}[d\tau_{\lambda\backslash\Lambda_R}]$$

we can easily compute

$$\int_E e^{J\sigma_b(\sigma_{b'}+\sigma_{b''})} m_o[d\sigma_b] = \frac{e^{J(\sigma_{b'}+\sigma_{b''})} + e^{-J(\sigma_{b'}+\sigma_{b''})}}{2}$$

[32]It is Fubini's theorem with positive measurable functions.

then the contribution of the spins in H_R does not appear in the integral anymore, because the set $\{(b', b''), b \in H_R\}$ is L_R. We would like to obtain now a coupling interaction between the spins in L_R: let's try to write

$$\frac{e^{J(\sigma_{b'}+\sigma_{b''})} + e^{-J(\sigma_{b'}+\sigma_{b''})}}{2} = Ke^{J'\sigma_{b'}\sigma_{b''}} \tag{26}$$

where K is a constant which cancels by normalisation.
On the event $\{\sigma_{b'} = +1, \sigma_{b''} = +1\}$, we should have

$$\cosh[2J] = Ke^{J'}$$

and on the events $\{\sigma_{b'} = -1, \sigma_{b''} = +1\}$ and $\{\sigma_{b'} = +1, \sigma_{b''} = -1\}$

$$1 = Ke^{-J'}$$

then, one could take $K = e^{J'}$ and $e^{2J'} = \cosh[2J]$ i.e $J' = \frac{1}{2}\ln(\cosh[2J])$ in the equation (15) which leads to[33]

$$\langle \sigma_{1,1} \rangle^{+,\omega'\tau_R} =$$

$$\frac{1}{\mathbf{Z}^{+,\omega',\tau_R}} \int_{\Omega_{L_R}} (\sigma_{1,1} e^{J' \sum_{<aa'> \subset L_R} \sigma_a \sigma_{a'} + J' \sum_{<aa'>, a \in L_R, a' \in \lambda \setminus \Lambda_R} \sigma_a \tau_{a'}}) m_{L_R}[d\sigma_{L_R}]$$

[33]We use the same notation for the partition function but we should multiply it by $K = e^{J'}$ to obtain this of this new Ising model.

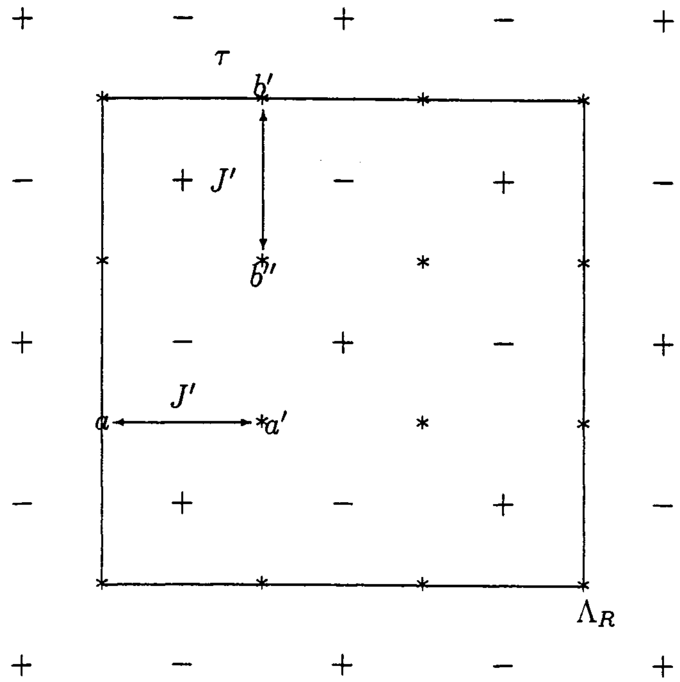

fig 4 : Ising model on $2\mathbb{Z}^2$ with coupling J'

One could, and should, remark that it is *exactly* the magnetisation of an Ising model with coupling J' on $2\mathbb{Z}^2$, with the boundary condition τ on $(\lambda \backslash \Lambda_R) \cap 2\mathbb{Z}^2$.
When the temperature is low enough, we know by the theorem (3.1) that a phase transition arises for this model. We will now do the same computation on a neighbourhood of the alternative configuration, and we will prove that on a same neighbourhood, a small variation of the ω' will lead to different selection of the extreme phase[34], and this will bring the essential discontinuity we want.

3.3.3 Computation of the magnetisation on the neighbourhood of the alternative configuration.

Recall that we have to prove an essential discontinuity in ω', i.e that no version of $\nu[\sigma'_{o,o}|\mathcal{F}_{\Lambda'^c}](\omega')$ can be continuous. We will have to work on sets of non zero ν-measure.

[34] An extreme phase is one of the extreme points of the convex set $\mathcal{G}(\gamma)$ (see [9]).

