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THE EXACT HAUSDORFF DIMENSION OF A BRANCHING SET

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Summary. We obtain a critical function for which the Hausdorff measure of a branching set generated by a simple Galton-Watson process is positive and finite. The results solve a conjecture of Hawkes (1981).

0. Introduction

Let $(\Omega, \mathcal{A}, P)$ be a probability space, $\mathcal{T}=\mathcal{T}(\omega)$ $(\omega\in\Omega)$ the genealogical tree of a one-type Galton-Watson process $(Z_n)\equiv(Z_n(\omega))$ $(n\geq 0, \omega\in\Omega)$ with a single founder member $Z_0\equiv 1$ and offspring distribution $Z\equiv Z_1$. The root of $\mathcal{T}$ is identified to the founder member which is represented by the null sequencer $\emptyset$. The vertices in the n-th level are represented by n-terms sequences $\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ of non-negative integers which correspond to the particles in the n-th generation of the branching process. The edges of $\mathcal{T}$ are formed by joining the vertices $\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_n)$ to their descendants $(\sigma,i)\equiv(\sigma_1,\sigma_2,\ldots,\sigma_n,i)$, $0\leq i<Z^{\sigma}$, where Z^{σ} is the number of children of σ. The tree $\mathcal{T}$ is then identified to a random set of finite sequences of non-negative integers. Let $K=K(\omega)$ be its boundary, i.e.the set of all the infinite sequences $(\sigma_1,\sigma_2,\ldots)$ such that $(\sigma_1,\sigma_2,\ldots,\sigma_n) \in \mathcal{T}$ for all n = 1,2,... The set K is called to be the *branching set* generated by a simple branching process [8]. Let $\mathbb{N}$ be the set of non-negative integers with the discrete topology, and $\mathbf{I} = \mathbb{N}^{\mathbb{N}}$ be the set of all sequences $\mathbf{i} = (i_1,i_2,\ldots)$ of the integers in $\mathbb{N}$ with the product topology. Then $\mathbf{I}$ is

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metrizable, and a possible choice of metric is given by $d(\mathbf{i},\mathbf{j})=2^{-|\mathbf{i}\wedge\mathbf{j}|}$, where $\mathbf{i}\wedge\mathbf{j}$ is the common sequece of $\mathbf{i}$ and $\mathbf{j}$, i.e., the maximal sequence $\mathbf{q}$ such that $\mathbf{q}<\mathbf{i}$ and $\mathbf{q}<\mathbf{j}$ (§2). We will always refer to this metric in this paper if it is not specified further. The branching set K is then a random compact subset of the metric space $(\mathbf{I},d)$ (§2).

If $p(s) = \sum_k p_k s^k$ is the generating function of the family distribution Z, and if $\mu = \sum kp_k < \infty$, then $W= \lim_n Z_n/\mu^n$ exists and, if

$$\sum p_k k \text{ Log } k < \infty, \qquad \text{(ZlogZ)}$$

then $\mathbb{E}(W) = 1$ and the extinction probability satisfies

$$P (Z_n \to 0) = P (W = 0) = P (K = \emptyset).$$

Throughout this paper, we suppose always that the moment condition (ZlogZ) is satisfied.

Our interest centers on the Hausdorff measures of the branching set K. In 1981, Hawkes first proved that if

$$\sum p_k k \log^2 k < \infty, \qquad (\text{Zlog}^2\text{Z})$$

the Hausdorff dimension of K satisfies

$$\dim K = \alpha := \log \mu / \log 2 \qquad (0.1)$$

almost surely on $K \neq \emptyset$. This result was also obtained under the weaker condition (ZlogZ) by several other authors in some different contexts (see for example [4,11,12,13]). As we shall see in §4, the fact that the condition (ZlogZ) suffices for the dimension result (0.1) can also be easily shown by a simple truncation from Hawkes (1981).

In the case where the offspring is of geometric distribution (i.e. $P(Z=i) = a(1-a)^{i-1}$ for some $a \in (0,1)$ and all $i \geq 1$), Hawkes showed moreover that $0 < \mathcal{H}^{\phi_1}(K) < \infty$ a.s., where $\mathcal{H}^{\phi_1}(K)$ denotes the Hausdorff measure of K with respect to the function

$$\phi_1(t) = t^{\alpha}(\log\log 1/ t)$$

Thus ϕ_1 is an exact dimension function of K, i.e. a function for which the

Hausdorff measure of K is almost surely positive and finite if $K \neq \emptyset$. Hawkes's proof on the last result was heavily dependent of the fact that the underlying distribution Z is geometric. In the case where it is of general distribution , He conjectured that the dimension function would be of the form $\tilde{\phi}(t) = t^{\alpha}R^{-1}(\log\log \frac{1}{t})$, if $R(x) := -\log P(W \geq x)$ is *regularly* varying at infinity (He did not precise the regularity condition). Our results here will show that this is endeed the case if, for example, the distribution of W decreases geometrically, or more generally

$$P(W \geq x) \sim Ce^{-\lambda x^{a}} \quad (C, \lambda, a>0,\ x \to \infty).$$

In fact, we shall be able to treat the case where

$$c_1 e^{-\lambda_1 x^{a}} \leq P(W \geq x) \leq c_2 e^{-\lambda_2 x^{a}} \qquad (\forall x \geq \Delta),$$

for some positive constants $\lambda_1, \lambda_2, c_1, c_2, a, \Delta > 0$. (§1, example 2) and the results we shall prove are much better than this: different dimension functions of the form t^{α}, $t^{\alpha}(\log \frac{1}{t})^{\beta}$, $t^{\alpha}(\log\log \frac{1}{t})^{\beta}$, etc. will be caculated explicitely according to the offspring distribution Z. For example, writing

$$\beta = 1 - \log \mu \,/\, \log \|Z\|_{\infty}\ , \tag{0.2}$$

where $\mu = \mathbb{E}Z$ and $\|Z\|_{\infty} = \text{ess sup } Z \leq \infty$ (thus $0 \leq \beta \leq 1$), the function

$$\phi_{\beta}(t) := t^{\alpha}(\log\log \frac{1}{t})^{\beta} \tag{0.3}$$

is an exact dimension function of K if the offspring distribution Z is bounded (i.e. $\|Z\|_{\infty} < \infty$, hence $0 \leq \beta < 1$) or it is not bounded ($\|Z\|_{\infty} = \infty$, hence $\beta = 1$) but it decreases geometrically,

$$c_1 a_1^{k} \leq P(Z=k) \leq c_2 a_2^{k},$$

say, where $c_1, c_2 > 0$, $0 < a_1 \leq a_2 < 1$ and k is sufficiently large (see §1, Theorem 3 and Example 3). The case where the reproductive distribution decreases more slowly, for example, the case where

$$c_1 k^{-\delta} \leq P(Z=k) \leq c_2 k^{-\delta} \quad (k \in \mathbb{N} \text{ sufficiently large})$$

for some constants c_1, $c_2 > 0$ and $\delta > 1$, will also be discussed (see §1,Theorem 6 and Example 4), and the convenable dimension functions will be found greater (This is rather natural: more slowly decreases the reproductive distribution, larger the branching set associated, and so greater the convenable gauge functions for the set.). However, in this case the *exact* dimension functions remain unknown, and the author conjectures that, quite probably, they would not exist (by an argument of density of an associated measure).

Finally, we point out that our results here are closely related to those of Graf, Mauldin and Williams (1988). In fact, the author has recently developed the ideas to Euclidian space, and thus improved the classical results of Gral et al.(1988) (see Liu 1993).

1. Main results and examples

The main results are stated in the following, where the moment condition (ZlogZ) is always supposed to be satisfied. For convenience, we establish our results for the spherical Hausdorff measure $\mu^f(.)$, where f=f(t) is a positive function defined for t>0 sufficiently small, non-decreasing and continuous on the right. However, all the conclusions hold for the ordinary Hausdoff measure $\mathcal{H}^f(.)$ since the two measures $\mu^f(.)$ and $\mathcal{H}^f(.)$ coincide on K (see §2, lemma 2.3).

