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FIXED POINTS OF A GENERALIZED SMOOTHING TRANSFORMATION AND APPLICATIONS TO BRANCHING PROCESSES

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Summary. Let $\{A_i\}$ be a sequence of nonnegative random variables such that $\tilde{N} := \sum_{i=1}^{\infty} 1_{\{A_i>0\}} < \infty$ almost surely. Let $\mathcal{M}$ be the class of all probability measures on $[0,\infty)$. Define a transformation T on $\mathcal{M}$ by letting $T\mu$ be the distribution of $\sum_{i=1}^{\infty} A_i Z_i$, where the Z_i are independent random variables with distribution μ, which are independent of $\{A_i\}$ as well. In earlier work, to study invariant measures of some infinite particle systems, Durrett and Liggett investigated the transformation T in the special case where $\|\tilde{N}\|_{\infty} := \text{ess.sup } \tilde{N} < \infty$. More special cases where considered by Mandelbrot, Kahane and Peyrière, and Guivarc'h in the study of a model for turbulence of Yaglom. In this paper, we study the transformation in general. The functional equation $\mu = T\mu$ then contains as special cases the well-known basic equations in general branching processes; these equations are closely related to the Kesten-Stigum theorem and the Seneta-Heyde norming for Galton-Watson processes. Assuming only $\mathbb{E}\tilde{N} < \infty$ and $\mathbb{E} \sum_{i=1}^{\infty} A_i \log^+ A_i < \infty$, we determine exactly when T has a nontrivial fixed point of finite or infinite mean and we prove that fixed points have some regular variation properties. The case where $\mathbb{E}\tilde{N} = \infty$ or $\mathbb{E} \sum_{i=1}^{\infty} A_i \log^+ A_i = \infty$ is also considered. If $\mathbb{E}\tilde{N}^{1+\delta} < \infty$ and $\mathbb{E}(\sum_{i=1}^{\infty} A_i)^{1+\delta} < \infty$ for some $\delta > 0$, we find *all* the fixed points and we prove that all nontrivial fixed points have stable-like tails. Convergence theorems are given to ensure that fixed points can be obtained by natural iterations with some appropriate initial distributions. Other limit theorems are also obtained when there is no nontrivial fixed point. The work answers in particular a question of Athreya for Bellman-Harris processes and completes a result of Biggins for branching random walks.

Key words: Smoothing transformation, Branching processes, Branching random walks, Mandelbrot's martingale, functional equation.
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1. Introduction and main results

Let $A_i \geq 0$ be a sequence of random variables (r.v.) with

$$\tilde{N} := \sum_{i=1}^{\infty} 1_{\{A_i>0\}} < \infty$$

almost surely (a.s.) and let N>0 be a random integer such that for all i>N, $A_i=0$ a.s. We note that when $\|\tilde{N}\|_\infty :=$ ess.sup $\tilde{N} < \infty$, N can be taken as a constant sufficiently large. Let $\mathcal{M}$ be the class of all probability measures on $[0,\infty)$. Define a transformation T on $\mathcal{M}$ by letting $T\mu$ be the distribution of $\sum_{i=1}^{\infty} A_i Z_i \equiv \sum_{i=1}^{N} A_i Z_i$, where the Z_i are independent r.v.'s with distribution μ, which are independent of $\{A_i\}$ as well. Of course T can be regarded as a (nonlinear) transformation on the class $\mathcal{L}$ of Laplace transforms ϕ of elements of $\mathcal{M}$:

$$(T\phi)(t) = \mathbb{E} \prod_{i=1}^{\infty} \phi(tA_i) \equiv \mathbb{E} \prod_{i=1}^{N} \phi(tA_i),$$

where (and throughout) the product is taken over all the indices i such that

$A_i>0$, and the empty product (which happens when $\tilde{N}=0$) is taken to be 1. We shall write alternatively $\prod_{i=1}^{\infty}$ or $\prod_{i=1}^{N}$, $\sum_{i=1}^{\infty}$ or $\sum_{i=1}^{N}$ according to convenience for context.

Kahane and Peyrière (1976), and Guivarc'h (1990) studied the fixed points of the transformation T in the case where N is constant and the A_i $(1\le i\le N)$ are independent and identically distributed. Their works were motivated by questions raised by Mandelbrot relating to a model for turbulence of Yaglom. Holley and Liggett (1981) studied the same problem in the case where N is constant and the A_i $(1\le i\le N)$ are fixed multiples of one random variable, and Durrett and Liggett (1983) considered the more general case where the N is constant but the A_i have arbitrary joint distribution. Their works were motivated by a number of problems in infinite particle systems. Closely related results are given in Kahane (1987), Ben Nasr (1987), Holley and Waymire (1992), Collet and Koukiou (1992), Franchi (1993) and Chauvin and Rouault (1993), etc.

If $1<m=\mathbb{E}N<\infty$ and $A_i=1/m$ $(1\le i\le N)$, then the equation $\phi=T\phi$ reduces to the Poincaré functional equation $\phi(u)=\mathbb{E}\phi^N(u/m)$, which arises in the Galton-Watson process. Similar equations [which are always special cases of our equation $\phi=T\phi$] arise in age-dependent branching processes or branching random walks. The study of these equations has been important, since it gives the limit behaviour of the population sizes of the associated processes. Many authors have contributed to it, see for example Harris (1948), Kesten-Stigum (1966), Seneta (1968, 1969 and 1974), Athreya (1971), Doney (1972 and 1973), Doney and Bingham (1974 and 1975) and Biggins (1977).

The transfomation T, in its various forms, was also used to study some fractal sets or flows in networks, implicitly or directly by Mauldin and Williams (1986), Falconer (1986 and 1987) and Liu (1993). So, the greatest advantage of the present work is perhaps that it reveals some intimate relations among the different subjects mentioned above: infinite particle systems, multiplicative chaos, branching processes, fractal geometry and flows in networks.

Let $\mathcal{F}$ be the set of all nontrivial fixed points of T:

$$\mathcal{F} = \{ \mu\in \mathcal{M}: T\mu=\mu \ \text{ and } \mu\neq\delta_0\}.$$

The elements μ of $\mathcal{F}$ will be identified with their Laplace transforms ϕ as well. We suppose throughout the paper that

$$P(\tilde{N}= 0 \text{ or } 1) <1 \quad\text{and}\quad P(\forall i\geq 1,\ A_i= 0 \text{ or } 1) <1. \tag{H0}$$

Otherwise, the situation is clear:

(a) If $P(\tilde{N} = 0 \text{ or } 1) = 1$, the equation $\mu=T\mu$ reduces to

$$Z \stackrel{d}{=} S\, Z$$

with $S= \sum_{i=1}^{N} A_i$ independent of Z, and, taking logarithms, we see that $\mathcal{F}\neq\emptyset$ if and only $S\equiv 1$ a.s.;

(b) If $P(\tilde{N}= 0 \text{ or } 1) <1$ and $P(\forall i\geq 1,\ A_i= 0 \text{ or } 1) =1$, then the equation reads

$$\phi(t)= f(\phi(t)) \quad (\forall t\geq 0),$$

where $f(t)= \sum_{k=0}^{\infty} P(\tilde{N}=k)t^k$. Therefore, if $\mathbb{E}\tilde{N}\leq 1$, then $\phi(t)=1$ for all $t\geq 0$; If $\mathbb{E}\tilde{N}>1$, then $\forall t\geq 0$, $\phi(t)=1$ or q, q being the unique fixed point in [0,1) of f. Since $\phi(0)=1$, ϕ is continuous and decreasing, we conclude that $\phi(t)\equiv 1$. So in both cases ($\mathbb{E}\tilde{N}\leq 1$ or $\mathbb{E}\tilde{N}>1$), T has only the trivial fixed point $\phi\equiv 1$.

For $x\in [0,\infty)$, write

$$S(x):= \sum_{i=1}^{\infty} A_i^x \equiv \sum_{i=1}^{N} A_i^x, \quad S:=S(1),$$

and

$$\rho(x):= \mathbb{E}S(x),$$

where (and throughout) the sum is taken over all the i such that $A_i>0$, and the empty sum (which happens when $\tilde{N}=0$) is taken to be 0. The function ρ is well defined on $[0,\infty)$ with values in $[0,\infty]$. We remark that

$$S(0)=\tilde{N} \text{ and } \rho(0)=\mathbb{E}\tilde{N}$$

by our notations. If

$$\mathbb{E}\tilde{N}<\infty \quad \text{and} \quad \mathbb{E}\sum_{i=1}^{N} A_i \log^+ A_i<\infty, \tag{H1}$$

where $\log^+ x = \max(0,\log x)$, then

$$\rho(x)<\infty \quad \text{and} \quad \rho'(x)= \mathbb{E}\sum_{i=1}^{N} A_i^x \log A_i <\infty$$

exists for all $x\in(0,1]$ (at the right point 1, $\rho'(1)$ denotes the left derivative); ρ is strictly convex on (0,1) since

$$\rho''(x)= \mathbb{E}\sum_{i=1}^{N} A_i^x \log^2 A_i <\infty$$

exists for all $x\in(0,1)$ by (H1), and is strictly positive by (H0). Sometimes we shall need the condition that for some $\delta>0$,

$$\mathbb{E}(\tilde{N}^{1+\delta})<\infty \text{ and } \mathbb{E}(S^{1+\delta})<\infty. \tag{H2}$$

It will be useful to remark that this condition is equivalent to

$$\mathbb{E}(\mathcal{S}^{1+\delta})<\infty, \tag{H2'}$$

where

$$\mathcal{S}:= \sum_{i=1}^{N} \max(A_i,1).$$

The minimal conditions were given in LIU (1994) for existence of nontrivial fixed points with finite mean :

Theorem 0. (*Existence of nontrivial solution with finite mean*) *Under the condition (H1), T has a nontrivial fixed point with finite mean if and only if*

$$\mathbb{E}S(1)\log^+ S(1) <\infty, \ \rho(1)=1 \text{ and } \rho'(1)<0.$$

This result reduces to the well-known Kesten-Stigum theorem in the context of Galton-Watson processes; it is due to Athreya (1971) for Bellman-Harris processes, to Doney for Crump-Mode processes (1972) and to Biggins (1977) for the general case under the condition $\mathbb{E}\sum A_i(\log^+ A_i)^2 <\infty$ instead of $\mathbb{E}\tilde{N}<\infty$. It was also obtained by Kahane and Peyrière (1976) if $\|\tilde{N}\|_\infty<\infty$ and A_i are i.i.d., and by Durrett and Liggett (1983) if $\|\tilde{N}\|_\infty<\infty$. In the context of age-dependent branching processes or branching random walks, a harder open problem was to know whether T has nontrivial fixed points of infinite mean in the case where $\mathbb{E}S(1)\log^+ S(1)=\infty$. [see for example Athreya (1971, p.598, problem (c)(i)) for Bellman-Harris processes.]

The following theorem solves this in a much more general setting.

Theorem 1. (*Existence of nontrivial solutions of finite or infinite mean*)

If $\mathbb{E}\tilde{N}>1$ *and* $\inf_{x\in[0,1]}\rho(x)\le 1$, *then* $\mathcal{F}\neq\emptyset$. *The converse holds subject to (H1).*

We remark that for the sufficiency part, we need neither $\mathbb{E}A_i\log^+ A_i<\infty$ nor $\mathbb{E}\tilde{N}<\infty$.

For the necessary condition in the case where (H1) does not hold, the following comparison methode will be useful:

Remark 1. *If* $\rho(1)>1$ *and*

$$\mathbb{E}A_i\log^+ A_i<\infty \text{ for all } i\ge 1, \text{ and } \liminf_{n\to\infty}\mathbb{E}\sum_{i=1}^{n} A_i\log A_i<0, \tag{H3}$$

then $\mathcal{F}=\emptyset$. *More generally, if for some constant integer* $n>0$ *and random variables* $0\le\tilde{A}_i\le A_i$ $(1\le i\le n)$,

$$\mathbb{E}\tilde{A}_i\log^+\tilde{A}_i<\infty \text{ for all } 1\le i\le n,\ \mathbb{E}\sum_{i=1}^{n}\tilde{A}_i>1 \text{ and } \mathbb{E}\sum_{i=1}^{n}\tilde{A}_i\log\tilde{A}_i\le 0, \tag{H4}$$

then $\mathcal{F}=\emptyset$.

This will be given in section 9. We remark that when $\mathbb{E}A_i\log^+ A_i<\infty$, the integral $\mathbb{E}A_i\log A_i$ is well defined and finite. The condition (H3) evidently holds in the context of branching processes (cf.§2.).

By Theorem 1, assuming (H1), we obtain the minimal conditions for existence of *any* nontrivial fixed points of T:

Corollary 1. *Under the condition (H1), the following assertions are equivalent:*

(a) $\mathcal{F}\neq\emptyset$. (*b*) $\mathbb{E}\tilde{N}>1$ and $\inf_{x\in[0,1]}\rho(x)\le 1$. (*c*) $\mathbb{E}\tilde{N}>1$ and *for some* $\alpha\in(0,1]$, $\rho(\alpha)=1$.

(*d*) *For some* $\alpha\in(0,1]$, $\rho(\alpha)=1$ and $\rho'(\alpha)\le 0$.

This follows from Theorem 1 since the conditions (b),(c) and (d) are equivalent each other subject to the hypothesis (H1). In fact, to see that (*c*) implies (*d*), it suffices to choose the least α for which $\rho(\alpha)=1$, using the convexity of ρ. The other implications are clear.

If $\|\tilde{N}\|_\infty<\infty$ and $\mathbb{E}A_i\log^+A_i<\infty$ ($\forall$i), the result was shown by Durrett and Liggett (1983,th.1). [Although Durrett and Liggett's work assumes $\mathbb{E}A_i^\gamma<\infty$ for some $\gamma>1$, the condition $\mathbb{E}A_i\log^+A_i<\infty$ ($\forall$i) suffices in the proofs of their Theorem 1.].

We recall that a function $\ell(x)\ge 0$ is called slowly varing at 0 (or ∞) if $\forall\lambda>0$ $\ell(\lambda u)/\ell(u)\to 1$ as $u\to 0$ (or ∞).

Theorem 2. *(Regular variation of nontrivial fixed point) Assume (H1)* and $\mathcal{F}\neq\emptyset$. *Let* α *be the unique point in (0,1] such that* $\rho(\alpha)=1$ *and* $\rho'(\alpha)\le 0$. *If* $\phi\in\mathcal{F}$, *then for some slowly varing function* $\ell(.)\ge 0$ *at 0,*

$$\lim_{t\to 0+}\frac{1-\phi(t)}{t^\alpha\ell(t)}=1 \quad \text{if } \alpha=1$$

and

$$\limsup_{t\to 0+}\frac{1-\phi(t)}{t^\alpha\ell(t)}\le 1 \quad \text{if } \alpha<1.$$

Moreover, if (H2) holds, then we can take $\ell(t)=c$ *if* $\rho'(\alpha)<0$ *and* $\ell(t)=c|\log t|$ *if* $\rho'(\alpha)=0$, *for some constant* $c>0$.

Remark 2. *In the case where* $\alpha<1$, *the function* $\ell(t)$ *can be constructed shch that for all* $t>0$ *sufficiently small,* $1-\phi(t)\le t^\alpha\ell(t)$.

Corollary 2. Under the conditions of Theorem 2, if $\mu\in\mathcal{F}$, then *the following conclusions hold:*

(i) $\mu(x,\infty)=O(x^{-\alpha}l(x))$ $(x\to+\infty)$. *(ii)* $\int_0^\infty x^a d\mu(x)<\infty$ *for all* $a\in[0,\alpha)$.

(iii) $\int_0^\infty x^\alpha d\mu(x)<\infty \Leftrightarrow \alpha=1$, $\rho'(1)<0$ and $\mathbb{E}S\log^+S<\infty$.

(iv) If $\alpha=1$, *then* $\int_0^x \mu(t,\infty)dt\sim\ell(x)$ $(x\to+\infty)$ *is slowly varing.*

This result was known only for Galton-Watson processes in the context of branching processes. For the proof, see Theorem 11.2.

Theorem 3. *(Convergence theorem) Suppose that* $\mathcal{F}\neq\emptyset$. *Let* α *be the unique point in (0,1] such that* $\rho(\alpha)=1$ *and* $\rho'(\alpha)\leq 0$. *Assume that either* $\alpha=1$ *or (H2) holds. If* $\phi\in\mathcal{F}$ *and* $\eta\in\mathcal{L}$ *are such that* $1-\phi(t)\sim 1-\eta(t)$ $(t\to 0+)$, *then*

$$\lim_{n\to\infty} T^n\eta=\phi.$$

The result means that if the behaviour of η matches that of ϕ at 0, then ϕ can be obtained by iterations of T with the initial element η. Other limit theorems will also be given. (cf. sections 7,8 and 9).

Corollary 3. (*uniqueness of nontrivial fixed points*) *Suppose that* $\mathcal{F}\neq\emptyset$. *Let* α *be the unique point in (0,1] such that* $\rho(\alpha)=1$ *and* $\rho'(\alpha)\leq 0$. *Assume that either* $\alpha=1$ *or (H2) holds. If* ϕ_1 *and* ϕ_2 *are nontrivial fixed points of T such that* $1-\phi_1(t)\sim 1-\phi_2(t)$ $(t\to 0+)$, *then* $\phi_1\equiv\phi_2$.

As a special case, if $\alpha=1$ and ϕ_1 *and* ϕ_2 *are nontrivial fixed points of T* with same finite mean, then $\phi_1\equiv\phi_2$.