Define $\Lambda'_R = \Lambda_{\frac{R-1}{2}}$ if R is odd (which is the case in fig 4) and $\Lambda'_R = \Lambda_{\frac{R}{2}}$ if R is even.

$$\mathcal{N}_R = \{\omega' \in \Omega', \omega'_{\Lambda'_R} = \omega'^{alt}_{\Lambda'_R}, \omega' \text{ arbitrary } outside\ \Lambda'_R\}$$

$$\mathcal{N}_{R,+} = \{\omega' \in \Omega', \omega'_{\Lambda'_R} = \omega'^{alt}_{\Lambda'_{R-1}}, \omega'_{\Gamma'_{R+1}} = +1, \omega' \text{ arbitrary } outside\ \Lambda'_R\}$$

$$\mathcal{N}_{R,-} = \{\omega' \in \Omega', \omega'_{\Lambda'_R} = \omega'^{alt}_{\Lambda'_{R-1}}, \omega'_{\Gamma'_{R+1}} = -1, \omega' \text{ arbitrary } outside\ \Lambda'_R\}$$

Where ω'^{alt} is the alternative configuration defined in the previous section and $\Gamma'_R = \Lambda'_R \backslash \Lambda'_{R-1}$.
$(\mathcal{N}_R, R > 0)$ is a basis of neighbourhoods for this alternative configuration. Moreover, $\mathcal{N}_{R,+}$ and $\mathcal{N}_{R,-}$ are open sets in the product topology, so we have $\nu(\mathcal{N}_{R,+}) = \nu(\mathcal{N}_{R,-}) > 0$.

Computation on $\mathcal{N}_{R,+}$:
We want to compute $\nu[\sigma'_{0,0}|\mathcal{F}_{\Lambda'^c}](\omega'_1)$ for $\omega'_1 \in \mathcal{N}_{R,+}$. We know from section (3.2) that for ν-almost (ω'_1), $\forall \Lambda' \in \mathcal{F}$

$$\nu[\sigma'_{0,0}|\mathcal{F}_{\Lambda'^c}](\omega'_1) = \mu[\sigma_{0,0}|\mathcal{F}_{\lambda^c}](\omega_1)$$

with[35] $\omega_1 \in T^{-1}(\omega'_1)$, $\sigma \in T^{-1}(\sigma')$ and $\lambda^c = T^{-1}(\Lambda'^c)$.
Let fix again $\Lambda' = \{0\}$.
Recall that we have $\nu(\mathcal{N}_{R,+}) > 0$, then $\mu[T^{-1}(\mathcal{N}_{R,+})] > 0$ and then we can find[36] $\omega_1 \in T^{-1}(\mathcal{N}_{R,+})$ for which the D.L.R equations are valids for μ. Then we can use the lemma (3.2) to prove that $\mu^{\omega'_1}$ is a Gibbs measure on $(\Omega_\lambda, \mathcal{F}_\lambda)$. We will note it[37] μ^+.
Let us do again what has been done for the alternative configuration in the section(3.3.2), changing in the notation ω' by '+'. We know that we have some Gibbs measure, which can be obtained as a weak limit of some (possibly stochastic with law ν_R) boundary conditions τ_R and we obtain again

$$\nu[\sigma'_{0,0}|\mathcal{F}_{\Lambda'^c}](\omega'_1) = \langle\sigma_{0,0}\rangle^+ = \lim_{R\to\infty} \langle\sigma_{0,0}\rangle^{+,\nu_R}$$

[35] We do not care which we take in $T^{-1}(\omega'_1)$ because we will only use it when it coincides with ω'_1 on the new configuration space.

[36] This justifies the claims of the footnotes 27 and 28.

[37] This is not the '+'-phase...which will be noted μ_+.

where

$$\langle\sigma_{o,o}\rangle^{+,\nu_R} = \int_\Omega \mu^+[\sigma_{o,o}|\mathcal{F}_{\Lambda_R}](\tau_R)d\nu_R(\tau_R)$$

Let us again assume that ν_R is a Dirac measure in τ_R, and try again to obtain a Gibbs measure on a more conventional lattice, the decorated lattice, by fixing the spin at the origin to be $'+'$.
We obtain an expression for $\langle\sigma_{0,0}\rangle^{+,\tau_R}$ similar to (22), except that we have an external magnetic field. For example, the integral in the numerator of (22) should be replaced by:

$$\int_{\Omega_\lambda} (\sigma_{0,0}e^{J(\sigma_{0,0}-1)(\sum_{\langle i0\rangle}\sigma_i)} \prod_{a\in L_R} A(\sigma,a)e^{\sum_{\langle ij\rangle,i\in\Lambda_R,j\in\lambda\setminus\Lambda_R} J\sigma_i\tau_j} e^{\sum_{i\in\Lambda_R} h_i\sigma_i})\kappa_R^\lambda(d\sigma_\lambda) \tag{27}$$

where

$$A(\sigma,a) = e^{\sum_{\langle ia\rangle\subset\Lambda_R} J\sigma_a\sigma_i}$$

where the external magnetic field is defined below and we obtain

$$\langle\sigma_{0,0}\rangle^{+,\tau_R} = \frac{1-\langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})}\rangle^{+,+,\tau_R}}{1+\langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})}\rangle^{+,+,\tau_R}} \tag{28}$$

where $\langle\cdot\rangle^{+,+,\tau_R}$ is the expectation under[38] $\mu^{+,+,\tau_R}$, the Gibbs measure on $(\Omega_{\lambda*},\mathcal{F}_{\lambda*})$ obtained from μ^{+,τ_R} by fixing the spin at the origin to be '+'(see section (3.3.2)).
Using the same techniques as in the section (3.3.2), we will just have to study the distribution in L_R, sublatice of $\mathbb{Z}^2$ consisting of the 'odd' sites contained in Λ_R, to compute $\langle\sigma_{1,1}\rangle^{+,+,\tau_R}$.