We first gather some preliminary results as follows:

Theorem 1. (*The dimension α of K and the α-dimensional measure associated*) *Let α be defined as in (0.1). Then*

(i) $\dim K(\omega) = \alpha$ *a.s. on* $K(\omega) \neq \emptyset$; *(ii)* $\mu^{\alpha}(K(\omega)) < \infty$ *a.s.;*

(iii) $0 < \mu^{\alpha}(K(\omega)) < \infty$ *a.s. on* $K(\omega) \neq \emptyset$ *if and only if Z is a.s. a constant.*

The dimension result was first found by Hawkes (1981) under the condition $(Z\log^2 Z)$. It was also proved by Falconer (1986, Corollary 5.7) and Lyons (1990, prop. 6.4) in different languages under the condition (ZlogZ), see also

Lyons and Pemantle (1992) or Lyons, Pemantle and Peres (1993, th.7.3). We shall see that, this can also be obtained easily by truncation from Hawkes (1981) (to relaxe the condition ($Z\log^2 Z$) as ($Z\log Z$), see §4, Corollary 4.1). The conclusion (ii) is easy; the conclusion (iii) is a special case of Falconer (1987, Lemma 4.4) (see §4).

Theorem 1 shows that in the non-degenerate case the α-dimensional measure of the branching set vanishes and so the function t^α is too small to measure the set. The following result is to give a criterion for a function of the form

$$\phi_\theta(t) := t^\alpha(\log\log \tfrac{1}{t})^\theta \tag{1.1}$$

to be an exact dimension function of K:

Theorem 2. (*A necessary and sufficient condition for $\mu^{\phi_\theta}(K)$ to be zero, positive and finite, or infinite*) *Let $0< \theta < +\infty$, ϕ_θ be the function defined by (1.1) and $r_{1/\theta} = r(W^{1/\theta})$ the radius of convergence of the moment generating function $\mathbb{E}(e^{tW^{1/\theta}})$ of $W^{1/\theta}$. Then*

$$\mu^{\phi_\theta}(K) = (r_{1/\theta})^\theta W \quad a.s. \tag{1.2}$$

In particular,

$$\mu^{\phi_\theta}(K) \text{ is zero, positive and finite, or infinite}$$

almost surely on $K \neq \emptyset$ if and only if

$$r_{1/\theta} \text{ is zero, positive and finite, or infinite}$$

respectively.

Remark. If $\mathbb{E}(Z^p)=\infty$, then $r_{1/\theta}=0$ for all $\theta\in(0,\infty)$, Theorem 2 is then interpreted as $\mu^{\phi_\theta}(K)=0$ a.s. for all $\theta\in(0,\infty)$. In this case Theorem 6 in the below will give more exact results.

Theorem 3. (*The exact dimension function: case $\|Z\|_\infty< +\infty$*) *Suppose that $\|Z\|_\infty< +\infty$. Let $\alpha\in(0,\infty)$ and $\beta\in[0,1)$ be defined as in (0.1) and (0.2). Then*

$$0 < \mu^{\phi_\beta}(K) < +\infty \tag{1.3}$$

almost surely on $K \neq \emptyset$, *where* $\phi_\beta(t) = t^\alpha(\mathrm{LogLog}\ \frac{1}{t})^\beta$. *More exactly, a.s.on* $K\neq\emptyset$,

$$\mu^{\phi_\beta}(K) = 1 \text{ if } \beta=0 \quad \text{and} \quad \mu^{\phi_\beta}(K) = (r_{1/\beta})^\beta W \text{ if } \beta>0, \tag{1.4}$$

where $0<r_{1/\beta}<\infty$.

Remark. We note that $\beta=0$ if and only if Z is a.s. a constant. This shows that in the deterministic case the term $(\mathrm{loglog}\ \frac{1}{t})^\beta$ disappears.

Theorem 4. *(Hausdorff measure of K: case* $\|Z\|_\infty = +\infty$*) Suppose that* $\|Z\|_\infty = +\infty$. *For* $\theta\in(0,\infty)$, *let* ϕ_θ *be defined by (1.1). Then*

(i) $\mu^{\phi_\theta}(K) = 0$ *a.s.* $\forall\ \theta < 1$;

(ii) $\mu^{\phi_1}(K) > 0$ *a.s. on* $K \neq \emptyset$ *if* $\mathbb{E}(e^{tZ}) < \infty$ *for some* $t>0$;

(iii) $\mu^{\phi_1}(K) < \infty$ *a.s. if* $\mathbb{E}(e^{tZ}) = \infty$ *for some* $t>0$;

Theorem 5. *(The exact dimension function: case* $\|Z\|_\infty = +\infty$*) Suppose that* $\|Z\|_\infty = +\infty$ *and that* $\mathbb{E}(e^{tZ})<\infty$ *for some but not all* $t>0$. *Then*

$$0 < \mu^{\phi_1}(K) < \infty \tag{1.5}$$

almost surely on $K \neq \emptyset$, *where*

$$\phi_1(t) = t^\alpha(\mathrm{loglog}\ \frac{1}{t}). \tag{1.6}$$

Moreover

$$\mu^{\phi_\beta}(K) = r_1 W \quad a.s., \tag{1.7}$$

where $r_1 = \sup\{t\geq 0: \mathbb{E}(e^{tW})<\infty\}$ *is positive and finite.*

Theorem 6. *For* $\theta\in(0,\infty)$, *put*

$$\psi_\theta(t) = t^\alpha(\log \frac{1}{t})^\theta. \tag{1.8}$$

Let

$$\gamma = \sup\{\ p\geq 1: \mathbb{E}Z^p<\infty\ \}\quad (1 \leq \gamma \leq \infty)\ , \tag{1.9}$$

then, almost surely on $K\neq\emptyset$,

(i) $\mu^{\psi_\theta}(K) = \begin{cases} 0 \text{ if } \theta<1/\gamma; \\ \infty \text{ if } \theta>1/(\gamma-1), \end{cases}$ *where* $1/\gamma$ *or* $1/(\gamma-1)$ *is interpreted as 0*

if $\gamma=\infty$;

(ii) $\mu^{\psi_{1/(\gamma-1)}}(K) = \infty$ *if* $1<\gamma<\infty$ *and* $\mathbb{E}(Z^{\gamma})=\infty$;

(iii) $\mu^{\psi_{1/\gamma}}(K) < \infty$ *a.s. if*

$$\limsup_{k\to\infty} \left\{ \sum_{\nu=[\log k]}^{k} P\left(W \ge \nu^{1/\gamma} \right) - \frac{1}{\gamma} \log k \right\} > -\infty. \tag{1.10}$$

Corollary. *(a) If* $\mathbb{E}(Z^{p})<\infty$ *for all* $p>1$, *that is, if* $\gamma=\infty$, *then for all* $\theta>0$, $\mu^{\psi_\theta}(K) = \infty$ *a.s.on* $K\neq\emptyset$. *(b) If* $\mathbb{E}(Z^{p})=\infty$ *for some but not all* $p>1$, *that is, if* $1<\gamma<\infty$, *then there exists* $\chi\in[1/\gamma,\ 1/(\gamma-1)]$ *such that a.s.on* $K\neq\emptyset$, $\mu^{\psi_\theta}(K) =0$ *if* $\theta<\chi$ *and* $\mu^{\psi_\theta}(K) =\infty$ *a.s. on* $K\neq\emptyset$ *if* $\theta>\chi$.

The assertion (b) holds also in the case where $\gamma=\infty$: χ is then interpreted as 0 and the result means $\mu^{\psi_\theta}(K) =0$ if $\theta<0$ and $\mu^{\psi_\theta}(K) =\infty$ if $\theta>0$. Thus (a) can be considered as a limit case of (b) with $\gamma=\infty$. There is another limit case with $\gamma=1$. The author believes that it would hold also with the interpretion that $\chi=\infty$ and $\mu^{\psi_\theta}(K) =0$ if $\theta\in(0,\infty)$.