Suppose that $\mathcal{F}\neq\emptyset$. Let α be the unique point in (0,1] for which $\rho(\alpha)=1$ and $\rho'(1)\leq 0$. Let X_α be a random variable with distribution determined by

$$\mathbb{E}f(X_\alpha) = \frac{1}{\rho(\alpha)}\mathbb{E}\sum\nolimits^* A_i^\alpha f(-\log A_i)$$

for nonnegative Borel functions f on $\mathbb{R}^1$, where $\sum^*$ denotes the summation over all the i's such that $A_i>0$. The problem is called of *lattice* type if there is an s>0 such that X_α is concentrated on the set $s\mathbb{Z}=\{zs\}$ $(z\in\mathbb{Z})$:

$$\sum_{z\in\mathbb{Z}} P(X_\alpha=zs)=1,$$

where $\mathbb{Z}=\{0,\pm 1, \pm 2,...\}$. Since

$$P(X_\alpha=zs) = \frac{1}{\rho(\alpha)}\mathbb{E}\sum\nolimits^* A_i^\alpha 1_{(-\log A_i=zs)},$$

by the definition of X_α, we see that the problem is of lattice type if and only if

$$\rho(\alpha) = \mathbb{E}\sum_i{}^{*} \sum_{z\in\mathbb{Z}} A_i^{\alpha}\, 1_{(-\log A_i = zs)}\,,$$

which reads also

$$\mathbb{E}\sum_i{}^{*} A_i^{\alpha}\, [1-1_{(-\log A_i = zs \text{ for some } z\in\mathbb{Z})}] = 0.$$

Therefore, *the problem is of lattice type if and only if there is an s>0 so that with probability one, each A_i is an integer multiple of s if A_i>0.* We will always take s to be the largest possible such number and will refer to it as the *span.* We set s=0 if the problem is non-lattice.

If s>0 and α<1, let $\mathcal{B}_{\alpha,s}$ be the collection of all strictly positive infinitely differentiable functions *p* on $\mathbb{R}^1$ which satisfy

(a) $p(x+s)=p(x)$ for all $x\in\mathbb{R}^1$, and

(b) $(-1)^k \frac{d^k}{d\theta^k}[\theta^{\alpha}p(-\log\theta)]\le 0$ for all k=1,2,...

If s=0 or α=1, let $\mathcal{B}_{\alpha,s}$ be the set of positive constant functions on $\mathbb{R}^1$. The class $\mathcal{B}_{\alpha,s}$ is relatively large for α<1 and s>0. For example, if a_n and b_n are numbers which satisfy

$$\sum_{n=1}^{\infty} \sqrt{(a_n^2+b_n^2)\prod_{j=0}^{\infty}[1+n^2/(j-\alpha)^2]} \le 1$$

(0<α<1) and if

$$p(x)=1+\sum_{n=1}^{\infty}(a_n \sin nx + b_n \cos nx)$$

then $p\in\mathcal{B}_{\alpha,2\pi}$ [cf. Durrett and Liggett (1983, th.5.2)].

Theorem 4. *(Totality of fixed points) Suppose that $\mathcal{F}\neq\emptyset$. Let α be the unique point in (0,1] for which ρ(α)=1 and ρ'(α)≤0. If (H2) holds, then there is a natural bijective correspondence between $\phi\in\mathcal{F}$ and $p\in\mathcal{B}_{\alpha,s}$ which is given by*

$$\lim_{t\to 0+} \frac{1-\phi(t)}{t^{\alpha}p(-\log t)} = 1 \qquad \text{if } \rho'(\alpha)<0$$

and

$$\lim_{t\to 0+} \frac{1-\phi(t)}{t^{\alpha}p(-\log t)\,|\log t|} = 1 \quad \text{if } \rho'(\alpha)=0.$$

The result means that subject to (H2), the fixed points can be identified by the behaviour of their Laplace transforms at 0. When $\alpha=1$ and $\rho'(\alpha)<0$, this just says that the fixed points are parametrized by their means.

corollary 4. *Under the conditions of Theorem 4, if $\alpha<1$ and the problem is nonlattice, then for all $\mu\in\mathcal{M}$, there is a constant $c>0$ such that, as $x\to\infty$,*

$$\mu(x,\infty) \sim cx^{-\alpha} \text{ if } \rho'(\alpha)<0,$$

and

$$\mu(x,\infty) \sim cx^{-\alpha}\log x \text{ if } \rho'(\alpha)=0.$$

2. Applications

2.1. Poincaré functional equation.

Let $N\geq 0$ be an integer-valued random variable with $m=\mathbb{E}N\in(1,\infty]$. Seneta (1968) showed that if $m<\infty$, then the Poincaré functioal equation

$$\phi(s) = \mathbb{E}\phi^N\left(\frac{s}{m}\right) \quad (s\geq 0) \tag{2.1}$$

has always a nontrivial solution in $\mathcal{L}$; he proved in sequal (see Seneta 1974 or Athreya 1971) that for any nontrvial solution $\phi\in\mathcal{L}$, there is a slowly varing function $\ell(s)$ at 0 so that $1-\phi(s) \sim s\ell(s)$ $(s\to 0)$.

Let us consider a slightly more general form of (2.1): for any given $a\in(0,\infty)$,(a may be <1,$=1$ or >1), we consider the functional equation

$$\phi(s) = \mathbb{E}\phi^N(as) \equiv T\phi(s) \quad (s\geq 0) \tag{2.1a}$$

in $\mathcal{L}$. This is the special case of our general transformation T with $A_i=a$ if $1\leq i\leq N$ and $A_i=0$ if $i>N$. Our theorems in section 1 apply, and we have

Theorem 2.1. *The Poincaré functional equation (2.1a) has a nontrivial solution in $\mathcal{L}$ if and only if $a\mathbb{E}N\leq 1$* (i.e., $\mathbb{E}N<\infty$ and $a\leq 1/\mathbb{E}N$)*; there is a nontrivial solution with finite mean if and only if $\mathbb{E}N\log^+N<\infty$. For any nontrvial solution $\phi\in\mathcal{L}$, there is a slowly varing function $\ell(s)$ at 0 so that $1-\phi(s) \sim s\ell(s)$ $(s\to 0)$. Each nontrivial solution ϕ can be obtained as a limit of iterations $T^n\psi$ $(n\to\infty)$ of T [defined in (2.1a)] with an arbitrary initial element $\psi\in\mathcal{L}$ satisfing $1-\psi(t) \sim 1-\phi(t)$ $(t\to 0)$.*

Proof. We have $\rho(x)=a^x\mathbb{E}N$. If a<1, then the hypothesis (H3) holds, so by Theorem 1 and Remark 1, the equation (2.1a) has a nontrivial solution if and only if $a\mathbb{E}N\le 1$. By Theorem 0, There is a nontrivial solution with finite mean if and only if $\mathbb{E}N\log^+N<\infty$. If $a=1$, then the hypothesis (H0) is not satisfied, and the conclusion comes from the discussion following that hypothesis. If a>1, then since $\mathbb{E}N>1$, we can find $\bar{a}<1$ such that $\bar{a}\mathbb{E}N>1$. Thus the result follows by Remark 1. ■

Let $Z_n(n\ge 0)$ be a supercritical Galton-Watson process with $Z_0=1$ and $Z_1=N$. The well-known Kesten-Stigum Theorem says that if $1<m=\mathbb{E}Z_1<\infty$, the random variables Z_n/m^n converge almost surely to a nondegenerate random variable W if and only if $\mathbb{E}Z_1\log^+Z_1<\infty$. This result is deepened by the Seneta-Heyde Theorem which says that, if $1<\mathbb{E}N<\infty$, then there is a sequence of constants $c_n>0$ $(c_n\to\infty)$ such that the random variables Z_n/c_n converge almost surely to a nontrivial random variable W whose Laplace transforms satisfies (2.1) (and $c_n\sim m^n$ if and only if $\mathbb{E}Z_1\log^+Z_1<\infty$). If $\mathbb{E}N=\infty$, it was proved by Seneta (1969) that such a sequence does not exist (even for convergence in distribution). We see that the last conclusion can also be derived by Theorem 2.1, noting that if $0<c_n\to\infty$ is such that Z_n/c_n converge in distribution to some nontrivial random variable W, then for some $0<a<1$, $c_n/c_{n+1}\to a$ (Seneta 1969,p.29) and consequently the Laplace transform of W satisfies (2.1a) (easy).

2.2. Crump-Mode process.

We consider a general branching process $\{Z(t):t\ge 0\}$ in the sense of Crump and Mode (1968-69) with a single ancestor $Z(0)=1$. Each individual reproduces independently; for any given parent individual the instants of birth of offspring are represented by the jumps of a counting process $\{N(t):t\ge 0\}$ with $N(0)=0$ and $N(\infty)<\infty$ which increases by one at the instants of birth of offspring; this process and the life time L of the parent may be dependent. We assume throughout that *either* $1<\mathbb{E}N(\infty)<\infty$, in which case there exists a unique positive α with

$$\mathbb{E}\int_0^\infty e^{-\alpha x}dN(x) = 1,$$

or $\mathbb{E}N(\infty)=\infty$ and $\exists$ a positive α (necessarily unique) satisfing the preceding identity. It was proved by Doney (1972) that the limit (in distribution)

$$W \stackrel{d}{:=} \lim_{t\to\infty} \frac{Z(t)}{\mathbb{E}Z(t)} \tag{2.2}$$

exists, and satisfies the functional equation

$$\phi(s) = \mathbb{E}\exp\{\int_0^\infty \log\phi(se^{-\alpha x})dN(x)\} \quad (s\geq 0), \tag{2.3}$$

where $\phi(s)=\mathbb{E}e^{-Ws}$. Writing $N=N(\infty)$ for the total number of offspring of a given parent, $t_1,t_2,\ldots,t_N$ for the successive instants of their births, and

$$A_i = e^{-\alpha t_i}, \quad (1\leq i\leq N) \tag{2.4}$$

we see that (2.3) can be reformulated as

$$\phi(s)= \mathbb{E} \prod_{i=1}^{N} \phi(sA_i) \equiv T\phi(s). \tag{2.3'}$$

Doney (1972) proved that (2.3) has a nontrivial solution ϕ in $\mathcal{L}$ with $1-\phi(s) \sim s$ $(s\to 0)$ if and only if $\mathbb{E}Y\log^+Y <\infty$, where

$$Y:= \int_0^\infty e^{-\alpha x}dN(x) \equiv \sum_{i=1}^{N} A_i. \tag{2.5}$$

Our Theorems in section 1 complete this as follows:

Theorem 2.2. *The functional equation (2.3) has always a nontrivial solution in* $\mathcal{L}$. *For any nontrivial solution* ϕ, *there is a slowly varing function* $\ell(s)$ *at 0 such that* $1-\phi(s) \sim s\ell(s)$ $(s\to 0)$. *For any given slowly varing function* $\ell(s)$ *at 0, there is at most one solution* ϕ *in* $\mathcal{L}$ *satisfing* $1-\phi(s) \sim s\ell(s)$ $(s\to 0)$. *Any solution* ϕ *can be obtained as a limit of iterations* $T^n\psi$ $(n\to\infty)$ *of* T *(defined in (2.3)') with an arbitrary initial element* $\psi\in\mathcal{L}$ *satisfing* $1-\psi(t) \sim 1-\phi(t)$ $(t\to 0)$.

Of course, the uniqueness in the above theorem can also be reformulated in the following way: if $\phi_1,\phi_2 \in\mathcal{L}$ are solutions satisfiny $1-\phi_1(s) \sim 1-\phi_2(s)$ $(s\to 0)$, then $\phi_1=\phi_2$.

The Bellman-Harris process is the particular case of a Crump-Mode process with

$$N(x)=\begin{cases} 0, & \text{if } x<L; \\ N, & \text{if } x\geq L, \end{cases}$$

where the offspring distribution N and the lifetime L are mutually independent. In this case, the functional equation (2.3) or (2.3)' reduces to

$$\phi(s)= \mathbb{E}\phi^N(se^{-\alpha L}) \equiv T\phi(s), \tag{2.6}$$

where α is the unique number in $(0,\infty)$ satisfing $\mathbb{E}Ne^{-\alpha L}=1$ (We suppose that $\mathbb{E}N>1$). Athreya (1971) proved that (2.6) has a nontrivial solution ϕ in $\mathcal{L}$ with $1-\phi(s)\sim s$ $(s\to 0)$ if and only if $\mathbb{E}N\log^+N<\infty$, and demanded whether it had always a nontrivial solution in $\mathcal{L}$ if $\mathbb{E}N\log^+N=\infty$. Our Theorem 2.1 answers this question in a more general setting.

2.3. Branching random walks

A branching random walk on the real line $\mathbb{R}^1$ can be described in the following way. An initial ancestor, who forms the zeroth generation, is created at the origin. His children form the first generation and their positions on the real line are described by the point process Z^1 on $\mathbb{R}^1$. Thus Z^1 is a random locally finite counting measure. The people in the nth generation give birth independently of one another and of the preceding generations to form the (n+1)th generation. The point process describing the displacements of the children of a person from that person's position has the same distribution as Z^1. Let $\{z_r^n\}$ be an enumeration of the positions of the people in the nth generation, and Z^n be the point process with the atoms $\{z_r^n\}$. Define

$$m(\theta):= \mathbb{E}\sum_r \exp(-\theta z_r^1) = \mathbb{E}\int e^{-\theta t}dZ^1(t).$$

We assume $m(0)>1$ and $m(\theta)<\infty$ for some fixed θ. The generation size $Z^n(-\infty,\infty)$ in the branching random walk form a supercritical Galton-Watson process. It is known (see Biggins 1977) that

$$W^n(\theta) := m(\theta)^{-n} \sum_r \exp(-\theta z_r^n)$$

is a martingale with respect the σ-field $\mathbb{F}_n$ generated by the births in the first n generations, and the limit

$$W(\theta) := \lim_{n\to\infty} W^n(\theta)$$

satisfies the functional equation

$$\phi(s) = \mathbb{E} \prod_r \phi(sA_r) \equiv T\phi(s) \tag{2.7}$$

with $\phi(s)=\mathbb{E}e^{-sW(\theta)}$ and

$$A_r = m(\theta)^{-1}\exp(-\theta z_r^1).$$

(Therefore $\mathbb{E}\sum A_r=1$). We consider the equation (2.7). If $\theta=0$, it reduces to the Poicaré functional equation (2.1). So we assume $\theta\neq 0$. Biggins (1977) obtained sufficient conditions for this equation to have nontrivial solutions of finite mean. Our results in section 1 will be applied to complete Biggins' theorem. We notice that if

$$m(0)<\infty \quad \text{and} \quad \mathbb{E} \sum_r |z_r^1| \exp(-\theta z_r^1)<\infty , \tag{2.8}$$

then

$$m(x)<\infty \quad \text{and} \quad m'(x) = -\mathbb{E} \sum_r z_r^1 \exp(-x z_r^1) \in (-\infty,\infty) \text{ exists}$$

for all $x\in(0,\theta]$ if $\theta>0$ and for all $x\in[\theta,0)$ if $\theta<0$; Also (H1) holds since

$$\rho(0)=\mathbb{E}\tilde{N}=m(0) \quad \text{and} \quad \mathbb{E}\sum A_r |\log A_r| \le \theta m(\theta)^{-1}\mathbb{E} \sum_r \exp(-\theta z_r^1)|z_r^1| + |\log m(\theta)|;$$

Finally for all $x\in(0,1]$,

$$\rho(x) = \mathbb{E} \sum_r A_r^x = m(\theta x)/[m(\theta)]^x \tag{2.9}$$

and

$$\rho'(x) = [\theta m'(\theta x) - m(\theta x)\log m(\theta)] /[m(\theta)]^x. \tag{2.10}$$

Therefore, by Corollary 1, Theorem 2 and Remark 2, and Theorem 3 and Remark 3, we obtain

Theorem 2.3. *Assume $m(0)>1$ and $m(\theta)<\infty$ for some fixed $\theta\neq 0$. Then the functional equation (2.7) has always a nontrivial solution in $\mathcal{L}$. Assume additionally*

(2.8) and let α *be the unique number in (0,1] such that*

$$m(\theta\alpha) = [m(\theta)]^{\alpha} \quad \text{and} \quad \theta m'(\theta\alpha) - m(\theta\alpha)\log m(\theta) \leq 0. \tag{2.11}$$

Then:

(a) For any nontrivial solution ϕ*, there is a slowly varing function* $\ell(s)$ *at 0 such that* $1-\phi(s) \sim s\ell(s)$ $(s\to 0+)$ *if* $\alpha=1$ *and* $1-\phi(s) \leq s^{\alpha}\ell(s)$ *for all sufficiently small* $s>0$ *if* $\alpha<1$*;*

(b) For any given slowly varing function $\ell(s)$ *at 0, there is at most one solution* ϕ *in* $\mathcal{L}$ *satisfing* $1-\phi(s) \sim s\ell(s)$ $(s\to 0)$*;*

(c) Any solution ϕ *can be obtained as a limit of iterations* $T^n\psi$ $(n\to\infty)$ *of T (defined in (2.7)) with an arbitrary initial element* $\psi\in\mathcal{L}$ *satisfing* $1-\psi(s) \sim 1-\phi(s)$ $(s\to 0)$*;*

(d) All solutions are of finite moments of order strictly inferior to α*; their* α*-th moments are finite if and only if*

$$\mathbb{E}\, W^1(\theta)\log^+ W^1(\theta) < \infty \quad \text{and} \quad \theta m'(\theta) - m(\theta)\log m(\theta) < 0 \tag{2.12}$$

(so $\alpha=1$). *In particular, there is a solution of finite (first) moment if and only if (2.12) holds.*

3. Sufficient conditions

We first determine the extinction probability $\mu\{o\}$ of any fixed point $\mu\in\mathcal{F}$. We remark that $\mu\{o\}\equiv\phi(\infty)$ if ϕ is the Laplace transform of μ.