[38]The first '+' means that the spin is fixed to be '+' at the origin, the second that the configuration after decimation is in $A_{R,+}$.

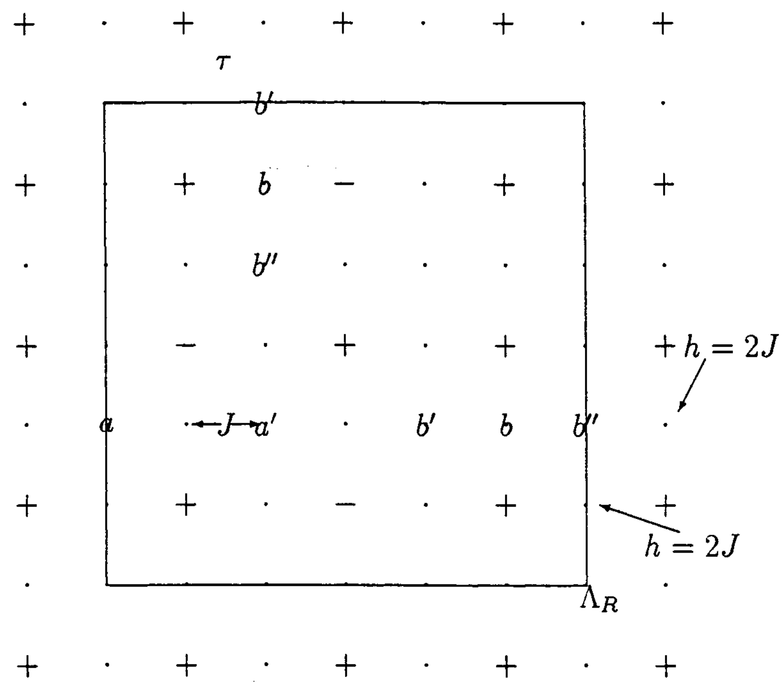

fig 5 : rise of a magnetic field on Γ_{R+1} and Γ_R

As in the case of the alternative configuration, we obtain an Ising model on the decorated lattice $(\Omega_{\lambda^*}, \mathcal{F}_{\lambda^*})$, with the important exception that an external magnetic field h_i appears on some sites i (see figure 4 above) because of the spins '+' on Γ'_R: the annihilation of the contribution of the 'even sites' does not occur here, it was only due to the alternative configuration. We obtain:
$h_i = 2J$ if $i \in \Gamma_R$ and is surrounded by two spins '+' from ω_1
$h_i = 3J$ if[39] $i \in \Gamma_{R+1}$ and its neighbourg $j \in \Gamma_{R+2}$ is s.t $\tau_j = +1$
$h_i = J$ if $i \in \Gamma_{R+1}$ and its neighbourg $j \in \Gamma_{R+2}$ is s.t $\tau_j = -1$
$h_i = 0$ otherwise.

We shall now use the two following lemmata to obtain an upper bound for $\langle \sigma_{1,1} \rangle^{+,+,\tau_R}$. These are well-known results, and a proof can be found in [14].

Lemma 3.4 (Griffiths' inequalities) *:*
Let us consider the Ising model on a lattice S with positive generalized external

[39] it is then surrounded by two spins '+' from ω_1.

magnetic field, i.e with a Hamiltonian defined by:
$\forall \Lambda \subset S, \forall \sigma_\Lambda \in \Omega_\Lambda$

$$\mathbf{H}_\Lambda(\sigma_\Lambda) = -J \sum_{\langle ij \rangle \subset \Lambda} \sigma_i \sigma_j - \sum_{i \in \Lambda} h_i \sigma_i \cdot$$

with $J > 0$ and $h_i \geq 0$, $\forall i \in S$.
Then

1.
$$\forall T \subset \Lambda, \ \langle \sigma_T \rangle_\Lambda \geq 0 \tag{29}$$

2.
$$\forall T, T' \subset \Lambda,$$
$$\langle \sigma_T \sigma_{T'} \rangle_\Lambda - \langle \sigma_T \rangle_\Lambda \langle \sigma_{T'} \rangle_\Lambda \geq 0 \tag{30}$$

◇

Remark 3.5

When the generalised magnetic is negative everywhere, we have the same kind of inequalities but we have to change '$\geq$' by '$\leq$' .