Conjecture. *In any cases* $(1\le\gamma\le\infty)$, $\mu^{\psi_\theta}(K) =0$ *a.s.if* $\theta<1/(\gamma-1)$ *and* $\mu^{\psi_\theta}(K) =\infty$ *a.s.on* $K\neq\emptyset$ *if* $\theta>1/(\gamma-1)$. *(In the limit cases where* $\gamma=1$ *or* ∞, *the result is interpreted as in the above.)*

Remark 7. All the results above hold with α replaced by $\alpha(M) := \log \mu / \log M$ if the distance $d(\mathbf{i},\mathbf{j}) = 2^{-|\mathbf{i}\wedge\mathbf{j}|}$ on $\mathbf{I}$ is replaced by $d_M(\mathbf{i},\mathbf{j}) := M^{-|\mathbf{i}\wedge\mathbf{j}|}$, where $M > 1$.

As applications of the theorems, we give some examples here:

Example 1. (*Embedding in euclidean space*) Suppose that the distribution of $Z = Z_1$ has compact support, that is $\|Z\|_\infty<\infty$ or $p_k= 0$ for k sufficiently large. Let M be an integer such that $M \ge \|Z\|_\infty$ (namely $p_k= 0$ for $k > M$). If $Z_1= k$

we choose at random k distinct integers $j_1, j_2, \ldots, j_k$ with $0 \le j_i \le M-1$ and let

$$I_1 = \bigcup_{i=1}^{Z_1} [j_i/M, (j_i+1)/M).$$

We now treat each interval in I_1 as the vertex of a tree and preceed inductively in the same fashion. At the n-th stage we have I_n as a union of Z_n intervals of length M^{-n}. The limit set $\bar{K} = \bigcap_0^\infty I_n$ can be described by the associated branching set K of the process under the mapping

$$f: K \longrightarrow \bar{K}, \ \mathbf{i} \longrightarrow \sum i_k M^{-k}.$$

If we consider covers of $\bar{K}$ by M-adic sets and if K carries the matric $d_M(\mathbf{i},\mathbf{j}) = M^{-|\mathbf{i}\wedge\mathbf{j}|}$, it is then easily seen that the Cantor set $\bar{K}$ has the same exact dimension function as K, given by $\phi_\beta(t) = t^{\alpha(M)}(\log\log \frac{1}{t})^\beta$, where

$$\alpha(M) = \log \mu \ / \ \log M \quad \text{and} \quad \beta = 1 - \log \mu \ / \ \log\|Z\|_\infty.$$

(Theorem 3 and Remark 7).

We give now a more explicit construction to explain this: divide the unit interval into three equal parts and retain each independently with probability p. Repeat this with the parts that remain, and so on. In this case M=3, $\mu=\mathbb{E}(Z)=3p$ and $\|Z\|_\infty = 3$. Then $\alpha=\log(3p)/\log 3 = 1+\log p/\log 3$ and $\beta= 1-\log(3p)/\log 3 = 1-\alpha$. The exact Hausdorff dimension function of the resulting fractal set is then $t^\alpha(\log\log\frac{1}{t})^{1-\alpha}$, where $\alpha=1+\log p/\log 3$.

Example 2. *(On the conjecture of Hawkes)* Hawkes (1981) conjectured that an exact dimension function of K would be of the form $h(t) = t^\alpha R^{-1}(\log\log\frac{1}{t})$ if $R(x) = -\log P(W\ge x)$ is *regular* at $+\infty$ (He did not precise the rerularity condition). We say that this is well the case if for example $P(W\ge x) = e^{-\lambda x^a}$ for some $\lambda > 0$ and $a > 0$, since in this case $R(x)= \lambda x^a$ and $h(t) = \frac{1}{\lambda} t^\alpha(\log\log\frac{1}{t})^{1/a}$, which is shown to be an exact dimension function of K by Theorem 2 (as $\mathbb{E}(W^k) = \lambda^{a+2-k/a}\,\Gamma(2-a+k/a)$ and consequently $r(W^a) = 1/\lambda$). In fact we can obtain a little more: if there exist some positive constants $\lambda_1, \lambda_2, c_1, c_2, a, \Delta > 0$ such that

$$c_1 e^{-\lambda_1 x^a} \le P(\ W \ge x\ ) \le c_2 e^{-\lambda_2 x^a} \qquad (\forall x \ge \Delta),$$

then $\phi_{1/a}(t) = t^{\alpha}(\log\log\frac{1}{t})^{1/a}$ is an exact dimension function of K. This is an immediate consequence of Theorem 2 if we note that

$$0 < \frac{1}{\lambda_1} \le r(W^a) \le \frac{1}{\lambda_2} < \infty.$$

Remark. In the case where $P(W\ge x)=x^{-\theta}$ ($\theta>1$ is a constant, $x\to\infty$), Hawkes's conjecture mean that the dimension function might be $\psi_{1/\theta}(t)= t^{\alpha}(\log \frac{1}{t})^{1/\theta}$. By Theorem 6, we know that, if there are some constants $\theta>1$, $0<c_1\le c_2<\infty$ such that $c_1x^{-\theta} \le P(W\ge x) \le c_2x^{-\theta}$ for sufficiently large $x>0$, then

$$\mu^{\psi_{1/\theta}}(K) < \infty \quad \text{and} \quad \mu^{\psi_{1/(\theta-1)}}(K) = \infty \text{ a.s. on } K \ne \emptyset,$$

where $\psi_{1/(\theta-1)}(t)= t^{\alpha}(\log \frac{1}{t})^{1/(\theta-1)}$, since $\mathbb{E}(W^p)= \int_0^{\infty} P(W\ge x^{1/p})dx \begin{matrix} <+\infty \text{ if } p<\theta \\ =+\infty \text{ if } p\ge\theta \end{matrix}$.

By a density argument, it appears probably that one would have $\mu^{\psi_b}(K)=0$ for all $b< 1/(\theta-1)$ (Thus a critical function would be $\psi_{1/(\theta-1)}$, rather than $\psi_{1/\theta}$), where $\psi_b(t)= t^{\alpha}(\log \frac{1}{t})^b$ for any $b\ge0$ (see Remark 4.1. and the conjecture after theorem 6).

Example 3. *(Case where the reproductive distribution decreases geometrically)* Suppose that there exist some constants $c_1>0$, $c_2>0$, and $0 < a_1\le a_2< 1$ such that

$$c_1a_1^k \le P(Z=k) \le c_2a_2^k$$

for all sufficiently large k, then the function $\phi_1(t) = t^{\alpha}(\log\log \frac{1}{t})$ is an exact dimension function of the branching set K.

This is immediate by Theorem 5, since $\mathbb{E}(e^{tZ}) < \infty$ if $t < \log \frac{1}{a_2}$ and $\mathbb{E}(e^{tZ}) = \infty$ if $t > \log \frac{1}{a_1}$. It covers of cause the case of geometric distribution.

Example 4. *(Case where the reproductive distribution decreases polynomially)* Suppose that

$$c_1 \frac{1}{k^\theta} \le P(Z=k) \le c_2 \frac{1}{k^\theta}$$

for some constants $c_1>0$, $c_2>0$, $\theta>2$ and sufficiently large k, then

$$\mu^{\psi_b}(K) = 0 \text{ if } b< 1/(\theta-1) \quad \text{and} \quad \mu^{\psi_b}(K) = \infty \text{ a.s. on } K\neq\emptyset \quad \text{if } b\ge 1/(\theta-2),$$

where $\psi_b(t)= t^\alpha(\log \frac{1}{t})^b$.

The result follows from Theorem 6 since $\mathbb{E}(Z^p) < \infty$ if $p < \theta-1$ and $\mathbb{E}(Z^p) = \infty$ if $p \ge \theta-1$.

2. Preliminaries

2.1. Sequences and trees

Let $\mathbb{N}$ be the set of non-negative integers with the discrete topology, $\mathbb{N}^k$ be the set of all k term sequences of the integers in $\mathbb{N}$, $T = \bigcup_{k=0}^{\infty}\mathbb{N}^k$ be the set of all finite sequences and $\mathbf{I} = \mathbb{N}^{\mathbb{N}}$ be the corresponding infinite sequences $\mathbf{i} = (i_1,i_2, \ldots)$ ($i_k \in \mathbb{N}$) with the product topology. We make the convention that $\mathbb{N}^0$ contains the null sequence $\emptyset$.