Theorem 3.1. *If* $\mathcal{F}\neq\emptyset$*, then (a)* $\mathbb{E}\tilde{N}>1$*, and (b) for any* $\mu\in\mathcal{F}$*,* $\mu\{0\}$ *is the unique fixed point in [0,1) of the function* $f(t) := \sum_{k=0}^{\infty} P(\tilde{N}=k)t^k \equiv \mathbb{E}t^{\tilde{N}}$.

Proof. Let $\{Z_i\}$ be independent random variables with distribution μ, which are independent of $\{A_i\}$ as well. Then the extinction probability $q:=\mu\{0\}$ $\in[0,1)$ satisfies

$$q= P(Z_1=0) = P\Big(\sum_{i=1}^{\infty} A_iZ_i=0\Big) = P(\forall i\ Z_i=0 \text{ if } A_i>0) = \sum_{k=0}^{\infty} P(\tilde{N}=k)q^k.$$

So f has a fixed point in [0,1). Since f is convex, $f(1)=1$ and $f'(1)=\mathbb{E}\tilde{N}$, it follows that $\mathbb{E}\tilde{N}>1$ or $f(t)=t$ for all t. In the latter case $\tilde{N}=1$ a.s., which is excluded by our hypothesis (H0). So $\mathbb{E}\tilde{N}>1$, and f has a unique fixed point in [0,1), which proves both parts of the theorem. ∎

We then remove the moment condition $\mathbb{E}A_i^\gamma<\infty$ ($\gamma>1$, i=1,2,...) of a result of Durrett and Liggett (1983, Theorem 1).

Lemma 3.2. *Suppose that* $\|\tilde{N}\|_\infty<\infty$. *If* $\rho(0)>1$ and $\inf_{x\in[0,1]} \rho(x)\leq 1$, *then* $\mathcal{F}\neq\emptyset$.

Proof. Since $\|\tilde{N}\|_\infty<\infty$, N can be taken as a constant. If for some $\gamma>1$ and all i=1,2,... $\mathbb{E}A_i^\gamma<\infty$, then (H1) holds and the given conditions are equivalent to $\rho(\alpha)=1$ and $\rho'(\alpha)\leq 0$ for some $\alpha\in(0,1]$. So the conclusion follows by Theorem 1 of Durrett and Liggett (1983).

To prove the result in the general case, we define

$$\tilde{A}_i(M)= A_i\wedge M \equiv \min(A_i,M)$$

for all M>0 and $i\geq 1$, and let ρ_M and $\tilde{T}_M$ be the corresponding function and smoothing transformation defined in terms of $\{\tilde{A}_i(M)\}$ just as ρ and T were defined in terms of $\{A_i\}$. Then $\rho_M(0)\equiv \rho(0)>1$. Since

$$\inf_{x\in[0,1]} \rho_M(x)\leq \inf_{x\in[0,1]} \rho(x)\leq 1,$$

$\tilde{T}_M$ has a nontrivial fixed point η_M by the preceding conclusion, and $\eta_M(\infty)=q\in[0,1)$ is independent of M by Theorem 3.1 since the function f(t) defined therein does not depend on M. Choose c>0 such that $\eta_M(c)=(q+1)/2$. This is possible since $\eta_M(0)=1$ and $\eta_M(\infty)=q$. So $\tilde{T}_M$ has a fixed point $\phi_M(t):=\eta_M(ct)$ with

$$\phi_M(1)=(q+1)/2.$$

By the selection and continuity theorem, we can choose a sequence $M_n\to\infty$ so that ϕ_{M_n} converge to a limit ϕ which is the Laplace transform of a possibly defective distribution μ (see for example Feller 1971, pp.267 and 431). Since

$$\phi_M(t)= \mathbb{E}\prod_{i=1}^{N}\phi_M(t(A_i\wedge M)),$$

evaluating this at $M=M_n$ and passing to limit as $n\to\infty$, we obtain by the dominated convergence theorem that

$$\phi(t)=\mathbb{E}\prod_{i=1}^{N}\phi(tA_i).$$

Letting $t\to 0$ we see that $\phi(0+)=f(\phi(0+))$, where $f(t)=\mathbb{E}t^{\tilde{N}}$. Since $\phi(0+)\geq\phi(1)=(q+1)/2\in(q,1)$, it follows that $\phi(0+)=1$ and $\phi(\infty)\leq\phi(1)<1$. Thus μ is not defective and $\mu\neq\delta_0$. Consequently $\phi\in\mathcal{F}$. ■

We now prove our main theorem for existence of nontrivial fixed points.

Theorem 3.3. If $\rho(0)>1$ *and* $\inf_{x\in[0,1]}\rho(x)\leq 1$, then $\mathcal{F}\neq\emptyset$.

Proof. The argument is similar to that used in the proof of Lemma 3.2. For $M=1,2,\ldots$, define $\bar{A}_i=A_i$ if $i\leq M$ and $\bar{A}_i=0$ if $i>M$. Let $\bar{T}_M$ be the corresponding smoothing transformation defined in terms of $\{\bar{A}_i\}$ and put

$$\rho_M(x)=\mathbb{E}\sum_{i=1}^{M\wedge N}A_i^x\ (x\geq 0),$$

where $M\wedge N:=\min(M,N)$. Then $\rho_M(x)$ increases to $\rho(x)$ for all $x\geq 0$ as M increases to ∞. Choose M sufficiently large such that $\rho_M(0)>1$. Since

$$\inf_{x\in[0,1]}\rho_M(x)\leq\inf_{x\in[0,1]}\rho(x)\leq 1,$$

$\bar{T}_M$ has a nontrivial fixed point η_M with $q_M:=\eta_M(\infty)\in[0,1)$ by Lemma 4.2. Note that q_M is the extinction probability of a supercritical Galton-Watson process with offspring distribution

$$\tilde{N}_M:=\sum_{i=1}^{M\wedge N}1_{\{A_i>0\}}$$

which increases with M, we see that q_M decreases as M increases. Thus the limit

$$q:=\lim_{M\to\infty}q_M \text{ exists with } q<1.$$

Choose $c>0$ such that $\eta_M(c)=(q_M+1)/2$. So $\bar{T}_M$ has a fixed point $\phi_M(t):=\eta_M(ct)$ with

$$\phi_M(1)=(q_M+1)/2.$$

By the selection and continuity theorems for distributions and Laplace transforms, we can choose a sequence $M_n \to \infty$ so that ϕ_{M_n} converge to a limit ϕ which is the Laplace transform of a possibly defective distribution μ. Since ϕ_M is a fixed point of $\tilde{T}_M$,

$$\phi_M(t) = \mathbb{E} \prod_{i=1}^{M \wedge N} \phi(tA_i).$$

Evaluating this at $M=M_n$ and passing to limit as $n\to\infty$, we obtain by the dominated convergence theorem that

$$\phi(t) = \mathbb{E} \prod_{i=1}^{N} \phi(tA_i).$$

Letting $t\to 0$ we see that $\phi(0+)=f(\phi(0+))$, where $f(t)=\mathbb{E}t^{\tilde{N}}$. Since

$$\phi(0+) \geq \phi(1) = \lim_{n\to\infty} \phi_{M_n}(1) = \lim_{n\to\infty} (q_{M_n}+1)/2 = (q+1)/2 \in (q,1),$$

it follows that $\phi(0+)=1$ and $\phi(\infty)\leq \phi(1)<1$. Thus μ is not defective and $\mu\neq\delta_0$. Consequently $\phi\in\mathcal{F}$. ■

We remark that we can prove in fact that q also verifies $q=\mathbb{E}q^{\tilde{N}}$ by passing to limit in $q_M=\mathbb{E}q_M^{\tilde{N}_M}$ as $M\to\infty$. So $q=\phi(\infty)$.

4. Necessary conditions. Basic properties of fixed points.

Throughout this section, we assume (H1). Let α be a point in (0,1] and X_α a random variable with distribution determined by

$$\mathbb{E}f(X_\alpha) = \frac{1}{\rho(\alpha)}\mathbb{E} \sum{}^{*} A_i^{\alpha} f(-\log A_i) \tag{4.1}$$

for nonnegative Borel functions f on $\mathbb{R}^1$, where $\sum^*$ denotes the summation over all the i's such that $A_i>0$. This is possible since the right hand side of (4.1) is a positive linear functional with unit norm. Given $\phi\in\mathcal{L}$ with $\phi\not\equiv 1$, define $D_\alpha(x)$ and $G_\alpha(x)$ by

$$\begin{aligned} &D_\alpha(x)= e^{\alpha x}[1-\phi(e^{-x})] \quad \text{and} \\ &G_\alpha(x)= e^{\alpha x}\, \mathbb{E}\left\{\prod_{i=1}^{N} \phi(e^{-x}A_i) -1+ \sum_{i=1}^{N} [1-\phi(e^{-x}A_i)]\right\}. \end{aligned} \tag{4.2}$$

For $\tilde{\phi}\in\mathcal{L}$, we define $\tilde{D}_\alpha$ and $\tilde{G}_\alpha$ analogously in terms of $\tilde{\phi}$.

Lemma 4.0. *(i) If $\tilde{\phi}=T\phi$, then $\tilde{D}_\alpha(x)=\rho(\alpha)\mathbb{E}D_\alpha(x+X_\alpha)-G_\alpha(x)$.*

(ii) $e^{-\alpha x}G_\alpha(x)\geq 0$ is a decreasing function of $x\in\mathbb{R}^1$.

(iii) If $\tilde{\phi}\geq\phi$, then $\tilde{G}_\alpha\leq G_\alpha$.

(iv) $G_\alpha(x)\leq e^{\alpha x}\mathbb{E}F(\min\{\tilde{N},\tilde{S}D_\alpha(x)e^{-\alpha x}\})$, where $\tilde{S}=\sum_{i=1}^{N}\max(A_i,1)$ and $F(u)=e^{-u}-1+u$.

(v) $\lim_{x\to\infty}\frac{G_\alpha(x)}{D_\alpha(x)}=0.$

Proof. With some slight modifications, the argument is the same as those in the proofs of lemmas 2.3,2.4 and 2.6 of Durrett and Liggett (1983,pp.282-284). In fact, part (i) holds since

$$\tilde{D}_\alpha(x)=e^{\alpha x}[1-\tilde{\phi}(e^{-x})]=e^{\alpha x}\left[1-\mathbb{E}\prod_{i=1}^{N}\phi(e^{-x}A_i)\right]=e^{\alpha x}\,\mathbb{E}\sum_{i=1}^{N}[1-\phi(e^{-x}A_i)]-G_\alpha(x)$$

$$=\mathbb{E}\sum_{i=1}^{N}A_i^\alpha D_\alpha(x-\log A_i)-G_\alpha(x)=\rho(\alpha)\mathbb{E}D_\alpha(x+X_\alpha)-G_\alpha(x).$$

Parts (ii) and (iii) follow from the fact that

$$\prod_{i=1}^{N}u_i-1+\sum_{i=1}^{N}(1-u_i)\geq\prod_{i=1}^{N}v_i-1+\sum_{i=1}^{N}(1-v_i)\quad\text{if } 0\leq u_i\leq v_i\leq 1.$$

For part (iv), use the inequality $u\leq e^{-(1-u)}$ to obtain

$$G_\alpha(x)\leq e^{\alpha x}\mathbb{E}\left\{\exp\left(-\sum_{i=1}^{N}[1-\phi(e^{-x}A_i)]\right)-1+\sum_{i=1}^{N}[1-\phi(e^{-x}A_i)]\right\}.$$

Since $\phi\in\mathcal{L}$, $\frac{1-\phi(u)}{u}$ is decreasing and $1-\phi(u)$ is increasing in u. Therefore

$$1-\phi(e^{-x}A_i)\leq\max(A_i,1)\,[1-\phi(e^{-x})].$$

Part (iv) now follows from the monotonicity of F on $[0,\infty)$ and the fact that

$$\sum_{i=1}^{N}[1-\phi(e^{-x}A_i)]\leq\min\{\tilde{N},\tilde{S}[1-\phi(e^{-x})]\}=\min\{\tilde{N},\tilde{S}D_\alpha(x)e^{-\alpha x}\}.$$

For part (v), note that since $\lim_{x\to\infty}D_\alpha(x)e^{-\alpha x}=\lim_{x\to\infty}[1-\phi(e^{-x})]=0$, it suffices, by changing variables $t=D_\alpha(x)e^{-\alpha x}$, to show that

$$\lim_{t\to 0}\frac{\mathbb{E}F(\tilde{S}t)}{t}=0.$$

But this follows from the dominated convergence theorem since $\frac{F(u)}{u}$ is bounded on $(0,\infty)$ and tends to zero as $u\to 0+$, and since $\hat{S}$ has a finite first moment by (H1). ∎

The proof of the following lemma follows that of Lemma (2.11) of Durrett and Liggett (1983,p.285). However, we shall give the details which are not only for the convenience of readers but will also be used later.

Lemma 4.1. *(a) Fix* $\alpha\in(0,1]$ *and let* $\mathcal{A}_\alpha$ *be the set of all functions g on* $\mathbb{R}^1$ *which satisfy*

(i) $g(o)=1$,

(ii) $g(y)e^{-\alpha y}$ *is decreasing in y, and*

(iii) $g(y)e^{-(1-\alpha)y}$ *is increasing in y. Then* $\mathcal{A}_\alpha$ *is uniformly bounded and equicontinuous on bounded sets, and, for all* $g\in\mathcal{A}_\alpha$ *and all* $y\in\mathbb{R}^1$,

$$Min\ \{e^{\alpha y},\ e^{-(1-\alpha)y}\} \le g(y) \le Max\ \{e^{\alpha y},\ e^{-(1-\alpha)y}\}. \qquad (4.3a)$$

If additionally g'(y) exists, then

$$-(1-\alpha)g(y) \le g'(y) \le \alpha g(y). \qquad (4.3b)$$

(b) Assume the problem is non-lattice and let $\mathcal{B}_\alpha$ *be the set of all functions g in* $\mathcal{A}_\alpha$ *which satisfy*

$$g(y) = \rho(\alpha)\mathbb{E}g(y+X_\alpha), \qquad \forall y\in\mathbb{R}^1. \qquad (4.4)$$

Then $\mathcal{B}_\alpha$ *is the set of all convex combinations of* $g_\beta(y) = e^{(\alpha-\beta)y}$ *for the (at most two)* β*'s which satisfy* $0\le\beta\le 1$ *and* $\rho(\beta)=1$.

Proof. (a) Fix $y\in\mathbb{R}^1$. By properties (ii) and (iii), if $\Delta y>0$, then

$$g(y+\Delta y)e^{-\alpha(y+\Delta y)} \le g(y)e^{-\alpha y} \quad \text{and} \quad g(y+\Delta y)e^{(1-\alpha)(y+\Delta y)} \ge g(y)e^{(1-\alpha)y}.$$

That is,

$$g(y)e^{-(1-\alpha)\Delta y} \le g(y+\Delta y) \le g(y)e^{\alpha\Delta y} \quad \text{if } \Delta y>0. \qquad (4.5a)$$

Similarly,

$$g(y)e^{\alpha\Delta y} \le g(y+\Delta y) \le g(y)e^{-(1-\alpha)\Delta y} \quad \text{if } \Delta y<0. \qquad (4.5b)$$

We remark that (4.5) can be rewritten as

$$g(y_1)\ \mathrm{Min}\{e^{\alpha\Delta y}, e^{-(1-\alpha)\Delta y}\} \le g(y_2) \le g(y_1)\ \mathrm{Max}\{e^{\alpha\Delta y}, e^{-(1-\alpha)\Delta y}\} \quad (4.6a)$$

for all $(y_1,y_2)\in\mathbb{R}^2$, where $\Delta y=y_2-y_1$. Taking $y_1=1$, (4.6a) gives (4.3a). As a consequence of (4.3a), $\mathcal{A}_\alpha$ is uniformly bounded on bounded sets. Again by (4.6a),

$$g(y_1)\ \mathrm{Min}\{e^{\alpha\Delta y}-1, e^{-(1-\alpha)\Delta y}-1\} \le g(y_2)-g(y_1)$$
$$\le g(y_1)\ \mathrm{Max}\{e^{\alpha\Delta y}-1, e^{-(1-\alpha)\Delta y}-1\}. \quad (4.6b)$$

Combing this with (4.3a), we see that $\mathcal{A}_\alpha$ is equicontinuous on bounded sets. Dividing (4.6a) by Δy and letting $\Delta y\to 0$ give (4.3b).

(b) It is easy to check that $g_\beta\in\mathcal{B}_\alpha$ if and only if $0\le\beta\le 1$ and $\rho(\beta)=1$. Thus the conclusion is immediate if $\mathcal{B}_\alpha=\emptyset$. Assume then that $\mathcal{B}_\alpha$ is not empty. By (a) and Ascoli's theorem, $\mathcal{B}_\alpha$ is a relatively compact subset of $C(\mathbb{R}^1)$ with the topology of uniform convergence on bounded sets of $\mathbb{R}^1$. We claim that $\mathcal{B}_\alpha$ is also closed. Let $g_n\in\mathcal{B}_\alpha$ be such that $g_n(y)\to g(y)$ $(n\to\infty)$ uniformly on bounded sets for some $g\in C(\mathbb{R}^1)$. Then it is easily seen that $g\in\mathcal{A}_\alpha$. Since

$$g_n(y+X_\alpha) \le \exp\{\alpha(y+X_\alpha)\} + \exp\{-(1-\alpha)(y+X_\alpha)\} \qquad \text{[by (4.3)]},$$
$$\mathbb{E}\exp\{\alpha X_\alpha\} = \rho(0)/\rho(1)<\infty \text{ and } \mathbb{E}\exp\{-(1-\alpha)X_\alpha\} = \rho(1)/\rho(\alpha)<\infty, \quad (4.7)$$

by the dominated convergence theorem, we see that $\forall y\in\mathbb{R}^1$,

$$\lim_{n\to\infty} \mathbb{E}g_n(y+X_\alpha) = \mathbb{E}g(y+X_\alpha). \quad (4.8)$$

Thus we can pass to the limit in $g_n(y) = \rho(\alpha)\ \mathbb{E}g_n(y+X_\alpha)$ to obtain (4.4). Hence $g\in\mathcal{B}_\alpha$, and so $\mathcal{B}_\alpha$ is closed. Thus $\mathcal{B}_\alpha$ is a compact convex subset of $C(\mathbb{R}^1)$ (the convexity is easy). Therefore $\mathcal{B}_\alpha$ is the closed convex hull of its extreme points by the Krein-Milman theorem.