Lemma 3.5 $\forall T \subset \Lambda$, $\forall i \in \Lambda$

$$\frac{\partial \langle \sigma_T \rangle_\Lambda}{\partial h_i} = \langle \sigma_T \sigma_i \rangle_\Lambda - \langle \sigma_T \rangle_\Lambda \langle \sigma_i \rangle_\Lambda \tag{31}$$

◇

Thus, with a positive generalized magnetic field, the magnetisation increases when increasing the parameter h_i when this is positive everywhere, and decreases when it increases when it is negative.
Now, we use those two lemmas to forget the magnetic field wich appeared on Γ_R. Thus a lower bound for $\langle \sigma_{1,1} \rangle^{+,+,\tau_R}$ will be the magnetisation of an Ising model on the decorated lattice with coupling J and with an external

magnetic field $h = 2J$ on[40] Γ_R, and this lower bound is valid *for all boundary condition* τ.
If we integrate out, as we have done for ω'^{alt}, with respect to the spins which have exactly two neighbours in the decorated lattice, we obtain that a lower bound for $\langle\sigma_{1,1}\rangle^{+,+,\tau_R}$ is the magnetization at volume Λ_R of an Ising model on $2\mathbb{Z}^2$, with coulping J' and with an external magnetic field h' on the boundary Γ_R and $2h'$ on the corners of this boundary. Doing the same computation as in section (3.3.2), we obtain:

$$J' = \frac{1}{2}\ln(\cosh[2J]) \ > 0$$

and

$$h' = \frac{1}{2}\ln(\cosh[2h]) = \frac{1}{2}\ln(\cosh[4J]) \ > 0$$

And we should emphasize that the bound is uniform in τ_R

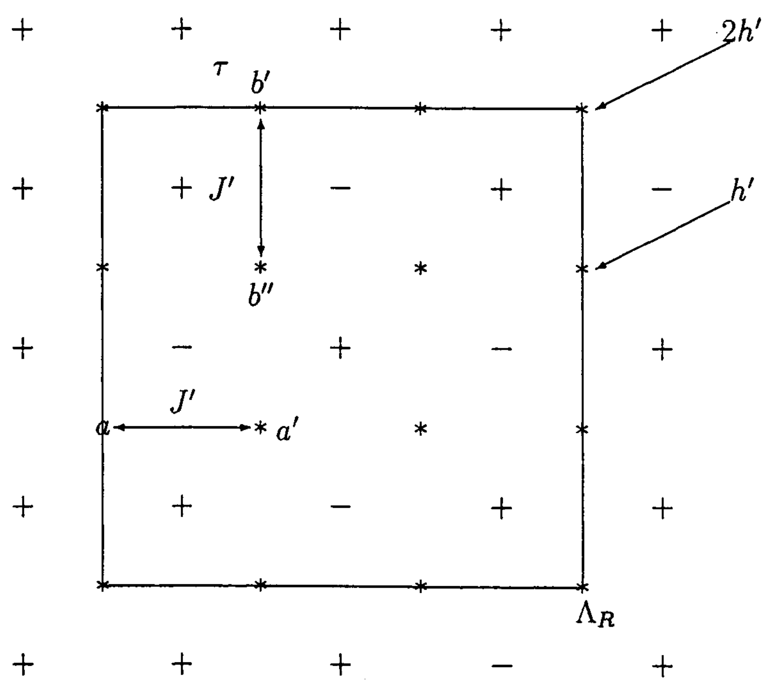

fig 6 : Ising model on $2\mathbb{Z}^2$ with magnetic field on Γ_R.

[40]We recall that $\Gamma_R \subset \lambda$ then the magnetic field only acts on the spins which are not fixed by ω'.

The action of this positive magnetic field on the boundary is exactly the same as the action of a boundary condition '+' on the following boundary Γ_{R+2} for the Ising model on $2\mathbb{Z}^2$. Let us assume that a phase transition arises for this model, i.e $J' > J_c$. If we use the results of the theorem (3.1), we can do the following computation.

$$\langle\sigma_{1,1}\rangle^{+,+,\nu_R} \geq \int_\Omega \inf_{\tau_R}[\langle\sigma_{1,1}\rangle^{+,+,\tau_R}]\, d\nu_R(\tau_R) = \inf_{\tau_R}[\langle\sigma_{1,1}\rangle^{+,+,\tau_R}]$$

then

$$\langle\sigma_{1,1}\rangle^{+,+,\nu_R} \geq \langle\sigma_{0,0}\rangle_{+,\Lambda_{R+2}} \overset{R\to\infty}{\longrightarrow} \mathrm{M}_o(J') > 0.$$

where $\langle\sigma_{0,0}\rangle_{+,\Lambda_{R+2}}$ is the magnetisation at volume Λ_{R+1} with '+' boundary condition mentionned above. In this computation, we use the property for the '+'-phase of the Ising model to be the weak limit of the Gibbs distribution with boundary condition '+'.

computation on $\mathcal{N}_{R,-}$:

Let us choose $\omega_2' \in \mathcal{N}_{R,-}$ and $\omega_2 \in T^{-1}(\mathcal{N}_{R,-})$ such that the D.L.R equations for μ are true for it. We know that this is possible because $\nu(\mathcal{N}_{R,-}) > 0$. We want to compute

$$\nu[\sigma_{0,0}'|\mathcal{F}'_{\Lambda'^c}](\omega_1') = \mu[\sigma_{0,0}|\mathcal{F}_{\lambda^c}](\omega_1)$$

for $\Lambda' = \{0\}$. We know by the previous paragraph that this the expectation under some Gibbs measure $\mu^{\omega_2'} = \mu^-$ on $(\Omega_\lambda, \mathcal{F}_\lambda)$, which can be selected as a weak limit of Gibbs measures with boundary condition τ_R of law ν_R and we obtain similarly[41]

$$\nu[\sigma_{0,0}'|\mathcal{F}'_{\Lambda'^c}](\omega_1') = \langle\sigma_{0,0}\rangle^- = \lim_{R\to\infty}\langle\sigma_{0,0}\rangle^{-,\nu_R}$$

where

$$\langle\sigma_{0,0}\rangle^{-,\nu_R} = \int_\Omega \mu^-[\sigma_{0,0}|\mathcal{F}_{\Lambda_R}](\tau_R)d\nu_R(\tau_R)$$