If $\mathbf{i} = (i_1,i_2, \ldots) \in \mathbb{N}^{\mathbb{N}}$, we write $\mathbf{i}|n = (i_1,i_2,\ldots,i_n)$ for the curtailement of $\mathbf{i}$ after n-terms; if $\sigma = (\sigma_1,\sigma_2,\ldots,\sigma_n) \in T$, we write $|\sigma| = n$ for the length of σ, and $\sigma^* = (\sigma_1,\sigma_2,\ldots,\sigma_n+1) \in T$ for the new sequence corresponding to σ obtained by augmenting the n-th composant σ_n to σ_n+1. If $\tau = (\tau_1,\tau_2, \ldots,\tau_m) \in T$ is another finite sequence, we write $\sigma*\tau \equiv (\sigma,\tau) = (\sigma_1,\sigma_2,\ldots,\sigma_n,\tau_1,\tau_2, \ldots,\tau_m)$ for the sequence obtained by juxtaposition of the terms of σ and τ. We partially order T by writing $\sigma < \tau$ (or $\tau > \sigma$) to mean that the sequence τ is an extension of σ, that is $\tau = \sigma*\tau'$ for some sequence $\tau' \in T$. We use a similar notation if $\sigma \in T$ and $\tau \in \mathbf{I}$. We remark that the null sequence $\emptyset < \mathbf{i}$ for any sequence $\mathbf{i}$. Finally, if $\mathbf{i}$ and $\mathbf{j}$ are two sequences of T or $\mathbf{I}$, we write $\mathbf{i}\wedge\mathbf{j}$ for the maximal sequence σ such that $\sigma<\mathbf{i}$ and $\sigma<\mathbf{j}$.

A tree $\mathcal{T}$ is a collection of finite sequences of non-negative integers such that (a) $\emptyset\in\mathcal{T}$; (b) If $\sigma\in\mathcal{T}$ then $\sigma*i\in\mathcal{T}$ if and only if $0\le i<Z^{\sigma}$ for some $Z^{\sigma}\in\mathbb{N}$; (c) $\sigma\in\mathcal{T}$ implies $\sigma'\in\mathcal{T}$ for any $\sigma'<\sigma$. See Neveu (1986). The sequences σ of $\mathcal{T}$ may be identified with the vertices of a directed graphe with σ joined to $\sigma*i$ in the obvious way. The null sequence $\emptyset$ corresponds to the root of the tree; Z^{σ} represent the number of edges going out from σ. The classical Galton-Watson branching process is then identified to a random tree. This tree will be also denoted by $\mathcal{T}$. The branching set $K = K(\omega)$ is then defined as its boundary, i.e., the set of infinite sequences $\mathbf{j}$ such that $\mathbf{i}\in\mathcal{T}$ for every finite curtailment $\mathbf{i}<\mathbf{j}$. However, we shall give a more careful definition of this set in the following.

2.2 Branching set and Hausdoff measures

Suppose that $p_k \ge 0$, that $\sum_{k=0}^{\infty} p_k = 1$ and let (Z^{σ}) $(\sigma \in T := \bigcup_{n=0}^{\infty} \mathbb{N}^n)$ be a countable family of independent random variables each distributed according to the law $P(Z = k) = p_k$. Here by convenience we write $\mathbb{N}^0 := \{\emptyset\}$ be the set of the null sequence. Put

$$\bar{C}_1 = \left\{ \mathbf{i} \in \mathbf{I} : 0 \le i_1 \le Z^{\emptyset}-1 \right\},$$

and by induction,

$$\bar{C}_{n+1} = \left\{ \mathbf{i} \in \bar{C}_n : 0 \le i_{n+1} \le Z^{(\mathbf{i}|n)}-1 \right\} \quad (n \ge 1).$$

The set

$$K = \bigcap_{n=1}^{\infty} \bar{C}_n$$

is then called the *branching set* generated by Z. Write

$$C_0 = \left\{ \emptyset \right\} \quad \text{and} \quad C_n = \left\{ (\mathbf{i}|n) : \mathbf{i} \in \bar{C}_n \right\} \quad (n \ge 1),$$

the Galton-Watson process can be defined by

$$Z_0 = 1 \quad \text{and} \quad Z_{n+1} = \sum_{\sigma\in C_n} Z^{\sigma} \quad (n \ge 0).$$

Thus $Z_1 \equiv Z^{\emptyset} \equiv Z$ by our notations. Let $\mathcal{T}=\mathcal{T}(\omega)$ be the genealogical tree of the process. Then $C_n=(\sigma\in\mathcal{T}: |\sigma|=n)$ $(n\in\mathbb{N})$ and $K=(\mathbf{i}\in\mathbf{I}: \mathbf{i}|n\in\mathcal{T}$ for all $n\geq 0)$.

As pointed out in the introduction, if $\mu = \sum kp_k < \infty$, then $W = \lim_n Z_n/\mu^n$ exists a.s. and, if $\sum p_k k \log k < \infty$, the extinction probability then satisfies

$$P (Z_n \to 0) = P (W = 0) = P (K = \emptyset),$$

and $\mathbb{E}(W) = 1$ (see for example Athrey-Ney 1972).

Suppose that (E,d) is a metric space, $A \subseteq E$, and $f=f(t)$ is a positive function defined for $t>0$ sufficiently small, non-decreasing and continuous on the right (which we call a *dimension function*). The Hausdoff measure of A with respect to the dimension function f is by definition

$$\mathcal{H}^f(A) = \lim_{\delta\to 0+} \mathcal{H}^f_\delta(A) \tag{2.1}$$

where

$$\mathcal{H}^f_\delta(A) = \inf \{ \textstyle\sum_{i=1}^{\infty} f(\text{diam } U_i): A \subset \bigcup_{i=1}^{\infty} U_i, \text{ diam } U_i \leq \delta \} \tag{2.2}$$

(diam (U_i) represents the diameter of U_i). It is not difficult to see that the quantity $\mathcal{H}^f(A)$ remains the same if in the definition we use covers of just open sets or just closed sets, or again just subsets of A, see for example Rogers (1970). If we use covers of just balls, we obtain the spherical Hausdorff measure:

$$\mu^f(A) = \lim_{\delta\to 0+} \mu^f_\delta(A) \tag{2.1'}$$

where

$$\mu^f_\delta(A) = \inf \{\textstyle\sum_{i=1}^{\infty} f(|U_i|): A\subset\bigcup_{i=1}^{\infty} U_i,\ |U_i|\leq\delta \text{ and } U_i \text{ are balls}\}. \tag{2.2'}$$

The two measures $\mathcal{H}^f(.)$ and $\mu^f(.)$ are in general not identical (see Besicovitch (1928, chapter 3) but equivalent for a large class of dimension functions (which are usually called *regular*):

Lemma 2.1. *Suppose that (E,d) is a metric space and $f(t)\geq 0$ is a positive function defined on $(0,T)$ $(T>0)$, non-decreasing and continuous on the right. If*

$$f(2t) \le cf(t) \quad \forall t\in(0, T/2) \tag{2.3}$$

for some c>0, then

$$\mathcal{H}^f(A) \le \mu^f(A) \le c\mathcal{H}^f(A) \text{ for all } A\subseteq E. \tag{2.4}$$

Proof. Clearly $\mathcal{H}^f_\delta(A) \le \mu^f_\delta(A)$ since any δ-cover of A by balls is a permissible covering in the definition of $\mathcal{H}^f_\delta(A)$. Also, if $\{U_i\}$ is a δ-cover of A, then $\{B_i\}$ is a 2δ-cover, where, for each i, B_i is choosen to be some ball containing U_i and of radius $|U_i|\le\delta$. Thus

$$\sum f(|B_i|) \le \sum f(2|U_i|) \le c \sum f(|U_i|),$$

and taking infima gives $\mu^f_{2\delta}(A) \le c\mathcal{H}^f(A)$. Letting $\delta \to 0$, it follows that $\mathcal{H}^f(A) \le \mu^f(A) \le c\mathcal{H}^f(A)$. □

We suppose that the condition (2.3) is always satisfied in this paper. It holds for example for $f(t)=t^\alpha$, $t^\alpha(\log\frac{1}{t})^\beta$, $t^\alpha(\log\log\frac{1}{t})^\beta$, etc. $(\alpha,\beta>0)$.