Suppose that g is an extreme point of $\mathcal{B}_\alpha$ and let

$$g_u(y) = \frac{g(u+y)}{g(u)}.$$

By (4.4) and the fact that $\mathbb{E}g(X_\alpha) = 1/\rho(\alpha)$,

$$g(y) = \frac{\mathbb{E}g(u+X_\alpha)}{\mathbb{E}g(X_\alpha)} = \frac{\int_{-\infty}^{\infty} g_u(y)g(u)P[X_\alpha\in du]}{\int_{-\infty}^{\infty} g(u)P[X_\alpha\in du]}.$$

Since $g_u \in \mathcal{B}_\alpha$ for each u and g is extremal, this implies that $g=g_u$ for all u in the support of the distribution of X_α. Therefore

$$g(u+y)=g(u)g(y) \tag{4.9}$$

for all y and all u in the support of X_α. Let

$$\mathbb{U}=\{u\in\mathbb{R}^1:\ \forall y\in\mathbb{R}^1,\ g(u+y)=g(u)g(y)\}.$$

Then $\mathbb{U}$ is a sub-group of $(\mathbb{R}^1,+)$ since (i) if $u\in\mathbb{U}$, then taking $y=-u$ in (4.9) gives $g(-u)=1/g(u)$, and consequently, for all $y\in\mathbb{R}^1$,

$$g(-u+y)=g(u+(-u+y))/g(u)=g(-u)g(y).$$

Thus $-u\in\mathbb{U}$; (ii) if $u_1\in\mathbb{U}$ and $u_2\in\mathbb{U}$, then $u_1+u_2\in\mathbb{U}$ because $\forall y\in\mathbb{R}^1$,

$$g(u_1+u_2+y)=g(u_1)g(u_2+y)=g(u_1)g(u_2)g(y)=g(u_1+u_2)g(y).$$

Therefore $\mathbb{U}$ is either dense in $\mathbb{R}^1$ or of the form $a\mathbb{Z}$ for some $a>0$. In fact, writing $a=\inf\{u\in\mathbb{U}:\ u>0\}$, we can easily verify that $\mathbb{U}$ is dense in $\mathbb{R}^1$ if $a=0$, and $\mathbb{U}=a\mathbb{Z}$ if $a>0$. Since $\mathbb{U}$ contains the support of X_α and the problem is nonlattice, we conclude that $\mathbb{U}$ is dense in $\mathbb{R}^1$, and so $\mathbb{U}=\mathbb{R}^1$ as g is continuous. It follows that $g=g_\beta$ for some β. This completes the proof of the lemma. ∎

Theorem 4.2. *Assume (H1) and $\mathcal{F}\neq\emptyset$. Then (a) there is an $\alpha\in[0,1]$ so that $\rho(\alpha)=1$, and (b) if $\phi\in\mathcal{F}$ and $\alpha\in(0,1]$ is such that $\rho(\alpha)=1$ and $\rho'(\alpha)\leq 0$, then*

$$\limsup_{x\to\infty}\frac{D_\alpha(x+y)}{D_\alpha(x)}\leq 1 \quad \text{if } \rho'(\alpha)<0$$

and

$$\lim_{x\to\infty}\frac{D_\alpha(x+y)}{D_\alpha(x)}=1 \quad \text{if } \rho'(\alpha)=0,$$

where $y>0$ is any multiple of s if the problem is of lattice type of span s, and arbitrary otherwise.

Proof. The argument follows that of the proof of Theorem 2.12 of Durrett-Liggett (1983,p.286). Fix an $\alpha\in(0,1]$ and put

$$g_x(y)=D_\alpha(x+y)/D_\alpha(x).$$

By Lemma 4.0,

$$D_\alpha(x)=\rho(\alpha)\mathbb{E}D_\alpha(x+X_\alpha)-G_\alpha(x).$$

Evaluating this at (x+y) and dividing by $D_\alpha(x)$ gives

$$g_x(y) = \rho(\alpha)\mathbb{E}g_x(y+X_\alpha) - \frac{G_\alpha(x+y)}{D_\alpha(x+y)}\, g_x(y). \tag{4.10}$$

Since $\phi\in\mathcal{L}$, $D_\alpha(y)e^{-\alpha y} = 1-\phi(e^{-y})$ is decreasing in y and $D_\alpha(y)e^{(1-\alpha)y}$ is increasing in y. Hence $g_x\in\mathcal{A}_\alpha$ for all $x\in\mathbb{R}^1$, where $\mathcal{A}_a$ is defined in Lemma 4.1. By that lemma, the collection $\{g_x(.), x\in\mathbb{R}^1\}$ is uniformly bounded and eqicontinuous on bounded subsets of $\mathbb{R}^1$, and hence is relatively compact in the topology of uniform convergence on bounded sets. Suppose $x_n\to\infty$ and $g_{x_n}(y)\to g(y)$ uniformly on bounded sets of $\mathbb{R}^1$ for some $g\in C(\mathbb{R}^1)$. As in the proof of (4.8), it is easily seen that $\lim_{n\to\infty}\mathbb{E}g_{x_n}(y+X_\alpha) = \mathbb{E}g(y+X_\alpha)$. Thus we may pass to the limit in (4.10) to obtain (4.4), using Lemma 4.0(v). Assume from now on that the problem is nonlattice. The lattice case is similar. Since $g\in\mathcal{B}_a$, by Lemma 4.1, there is a $\beta\in[0,1]$ for which $\rho(\beta)=1$. This proves part (a) of the theorem. For part (b), suppose now that $\rho(\alpha)=1$ and $\rho'(\alpha)\le 0$. Again by Lemma 4.1,

$$g(y) = \lambda + (1-\lambda)e^{-(\beta-\alpha)y} \tag{4.11}$$

for some $\lambda\in[0,1]$, $\beta\in[0,1]$ with $\rho(\beta)=1$, and all $y\in\mathbb{R}^1$. Since $\rho(.)$ is convex, we have $\beta\ge\alpha$ if $\rho'(\alpha)<0$ and $\beta=\alpha$ if $\rho'(\alpha)=0$. It follows from (4.11) that $g(y)\le 1$ if $y>0$ when $\rho'(\alpha)<0$ and $g(y)=1$ when $\rho'(\alpha)=0$. Since this is true for all limit points of $g_x(y)$ as $x\to\infty$, the proof of the theorem is complete. ∎

Corollary 4.2. *Suppose that* $\phi\in\mathcal{F}$, $\alpha\in(0,1]$, $\rho(\alpha)=1$, $\rho'(\alpha)\le 0$, *then*

(a) $\limsup_{x\to\infty} \frac{1}{x}\log D_\alpha(x) \le 0$, *and*

(b) $G_\alpha(x)$ *is directly Riemann integrable on* $\mathbb{R}^1$ *if* $\mathbb{E}S^{1+\delta}<\infty$ *for some* $\delta>0$.

Proof. Part (a) follows from Theorem 4.5 and the monotonicity of $e^{-\alpha x}G_\alpha(x)$. To see this, let $y_0>0$ be such that $\limsup_{x\to\infty}\frac{D_\alpha(x+y_0)}{D_\alpha(x)}\le 1$. $\forall\varepsilon>0$, $\exists x_0=x_0(y_0,\varepsilon)$ such that $\forall x\ge x_0$, $D_\alpha(x+y_0)\le(1+\varepsilon)D_\alpha(x)$. Iterating this gives that for all m=1,2,...

$$\log D_\alpha(x_0+my_0)\le m\log(1+\varepsilon) + \log D_\alpha(x_0).$$

For all y>0, choose $m\in\mathbb{N}$ such that $x_0+my_0\le y<x_0+(m+1)y_0$. Thus

$$D_\alpha(x_0+y)e^{-\alpha(x_0+y)} \le D_\alpha(x_0+my_0)e^{-\alpha(x_0+my_0)},$$

and

$$\frac{1}{y}\log D_\alpha(x_0+y) \le \frac{1}{y}\{\ \log D_\alpha(x_0+my_0) + \alpha(y-my_0)\ \}$$

$$\le \frac{m\log(1+\varepsilon)}{x_0+my_0} + \frac{1}{y}\{\ \log D_\alpha(x_0) + \alpha(y-my_0)\ \}.$$

Letting $y\to\infty$ gives $\limsup_{y\to\infty}\frac{1}{y}\log D_\alpha(x_0+y)\le\log(1+\varepsilon)$. Thus (a) holds.

For part (b), again since $e^{-\alpha x}G_\alpha(x)$ is decreasing in x, it suffices to show that $G_\alpha(x)$ is integrable on $\mathbb{R}^1$ (see the proof of Corollary 2.17 of Durrett-Liggett 1983,p.287). By Lemma 4.0(iv),

$$G_\alpha(x)\le e^{\alpha x}\mathbb{E}F(\min\{\tilde{N},\tilde{S}D_\alpha(x)e^{-\alpha x}\}) \le e^{\alpha x}\mathbb{E}F(\tilde{N})\le e^{\alpha x}\mathbb{E}\tilde{N},$$

where $F(u)= e^{-u}-1+u$. It follows that $\int_{-\infty}^0 G_\alpha(x)dx<\infty$. To deal with integrability at $+\infty$, for all $\varepsilon>0$ so small that $0<\varepsilon/(\alpha-\varepsilon)<\min(1,\delta)$, choose x_0 so that for $x\ge x_0$, $D_\alpha(x)\le e^{\varepsilon x}$. Again by Lemma 4.0(iv),

$$\int_{x_0}^\infty G_\alpha(x)dx \le \int_{x_0}^\infty e^{\alpha x}\mathbb{E}F(\tilde{S}D_\alpha(x)e^{-\alpha x})dx$$

$$\le \int_{x_0}^\infty e^{\alpha x}\mathbb{E}F(\tilde{S}e^{-(\alpha-\varepsilon)x})dx = \frac{1}{\alpha-\varepsilon}\int_0^a \frac{\mathbb{E}F(\tilde{S}u)}{u^{2+\beta}}du,$$

where $a=\exp\{-(\alpha-\varepsilon)x_0\}$ and $\beta=\varepsilon/(\alpha-\varepsilon)$. Since $0<\beta<\min(1,\delta)$ and the last integral is finite if (and only if) $\mathbb{E}\tilde{S}^{1+\beta}<\infty$ (see for example Bingham and Doney 1974, p.718, Theorem B),we see that $\int_{x_0}^\infty G_\alpha(x)dx<\infty$, which ends the proof of (b). ■

The following result is the key to identifing the elements of $\mathcal{F}$.

Theorem 4.3. *Assume (H2) and $\mathcal{F}\neq\emptyset$. Let α be the unique point in (0,1] for which $\rho(\alpha)=1$ and $\rho'(\alpha)\le 0$. If $\phi\in\mathcal{F}$, then there is a $p\in\mathcal{B}_{\alpha,s}$ so that*

$$\lim_{t\to 0+}\frac{1-\phi(t)}{t^\alpha p(-\log t)}=1 \qquad \text{if } \rho'(\alpha)<0$$

and

$$\lim_{t\to 0+}\frac{1-\phi(t)}{t^\alpha p(-\log t)|\log t|}=1 \quad \text{if } \rho'(\alpha)=0.$$

Proof. With some obvious modifications, the proof is the same as that of Theorem 2.18 of Durrett and Liggett (1983, pp.288-292), where the crux is their Lemma 2.3, Theorem 2.12(b) and Corollary 2.17, which correspond our Lemma 4.0, Theorem 4.2(b) and Corollary 4.2(b) respectively. However, since the argument is very interesting and not evident, we present it as follows for the convenience of readers. Let S_n be the random walk with $S_0=0$ whose increments have distribution X_α. Since $\rho(\alpha)=1$ and $\phi\in\mathcal{F}$, Lemma 4.0 gives

$$D_\alpha(x)= \mathbb{E}D_\alpha(x+X_\alpha) -G_\alpha(x). \tag{4.12}$$

Let us begin with the transient case where $\mathbb{E}X_\alpha=-\mathbb{E}\Sigma^* A_i^\alpha \log A_i=-\rho'(\alpha)>0$. Iterating (4.12) and passing to the limit, we see that

$$D_\alpha(x)= \lim_{n\to\infty} \mathbb{E}D_\alpha(x+S_n) - \sum_{k=0}^{\infty} \mathbb{E}G_\alpha(x+S_k). \tag{4.13}$$

Here the sum is finite and tends to 0 as $x\to+\infty$ by Corollary (4.2) and the renewal theorem, while the limit

$$p(x):=\lim_{n\to\infty} \mathbb{E}D_\alpha(x+S_n) \geq D_\alpha(x)>0$$

exists because $\mathbb{E}D_\alpha(x+S_n)$ in increasing in n by (4.12). Since $D_\alpha(x)e^{-\alpha x}$ is decreasing in x and $D_\alpha(x)e^{-(1-\alpha)x}$ is increasing in x, the function p(x)/p(0) is in $\mathcal{A}_\alpha$ (defined in Lemma 4.1), and so p(x) in continuous on $\mathbb{R}^1$. Since

$$p(x)= \mathbb{E}p(x+X_\alpha),$$

p(x)/p(0) is of the form (4.11) with some $\lambda\in[0,1]$ and $\beta\in[\alpha,1]$ in the nonlattice case and is s-periodic in the lattice case (cf.Lemma 4.1(b) or Choquet's theorem for harmonic functions). Since $D_\alpha(x)\leq e^{\alpha x}$, $\limsup_{x\to-\infty} p(x) <\infty$. Therefore p(x)/p(0)=1 (i.e.p(x) is constant) in the nonlattice case and p(x) is s-periodic in the lattice case. Putting $\theta=e^{-x}$ and recalling the definition of D_α, it follows from (4.13) that

$$\lim_{\theta\to 0+} \frac{1-\phi(\theta)}{\theta^{\alpha} p(-\log\theta)} = 1.$$

To check that in the lattice case that

$$(-1)^k \frac{d^k}{d\theta^k}[\theta^{\alpha} p(-\log\theta)] \le 0$$

for all k=1,2,...and $\theta>0$, use the periodicity of p to write

$$\theta^{\alpha}p(-\log\theta) = \theta^{\alpha}p(-\log\theta+ns)$$

$$= e^{n\alpha s}\left[\frac{\theta^{\alpha}e^{-n\alpha s}p(-\log\theta+ns)}{1-\phi(\theta e^{-ns})}\right][1-\phi(\theta e^{-ns})].$$

Therefore $\theta^{\alpha}p(-\log\theta) = \lim_{n\to\infty} e^{n\alpha s}[1-\phi(\theta e^{-ns})]$. Since $\phi\in\mathcal{L}$, it follows that the derivatives of $\theta^{\alpha}p(-\log\theta)$ have the correct signs. Note that since $\frac{1-\phi(\theta)}{\theta}$ is monotone, if $\alpha=1$ then p is both monotone and periodic, and hence constant.

Turning now to the recurrent case $\mathbb{E}X_{\alpha}=-\rho'(\alpha)=0$. Let τ be the first time that S_n enters $(0,\infty)$, so that S_{τ} is the strict ascending ladder variable associated with X_{α}. Since $X_{\alpha}\neq 0$, $\tau<\infty$ a.s. By (4.12),

$$D_{\alpha}(x+S_n) - \sum_{k=0}^{n-1} G_{\alpha}(x+S_k)$$

is a martingale. By the martingale stopping theorem,

$$\mathbb{E}D_{\alpha}(x+S_{\tau\wedge n}) - \mathbb{E}\sum_{k=0}^{\tau\wedge n-1} G_{\alpha}(x+S_k) = D_{\alpha}(x). \tag{4.14}$$

By (4.7) and (3.6a) of Chap.XII of Feller (1971),

$$\mathbb{E}e^{\alpha S_{\tau}} < \infty. \tag{4.15}$$

Therefore, since $D_{\alpha}(x)\le e^{\alpha x}$ and $S_n\le S_{\tau}$ for $n<\tau$, we may pass to the limit in (4.14) to obtain

$$\mathbb{E}D_{\alpha}(x+S_{\tau})-D_{\alpha}(x) = R(x) := \mathbb{E}\sum_{k=0}^{\tau-1} G_{\alpha}(x+S_k) = \sum_{k=0}^{\infty} \mathbb{E}G_{\alpha}(x+T_k), \tag{4.16}$$

where T_k is the random walk whose increments have the distribution of the weak descending ladder variable for the original random walk S_n, and the last equality holds by the duality lemma of Sect.XII.2 of Feller (1971).