We can write in the same way as in the previous paragraph for the formula (28):

$$\langle\sigma_{0,0}\rangle^{-,\tau_R} = \frac{1 - \langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})}\rangle^{+,-,\tau_R}}{1 + \langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})}\rangle^{+,-,\tau_R}}$$

[41] The $\langle\cdot\rangle^-$ is just a notation. This is not the expectation under any '−'-phase..

where $\langle\cdot\rangle^{+,-,\tau_R}$ is the expectation under $\mu^{+,-,\tau_R}$, the Gibbs measure on $(\Omega_{\lambda^\star,\mathcal{F}_{\lambda^\star}})$ obtained from μ^{-,τ_R} by fixing the spin at the origin to be '+'(see section (3.3.2)).
Now let us study this measure by computing $\langle\sigma_{0,0}\rangle^{-,\tau_R}$, as we have done for $\mathcal{N}_{R,+}$, and compute first $\langle\sigma_{1,1}\rangle^{+,-,\tau_R}$ as above. The spin at the origin is still fixed to be '+' and we have similarly situation as this in the $\mathcal{N}_{R,+}$-case: this magnetisation is this of an Ising model on the decorated lattice with coupling $J > 0$, with an external magnetic field. The only changes are the values of this magnetic field and the places it acts.

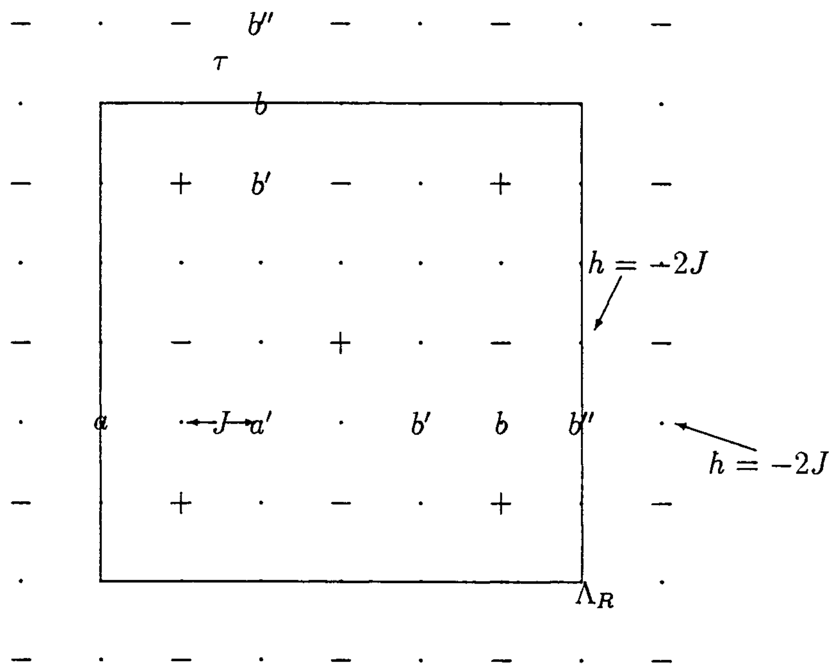

fig 5b : rise of a magnetic field on Γ_R and Γ_{R+1}

We have
$h_i = -2J$ if $i \in \Gamma_R$ and is surrounded by two spins '−' from ω_1

$h_i = -3J$ if[42] $i \in \Gamma_{R+1}$ and its neighbourg $j \in \Gamma_{R+2}$ is s.t $\tau_j = -1$
$h_i = -J$ if $i \in \Gamma_{R+1}$ and its neighbourg $j \in \Gamma_{R+2}$ is s.t $\tau_j = +1$
$h_i = 0$ otherwise. We have then 2 differences: the magnetic field is here negative, and their a shift on its location on Γ_R. But we can use the remark (3.5) about the Griffiths'inequalities, and use them in the same manner to forget the magnetic field on Γ_R in order to obtain an upper bound for $\langle\sigma_{1,1}\rangle^{+,-,\tau_R}$ valid for all boundary condition τ_R. This upper bound is the magnetisation of an Ising model on the decorated lattice with coupling $J > 0$ and with an external magnetic field $h = -2J$ on Γ_{R+1}.
Now, if we proceed as in the previous paragraph, when we consider $J' > J_c$ we obtain the following upper bound:

$$\langle\sigma_{1,1}\rangle^{+,-,\nu_R} \leq \langle\sigma_{0,0}\rangle_{-,\Lambda_{R+2}} \overset{R\to\infty}{\longrightarrow} -\mathrm{M}_o(J') < 0.$$