If $0 < \mu^f(A) < \infty$, we say that f is an *exact dimension function* of A, or simply an *exact dimension* of A, or an *exact measure function* of A. If $f(t) = t^a$ (a>0), we write $\mu^a(A)$ instead of $\mu^f(A)$, and we call it the a-dimensional (spherical) Hausdorff measure of A.The Hausdorff dimension of A is defined as

$$\dim A = \sup \{ a > 0 \mid \mu^a(A) = +\infty \} \equiv \inf \{ a > 0 \mid \mu^a(A) = 0 \}$$

Then $\mu^a(A) = +\infty$ if $a < \dim A$ and $\mu^a(A) = 0$ if $a > \dim A$. All the statements in this paragraph hold if the spherical Hausdorff measure $\mu^f(.)$ is replaced by the ordinary one $\mathcal{H}^f(.)$ since they are equivalent.

For our purposes, we shall take $E=\mathbf{I}$ with the metric d defined by $d(\mathbf{i},\mathbf{j}) = 2^{-|\mathbf{i}\wedge\mathbf{j}|}$. First of all, we remark that the balls of $\mathbf{I}$ are of the form

$$B(\sigma) := \{\mathbf{i}\in\mathbf{I}: \mathbf{i}>\sigma\} \quad (\sigma\in T),$$

which constitute a basis of the topology of $\mathbf{I}$. Secondly we have

Lemma 2.2 *K(ω) is a. s. a separable compact topological space if Z<∞ a.s.*

proof. Since it is evident that K has a.s. a countable topological basis, it suffices to prove that K is a.s. compact. We remark that K can be regarded as a subset of the product space $E = \prod_{n=1}^{\infty} E_n$, with product topology, where $E_n = \{0,1,\dots,Z_n-1\}$ carries the discrete topology. Since E_n is compact, so is E by Tychonoff's theorem. As K is a closed subspace of E, it is then compact. □

Finally we claim that on $(\mathbf{I},d)$, the sherical Hausdorff measure $\mu^f(.)$ coincides with the ordinary Hausdorff measure $\mathcal{H}^f(.)$, although it is not so for a general metric space.

Lemma 2.3 *For any dimension function f, we have*

$$\mu^f(A) = \mathcal{H}^f(A), \ \forall A \subseteq \mathbf{I}.$$

Proof. The same method as in the proof of Lemma 2.1, noting that if $\{U_i\}$ is a δ-cover of A, so is $\{B_i\}$, where, for each i, B_i is choosen to be some ball of a center $a_i \in U_i$ and of radius $\mathrm{diam}(U_i) \le \delta$. In fact, for each i, taking $a_i \in U_i$ and $k_i = \inf_{\mathbf{i},\mathbf{j} \in U_i} |\mathbf{i} \wedge \mathbf{j}|$, the ball $B_i := B(a_i|k_i)$, where $B(a_i|k_i) = \{a_i\}$ if $k_i = +\infty$, satisfies our needs: firstly, if $x \in U_i$, then $|x \wedge a_i| \ge k_i$ by the definition of k_i. Thus $x > a_i|k_i$ or namely $x \in B(a_i|k_i)$. This shows that $U_i \subseteq B_i$. Secondly, $\mathrm{diam}\ (B_i) = 2^{-k_i} = \sup_{\mathbf{i},\mathbf{j} \in U_i} 2^{-|\mathbf{i} \wedge \mathbf{j}|} = \mathrm{diam}(U_i)$. Therefore $\{B_i\}$ is a δ-cover of balls of A. The proof is then completed. □

2.3. A random measure μ_ω on $\mathbf{I} = \mathbb{N}^{\mathbb{N}}$ and the Q-measure on $\Omega \times \mathbf{I}$

If $\sigma \in C_n$, we let

$$Z_{\sigma,p} = \sum_{\tau \in C_p,\, \tau > \sigma} Z^\tau$$

denote the number of descendents of σ in the generation p. We define

$$W_\sigma = \lim_{p \to \infty} \frac{Z_{\sigma,p}}{\mu^{p-n}} \quad , \text{ if } \sigma \in C_n$$

and choose W_σ as an independent copy of W if $\sigma \in \mathbb{N}^n - C_n$ such that (W_σ) $(\sigma \in \mathbb{N}^n - C_n)$ is a family of independent random variables, and as a family, independent of the family (W_τ) $(\tau \in C_n)$. Then (W_σ) $(\sigma \in T)$ is a family of random variables, each distributed as $W = \lim_{n\to\infty} Z_n/\mu^n$, and W_σ and W_τ are independent if neither $\sigma < \tau$ nor $\tau < \sigma$. It is easily verified that

$$\mu^{-|\sigma|}W_\sigma = \sum_{i=0}^{Z_\sigma - 1} \mu^{-|\sigma*i|} W_{\sigma*i}$$

almost surely, where the sum is interpreted as 0 if $Z_\sigma=0$. So if we write

$$1_\sigma = \begin{cases} \mu^{-|\sigma|} & \text{if } \sigma \in C_n \ ; \\ 0 & \text{if } \sigma \in \mathbb{N}^n - C_n \ , \end{cases}$$

then

$$W_\sigma 1_\sigma = \sum_{i=0}^{\infty} 1_{\sigma*i} W_{\sigma*i} \ . \tag{2.5}$$

For $\sigma \in T$, let

$$B(\sigma) = \left\{ \tau \in \mathbf{I} : \sigma < \tau \right\}$$

be a ball in $\mathbf{I}$ of radius $2^{-|\sigma|}$ and define

$$\mu_\omega(B(\sigma)) = 1_\sigma W_\sigma(\omega) \equiv 1_{\sigma\in\mathcal{T}}\, \mu^{-|\sigma|} W_\sigma .$$

An arbitrary clopen set $A \subseteq \mathbf{I}$ can be written as

$$A = \bigcup_{i=1}^{k} B(\sigma_i) \ .$$

Let $k_0 = \max\ (|\sigma_i| : i = 1,\ldots,k)$ and

$$E = \left\{ \tau \in \mathbb{N}^{k_0} : \ \exists\, i \in (1,\ldots,k) \text{ such that } \sigma_i < \tau \right\},$$

then $$A = \bigcup_{\tau\in E} B(\tau) \quad \text{and} \quad B(\tau) \cap B(\tau') = \phi \text{ if } \tau \neq \tau' .$$

Define $$\mu_\omega(A) = \sum_{\tau\in E} \mu_\omega(B(\tau)),$$

By (2.5) μ_ω is a well-defined finitely additive measure on the field of all clopen subsets of $\mathbf{I}$ and, therefore, uniquely extends to a Borel measure on $\mathbf{I}$. This measure will be called μ_ω again. We remark that it is concentrated on the branching set $K(\omega)$, and $\mu_\omega(K(\omega))=W(\omega)$.

It will prove very useful to consider the product space $\Omega \times \mathbf{I}$ with the product σ-field and with probability law Q defined by

$$Q(A) = \mathbb{E} \int 1_A(\omega, \mathbf{i})\ d\mu_\omega(\mathbf{i}),$$

see for example Proposition 4.1. in section 4.

3. Moments results on branching processes

Let (Z_n) ($n \geq 0$, $Z_0 = 1$) be a branching process with family distribution Z_1 and

$$W = \lim_{n\to\infty} Z_n/\mu^n. \tag{3.1}$$

We shall need some results on the order of growth of the moments $\mathbb{E}(W^k)$ of W, which themselves are interesting.

Lemma 3.1 (*Comparison theorem for radius of convergence of W and Z*)

Denote by $r(Z_1)$ the radius of convergence of the moment generating function $\mathbb{E}\,[e^{tZ_1}]$ of Z_1 and $r(W)$ that of W, then

r(W) is zero, positive and finite, or infinite

if and only if the same is true for $r(Z_1)$.