By the renewal theorem and Corollary 4.2(b), there is a strictly positive continuous function p(x) which is constant in the nonlattice case and periodic of period s in the lattice case so that

$$\lim_{x\to\infty} [R(x)-p(x)]=0 \tag{4.17a}$$

in the nonlattice case and

$$\lim_{n\to\infty} [R(x+ns)-p(x)]=0 \tag{4.17b}$$

in the lattice case. Consider now the nonlattice case only, since the lattice case is handled similarly with derivatives and integrals being replaced by diferences and sums respectively. Integrating (4.16) and using (4.3), (4.15) and Fubini's theorem, we obtain

$$\int_0^x R(z)dz = \int_0^x [\mathbb{E}\int_0^{S_\tau} D_\alpha'(z+y)dy]dz = \mathbb{E}\int_0^{S_\tau} [\int_0^x D_\alpha'(z+y)dz]dy$$

$$= \mathbb{E} \int_0^{S_\tau} D_\alpha(x+y)dy -c, \tag{4.18}$$

where $c=\mathbb{E}\int_0^{S_\tau} D_\alpha(y)dy= \int_0^\infty P(S_\tau\geq y)D_\alpha(y)dy<\infty$. Since $D_\alpha(x)e^{-\alpha x}$ is decreasing in x, $D_\alpha(x+y)/D_\alpha(x) \leq e^{\alpha y}$ for $y\geq 0$ (cf.also (4.3)). Therefore, dividing (4.18) by $D_\alpha(x)$ and using (4.15), Theorem 4.2 (b) and the dominated convergence theorem, we see that

$$\lim_{x\to\infty} \frac{\int_0^x R(z)dz+c}{D_\alpha(x)} = \mathbb{E}S_\tau ,$$

which is positive and finite. Therefore, by (4.17),

$$\lim_{x\to\infty} \frac{D_\alpha(x)}{x} = \frac{1}{\mathbb{E}S_\tau} \lim_{x\to\infty} \frac{1}{x}\int_0^x p(z)dz \tag{4.19}$$

exists and is positive and finite. In the lattice case, the corresponding conclusion is that

$$\lim_{n\to\infty} \frac{D_\alpha(x+ns)}{x+ns} = \frac{p(x)}{\mathbb{E}S_\tau} \tag{4.20}$$

for each $x\in\mathbb{R}^1$. By the monotonicity and (4.19), $D_\alpha(x)/x$ is bounded at $+\infty$. Hence by (4.3b)(with $g(.)=D_\alpha(.)/D_\alpha(0)$), $D_\alpha(x)/x$ has a uniformly bounded derivative at $+\infty$. Hence the family of the functions $u_n(x)=D_\alpha(x+ns)/(x+ns)$ $(n\geq 1)$ is equicontinuous on $[0,s]$, so pointwise convergence implies uniform convergence. Therefore (4.20) implies

$$\lim_{x\to\infty} \frac{D_\alpha(x)}{x\,p(x)} = \frac{1}{\mathbb{E}S_\tau}. \tag{4.21}$$

Part (b) of the theorem now follows from (4.19) or (4.20) with $p(x)$ replaced by $p(x)/\mathbb{E}S_\tau$, by putting $\theta=e^{-x}$. The verification that $p(x)\in\mathcal{B}_{\alpha,s}$ is the same as in the transient case, which was dealt with earlier in this proof. ■

5. Stable transformation. Canonical fixed points.

The idea of stable transformation plays an essential role for the study of smoothing transformation. It is due independently to Durrett and Liggett (1983) and Guivarc'h (1990) with some different points of view. The formalism here is slightly different from theirs.

For $\alpha\in(0,1)$, define a transformation S_α: $\mathcal{L} \to \mathcal{L}$ by

$$(S_\alpha\phi)(t)= \phi(t^\alpha) \quad (\forall\phi\in \mathcal{L}).$$

To see that $S_\alpha\phi \in\mathcal{L}$, it suffices to note that if Y and Z are independent random variables with Laplace transforms e^{-t^α} and ϕ respectively, then $YZ^{1/\alpha}$ has Laplace transform $S_\alpha\phi$. For convenience, if $\alpha=1$, S_α is naturally taken to be the identical transformation.

Definition 5.1. *The transformation S_α defined above is called a stable transformation.*

For $\alpha>0$, let T_α be the transformation analogous to T obtained by replacing A_i with A_i^α. The importance of the stable transformation is due to the following interesting conjugate relation.

Theorem 5.2. (Conjugate relation) *For* all $\alpha\in(0,1)$,

$$TS_\alpha = S_\alpha T_\alpha.$$

Proof. For any $\phi \in \mathcal{L}$,

$$(TS_\alpha\phi)(t) = \mathbb{E}\prod_{i=1}^{N}(S_\alpha\phi)(tA_i) = \mathbb{E}\prod_{i=1}^{N}\phi(t^\alpha A_i^\alpha) = (T_\alpha\phi)(t^\alpha) = (S_\alpha T_\alpha\phi)(t) \quad \blacksquare$$

Corollary 5.3. *For $\alpha \in (0,1)$, if $\phi \in \mathcal{L}$ is a fixed point of T_α, then $S_\alpha\phi$ is a fixed point of T.*

Proof. By theorem 4.1, if $\phi = T_\alpha\phi$, then $T(S_\alpha\phi) = S_\alpha(T_\alpha\phi) = S_\alpha\phi$. $\blacksquare$

Corollary 5.4. *Suppose that $\mathbb{E}\tilde{N} < \infty$. If for some $\alpha \in (0,1]$, $\mathbb{E}S(\alpha)\log^+ S(\alpha) < \infty$, $\rho(\alpha)=1$ and $\psi'(\alpha)<0$, then for all constant $c>0$, T has a fixed point ϕ with $1-\phi(t) \sim ct^\alpha$ $(t \to 0)$.*

Proof. Under the given conditions, the transformation T_α has a fixed point ϕ_α with mean 1 by Theorem 0. So T has a fixed point $\phi := S_\alpha\phi_\alpha$ with $1-\phi(t) \sim t^\alpha$ $(t \to 0)$ by Corollary 5.3. Since for all constant $c>0$, $\tilde{\phi}(t) := \phi(ct)$ is a fixed point whenever ϕ is, the proof is finished. $\blacksquare$

To make clear the totality of fixed points of T, let us introduce after Guivarc'h (1990) the notion of *canonical fixed points of T*:

Definition 5.5. *A nontrivial fixed point ϕ of T is termed canonical if it can be expressed in the form $\phi = S_\alpha\psi$ for some fixed point ψ of T_α, where $\alpha \in (0,1]$ is the unique point such that $\rho(\alpha)=1$ and $\rho'(\alpha) \le 0$.*

Thus if $\alpha<1$, the canonical fixed points of T are exactly all those which can be obtained from the fixed points of T_α by the stable transfomation. So the study of canonical fixed points (and only those) in the case where $\alpha<1$ can be transfered to the study for the case where $\alpha=1$. If $\alpha=1$, all fixed points of T are called canonical.

A natural question is to ask, for the case where $\alpha<1$, whether there are fixed points which are not canonical. Our theorem of totality of fixed points will show that the answer is positive in the lattice case, and negative in the non-lattice case (cf.Sect.10).

6. Regular variation of fixed points under first moment conditions

We shall use the following simple result on slowly varing functions.

Lemma 6.1. *If $g(u)\geq 0$ is monotone, then $\lim_{u\to\infty} g(\lambda u)/g(u)=1$ for some $0<\lambda\neq 1$ if and only if it holds for all $\lambda>0$ (that is, $g(u)$ is slowly varing).*

Proof. It suffices to prove that if $\lim_{u\to\infty} g(\lambda u)/g(u)=1$ for some $0<\lambda=\lambda_0\neq 1$, then it holds for all $\lambda>0$. Since

$$\frac{g(\lambda_0^2 u)}{g(u)}=\frac{g(\lambda_0^2 u)}{g(\lambda_0 u)}\,\frac{g(\lambda_0 u)}{g(u)},$$

we see that $\lim_{u\to\infty} g(\lambda_0^2 u)/g(u)=1$. Iterating this, we have $\lim_{u\to\infty} g(\lambda_0^m u)/g(u)=1$ for all m=1,2,... Putting $u_1=\lambda_0^m u$ for fixed m=1,2,..., we obtain $\lim_{u_1\to\infty} g(u_1)/g(\lambda_0^{-m}u_1)=1$, namely $\lim_{u\to\infty} g(\lambda_0^{-m}u)/g(u)=1$. Hence $\lim_{u\to\infty} g(\lambda_0^m u)/g(u)=1$ for all integers $m\in\mathbb{Z}$. For each fixed $\lambda>0$, choose $m\in\mathbb{Z}$ such that $\lambda_0^{m-1}<\lambda\leq\lambda_0^m$ if $\lambda_0>1$, and $\lambda_0^{m-1}>\lambda\geq\lambda_0^m$ if $\lambda_0<1$. By the monotonicity of g(u), $g(\lambda u)/g(u)$ varies between $g(\lambda_0^{m-1}u)/g(u)$ and $g(\lambda_0^m u)/g(u)$. Thus $\lim_{u\to\infty} g(\lambda u)/g(u)=1$. ∎

Theorem 6.2. *Assume (H1) and $\mathcal{F}\neq\emptyset$. Let α be the unique point in (0,1] such that $\rho(\alpha)=1$ and $\rho'(\alpha)\leq 0$. If $\phi\in\mathcal{F}$, then for some slowly varing function $\ell(.)\geq 0$ at 0,*

$$\lim_{t\to 0+}\frac{1-\phi(t)}{t^\alpha \ell(t)}=1 \quad \text{if } \alpha=1 \text{ or } \phi \text{ is canonical,}$$

and

$$\limsup_{t\to 0+}\frac{1-\phi(t)}{t^\alpha \ell(t)}\leq 1 \text{ if } \alpha<1.$$

Moreover, if (H2) holds, then we can take $\ell(t)=c$ if $\rho'(\alpha)<0$ and $\ell(t)=c|\log t|$ if $\rho'(\alpha)<0$, for some constant $c>0$.

Remark. *In the case where $\alpha<1$, the function $\ell(t)$ can be constructed shch that $1-\phi(t)\leq t^\alpha\ell(t)$* for all sufficiently small t>0.

Proof. If (H2) holds, the result is immediate by Theorem 4.3. Assume only (H1) and write

$$A_\alpha(t) := \frac{1-\phi(t)}{t^\alpha}.$$

Then $D_\alpha(x) = A_\alpha(e^{-x})$. *By Theorem 4.1,* $\limsup_{x\to\infty} \frac{D_\alpha(x+y)}{D_\alpha(x)} \le 1$ *for some* $y>0$. *Hence*

$$\limsup_{t\to 0+} \frac{A_\alpha(\lambda t)}{A_\alpha(t)} \le 1 \quad \text{for some } 0<\lambda<1.$$

If $\alpha=1$, then A_α is decreasing since $\phi\in\mathcal{L}$. Thus $A_\alpha(\lambda t)\ge A_\alpha(t)$ for $0<\lambda<1$ and all $t>0$. Therefore

$$\lim_{t\to 0+} \frac{A_\alpha(\lambda t)}{A_\alpha(t)} = 1 \quad \text{for some } 0<\lambda<1.$$

So A_α is slowly varing by Lemma 6.1. If $\alpha<1$ and ϕ is canonical, then $\phi(t)= \psi(t^\alpha)$ for some fixed point ψ of T_α. Using the conclusion for T_α, we see that $\frac{1-\psi(t)}{t}$ is slowly varing at 0, and so is A_α. In the general case, we have

$$\limsup_{x\to\infty} \frac{1}{x} \log D_\alpha(x) \le 0$$

by Corollary 4.2. Thus the function

$$h(y) := \max\{1, D_\alpha(\log y)\} \quad (y>0)$$

is of order 0 in that

$$\lim_{y\to\infty} \frac{\log h(y)}{\log y} = 0.$$

So there exists a slowly varing function $\ell_0(y)$ at ∞ such that

$$\limsup_{y\to\infty} \frac{h(y)}{\ell_0(y)} = 1$$

with $h(y)\le \ell_0(y)$ for all large y [see Bingham, Goldie and Teugels (1987),p.81, Theorem 2.3.11. But there is an error in the statement of that theorem: the assertion $\limsup_{x\to\infty} f(x)/g(x)=1$ therein should be $\limsup_{x\to\infty} g(x)/f(x)=1$.]. Therefore

$$\limsup_{y\to\infty} \frac{D_\alpha(\log y)}{\ell_0(y)} \le 1$$

with $D_\alpha(\log y) \le \ell_0(y)$ for sufficiently large $y>0$. This means

$$\limsup_{t\to 0+} \frac{A_\alpha(t)}{\ell_0(1/t)} \le 1$$

with $A_\alpha(t) \le \ell_0(1/t)$ for all small $t>0$. The proof is finished by taking

$\ell(t)=\ell_0(1/t)$. ■

7. Iterations: convergence to fixed points

We shall prove our main convergence theorem in this section. The method can be compared to that of Durrett and Liggett (1983) who introduuced an associated branching random walk. The treatment here is direct and elementary.

For all sequences $\sigma\in\bigcup_{i=1}^{\infty}\mathbb{N}^i$ of positive integers, we denote by $|\sigma|$ its length, and let

$$(A_{\sigma,1},\ A_{\sigma,2},\ldots)$$

be independent copies of $(A_1,A_2,\ldots)$.

For a probability measure $\mu\in\mathcal{M}$ with Laplace transform ϕ, we have

$$T^n\mu = \text{distribution of } \sum_{|\sigma|=n}\ell_\sigma Z_\sigma,$$

where $\ell_\sigma := A_{\sigma_1}A_{\sigma_1\sigma_2}\ldots A_{\sigma_1\sigma_2\ldots\sigma_n}$ if $\sigma=\sigma_1\sigma_2\ldots\sigma_n$, $\{Z_\sigma:|\sigma|=n\}$ are independent random variables with distribution μ, which are independent of $\{A_\sigma:|\sigma|\le n\}$ as well. The sum is taken over all σ such that $\ell_\sigma>0$. In terms of Laplace transforms, the iteration formula reads

$$T^n\phi(t) = \mathbb{E}\prod_{|\sigma|=n}\phi(t\ell_\sigma).$$

For $n\in\mathbb{N}$, define

$$\ell_n := \max_{|\sigma|=n}\ell_\sigma.$$

The following interesting result may probably be classical in the theory of branching random walks if we take logarithms.

Lemma 7.1. *If $P(\max_{i\ge1}A_i=1)<1$, then*

$$P(\limsup_{n\to\infty}\ell_n=0 \text{ or } \infty)=1 \text{ and } P(\liminf_{n\to\infty}\ell_n=0 \text{ or } \infty)=1.$$

Proof. It is easy to check that

$$\ell_{n+1} = \max_{i\geq 1} A_i \ell_{n,i},$$

where $\{\ell_{n,i}\}$ $(i\geq 1)$ are independent copies of ℓ_n, which are independent of $\{A_i\}$ as well. Letting $n\to\infty$ gives

$$\limsup_{n\to\infty} \ell_n \stackrel{d}{=} (\max_{i\geq 1} A_i) \limsup_{n\to\infty} \ell_n ,$$

where $\limsup_{n\to\infty} \ell_n$ is independent of $\max_{i\geq 1} A_i$. Taking logarithms, we see that either $P(\max_{i\geq 1} A_i=1) =1$ or $P(\limsup_{n\to\infty} \ell_n =0 \text{ or } \infty) =1$. The assertion for liminf follows similarly. ∎

Lemma 7.2. *If for some* $\alpha\in(0,\infty)$, $\rho(\alpha)\leq 1$, *then*

$$P(\lim_{n\to\infty} \ell_n =0) =1.$$

Proof. It is easily verified that $\{Y_n;\mathbb{F}_n\}$ forms a martingale, where

$$Y_n := \rho(\alpha)^{-n} \sum_{|\sigma|=n} \ell_\sigma^\alpha ,$$

and

$$\mathbb{F}_n := \sigma(A_\tau: |\tau|\leq n).$$

The martingale convergence theorem ensures that Y_n converges almost surely to a finite random variable Y.

If $\rho(\alpha)<1$, the conclusion follows since

$$\limsup_{n\to\infty} \ell_n^\alpha \leq \limsup_{n\to\infty} \sum_{|\sigma|=n} \ell_\sigma^\alpha = \limsup_{n\to\infty} Y_n\rho(\alpha)^n =0 \quad \text{almost surely.}$$

If $\rho(\alpha)=1$, the same argument as above shows that $\limsup_{n\to\infty} \ell_n^\alpha <\infty$ almost surely.

So by Lemma 6.1, either $P(\max_{i\geq 1} A_i=1) =1$ or $P(\limsup_{n\to\infty} \ell_n =0) =1$. But if $P(\max_{i\geq 1} A_i=1) =1$ and (H0), then $\rho(x)$ is strictly decreasing on $[\alpha,\infty)$, so we can choose $\tilde{\alpha}>\alpha$ such that $\rho(\tilde{\alpha})<1$, and then $\limsup_{n\to\infty} \ell_n^{\tilde{\alpha}}=0$ by the preceding argument.

Therefore, in all cases, we have $P(\limsup_{n\to\infty} \ell_n =0) =1$. ∎

The following comparison result will be frequently used. It says that inequalities for *small* $t>0$ can be transfered to inequalities for *all* $t>0$.