where $\langle\cdot\rangle_{-,\Lambda_{2R+2}}$ is the magnetisation of an Ising model on $2\mathbb{Z}^2$ at volume Λ_{2R+2} with the $'-'$ boundary condition, which appears because of the negative magnetic field on the boundary.
We obtain a magnetisation opposite to the one previously obtained because changing ω_1' into ω_2' leads to the selection of a different phase.
Let's now compute $\langle\sigma_{0,0}\rangle^{-,\nu_R}$ in a different way, in order to compare it to $\langle\sigma_{0,0}\rangle^{+,\nu_R}$ much easily.
We come back to the beginning, i.e section (3.3.2). We had

$$\langle\sigma_{0,0}\rangle^{-,\nu_R} \stackrel{\text{def}}{=} \int_\Omega \mu^-[\sigma_{0,0}|\mathcal{F}_{\Lambda'^c}](\tau_R)d\nu_R[\tau_R]$$

We still have the equation (22) except that a negative magnetic field arises at the boundary. Let us do a change of variable in the integral (27), changing σ into $-\sigma$ ('spin-flip'), we obtain that the integral becomes (multiplicated by $(-1)^{|\Lambda_R|}$ which cancels with the partition function)

$$\int_{\Omega_\lambda}(-\sigma_{0,0}e^{J(-\sigma_{0,0}-1)(\sum_{\langle i0\rangle}\sigma_i)}\prod_{a\in L_R}A(\sigma,a)e^{\sum_{\langle ij\rangle,i\in\Lambda_R,j\in\lambda\setminus\Lambda_R}J\sigma_i\tau_j}e^{\sum_{i\in\Lambda_R}h_i\sigma_i})\kappa_R^\lambda(d\sigma_\lambda)$$

where

$$A(\sigma,a) = e^{\sum_{\langle ia\rangle\subset\Lambda_R}J\sigma_a\sigma_i}$$

[42] it is then surrounded by two spins '$-$' from ω_1.

and if we integrate out with respect to the origin first, this leads to:

$$1 - \int_{\Omega_{\lambda^*}} e^{+2J(\sum_{\langle i0 \rangle}\sigma_i)} \prod_{a \in L_R} A(\sigma, a) e^{\sum_{\langle ij \rangle, i\in\Lambda_R, j \in \lambda\setminus\Lambda_R} J\sigma_i\tau_j} e^{\sum_{i\in\Lambda_R} h'_i\sigma_i} \kappa_R^{\lambda^*}(d\sigma_{\lambda_*})$$

where $h'_i = -h_i$. We obtain the following[43]:

$$\langle \sigma_{0,0} \rangle^{-,\nu_R} = \frac{1 - \langle e^{+2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})} \rangle^{+,+,\nu_R}}{1 + \langle e^{+2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})} \rangle^{+,+,\nu_R}}$$

We shall now use this for the final computation.

3.4 Essential discontinuity of the conditional probabilities for ν:

Let us consider now $\omega'_1 \in \mathcal{N}_{R,+}$ *and* $\omega'_2 \in \mathcal{N}_{R,-}$ as above.
Then we have for some boundary condition of law ν_R

$$\nu[\sigma'_{0,0}|\mathcal{F}'_{\{0\}^c}](\omega'_1) - \nu[\sigma'_{0,0}|\mathcal{F}'_{\{0\}^c}](\omega'_2) = \lim_{R\to\infty} (\langle \sigma_{0,0} \rangle^{+,\tau_R} - \langle \sigma_{0,0} \rangle^{-,\tau_R})$$

where, with $\star =' +'$ *or* $'-'$,

$$\langle \sigma_{0,0} \rangle^{\star,\nu_R} = \frac{1 - \langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})} \rangle^{+,\star,\nu_R}}{1 + \langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})} \rangle^{+,\star,\nu_R}}$$

using the previous section, we have:

$$\langle \sigma_{0,0} \rangle^{+,\nu_R} - \langle \sigma_{0,0} \rangle^{-,\nu_R} =$$

$$\frac{1 - \langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})} \rangle^{+,+,\nu_R}}{1 + \langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})} \rangle^{+,+,\nu_R}} - \frac{1 - \langle e^{+2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})} \rangle^{+,+,\nu_R}}{1 + \langle e^{+2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})} \rangle^{+,+,\nu_R}} =$$

$$\frac{2(\langle e^{+2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})} \rangle^{+,+,\nu_R} - \langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})} \rangle^{+,+,\nu_R})}{D(R)}$$

where the denominator $D(R) =$

$$(1 + \langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})} \rangle^{+,+,\nu_R})(1 + \langle e^{+2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})} \rangle^{+,+,\nu_R})$$

[43] One could think that the spin-flip would change the origin into '−'. It is true, but the previous computation proves that we obtain these expectation with respect to the measure we had when the origin was fixed to be '+'.