Proof. We first note that

$$\mathbb{E}\,[\,e^{tZ_1}] = 1 + t\,\mathbb{E}[Z_1] + \frac{t^2}{2!}\,\mathbb{E}[Z_1^2] + \ldots$$

$$\mathbb{E}\,[\,e^{tW}\,] = 1 + t\,\mathbb{E}[W] + \frac{t^2}{2!}\,\mathbb{E}[W^2] + \ldots$$

and that

$$\mathbb{E}[W^n] = \mathbb{E}[\mathbb{E}(W^n|\mathbb{F}_1)] \geq \mathbb{E}\,[\,\mathbb{E}(W|\mathbb{F}_1)^n] = \mathbb{E}[(Z_1/\mu)^n] = \mu^{-n}\mathbb{E}[Z_1^n],$$

where $\mathbb{F}_1$ is the σ-algebra generated by Z_1. We have then immediately that

$$r(W) \leq \mu\, r(Z_1)$$

by the well known formular on the radius of convergence of Taylor series. This shows that $r(Z_1) < +\infty$ implies $r(W) < +\infty$, and $r(Z_1)=0$ implies $r(W)=0$.

We then prove that $r(Z_1) > 0$ implies $r(W) > 0$. Put

$$p(t) = \mathbb{E}[t^{Z_1}],$$

then

$$\mathbb{E}[t^{Z_n}] = (p^\circ)^n(t),$$

where $(p^\circ)^1(t)=p(t)$ and $(p^\circ)^{k+1}(t) = p((p^\circ)^k(t))$ $(k\geq 1)$. We shall prove that there exists $t>0$ such that $E(e^{tW})<\infty$. To this end, we shall find $t>0$ such that

$$E(e^{tZ_n/\mu^n}) \le C$$

for some $C>0$ and all $n\ge 0$. Since $E(e^{xZ_1})<\infty$ for some $x>0$, we can choose by induction $r_1, r_2, \ldots, r_n, \ldots$ such that $r_1 > 1$ and that

$$p(r_2) = r_1 , p(r_3) = r_2 , \ldots , p(r_n) = r_{n-1} , \ldots$$

Now

$$r_{n-1}-1 = p(r_n)-1 = \int_1^{r_n} p'(t)dt \le (r_n-1)p'(r_n),$$

where $p'(t)$ represents the derivative of $p(t)$. Hence

$$r_n-1 \ge [p'(r_n)p'(r_{n-1})\ldots p'(r_2)]^{-1}(r_1-1).$$

Since

$$\mathbb{E}[e^{tZ_n/\mu^n}] = p^{\cdot n}(e^{t/\mu^n}) \le p^{\cdot n}(r_n) = r_1$$

if $e^{t/\mu^n} \le r_n$, and $p^{\cdot n}(r_n) = r_1$ by the definition of r_n, the proof will be completed if we can choose $t > 0$ such that

$$t/\mu^n \le \log\ (1+[p'(r_n)\ldots p'(r_2)]^{-1}(r_1-1))$$

for all $n \ge 2$. But the latter is implied by

$$t/\mu^n \le \frac{1}{2}[p'(r_n)\ldots p'(r_2)]^{-1}(r_1-1)$$

as $\log(1+x) \ge \frac{1}{2} x$ for $0 \le x \le 1$, it then suffices to prove that

$$\prod_{n=2}^{\infty} \frac{p'(r_n)}{\mu} = \prod_{n=2}^{\infty} (1+ \frac{1}{\mu} \int_1^{r_n} p''(t)dt\ ) < +\infty.$$

We see that this is so because

$$\int_1^{r_n} p''(t)dt \le (r_n-1)p''(r_n) \le (r_n-1)p''(r_1)$$

and

$$r_n-1 \le \frac{1}{p'(1)}\ (p(r_n)-1) = \frac{1}{\mu}\ (r_{n-1}-1) \le \ldots \le \frac{1}{\mu^{n-1}}(r_1-1).$$

We finally prove that $r(Z_1)=\infty$ implies $r(W)=\infty$. To see this, we recall the functional equation

$$\Phi(\mu t)= G(\Phi(t)),$$

where $\Phi(t)=\mathbb{E}[e^{tW}]$ and $G(t)=\mathbb{E}[e^{tZ_1}]$. Since we have shown that $r(Z_1)>0$ implies $r(W)>0$, we know at least $\Phi(t)<\infty$ for some $t>0$. From the functional equation and the fact that $G(t)<\infty$ for all $t>0$ ($r(Z_1)=\infty$), we know immediately $\Phi(t)<\infty$ for all $t>0$. The proof is then completed. □

The following result can be compared by that of Kahane-Peyrière (1976) which was concerned to a model of turbulence of Mandelbrot.

Lemma 3.2. *Suppose that $\|Z_1\|_\infty < +\infty$ and write*

$$\beta = 1 - \log \mu \,/\, \log \|Z_1\|_\infty .$$

Then

(i)
$$\lim_{k\to+\infty} \frac{\log \mathbb{E}(W^k)}{k \operatorname{Log} k} = \beta;$$

(ii) For $0<\theta<\infty$, denote by r_θ the radius of convergence of the moment generating function $\mathbb{E}(e^{tW^\theta})$ of W^θ. If in addition to $\|Z_1\|_\infty<\infty$, Z_1 is not a.s.a constant, then $0<\beta<1$ and

$$0 < r_{1/\beta} < +\infty,$$

or namely $\mathbb{E}(e^{tW^{1/\beta}}) < +\infty$ for some but not all $t>0$.

Proof. (i) We first prove that

$$\liminf_{k\to+\infty} \frac{\log \mathbb{E}(W^k)}{k \operatorname{Log} k} \geq \beta.$$

For convenience, write $n = \|Z_1\|_\infty$. Then $p_n = P(Z_1=n) > 0$. Since

$$W = \frac{1}{\mu} \sum_{i=1}^{Z_1} W_i \;, \tag{3.2}$$

where $W_i\,(i\geq1)$ are independent from each other and from Z_1, and have the same distribution as W, we have

$$\begin{aligned}\mathbb{E}[W^k|\; Z_1=n] &= \frac{n}{\mu^k}\,\mathbb{E}[W^k] + \frac{1}{\mu^k} \sum_{\substack{k_1+k_2+\dots+k_n=k\\ 0\leq k_i\leq k-1}} \frac{k!}{k_1!\dots k_n!} \prod_{i=1}^{n} \mathbb{E}[W^{k_i}] \\ &\geq \frac{n}{\mu^k}\,\mathbb{E}[W^k] + \frac{1}{\mu^k}(n^k-n) \inf \prod_{i=1}^{n} \mathbb{E}[W^{k_i}], \end{aligned} \tag{3.3}$$

where $\mathbb{E}(W^k|\; Z_1 = n)$ denotes the expectation of W^k conditioned on $Z_1= n$ and the inferior is taken over all $(k_1,k_2,\dots,k_n)$ such that $k_1+ k_2+\dots+ k_n=$

k and that $0 \le k_i \le k-1$. If $k = n\tilde{k}$, this inferior is $(\mathbb{E}[W^{\tilde{k}}])^n$. Hence

$$\mathbb{E}[W^{n\tilde{k}} \mid Z_1=n] \ge \frac{n}{\mu^k}\mathbb{E}[W^{n\tilde{k}}] + \mu^{-n\tilde{k}}(n^{n\tilde{k}}-n)(\mathbb{E}[W^{\tilde{k}}])^n$$
$$\ge (\frac{n}{\mu})^{n\tilde{k}}(\mathbb{E}[W^{\tilde{k}}])^n,$$

or $\mathbb{E}[W^{n\tilde{k}}] \ge p_n(\frac{n}{\mu})^{n\tilde{k}}(\mathbb{E}[W^{\tilde{k}}])^n$. Therefore

$$\frac{1}{n\tilde{k}} \log \mathbb{E}[W^{n\tilde{k}}] \ge \log \frac{n}{\mu} + \frac{1}{\tilde{k}} \log \mathbb{E}[W^{\tilde{k}}] + \frac{\log p_n}{n\tilde{k}}.$$