Lemma 7.3. *Suppose that for some $\alpha\in(0,\infty)$, $\rho(\alpha)\le 1$. If $\phi,\bar{\phi}\in\mathcal{L}$ are such that for some $t_0>0$ and all $0<t\le t_0$*

$$\phi(t) \le \bar{\phi}(t),$$

then for all $t>0$,

$$\limsup_{n\to\infty} T^n\phi(t) \le \limsup_{n\to\infty} T^n\bar{\phi}(t)$$

and

$$\liminf_{n\to\infty} T^n\phi(t) \le \liminf_{n\to\infty} T^n\bar{\phi}(t).$$

Proof. Let $t>0$ be fixed. Since $P(\ell_n\to 0)=1$ by Lemma 6.2, for arbitrary $\varepsilon>0$, we can choose $n_0\in\mathbb{N}$ sufficiently large such that for all $n\ge n_0$,

$$P(t\ell_n >t_0) <\varepsilon.$$

Therefore, for all $n\ge n_0$

$$\begin{aligned} T^n\phi(t) &= \mathbb{E} \prod_{|\sigma|=n} \phi(t\ell_\sigma) \\ &= \mathbb{E}\, 1_{\{t\ell_n\le t_0\}} \prod_{|\sigma|=n} \phi(t\ell_\sigma) + \mathbb{E}\, 1_{\{t\ell_n> t_0\}} \prod_{|\sigma|=n} \phi(t\ell_\sigma) \\ &\le \mathbb{E}\, 1_{\{t\ell_n\le t_0\}} \prod_{|\sigma|=n} \bar{\phi}(t\ell_\sigma) + P(t\ell_n >t_0) \\ &\le \mathbb{E} \prod_{|\sigma|=n} \bar{\phi}(t\ell_\sigma) + \varepsilon \\ &= T^n\bar{\phi}(t) + \varepsilon. \end{aligned}$$

The conclusion then follows by letting $n\to\infty$ and then $\varepsilon\to 0$. ∎

The following theorem is our main convergence result.

Theorem 7.4. *Assume (H1) and $\mathcal{F}\ne\emptyset$. Let α be the unique point in $(0,1]$ such that $\rho(\alpha)=1$ and $\rho'(\alpha)\le 0$. If $\alpha<1$, we assume additionally that either ϕ is a* *canonical fixed point or (H2) holds. If* $\phi\in\mathcal{F}$ *and* $\eta\in\mathcal{L}$ *are such that* $1-\phi(t)\sim 1-\eta(t)$ $(t\to 0+)$, *then*

$$\lim_{n\to\infty} T^n\eta=\phi.$$

Proof. If (H2) holds, then by Theorem 4.3,

$$\liminf_{t\to 0}\frac{1-\phi(ct)}{1-\phi(t)}=c^{\alpha}\min_{0\le x\le s}\frac{p(-\log c+x)}{p(x)},$$

and

$$\limsup_{t\to 0}\frac{1-\phi(ct)}{1-\phi(t)}=c^{\alpha}\max_{0\le x\le s}\frac{p(-\log c+x)}{p(x)}$$

for $c>0$, where $p\in\mathcal{B}_{\alpha,s}$. Since $p\in\mathcal{B}_{\alpha,s}$, $\theta^{\alpha}p(-\log\theta)$ is strictly increasing on $[0,\infty)$. Therefore

$$\liminf_{t\to 0}\frac{1-\phi(ct)}{1-\phi(t)}>1 \text{ if } c>1$$

and

$$\limsup_{t\to 0}\frac{1-\phi(ct)}{1-\phi(t)}<1 \text{ if } c<1.$$

The last assertion holds also under the weaker assumption (H1) in the case where $\alpha=1$ or ϕ is a canonical fixed point, since we then have

$$\frac{1-\phi(ct)}{1-\phi(t)}\to c^{\alpha}\quad (t\to 0+),$$

by Theorem 6.2. Using $1-\phi(t)\sim 1-\eta(t)$ $(t\to 0+)$, we have in all the given cases,

$$\liminf_{t\to 0+}\frac{1-\phi(ct)}{1-\eta(t)}>1 \text{ if } c>1$$

and

$$\limsup_{t\to 0+}\frac{1-\phi(ct)}{1-\eta(t)}<1 \text{ if } c<1.$$

Fix $\theta>0$ and $c>1$. Put

$$\underline{\phi}(x)=\phi(ct),\quad\text{and}\quad \bar{\phi}(x)=\phi(c^{-1}t).$$

Then $\underline{\phi},\bar{\phi}\in\mathcal{F}$ and for some $t_0>0$ and all $0<t\le t_0$,

$$\underline{\phi}(t)\le\eta(t)\le\bar{\phi}(t).$$

By Lemma 7.3,

$$\liminf_{n\to\infty} T^n\underline{\phi}(\theta)\le\liminf_{n\to\infty} T^n\eta(t)\le\limsup_{n\to\infty} T^n\eta(t)\le\limsup_{n\to\infty} T^n\bar{\phi}(\theta).$$

Since $T\underline{\phi}=\underline{\phi}$ and $T\bar{\phi}=\bar{\phi}$,

$$\phi(\theta) \le \liminf_{n\to\infty} T^n\eta(t) \le \limsup_{n\to\infty} T^n\eta(t) \le \phi(\theta).$$

That is, all limit points of $T^n\eta(\theta)$ lie between $\phi(c\theta)$ and $\phi(c^{-1}\theta)$. Since $c>1$ is arbitrary, $\lim_{n\to\infty} T^n\eta(\theta) = \phi(\theta)$. ■

Corollary 7.5. *Assume (H1). Suppose that for some* $\alpha\in(0,1]$, $\mathbb{E}S(\alpha)\log^+S(\alpha)<\infty$, $\rho(\alpha)=1$ *and* $\rho'(\alpha)<0$. *If* $\eta\in\mathcal{L}$ *is such that* $1-\eta(t) \sim ct^{\alpha}$ $(t\to0+)$ *for some constant* $c>0$, *then* $T^n\eta$ *converge to a nontrivial fixed point* ϕ *with* $1-\phi(t) \sim ct^{\alpha}$ $(t\to0+)$.

Proof. Under the given conditions, T has a nontrivial fixed point ϕ with $1-\phi(t) \sim ct^{\alpha}$ $(t\to0+)$. It follows by Theorem 7.4 that $\lim_{n\to\infty} T^n\eta = \phi$. ■

Theorem 7.4. deals with the case where $\mathcal{F}\neq\emptyset$. In section 9, we shall treat the case where $\mathcal{F}=\emptyset$.

8. Limit theorems: an extention of the Kesten-Stigum theorem.

In this section, we suppose that $\rho(1)=1$. The following result was proved in Liu (1994).

Lemma 8.0. *Assume (H1) and* $\rho(1)=1$. *Then* $T^n\delta_1$ *converge to a fixed point* ν *with finite mean, and* $\nu\neq\delta_0$ *if and only if*

$$\mathbb{E}S\log^+S<\infty \text{ and } \rho'(1)<0. \tag{8.1}$$

We shall generalize this to the case where the δ_1 is replaced by any elements of $\mathcal{M}$ with finite mean.

We say that a probability measure ε on $[-\infty,\infty]$ is stochastically inferior to another η and we denote by

$$\varepsilon << \eta \quad \text{if } \forall t\in\mathbb{R}^1,\ \varepsilon(t,\infty] \le \eta(t,\infty].$$

This condition is equivalent to the existence of random variables X and Y of distributions ε and η respectively satisfing $X\le Y$. It follows immediately that

$$T\varepsilon << T\eta \quad \text{if} \quad \varepsilon << \eta \tag{8.2}$$

by the definition of T.

For a number $\alpha\in(0,1]$ and two probability measures ε and η on $[0,\infty]$, we denote by $d_\alpha(\varepsilon,\eta)$ the (largest) lower bound of the integrals $\int|x-y|^\alpha d\theta(x,y)$, where θ is a probability measure on $\mathbb{R}^2$ with projections ε and η. We write d for d_1. If ε and η have moments of order α, d_α defines a distance. Clearly d_α is the lower bound of $\mathbb{E}|X-Y|^\alpha$, where X and Y are random variables with distributions ε and η respectively. Finally, the lower bound which defines d_α can be attained and, if F_ε and F_η denote the distribution functions of ε and η, we have

$$d(\varepsilon,\eta) = \int_0^\infty |F_\varepsilon(x)-F_\eta(x)|dx.$$

(Fortet et Mourier 1953; see also Royer 1984 or Guivarc'h 1990, pp.270-271.)

The following result shows that the smoothing transformation T is a contraction in some sense.

Proposition 8.1. *Let* $\alpha\in(0,1]$ *and* $\rho(\alpha)<\infty$.

(i) If $\varepsilon\in\mathcal{M}$ *is of finite moment of order* α, *then so is* $T\varepsilon$. *More precisely, if* $\int x^\alpha d\varepsilon(x)<\infty$, *then*

$$\int x^\alpha d(T\varepsilon)(x) \le \rho(\alpha) \int x^\alpha d\varepsilon(x). \tag{8.3}$$

If additionally $\alpha<1$, *then the equality in (8.3) happens only if* $\varepsilon=\delta_0$.

(ii) If $\varepsilon,\varepsilon'\in\mathcal{M}$ *have finite moments of order* α, *then*

$$d_\alpha(T\varepsilon,T\varepsilon') \le \rho(\alpha)d_\alpha(\varepsilon,\varepsilon'). \tag{8.4}$$

Moreover, if either

(a) $\alpha<1$ *or*

(b) $\alpha=1$ *with* $\rho(\alpha)=1$ *and* $\int xd\varepsilon(x) = \int xd\varepsilon'(x)$,

then the equality in (8.4) happens only if $\varepsilon=\varepsilon'$.

Proof. (i). Let $\{Z_i\}$ be independent random variables with distribution ε, which are independent of $\{A_i\}$ as well. Then $Z:= \sum A_iZ_i$ has distribution $T\varepsilon$. Since $\alpha\in(0,1]$,

$$Z^\alpha \le \sum A_i^\alpha Z_i^\alpha. \tag{8.5}$$

By (H0), the inequality is strict with positive probability if $\alpha<1$ and $P(Z>0)>0$. Taking expectations gives the conclusion of part (i).

For part (ii), we choose two random variables Z and Z' with distributions ε and ε', which are independent of $\{A_i\}$, such that

$$d_\alpha(\varepsilon,\varepsilon') = \mathbb{E}|Z-Z'|^\alpha.$$

Let $\{(Z_i,Z_i')\}$ be independent copies of (Z,Z') which are also independent of $\{A_i\}$, and put

$$\tilde{Z} = \sum A_iZ_i, \quad \text{and} \quad \tilde{Z}' = \sum A_iZ_i',$$

then $\tilde{Z}$ and $\tilde{Z}'$ have distributions $T\varepsilon$ and $T\varepsilon'$. Since

$$d_\alpha(T\varepsilon,T\varepsilon') \le \mathbb{E}|\tilde{Z}-\tilde{Z}'|^\alpha,$$

$$\tilde{Z}-\tilde{Z}' = \sum A_i(Z_i-Z_i'),$$

and

$$|\sum A_i(Z_i-Z_i')|^\alpha \le \sum A_i^\alpha|Z_i-Z_i'|^\alpha, \tag{8.6}$$

where [again by hypothesis (H0)] the inequality is strict with positive probability if $\alpha<1$ and $P(|Z-Z'|>0)>0$, the conclusion for the case where $\alpha<1$ follows by taking expectations in (8.6). If $\alpha=1$, the conclusion was proved in Liu (1994). ∎

Theorem 8.2. *Assume (H1) and $\rho(1)=1$. If $\mu\in\mathcal{M}$ is of finite mean with $\mu\neq\delta_0$, then $T^n\mu$ converge to a fixed point ν with finite mean, and $\nu\neq\delta_0$ if and only if (8.1) holds.*

Proof. If (8.1) holds, then T has a fixed point $\nu\in\mathcal{F}$ with finite mean. By a scale change, we can suppose that ν has the same mean as μ. Thus $T^n\mu \to \nu$ by Theorem 7.4. It remains to prove that if (8.1) does not hold then $T^n\mu \to \delta_0$. Suppose that (8.1) does not hold. If μ is of compact support contained in $[0,b]$, then $\mu << \delta_b$ and $T^n\mu << T^n\delta_b \to \delta_0$ by Lemma 8.0.

If μ is not of campact support, it can be approximated as near as we want by measures of compact support, in the sense of d. If μ' is such an approximation, we have

$$d(T^n\mu, T^n\mu') \le d(\mu,\mu')$$

from which we obtain clearly that

$$\lim_{n\to\infty} T^n\mu = \delta_0. \qquad \blacksquare$$

9. Limit theorems: case where $\operatorname{Inf}_{x\in[0,1]} \rho(x) > 1$.

For simplicity, we shall mainly consider the case where $\rho'(1)\le 0$ (suppose that it exists) and $\rho(1)>1$. Thus $\operatorname{Inf}_{x\in[0,1]}\rho(x) = \rho(1)<1$. We recall that for any $\mu\in\mathcal{M}$,

$$T^n\mu = \text{distribution of } \sum_{|\sigma|=n} \ell_\sigma Z_\sigma,$$

where $\{Z_\sigma : |\sigma|=n\}$ are independent random variables with distribution μ, which are independent of $\{A_\sigma : |\sigma|\le n\}$ as well (cf. Section 7). For convenience, let us write

$$Z^{(n)} \equiv Z^{(n)}(\mu) := \sum_{|\sigma|=n} \ell_\sigma Z_\sigma. \tag{9.1}$$

As usual, let q be the unique fixed point in [0,1) of the function

$$f(t) = \sum_{k=0}^{\infty} P(\tilde{N}=k)t^k. \tag{9.2}$$

Then q is the extinction probability of the Galton-Watson process given by attaching an individual to the vertices σ for which $\ell_\sigma>0$. Hence

$$P(\ell_\sigma = 0 \text{ if } |\sigma| \text{ is sufficiently large}) = q. \tag{9.3}$$

It follows that

$$P\left(\lim_{n\to\infty} Z^{(n)}(\mu) = 0\right) \ge q. \tag{9.4}$$

As an immediate consequence, for all $\varepsilon>0$,

$$\liminf_{n\to\infty} T^n\mu\,[0,\varepsilon] \ge q. \tag{9.5}$$

Let ϕ be the Laplace transform of μ, then (9.5) implies that, for all $t>0$,

$$\liminf_{n\to\infty} T^n\phi(t) \ge q. \tag{9.6}$$

This follows from (9.5) since

$$T^n\phi(t) = \mathbb{E}[\exp(-Z^{(n)}t)] \geq e^{-\varepsilon t}P(Z^{(n)}\leq\varepsilon),$$

letting $n\to\infty$ and then $\varepsilon\to 0$ gives (9.6).

If $\mu=\delta_1$, it is easily seen that $Z^{(n)}/\rho(1)^n$ is a martingale. If the limit of this martingale is not degenerate, then $Z^{(n)} = Z^{(n)}(\delta_1) \to \infty$ with probability $1-q$ (since $\rho(1)>1$), and consequently for all $x>0$,

$$\liminf_{n\to\infty} T^n\delta_1(x,\infty) \geq 1-q. \tag{9.7}$$

By (9.5) and (9.7), we see that for all $\varepsilon>0$,

$$\lim_{n\to\infty} T^n\delta_1[0,\varepsilon] = q. \tag{9.8}$$

This discussion introduces us to the following

Theorem 9.1. *Assume (H1), $\rho(1)>1$ and $\rho'(1)\leq 0$. In the case where $\rho'(1)=0$, we assume additionally that $\mathbb{E}S\log^+S<\infty$. If $\delta_0\neq\mu\in\mathcal{M}$, then*

$$\lim_{n\to\infty} T^n\mu = \delta_{0,\infty}, \tag{9.9}$$

where $\delta_{0,\infty}\{0\}=q$ and $\delta_{0,\infty}\{\infty\}=1-q$, q being the unique fixed point in $[0,1)$ of the function $f(t)=\sum_{k=0}^{\infty} P(\tilde{N}=k)t^k$.

Of course, we can restate the result in terms of Laplace transforms:

Theorem 9.1'. *Under the hypothesis of Theorem 9.1, if $1\neq\phi\in\mathcal{L}$, then for all $t>0$,*

$$\lim_{n\to\infty} T^n\phi(t) = q \tag{9.9'}$$

Proof of Theorem 9.1. For simplicity, we assume $q=0$. The general case follows from similar lines (see also the discussion preceding Theorem 9.1). We should then prove that

$$\lim_{n\to\infty} T^n\mu = \delta_\infty. \tag{9.10}$$

We distinguish several cases:

(a) $\mathbb{E}S\log^+S<\infty$ and μ is of compact support. Write $\bar{A}_i= A_i/\rho(1)$ and define $\bar{T}$ and $\bar{\rho}$ in terms of $\{\bar{A}_i\}$ just as T and ρ were defined in terms of $\{A_i\}$. Then $\bar{T}=$

$T/\rho(1)$. Since $\bar\rho(1)=1$ and

$$\bar\rho'(1) = \mathbb{E}\Sigma \frac{A_i}{\rho(1)}\log\frac{A_i}{\rho(1)} = \frac{\rho'(1)}{\rho(1)} -\log\rho(1) <0,$$

$T^n\mu$ converge to some $\nu\in\mathcal{M}$ with $\nu\{0\}=q=0$ by Theorem 8.2. Since $\rho(1)>1$, this gives (9.10).

(b) $\mathbb{E}S\log^+S<\infty$ and μ is stochastically superior to some $\mu'\neq\delta_0$ of compact support. Since T preserves stochastic inequalities, (9.10) holds also.

(c) $\mathbb{E}S\log^+S=\infty$ and $\rho'(1)<0$. For M=1,2,..., write $A_i(M)=A_i$ if $i\le M$ and $A_i(M)=0$ if $i>M$. Define T_M and ρ_M in terms of $\{A_i(M)\}$ just as T and ρ were defined in terms of $\{A_i\}$. Choose M sufficiently large such that $\rho_M(1)>1$ and $\rho_M'(1)<0$. Thus $T_M^n\mu\to\delta_\infty$. Since $T\mu >> T_M\mu$, $T^n\mu\to\delta_\infty$. ∎

By the method of the proof, we have in fact the following comparison result which may apply when (H1) does not hold or $\rho'(1)\ge0$, or even $\rho'(1)$ does not exist.