is positive. let us study the sign if the numerator $N(R) =:$

$$2(\langle e^{+2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})}\rangle^{+,+,\nu_R} - \langle e^{-2J(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})}\rangle^{+,+,\nu_R})$$

$$\begin{aligned} &= 2(\langle\sum_{k=0}^{\infty}\frac{(2J)^k}{k!}(\sigma_{0,1} \quad + \quad \sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})^k\rangle^{+,+,\nu_R} \\ &\qquad - \quad \langle\sum_{k=1}^{\infty}\frac{(-2J)^k}{k!}(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})^k\rangle^{+,+,\nu_R}) \\ &= 8J\langle\sigma_{0,1} \quad + \quad \sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0}\rangle^{+,+,\nu_R} \\ &\qquad + \quad 4J\sum_{k=1}^{\infty}\frac{(2J)^{2k+1}}{(2k+1)!}\langle(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})^{2k+1}\rangle^{+,+,\nu_R} \end{aligned}$$

and then we have

$$N(R) \ \geq \ 32J\langle\sigma_{0,1}\rangle^{+,+,\nu_R}$$

because under this phase, we have a positive generalized external magnetic field, and we know by Griffths'inequality(1) that for k odd,

$$\langle(\sigma_{0,1}+\sigma_{1,0}+\sigma_{0,-1}+\sigma_{-1,0})^k\rangle^{+,+,\nu_R} \ \geq 0$$

If J is such that $J' > J_c$, we know that $\langle\sigma_{0,1}\rangle^{+,+,\nu_R} \overset{R\to\infty}{\longrightarrow} \mathbf{M}_o(J') > 0$, then we obtain

$$\begin{aligned} \lim_{R\to\infty}(<\sigma_{0,0}>^{+,\nu_R} - <\sigma_{0,0}>^{-,\nu_R}) \quad &\leq \quad \lim_{R\to\infty}(32J<\sigma_{0,1}>^{+,+,\nu_R}) \\ &= \quad 16JM_o(J') > \ 0 \end{aligned}$$

This proves the non-quasilocality of the decimated measure. If we express this in a topological way, we obtain the

Lemma 3.6 (essential discontinuity) *Let $J' > J_c$ and let ω'^{alt} be the alternative configuration.*
$\forall\epsilon > 0$, $\forall\mathcal{N}$ neighbourhood of ω'^{alt}, $\exists R_o > 0$ such that $\forall R > R_o$, we can find $\mathcal{N}_{R,+}$, $\mathcal{N}_{R,-} \subset \mathcal{N}$ with $\nu[A_{R,+}] = \nu[A_{R,-}] > 0$ and for ν-almost $\omega'_1 \in \mathcal{N}_{R,+}$, for ν-almost $\omega'_2 \in \mathcal{N}_{R,-}$,

$$\nu[\sigma'_{0,0}|\mathcal{F}'_{\{0\}^c}](\omega'_1) \ - \ \nu[\sigma'_{0,0}|\mathcal{F}'_{\{0\}^c}](\omega'_2) \ > \epsilon$$

Thus, no version of the conditional probabilities of ν given $\mathcal{F}'_{\{0\}^c}$ can be continuous.

Hence the following theorem expresses the non-Gibbsianness of the decimated measure at low temperature, for the 2d-Ising model.

Theorem 3.2 *Let μ be any Gibbs measure for the Ising model on $\mathbb{Z}^2$, with coupling $J > \frac{1}{2} \arg\cosh[e^{2J_c}]$ and without any external magnetic field. Then the decimated measure cannot be consistent with any absolutely convergent potential: It is not a Gibbs state.*

4 The Dobrushin program

This failure of Gibbsiannity does not come from the pathologies of the decimation transformation, which 'forget' a lot of spins. It has been proved in [5] that non-Gibbsianness arose in many other transformations of the renormalization group, such that some Kadanoff transformation for the Ising model or some cases of the majority rule transformation for the Ising model. The proofs are based on this developped in the previous section.
The requirement of being consistent with an absolutely convergent potential appears to be too strong. Using this, sveral authors have tried to restore this formalism by requiring weaker conditions (see [3, 1, 12, 13, 7] for example). We describe here the restoration of Maes et al. ([12, 13, 11]), and we shall apply it to this decimated measure elsewhere. The example mainly used by Maes et al. is very similar: it is the projection of the Ising model on the real line.

4.1 Almost Gibbsianness

Let $(\Omega, \mathcal{F}, \lambda)$ be a probability space as described in section 2, and let ν be any probability measure on $(\Omega, \mathcal{F})$.

Definition 4.1 [Bad configuration] A configuration $\omega \in \Omega$ is 'bad' for ν if for some $i \in S, \epsilon > 0, \forall \Lambda \in \mathcal{S}, \exists \Lambda' \supset \Lambda, \Lambda' \in \mathcal{S},$ *and* $\tau, \tau' \in \Omega$ such that:

$$| \ \nu[\omega_i | \omega_{\Lambda \setminus i} \tau_{\Lambda' \setminus i}] \ - \nu[\omega_i | \omega_{\Lambda \setminus i} \tau'_{\Lambda' \setminus i}] \ | \geq \epsilon$$

◇

Thus the alternative configuration of the previous section was a bad configuration for the decimated measure.

Lemma 4.1 *If ω is a 'bad' configuration for ν and if γ is a specification consistent with μ then $\exists\, i \in S$ such that $\gamma_{\{i\}}(\sigma|\cdot)$ is discontinuous in ω.* $\diamond$

We call S_ν the set of all the configurations which are bad for ν.