Choosing $\tilde{k} = n^r$ ($r \in \mathbb{N}$) and using this inequality repeatedly, we see that

$$n^{-(r+1)} \log \mathbb{E}\,[W^{n^{r+1}}] \ge (r+1) \log \frac{n}{\mu} + \log \mathbb{E}[W] + \frac{\log p_n}{n} \sum_{i=0}^r \frac{1}{n^i}.$$

Thus

$$n^{-r} \log \mathbb{E}\,[W^{n^r}] \ge r \log \frac{n}{\mu} + C(n) \tag{3.4}$$

($\forall r \ge 0$), where $C(n) > -\infty$ is a constant independent of r. Hence

$$\liminf_{r\to\infty} \frac{\log \mathbb{E}[W^{n^r}]}{n^r \log n^r} \ge 1 - \log \mu \,/\, \log n.$$

Now for each $k \in \mathbb{N}$ sufficiently large, choose $r \in \mathbb{N}$ such that $n^r \le k < n^{r+1}$. Thus

$$\frac{\log \mathbb{E}(W^k)}{k \log k} = \frac{\log[\mathbb{E}(W^k)]^{1/k}}{\log k} \ge \frac{\log[\mathbb{E}(W^{n^r})]^{1/n^r}}{\log n^{r+1}} = \frac{\log \mathbb{E}[W^{n^r}]}{n^r \log n^{r+1}}$$

and consequently

$$\liminf_{k\to\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} \ge \liminf_{r\to\infty} \frac{\log \mathbb{E}[W^{n^r}]}{n^r \log n^{r+1}} = \liminf_{r\to\infty} \frac{\log \mathbb{E}[W^{n^r}]}{n^r \operatorname{Log} n^r} \ge 1 - \log \mu \,/\, \log n.$$

We now prove that

$$\limsup_{k\to+\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} \le \beta.$$

Write again $n = \|Z_1\|_\infty$, then

$$W \le \frac{1}{\mu} \sum_{i=1}^n W_i\ ,$$

where W_i ($i \ge 1$) are independent from each other, and have the same distribution as W. Hence

$$\mathbb{E}[W^k] \le \frac{n}{\mu^k}\mathbb{E}[W^k] + \frac{1}{\mu^k}\sum_{\substack{k_1+k_2+\ldots+k_n=k\\ 0\le k_i\le k-1}} \frac{k!}{k_1!\ldots k_n!}\prod_{i=1}^{n}\mathbb{E}[W^{k_i}].$$

Thus

$$\mathbb{E}[W^k] \le \frac{1}{\mu^k-n}\sum_{\substack{k_1+k_2+\ldots+k_n=k\\ 0\le k_i\le k-1}} \frac{k!}{k_1!\ldots k_n!}\prod_{i=1}^{n}\mathbb{E}[W^{k_i}]. \tag{3.5}$$

Write $B_k = \sup_{l<k} (\mathbb{E}[W^l]/l!)^{1/l}$, then

$$B_{k+1}^k \le \sup\left(\frac{k^n}{\mu^k-n}B_k^k,\ B_k^k\right) \tag{3.6}$$

since the number of the terms in the sum $\sum$ is inferior to k^n. Therefore B_k is bounded. This shows that $\mathbb{E}(e^{tW}) < +\infty$ for sufficiently small $t > 0$. Again from the recursive relation, we obtain

$$\mathbb{E}(e^{W\mu t}) \le (\mathbb{E}(e^{Wt}))^n.$$

So $\mathbb{E}(e^{tW}) < +\infty$ for all $t > 0$ and

$$\mathbb{E}[e^{W\mu^k t}] \le (\mathbb{E}(e^{Wt}))^{n^k} = (\mathbb{E}(e^{Wt}))^{\mu^{kK}},$$

where $k \in \mathbb{N}$ and

$$K = \log n / \log \mu.$$

Put

$$\psi(t) = \log \mathbb{E}(e^{tW}),$$

then $\psi(\mu t) \le n\psi(t)$, and consequently $\psi(\mu^k) \le n^k\psi(1)$. For each $k\in\mathbb{N}$, choose an integer $i \ge 0$ such that

$$\mu^i \le k^{1/K} < \mu^{i+1}.$$

Thus

$$\mathbb{E}[e^{k^{1/K}W}] \le \mathbb{E}[e^{\mu^{i+1}W}] \le (\mathbb{E}(e^W))^{n^{i+1}} = (\mathbb{E}(e^W))^{\mu^{(i+1)K}} \le (\mathbb{E}(e^W))^{\mu^K k}.$$

On the other hand, by Markov's inequality we have

$$\mathbb{E}(W^k) = \int_0^\infty p(W^k>t)\,dt = \int_0^\infty p(\,e^{k^{1/K}W} > e^{k^{1/K}t^{1/k}})\,dt$$

$$\le \mathbb{E}[e^{k^{1/K}W}]\int_0^\infty e^{-k^{1/K}t^{1/k}}\,dt$$

$$= \mathbb{E}[e^{k^{1/K}W}]k!/\,k^{k/K}$$

Thus

$$\mathbb{E}(W^k) \le (\mathbb{E}(e^W))^{\mu^K k} k!/\ k^{k/K}$$

Writing

$$B = (\mathbb{E}(e^W))^{\mu^K} \equiv (\mathbb{E}(e^W))^n$$

($K = \log n / \log \mu$), then

$$\mathbb{E}(W^k) \le B^k (k!)/\ k^{k/K} \tag{3.7}$$

for all $k \ge 1$. Since $1-1/K = \beta$, it then follows that

$$\limsup_{k\to+\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} \le \beta .$$

Thus (i) is proved.

(ii) We first note that $0< \beta <1$ since Z_1 is not a.s. a constant. The first part of the proof above shows that

$$\mathbb{E}[W^{n^r}] \ge [(\frac{n}{\mu})^r e^{C(n)}]^{n^r}$$

for all $r\ge 0$ (cf.(3.4). For $k \in \mathbb{N}$ sufficiently large, choose $r \in \mathbb{N}$ such that $n^r \le k/\beta < n^{r+1}$. Using Stirling's formular gives

$$\limsup_{k\to\infty} \left(\frac{\mathbb{E}(W^{k/\beta})}{k!}\right)^{1/k} \ge \limsup_{k\to\infty} \frac{\mathbb{E}[W^{n^r}])^{1/(n^r\beta)}}{k/e}$$

$$\ge \limsup_{r\to\infty} \frac{(\frac{n}{\mu})^{r/\beta} e^{C(n)/\beta}}{\beta\, n^{r+1}/\ e} = \frac{e^{C(n)+1}}{\beta n}$$

since $(n/\mu)^{r/\beta} = n^r$. Thus $r(W^{1/\beta}) < \infty$.

Using $\mathbb{E}(W^k) \le B^k (k!)/\ k^{k/K}$ (proof of (i)), the same method as above applies, yielding that

$$\limsup_{k\to\infty} \left(\frac{\mathbb{E}(W^{k\theta})}{k!}\right)^{1/k} \le \limsup_{k\to\infty} \frac{(\mathbb{E}(W^{[k\theta]+1}))^{\theta/([k\theta]+1)}}{k!^{1/k}}$$

$$\le \limsup_{r\to\infty} \frac{B^{\theta}([k\theta]+1)!^{\theta/([k\theta]+1)}}{k!^{1/k}([k\theta]+1)^{\theta/K}} = B^{\theta}\theta < +\infty,$$

where $\theta = 1/\ \beta$. This shows that $r(W^{1/\beta}) > 0$. The proof is completed. □

Lemma 3.3. *Suppose that $\|Z_1\|_\infty = +\infty$ and denote by $r_\theta \equiv r(W^\theta)$ the radius of convergence of the moment generating function $\mathbb{E}(e^{tW^\theta})$ of W^θ. Then*

(i) $\liminf_{k\to+\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} \geq 1$ *and* $r(W^\theta) = 0 \ \forall\, \theta > 1$;

(ii) $\lim_{k\to+\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} = 1$ *and* $r(W) > 0$ *if additionally* $\mathbb{E}(e^{tZ}) < \infty$ *for some* $t>0$.