Corollary 9.2. *Assume that for some $\bar N\le N$ and $\bar A_i\le A_i$ $(1\le i\le\bar N)$ such that either*

$$\mathbb{E}\bar N<\infty,\ \mathbb{E}\sum_{i=1}^{\bar N}\bar A_i\log^+\bar A_i<\infty,\ \mathbb{E}\sum_{i=1}^{\bar N}\bar A_i>1 \text{ and } \mathbb{E}\sum_{i=1}^{\bar N}\bar A_i\log\bar A_i<0, \tag{9.11a}$$

or

$$\mathbb{E}\bar N<\infty,\ \mathbb{E}\left[\sum_{i=1}^{\bar N}\bar A_i\right]\log^+\left[\sum_{i=1}^{\bar N}\bar A_i\right]<\infty,\ \mathbb{E}\sum_{i=1}^{\bar N}\bar A_i>1 \text{ and } \mathbb{E}\sum_{i=1}^{\bar N}\bar A_i\log\bar A_i=0, \tag{9.11b}$$

then (9.9) holds for all $\mu\in\mathcal{M}-\{\delta_0\}$.

In particular, we have

Corollary 9.3. *If for some constant integer n>0 and random variables $0\le\bar A_i\le A_i$,*

$$\mathbb{E}\bar A_i\log^+\bar A_i<\infty \text{ for all } 1\le i\le n,\ \mathbb{E}\sum_{i=1}^{n}\bar A_i>1 \text{ and } \mathbb{E}\sum_{i=1}^{n}\bar A_i\log\bar A_i\le0, \tag{9.12}$$

then (9.9) holds for all $\mu\in\mathcal{M}-\{\delta_0\}$.

We remark that the condition (9.12) holds for some $n\in\mathbb{N}$ and $\bar A_i=A_i$ $(1\le i\le n)$ if

$$\mathbb{E}A_i\log^+A_i<\infty \text{ for all } i\ge1,\ \rho(1)>1 \text{ and } \liminf_{n\to\infty}\mathbb{E}\sum_{i=1}^{n}A_i\log A_i<0. \tag{9.13}$$

The result applies for example in the context of branching processes with

$\tilde{E}N=\infty$. The following result gives a neccessary condition for $\mathcal{F}\neq\emptyset$ which applies when (H1) does not hold.

Corollary 9.4. *If $\rho(1)>1$ and (9.13) holds, then $\mathcal{F}=\emptyset$. More generally, if for some $\tilde{N}\leq N$ and $\tilde{A}_i\leq A_i$ $(1\leq i\leq\tilde{N})$ such that eithe (9.11a) or (9.11b) holds, then $\mathcal{F}=\emptyset$.*

10. Totality of fixed points; more on convergences

Theorem 10.1. *Suppose that (H2) holds, that $\rho(\alpha)=1$ and $\rho'(\alpha)\leq 0$ for some $\alpha\in(0,1]$. If $p\in\mathcal{B}_{\alpha,s}$, then there is a unique $\phi\in\mathcal{F}$ so that*

$$\lim_{t\to 0+}\frac{1-\phi(t)}{t^{\alpha}p(-\log t)}=1 \qquad \text{if } \rho'(\alpha)<0 \tag{10.1a}$$

and

$$\lim_{t\to 0+}\frac{1-\phi(t)}{t^{\alpha}p(-\log t)\,|\log t|}=1 \qquad \text{if } \rho'(\alpha)=0. \tag{10.1b}$$

Proof. The uniqueness comes from Theorem 7.4. If the problem is non-lattice or $\alpha=1$, then $\mathcal{B}_{\alpha,s}$ consists only of constants, so this result follows from Theorems 3.3 and 4.3. If $\alpha<1$ and $s>0$, the argument is the same as that of the proof of Theorem 5.1 of Durrett and Liggett (1983,pp.297-298), by using again our Theorems 3.3 and 4.3 instead of their Theorems 3.1,3.5 and 2.18. It proceeds as follows. Let $g(\theta)=e^{-\theta}$ if $\rho'(\alpha)<0$ and

$$g(\theta)=\frac{2}{\pi}\int_0^{\infty}\frac{e^{-\theta x}}{1+x^2}dx,$$

which is asymptotic to $1-\theta|\log\theta|$ as $\theta\to 0+$, if $\rho'(\alpha)=0$. Then by criterion 2 of Sect. XIII.4 of Feller (1971), since $p\in\mathcal{B}_{\alpha,s}$, it follows that the function ψ defined by

$$\psi(\theta)=g[\theta^{\alpha}p(-\log\theta)] \text{ if } \rho'(\alpha)<0 \text{ and } \psi(\theta)=g[\theta^{\alpha}p(-\log\theta)/\alpha] \text{ if } \rho'(\alpha)<0$$

is in $\mathcal{L}$. It is easy to check the property (10.1) holds for ψ (instead of ϕ therein). By Theorem 3.3, we can take $\tilde{\psi}\in\mathcal{F}$, and by Theorem 4.3, the property (10.1) holds for $\tilde{\psi}$ and some $\tilde{p}\in\mathcal{B}_{\alpha,s}$ (instead of ϕ and p). Since $\theta^{\alpha}p(-\log\theta)$ is strictly increasing on $(0,\infty)$ and tends to 0 or ∞ as θ tends to 0 or ∞

respectively, the equation

$$u^{\alpha}\tilde{p}(-\log u) = \theta^{\alpha}p(-\log\theta)$$

defines a function $u=u(\theta)$. By the periodicity of p and $\tilde{p}$, $u(\theta e^{s})=u(\theta)e^{s}$. Hence $u(\theta A_i)=u(\theta)A_i$ (because $\log A_i=ns$ for some $n\in\mathbb{Z}$) and $\upsilon(\theta)/\theta$ is bounded away from 0 and ∞ on $(0,\infty)$. Therefore, if we define $\phi(\theta)=\tilde{\psi}[u(\theta)]$, then ϕ satisfies $T\phi=\phi$ and (10.1). It remains to show that $\phi\in\mathcal{L}$. Since (10.1) holds for both ϕ and ψ, $1-\phi(\theta) \sim 1-\psi(\theta)$ as $\theta\to 0+$. The proof of Theorem 7.4 shows that $\lim_{n\to\infty} T^n\psi=\phi$, thus completing the proof since $\psi\in\mathcal{L}$. ∎

Let us now come back to the problem of convergence.

Theorem 10.2. *Suppose that (H2) holds, that $\rho(\alpha)=1$ and $\rho'(\alpha)\le 0$ for some $\alpha\in(0,1]$. If $\psi\in\mathcal{L}$ is such that for some $p\in\mathcal{B}_{\alpha,s}$, (10.1) holds with ϕ being replaced by ψ, then $T^n\psi$ converge to some $\phi\in\mathcal{F}$ which also satisies (10.1). In particular, if for some $\psi\in\mathcal{L}$ and constant $c>0$,*

$$1-\psi(t) \sim ct^{\alpha} \ (t\to 0) \quad \text{if } \rho'(\alpha)<0$$

and

$$1-\psi(t) \sim ct^{\alpha}|\log t| \ (t\to 0) \quad \text{if } \rho'(\alpha)=0,$$

then $T^n\psi$ converge to a canonical fixed point $\phi\in\mathcal{F}$ which satisfies

$$1-\phi(t) \sim ct^{\alpha} \ (t\to 0) \quad \text{if } \rho'(\alpha)<0 \tag{10.2a}$$

and

$$1-\phi(t) \sim ct^{\alpha}|\log t| \ (t\to 0) \quad \text{if } \rho'(\alpha)=0. \tag{10.2b}$$

The canonical fixed points are exactly those ϕ in $\mathcal{F}$ which satisfies (10.2). If the problem is nonlattice, all fixed points are canonical.

Proof. By Theorem 10.1, we can find $\phi\in\mathcal{F}$ such that (10.1) holds. Thus $1-\psi(t) \sim 1-\phi(t)$ $(t\to 0+)$. By Theorem 7.4, $\lim_{n\to\infty} T^n\psi=\phi$. By Theorem 4.3, under the condition (H2), a canonical fixed point ϕ satisfies (10.2). By the uniqueness in Theorem 10.1, for each constant $c>0$, there is only one fixed point which satisfies (10.2). So any fixed point satisfying (10.2) is canonical. If the problem is nonlattice, $\mathcal{B}_{\alpha,s}$ consists only of constants, thus completing the proof of the theorem. ∎

We now consider another natural question: if the initial element $\psi\in\mathcal{L}$ does not satisfy (10.1), what can we say about the iterations $T^n\psi$? Let us consider some regular cases to get some ideas on this question. We distinguish the cases according as $\rho'(\alpha)<0$ or $\rho'(\alpha)=0$. We recall that q is the unique fixed point in [0,1) of the function $f(t)=\sum_{k=0}^{\infty} P(\tilde{N}=k)t^k$.

Theorem 10.3. *Assume (H1) and, for some $\alpha\in(0,1]$, $\rho(\alpha)=1$ and $\rho'(\alpha)\le 0$. Let $\psi\in\mathcal{L}$.*

(a) If for some $a\in(0,\alpha)$, $c>0$, $t_0>0$ and all $t<t_0$,

$$1-\psi(t) \ge ct^a, \tag{10.3}$$

then $\forall t>0$,

$$\lim_{n\to\infty} T^n\psi(t) = q. \tag{10.4}$$

(b) If for some $b\in(\alpha,\infty)$, $c>0$, $t_0>0$ and all $t<t_0$,

$$1-\psi(t) \le ct^b, \tag{10.5}$$

then $\forall t>0$,

$$\lim_{n\to\infty} T^n\psi(t) = 1. \tag{10.6}$$

(c) In the case where $\rho'(\alpha)=0$, (10.6) also holds under the weaker condition that for some $c>0$, $t_0>0$ and all $t<t_0$,

$$1-\psi(t) \le ct^\alpha. \tag{10.7}$$

In particular, if $1-\psi(t) \sim ct^\alpha$ $(t\to 0+)$ for some $c>0$, then $\lim_{n\to\infty} T^n\psi(t) = 1$.

Proof. (a) Let $d\in(a,\alpha)$. Then $\rho(d)>1$ and $\rho'(b)<0$. By the conjugate relation $TS_d=S_dT_d$, we have

$$T^nS_d = S_dT_d^n . \tag{10.8}$$

Since $\rho(d)>1$ and $\rho'(d)<0$, by Theorem 9.1',

$$\lim_{n\to\infty} T_d^n\phi_0(t) = q$$

for all $t>0$, where $\phi_0(t) = e^{-t}$. Writing

$$\psi_d(t)= e^{-t^d},$$

by (10.8) we see that

$$\lim_{n\to\infty} T^n\psi_d(t) = q$$

for all t>0. Since

$$1-\psi_d(t) \sim t^d \ (t\to 0),$$

by (10.3) we can choose $t_0>0$ sufficiently small such that, for all $0<t<t_0$,

$$1-\psi_d(t) \leq 1-\psi(t).$$

Hence by Lemma 7.3, for all t>0,

$$\limsup_{n\to\infty} T^n\psi(t) \leq \limsup_{n\to\infty} T^n\psi_d(t) = q.$$

Since we have always $\liminf_{n\to\infty} T^n\psi(t) \geq q$ [see (9.6)], (10.4) follows.

(b) If $\rho'(\alpha)=0$, the conclusion follows from part (c) of the theorem. So we only consider the case $\rho'(\alpha)<0$ for the moment. If $\alpha=1$, the condition (10.5) can happen only if $\psi\equiv 1$, since ψ then corresponds to a measure with mean 0.Thus we can suppose that $\alpha<1$. Let $d\in(\alpha,b\wedge 1)$ be sufficiently near to α such that $\rho(d)<1$. This is possible since $\rho(\alpha)=1$ and $\rho'(\alpha)<0$. Note that $[\rho(\alpha)^{-1}T_d]^n\delta_1$ converge to some $\mu\in\mathcal{M}$ and $\rho(d)<1$, $T_d^n\delta_1\to\delta_0$. So by (10.8),

$$\lim_{n\to\infty} T^n\psi_d = 1,$$

where $\psi_d(t)= e^{-t^d}$. By (10.5),

$$1-\psi_d(t) \geq 1-\psi(t)$$

for t>0 sufficiently small. So by Lemma 7.3,

$$\liminf_{n\to\infty} T^n\psi(t) \geq \liminf_{n\to\infty} T^n\psi_d(t)=1.$$

Therefore $\lim_{n\to\infty} T^n\psi(t)=1$.

(c) We shall use again the conjugate relation $TS_\alpha=S_\alpha T_\alpha$. Since $\rho(\alpha)=1$ and $\rho'(\alpha)=0$, by Theorem 8.2,

$$\lim_{n\to\infty} T_\alpha^n\phi_0(t) = 1$$

for all t>0, where $\phi_0(t) = e^{-t}$. Writing $\psi_\alpha(t)= e^{-t^\alpha}$ and using

$T^n S_\alpha = S_\alpha T^n_\alpha$, we obtain

$$\lim_{n\to\infty} T^n \psi_\alpha(t) = 1$$

for all $t>0$. By a scale change, we see that for all constant $\tilde{c}>0$ and and all $t>0$,

$$\lim_{n\to\infty} T^n \tilde{\psi}(t) = 1,$$

where $\tilde{\psi}(t) = \psi_\alpha(\tilde{c}t)$. Since

$$1-\tilde{\psi}(t) \sim \tilde{c}^\alpha t^\alpha \ (t\to 0),$$

by (10.7) we can choose $\tilde{c}$ sufficiently large and $t_0>0$ sufficiently small such that, for all $0<t<t_0$,

$$1-\tilde{\psi}(t) \geq 1-\psi(t).$$

Hence by Lemma 7.3, for all $t>0$,

$$\liminf_{n\to\infty} T^n \psi(t) \geq \liminf_{n\to\infty} T^n \tilde{\psi}(t) = 1.$$

This gives (10.6). ■

Corollary 10.6. *Assume (H1) and for some* $\alpha\in(0,1]$, $\rho(\alpha)=1$ *and* $\rho'(\alpha)<0$. *Let* $\nu\in\mathcal{M}$.

(a) If ν *is such that for some* $a\in(0,\alpha)$, $c>0$, $x_0>0$ *and all* $x>x_0$,

$$\nu(x,\infty) \geq cx^{-a}, \tag{10.9}$$

then

$$\lim_{n\to\infty} T^n \nu = \delta_{0,\infty}. \tag{10.10}$$

(b) If $\alpha<1$ and ν *is such that for some* $b\in(\alpha,\infty)$, $c>0$, $x_0>0$ *and all* $x>x_0$,

$$\nu(x,\infty) \leq cx^{-b}, \tag{10.11}$$

then

$$\lim_{n\to\infty} T^n \nu = \delta_0 . \tag{10.12}$$

(c) In the case where $\alpha<1$ *and* $\rho'(\alpha)=0$, *(10.12) also holds under the weaker condition that for some* $c>0$, $x_0>0$ *and all* $x>x_0$,

$$\nu(x,\infty) \leq cx^{-\alpha}. \tag{10.13}$$

In particular, if $\nu(x,\infty) \sim cx^{-\alpha}$ $(x\to\infty)$ *for some* $c>0$, *then* $\lim_{n\to\infty} T^n \nu = \delta_0$.

Proof. This follows immediately from Theorem 10.5 and Lemma 11.1 in the next section. ∎

11. Moments and Tails

The following Tauberian Theorem has been used in the Sect.10, and will also be used later.