Definition 4.2 [Almost Gibbsian measure] A probability measure ν on $(\Omega, \mathcal{F})$ is *almost Gibbsian* if there exists a specification γ such that $\nu \in \mathcal{G}(\gamma)$ and $\nu(\Omega_\gamma) = 1$ where

$$\Omega_\gamma = \{\omega \ : \ \forall \Lambda \in \mathcal{S},\ \forall F \in \mathcal{F},\ \gamma_\Lambda(F|\cdot) \text{ is continuous in } \omega\}$$

$\diamond$

Remark 4.1

- A Gibbs measure is almost Gibbsian ! $(\Omega_\gamma = \Omega)$
- We don't deal with potentials here.

Theorem 4.1

- $S_\nu = \emptyset \iff$ *ν is Gibbsian*
- $\nu[S_\nu] = 0 \iff$ *ν is almost Gibbsian*
- $\nu[S_\nu] > 0 \iff$ *ν is not almost Gibbsian*

$\diamond$

4.1.1 Weakly Gibbsian states

Definition 4.3 [Weakly Gibbsian measure] A probability measure ν on $(\Omega, \mathcal{F})$ is *weakly Gibbsian* if there exists a potential Φ and a tail-measurable set Ω_Φ such that

1. Φ is absolutely convergent on Ω_Φ
2. $\nu[\Omega_\Phi] = 1$
3. For every $\mathcal{F}_\Lambda$-measurable function f, its expectation value is given by

$$\nu[f] = \int_\Omega \frac{1}{\mathbf{Z}_\Lambda(\omega_{\Lambda^c})}\, f(\omega_\Lambda)\, e^{-\mathbf{H}_\Lambda^\Phi(\omega)}\, d\nu[\omega_{\Lambda^c}]$$

$\diamond$

As Ω_Φ is a tail-measurable set, then $\mathbf{Z}_\Lambda(\omega_{\Lambda^c})$ is well defined $\forall \omega \in \Omega_\Phi$ and we can write the integral above.
We have the

Theorem 4.2 *If ν is almost Gibbsian then ν is weakly Gibbsian and we can choose $\Omega_\Phi = \Omega_\gamma$ such that $\nu \in \mathcal{G}(\gamma)$* $\diamond$

The converse is not true in general. Some examples and the basics properties of the weakly Gibbsian states can be found in [12, 13, 11]. A variational principle is given in [11] but apparently, no large deviation principle has been given yet.
the next stage of our work is to the investigate the almost Gibbsianness and weak Gibbsianness of the decimated measure.

References

[1] J. Bricmont, A. Kupiainen, R. Lefevere. Renormalization Group Pathologies and the Definition of Gibbs States, preprint, UC Louvain, 1997.

[2] B.Prum. Processus sur un réseau et mesures de Gibbs. Techniques stochastiques, Masson, Paris etc. 1986.

[3] R.L Dobrushin and S.B Shlosman. Gibbsian description of 'non Gibbsian' field, Russian Math Surveys **52**, 285-297, 1997.

[4] J.Dixmier. Topologie Générale. Presses Universitaire de France Mathématiques, 1981.

[5] A. C. D. van Enter, R. Fernández, and A. D. Sokal. Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory. *J. Stat. Phys.*, **72**:879–1167, 1993.

[6] W.Feller. An Introduction to Probability Theory and Its Applications, volume I. Wiley publications in statistics, 1957.

[7] R. Fernández and C-E.Pfister. Global specifications and non-quasilocality of projections of Gibbs measures. EPFL preprint, 1996.

[8] D.Griffeath. Introduction to random fields J.G.Kenneny, J.L.Snell, A.W.Knapp eds, Graduate Texts in Mathematics, vol 40, Springer Verlag, New york,Heildelberg, Berlin, 1976.

[9] H.O. Georgii. Gibbs Measures and Phase Transitions. Walter de Gruyter (de Gruyter Studies in Mathematics, Vol. 9), Berlin–New York, 1988.

[10] J.L.Kelley. General Topology. University Series in Higher Mathematics. D.Van Nostrand Company inc. Toronto etc., 1955.

[11] J.Lörinczi and C.Maes. Weakly gibbsian measures for lattice spin systems. preprint K-U.Leuven, 1997.

[12] C. Maes and K. Vande Velde. Relative energies for non-Gibbsian states. preprint K-U Leuven, 1996.

[13] C. Maes, F. Redig and A. Van Moffaert. Almost Gibbsian versus Weakly Gibbsian. preprint K-U Leuven, 1997.

[14] V.A.Malyshev and R.A Minlos. Gibbs Random Fields, cluster expansions. Mathematics and its applications, Kluwer Academic Publishers, 1991.

[15] J.Neveu, Bases Mathématiques du calcul des Probabilités. Masson et Cie, 1970.

[16] C.Preston. Random Fields. Lecture Notes in Mathematics 534, Springer-Verlag, 1976.

[17] A.D. Sokal. Existence of compatible families of proper regular conditional probabilities. Z.Wahrsch. verw. Geb **56**:537-548, 1981.

[18] W.G.Sullivan Potentials for almost Markovian random fields, *Commun. Math. Phys.*, **33**:61-74, 1976.