Lemma 11.1. *Let $\mu\in\mathcal{M}$ and ϕ be its Laplace transform. Then*

(a) for all $t>0$,

$$1-\phi(t) \geq (1-e^{-1})\ \mu(x,\infty) \quad \text{with } x=1/t.$$

(b) For all $\alpha\in[0,1)$ and any slowly varing function $\ell(x)\geq 0$ at ∞,

$$\Gamma(1-\alpha)\ \liminf_{x\to\infty} \frac{\mu(x,\infty)}{x^{-\alpha}\ell(x)} \leq \liminf_{t\to 0} \frac{1-\phi(t)}{t^{\alpha}\ell(1/t)}$$

$$\leq \limsup_{t\to 0} \frac{1-\phi(t)}{t^{\alpha}\ell(1/t)} \leq \Gamma(1-\alpha)\ \limsup_{x\to\infty} \frac{\mu(x,\infty)}{x^{-\alpha}\ell(x)}.$$

Proof. For part (a), it suffices to note that for all t and x >0,

$$\begin{aligned} 1-\phi(t) &= t\int_0^\infty e^{-ty}\mu(y,\infty)\ dy \\ &\geq t\int_0^y e^{-ty}\mu(y,\infty)dy \geq \mu(x,\infty)t\int_0^x e^{-ty}dy \\ &= \mu(x,\infty)\ (1-e^{-tx}). \end{aligned}$$

For part (b), let us write

$$\begin{matrix}\bar{a}\\ \underline{a}\end{matrix} = \lim_{t\to 0}\begin{matrix}\sup\\ \inf\end{matrix} \frac{1-\phi(t)}{t^{\alpha}\ell(1/t)} \quad \text{and} \quad \begin{matrix}\bar{b}\\ \underline{b}\end{matrix} = \lim_{x\to\infty}\begin{matrix}\sup\\ \inf\end{matrix} \frac{\mu(x,\infty)}{x^{-\alpha}\ell(x)}.$$

We first prove that $\Gamma(1-\alpha)\underline{b} \leq \underline{a}$. By the definition of $\underline{b}$, for all $b<\underline{b}$, there is some $x_b>0$ such that for all $x\geq x_b$,

$$\mu(x,\infty) \geq bx^{-\alpha}\ell(x).$$

Thus

$$\begin{aligned} 1-\phi(t) &= t\int_0^\infty e^{-ty}\mu(y,\infty)\ dy \\ &\geq bt\int_{x_b}^\infty e^{-ty}y^{-\alpha}\ell(y)dy = bt^{\alpha}\int_{tx_b}^\infty e^{-x}x^{-\alpha}\ell(x/t)dx. \end{aligned}$$

Since for all fixed $x>0$, $\ell(x/t)/\ell(1/t) \to 1$ as $t\to 0$, the dominated convergence theorem gives

$$\liminf_{t\to 0} \frac{1-\phi(t)}{t^{\alpha}\ell(1/t)} \geq b\ \Gamma(1-\alpha).$$

Letting $b\to\underline{b}$, we obtain that $\underline{a} \geq \underline{b}\Gamma(1-\alpha)$. We now prove that $\bar{a}\leq\bar{b}\Gamma(1-\alpha)$. We can suppose that $\bar{b}<\infty$ since otherwise there is nothing to prove. For all $b'>\bar{b}$, there is some $x_{b'}>0$ such that for all $x\geq x_{b'}$,

$$\mu(x,\infty) \leq b'x^{-\alpha}\ell(x).$$

Therefore

$$1-\phi(t) = t\int_0^\infty e^{-ty}\mu(y,\infty)dy \leq tx_{b'}+ b't \int_{x_{b'}}^\infty e^{-ty}y^{-\alpha}\ell(y)dy$$

$$= tx_{b'} + b't^{\alpha} \int_{tx_{b'}}^\infty e^{-x}x^{-\alpha}\ell(x/t)dx.$$

Since $t/t^{\alpha}\ell(1/t)\to 0$ as $t\to 0$, the same argument as above shows that $\bar{a}\leq b'\Gamma(1-\alpha)$ and then $\bar{a}\leq\bar{b}\Gamma(1-\alpha)$. This ends the proof of the Lemma. ∎

Theorem 11.2. *Assume (H1) and for some $\alpha\in(0,1]$, $\rho(\alpha)=1$ and $\rho'(1)\leq 0$. If $\mu\in\mathcal{F}$,* then

(i) $\mu(x,\infty) = O(x^{-\alpha}l(x))\ (x\to+\infty)$;

(ii) $\int_0^\infty x^a d\mu(x)<\infty$ *for all* $a\in[0,\alpha)$;

(iii) If $\alpha=1$, *then* $\int_0^\infty xd\mu(x)=\infty$ *if and only if* $\mathbb{E}S\log^+S=\infty$ *or* $\rho'(1)=0$;

If $\alpha<1$, *then* $\int_0^\infty x^{\alpha}d\mu(x)=\infty$.

(iv) If $\alpha=1$, *then* $\int_0^x \mu(t,\infty)dt \sim \ell(x)\ \ (x\to+\infty)$ *is slowly varing.*

Proof. Part (i) follows from Theorem 6.2 and Lemma 11.1. Part (ii) follows from part (i) since $\int_0^\infty x^a d\mu(x) = a\int_0^\infty x^{a-1}\mu(x,\infty)dx$. For part (iii), the conclusion for $\alpha=1$ comes from Theorem 0; The conclusion for $\alpha<1$ follows by Proposition 8.1, since the equality in (8.3) holds with $\varepsilon=\mu$. Part (iv) follows from Theorem 6.2 and the general Tauberian theorem. ∎

Of course, in the case where $\alpha<1$, Theorem 6.2 can be applied to obtain tail behaviour of a fixed point $\mu\in\mathcal{F}$. For example, it is easily seen that we have

Theorem 11.3. *Assume (H2). Suppose that for some $\alpha\in(0,1)$, $\rho(\alpha)=1$ and $\rho'(\alpha)=0$. If the problem is nonlattice, then for all $\mu\in\mathcal{F}$, there is a constant $c>0$ such that, as $x\to\infty$,*

$$\mu(x,\infty) \sim cx^{-\alpha} \qquad \text{if } \rho'(\alpha)<0$$

and

$$\mu(x,\infty) \sim cx^{-\alpha}\log x \quad \text{if } \rho'(\alpha)=0.$$

Some similar results can also be derived in the lattice case.

In the case where $\rho(1)=1$ and $\rho'(1)<0$, we need further informations about moments of order greater than 1. The following result was given in Liu (1994):

Theorem 11.4. *Assume (H1). Suppose that $\rho(1)=1$ and $\rho'(1)<0$. Let $Z\geq0$ be a solution of (E). Then for all $a>1$,*

$$\mathbb{E}Z^a<\infty \text{ if and only if } \mathbb{E}S^a<\infty \text{ and } \rho(a)<1,$$

provided that one of the following conditions holds:

(a) $1<a\leq2$; (b) $a=2,3,\ldots$; (c) $\|\mathrm{Max}_i\, A_i\|_\infty<\infty$; (d) $\|\tilde{N}\|_\infty<\infty$.

Put

$$\bar{\alpha} = \sup\{a\geq1:\ \rho(a)\leq1\},$$

and

$$\beta = \sup\{a\geq1:\ \mathbb{E}S^a<\infty\}.$$

Since $\mathbb{E}\,[\mathrm{Max}_i\, A_i]^a \leq \rho(a)$, $\|\mathrm{Max}_i\, A_i\|_a\leq [\rho(a)]^{1/a}$ (where $\|.\|_a$ denotes the norm in L^a), letting $a\to\infty$ we see that

$$\bar{\alpha} =\infty \text{ if and only if } \|\mathrm{Max}_i\, A_i\|_\infty \leq1.$$

By Theorem 11.4, at least at the case where $\|\mathrm{Max}_i\, A_i\|_\infty<\infty$ or $\|\tilde{N}\|_\infty<\infty$, the number $\bar{\alpha}\wedge\beta(\leq\infty)$ is the critical value for existence of moments of $\mu\in\mathcal{F}$:

$$\int_0^\infty x^a d\mu(x)<\infty \text{ if } a<\bar{\alpha}\wedge\beta,$$

and

$$\int_0^\infty x^a d\mu(x)=\infty \text{ if } a>\bar{\alpha}\wedge\beta.$$

Of course, the tail behavior of $\mu\in\mathcal{F}$ differs according as $\bar{\alpha}\wedge\beta<\infty$ or $=\infty$. In the first case, the situation differs also according as $\bar{\alpha}>\beta$ or $\bar{\alpha}\leq\beta$.

Case 1: $1<\beta<\bar{\alpha}$. We have then $\beta<\infty$ and $\rho(\beta)<1$. Following Bingham and Doney (1975), we obtain the following comparison result:

Theorem 11.5. *Suppose that $\rho(1)=1$, $\rho'(1)<0$, $1<\beta<\bar{\alpha}$, and $\|\operatorname{Max}_i A_i\|_\infty<\infty$. Let $Z\geq 0$ be the unique fixed point of T satisfing $\mathbb{E}Z=1$. Then for all slowly varing function $\ell(x)$ at ∞, the following assertions hold:*

(i) If $\beta>1$ is not an integer, then

(a): $$P(S>x) \sim x^{-\beta}\ell(x)\ (x\to\infty) \iff P(Z>x) \sim x^{-\beta}\ell(x)/[1-\rho(\beta)]\ (x\to\infty),$$

and (b): $$\mathbb{E}S^{\beta}\ell(S)<\infty \iff \mathbb{E}Z^{\beta}\ell(Z)<\infty.$$

(ii) If $\beta>1$ is an integer, then

(a): $$\mathbb{E}S^{\beta}1_{\{S\leq x\}} \sim \ell(x)\ (x\to\infty) \iff \mathbb{E}Z^{\beta}1_{\{Z\leq x\}} \sim \ell(x)/[1-\rho(\beta)]\ (x\to\infty)$$

provided that $\lim_{x\to\infty}\ell(x)=\infty$;

(b): $$\mathbb{E}S^{\beta}1_{\{S>x\}} \sim \ell(x)\ (x\to\infty) \iff \mathbb{E}Z^{\beta}1_{\{Z>x\}} \sim \ell(x)/[1-\rho(\beta)]\ (x\to\infty)$$

provided that $\mathbb{E}S^{\beta}<\infty$; and

(c): $$\mathbb{E}S^{\beta}\ell^{*}(S)<\infty \iff \mathbb{E}Z^{\beta}\ell^{*}(Z)<\infty,$$

where $\ell^{*}(x):=\int_0^x \frac{\ell(t)}{t}dt$.

(iii) If for each $1<a<b<\infty$, $\limsup_{x\to\infty}\ell^{*}(b^n)/\ell^{*}(a^n)<\infty$, then

$$\mathbb{E}S\ell^{**}(S)<\infty \iff \mathbb{E}Z\ell^{*}(Z)<\infty.$$

(iv) In particular, for all $a>0$, taking $\ell(x)=\log^{a-1}x$ for $x>1$ in (i-iii), we have

$$\mathbb{E}S^{\beta}(\log^{+}S)^{a}<\infty \iff \mathbb{E}Z^{\beta}(\log^{+}Z)^{a}<\infty,$$

where $\beta>1$ (is integer or not), and

$$\mathbb{E}S(\log^{+}S)^{1+a}<\infty \iff \mathbb{E}Z(\log^{+}Z)^{a}<\infty.$$

Proof. With some slight changes, the arguments of Bingham and Doney (1975) for the proof of their Theorem 2 applies in the present setting. The point is that, their proof is based on their functional equation (**) (Bingham and Doney 1975, p.70), which corresponds to our functional equation $\phi=T\phi$ with $\operatorname{Max}_i A_i\leq 1$. A check of the details of their proof shows that the condition that

$\underset{i}{Max} A_i \leq M$ for some constant $M>0$ suffices. ∎

Case 2: $1<\bar{\alpha}\leq\beta$ and $\bar{\alpha}<\infty$. The following result was given by Guivarc'h (1990):

Theorem 11.6. *(Guivarc'h 1990) Suppose that $\rho(1)=1$, $\rho'(1)<0$, $n:=\|\tilde{N}\|_\infty<\infty$, and $A_i (1\leq i\leq n)$ are independent and identically distributed. If the problem is nonlattice and $\rho(\chi)=1$ for some $\chi>1$, then for all $\mu\in\mathcal{F}$, there is a constant $c>0$ such that*

$$\mu(x,\infty) \sim cx^{-\chi} \quad (x\to\infty).$$

The case where $\bar{\alpha}\wedge\beta<\infty$ is composed of the cases 1 and 2. It remains the case where $\bar{\alpha}\wedge\beta=\infty$, or eqivalently $\bar{\alpha}=\beta=\infty$.

Case 3: $\bar{\alpha}=\beta=\infty$. We have then $\underset{i}{Max}\, A_i\leq 1$ almost surely. This case was studied in Liu (1993 and 1994b). For example, we have

Theorem 11.7.*(Liu 1993 and 1994a) Suppose that $\underset{i}{Max}\, A_i\leq 1$ almost surely, $\rho(1)=1$ and $\rho'(1)<0$. Let*

$$\gamma:=\inf\{a\in[0,1): \|S(\tfrac{1}{1-a})\|_\infty \leq 1\},$$

where $\inf\emptyset=1$. If $\|\tilde{N}\|_\infty<\infty$, $0<\gamma<1$, and $Z\geq 0$ is the unique solution of (E) with $\mathbb{E}Z=1$, then (a) $\lim_{k\to\infty}\frac{\log\mathbb{E}Z^k}{k\log k}=\gamma$; and (b) for some constant $A>0$ and all $\varepsilon>0$,

$$\exp(-x^{(1/\gamma)+\varepsilon}) \leq P(Z>x) \leq \exp(-Ax^{1/\gamma})$$

for all $x>0$ sufficiently large.

Liu (1993 and 1994a) gave also sufficient conditions under which for some constant $A_1>0$ and all $x>0$ sufficiently large,

$$\exp(-A_1 x^{1/\gamma}) \leq P(Z>x).$$

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References

Asmussen,S.and Hering,H.(1983): Branching processes. Birkhäuser.

Athreya,K.B.(1971): A note on a functional equation arising in Galton-Watson branching processes. J.Appl. Prob., 8, 589-598.

Athreya,K.B.and Ney,P.E.(1972): Branching processes. Springer, Berlin.

Ben Nasr, F. (1987): Mesures aléatoires de Mandelbrot associées a des substitutions. C.R. Acad. Sci. Paris, Sér.I, 304, 255-258.

Biggins,J.D. (1977): Martingale convergence in the branching random walk. J.Appl. Prob. 14, 25-37.

Bingham,N.H. and Doney,R.A. (1974): Asympototic properties of supercritical branching processes I: The Galton-Watson process. Adv. Appl. Prob., 6, 711-731.

Bingham,N.H. and Doney,R.A. (1975): Asympototic properties of supercritical branching processes II: Crump-Mode and Jirina processes. Adv. Appl. Prob., 7, 66-82.

Chauvin,B. and Rouault,A.(1993): Boltzmann-Gibbs weights in the branching random walk. Prépublication no.175 du Laboratoire de Probabilités de l'Université Paris VI.

Collet,P. and Koukiou,F. (1992): Large deviations for multiplicative chaos. Commun. Math. Phys. 147, 329-342.

Crump,K. and Mode,C.J. (1968): A general age-dependent branching process (I). J.Math. Anal. Appl., 24,497-508.

Crump,K. and Mode,C.J. (1969): A general age-dependent branching process (II). J.Math. Anal. Appl., 25, 8-17.

Doney,R.A.(1972): A limit theorem for a class of supercritical branching processes. J.Appl. Prob.,9,707-724.

Doney,R.A.(1973): On a functional equation for general branching processes. J.Appl. Prob.,10, 497-508.

Durrett,R.and Liggett,T.(1983): Fixed points of the smoothing transformation. Z. Wahrscheinlichkeitstheorie verw. Gebiete, 64, 275-301.

Falconer,K.J. (1986): Random fractals. Math.Proc.Camb.Phil.Soc., 100, 559-582.

Falconer,K.J. (1987): Cut-set processes and tree processes. Math. Proc. Amer. Math. Soc. (2),101, 337-346.

Feller,W.(1971):An introduction to probability theory and its applications. Vol.2, 2nd ed. John Wiley & Sons.

Fortet,R. et Mourier,E (1953): Convergence de la répartition empirique vers la répartion théorique. Ann. Scient. Ecole Normale Sup. Série 3, tome 70, 266-285.

Franchi,J.(1993): Chaos multiplicatif: un traitement simple et complet de la fonction de la partition. Prépublication no. 148 du Laboratoire de Probabilités de l'Université Paris VI.

Graf,S., Mauldin, R.D.and Williams,S.C. (1988): The exact Hausdorff dimension in random recursive constructions. Mem. Amer. Math. Soc. 71, n°381.

Guivarc'h, Y.(1990): Sur une extension de la notion semi-stable, Ann. Inst. Henri Poincaré, (Prob.et Stat.), 26, 261-285.

Heyde,C.C.(1970): Extension of a result of Seneta for the supercritical Galton-Watson process. Ann. Math. Statist. 41, 739-742.

Holley,R. and Liggett,T. (1981): Generalized potlatch and smoothing processes. Z. Wahrscheinlichkeitstheorie verw. Gebiete, 55, 165-195.

Holley,R. and Waymire,E.C. (1992): Multifractal dimensions and scaling exponents for strongly bounded random cascades. Ann. Appl. Prob. 2, 819-845.

Kahane,J.P.(1987): Multiplications aléatoires et dimensions de Hausdorff. Ann. Inst. Henri Poincaré, Sup.au no.2, vol.23, p.289-296.

Kahane,J.P. and Peyrière J. (1976): Sur certaines martingales de Benoit Mandelbrot. Adv. Math., 22, 131-145.

Kesten,H. and Stigum,B.P.(1966): A limit theorem for multidimensional Galton-Watson processes, Ann. Math. Statist. 37, 1211-1223.

Liu,Q. (1993): Sur quelques problèmes à propos des processus de branchement, des flots dans les réseaux et des mesures de Hausdorff asociées. Thèse, Université Paris 6.

Liu,Q. (1994): Sur une equation fonctionnelle et ses applications: une extension du théorème de Kesten-Stigum concernant des processus de branchement. Prépublication, Université Rennes 1.

Liu,Q. (1994a): Growth of entire characteristic functions and tails in tree martingales, Prépublication, Université Rennes 1.

Mandelbrot,B. (1974): Multiplications aléatoires et distributions invariantes par moyenne pondérée aléatoire. C.R. Acad. Sci. Paris, vol.278, Série A, 325-346.

Mandelbrot,B. (1974): Multiplications aléatoires et distributions invariantes par moyenne pondérée aléatoire: Quelques extensions. C.R. Acad. Sci. Paris, vol.278, Série A, 355-358.

Mauldin,R.A. and Williams,S.C.(1986): Random constructions, asymptotic geometric and topological properties. Trans.Amer.Math.Soc., 295, 325-346.

Royer,G.(1984): Distance de Fortet-Mourier et fonctions log-concaves. Annales Scientifiques de Clermont-Ferrand, n°.78.

Seneta,E.(1968): On recent theorems concerning the supercritical Galton-Watson process. Ann. Math. Stat. 39, 2098-2102.

Seneta,E.(1969): Functional equations and the Galton-Watson process. Adv. Appl. Prob. 1, 1-42.

Seneta,E.(1973): A Tauberian theorem of E.Landau and W.Feller, Ann. Prob. 1, 1057-1058.

Seneta,E.(1974): Regular varing functions in the theory of simple branching processes. Adv. Appl.Prob.6,408-420.

Waymire,E.C. and Williams,S.C.(1993): Multiplicative cascades: Dimension spectra and dependence. Preprint, Origon State University and Utah State University.