Proof.(i) Let $\theta > 1$. Choose $n \in \mathbb{N}$ sufficiently large such that $P(Z_1=n) > 0$ and $\theta\beta_n > 1$, where $\beta_n = 1 - \log \mu / \log n$. The proof of part (i) of the above lemma shows that

$$\liminf_{r\to\infty} \frac{\log \mathbb{E}[W^{n^r}]}{n^r \log n^r} \geq \beta_n .$$

Hence $\liminf_{k\to+\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} \geq \beta_n$ and the inequality in (i) follows by letting $n \longrightarrow \infty$. Moreover $\forall \varepsilon>0$

$$\log \mathbb{E}[W^{n^r}] \geq (\beta_n - \varepsilon) n^r \log n^r$$

for sufficiently large r. We choose $\varepsilon > 0$ such that $\theta(\beta_n-\varepsilon) > 1$. For $k \in \mathbb{N}$ sufficiently large, choose $r \in \mathbb{N}$ such that $n^r \leq k\theta < n^{r+1}$. The Stirling's formular gives then

$$\limsup_{k\to\infty} \left(\frac{\mathbb{E}(W^{k\theta})}{k!}\right)^{1/k} \geq \limsup_{k\to\infty} \frac{\mathbb{E}[W^{n^r}])^{\theta/n^r}}{k/e} \geq \limsup_{r\to\infty} \frac{(n^r)^{\theta(\beta_n-\varepsilon)}}{n^{r+1}/(e\theta)} = +\infty .$$

Thus $r_\theta = 0$, as desired.

(ii) Suppose that $\mathbb{E}(e^{tZ}) < \infty$ for some $t>0$. Then $r(Z) > 0$ and hence $r(W) > 0$ by lemma 3.1. Consequently $\limsup_{k\to\infty} \left(\frac{\mathbb{E}(W^k)}{k!} \right)^{1/k} < \infty$ and $\limsup_{k\to\infty} \frac{\mathbb{E}(W^k)^{1/k}}{k/e} < \infty$ by Stirling's formula. The last result gives immediately $\limsup_{k\to+\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} \leq 1$ which ensures our conclusion combining with the inequality in (i). □

Remark. The result $\lim_{k\to+\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} = 1$ does not hold in general. For example, if $\mathbb{E}(W^k)=+\infty$ for some $k>0$, then it is evident that $\lim_{k\to+\infty} \frac{\log \mathbb{E}(W^k)}{k \log k} = +\infty$.

4. Proof of Theorems

The theorems will be proved in a series of propositions and lemmas, using the preliminaries in the sections 2 and 3.

We first recall that we have defined a random measure $\mu_\omega(.)$ on $\mathbf{I}$, concentrated on the branching set $K(\omega)$, such that

$$\mu_\omega(B(\sigma)) = 1_\sigma W_\sigma(\omega) \equiv 1_{\sigma\in\mathcal{T}}\ \mu^{-|\sigma|}W_\sigma,$$

where $B(\sigma) = \{\tau\in\mathbf{I}: \sigma<\tau\}$, (W_σ) $(\sigma \in T)$ is a family of random variables, each distributed as $W = \lim_{n\to\infty} Z_n/\mu^n$, and W_σ and W_τ are independent if neither $\sigma < \tau$ nor $\tau < \sigma$. We have also defined a probability measure on the product space $\Omega\times\mathbf{I}$ by

$$Q(A) = \mathbb{E}\int 1_A(\omega,\mathbf{i})\ d\mu_\omega(\mathbf{i}),$$

see section 2. To obtain some density theorems about the measure μ_ω, we shall need the distributions of the random variables $\hat{W}_n(\omega,\mathbf{i}) := W_{\mathbf{i}|n}$ on $\Omega\times\mathbf{I}$ $(n\geq 1)$.

Lemma 4.0. *Let $f: \mathbb{R}\to[0,\infty)$ be a Borel measurable function, then*

$$\mathbb{E}_Q f(W_{\mathbf{i}|n}) = \mathbb{E}\, Wf(W),$$

where $\mathbb{E}_Q$ represents the expectation with respect to Q.

Proof. From the definition of Q and the structure of the Galton-Watson tree, we have

$$\mathbb{E}_Q f(W_{\mathbf{i}|n}) = \mathbb{E}\sum_{\sigma\in C_n} f(W_\sigma)\mu^{-n}W_\sigma = \mathbb{E}\sum_{\sigma\in C_{n-1}} \mu^{-n}\sum_{0\leq i<Z^\sigma} f(W_\sigma)W_\sigma$$

$$= \mathbb{E}\sum_{\sigma\in C_{n-1}} \mu^{-n}\mu\ \mathbb{E}(f(W)W) = \mathbb{E}Wf(W). \qquad \square$$

We can now obtain our density results about the measure μ_ω. We recall that $\alpha = \log\mu/\log 2$ and we remark that $|B(\mathbf{i}|n)| := \operatorname{diam} B(\mathbf{i}|n) = 2^{-n}$.

The first result (4.0) in the following proposition was first obtained by Hawkes (1981). For convenience, we shall give a simple proof here.

Proposition 4.1 (*Density theorem about the measure μ_ω*)

(0) If $\mathbb{E}\ Z\log^2 Z<\infty$, then

$$\lim_{n\to\infty} \frac{\log \mu_\omega B(\mathbf{i}|n)}{n} = -\log \mu \quad \text{for } P\text{-a.e. } \omega \text{ and } \mu_\omega\text{- a.e. } \mathbf{i} \tag{4.0}$$

or equvalently

$$\lim_{n\to\infty} \frac{\log \mu_\omega B(\mathbf{i}|n)}{\log |B(\mathbf{i}|n)|} = \alpha \quad \text{for } P\text{-a.e. } \omega \text{ and } \mu_\omega\text{- a.e. } \mathbf{i}. \tag{4.0}'$$

(i) If $\mathbb{E}(e^{tW^\theta}) < \infty$ *for some* $\theta\in(0,\infty)$ *and* $t\in(0,\infty)$, *then*

$$\limsup_{n\to\infty} \frac{\mu^n \mu_\omega B(\mathbf{i}|n)}{(\log n)^{1/\theta}} \leq t^{-1/\theta} \quad \text{for } P\text{-a.e. } \omega \text{ and } \mu_\omega\text{- a.e. } \mathbf{i}, \tag{4.1}$$

or equivalently

$$\limsup_{n\to\infty} \frac{\mu_\omega B(\mathbf{i}|n)}{\phi_{1/\theta}(|B(\mathbf{i}|n)|)} \leq t^{-1/\theta} \quad \text{for } P\text{-a.e. } \omega \text{ and } \mu_\omega\text{- a.e. } \mathbf{i}, \tag{4.1}'$$

where $\phi_{1/\theta}(t) = t^\alpha(\log\log \frac{1}{t})^{1/\theta}$. *Consequently (with no condition)*

$$\limsup_{n\to\infty} \frac{\mu_\omega B(\mathbf{i}|n)}{\phi_{1/\theta}(|B(\mathbf{i}|n)|)} \leq r_\theta^{-1/\theta} \quad \text{for } P\text{-a.e. } \omega \text{ and } \mu_\omega\text{- a.e.} \mathbf{i}, \tag{4.1}''$$

where $r_\theta = \sup\{t\geq 0: \mathbb{E}(e^{tW^\theta})<\infty\}$, $r_\theta^{-1/\theta}$ *is interpreted as 0 if* $r_\theta=\infty$, *and* ∞ *if* $r_\theta=0$.

(ii) If $\mathbb{E}(W^{\theta+1}) < \infty$ *for some* $\theta\in(0,\infty)$, *then*

$$\lim_{n\to\infty} \frac{\mu^n \mu_\omega B(\mathbf{i}|n)}{n^{1/\theta}} = 0 \quad \text{for } P\text{-a.e. } \omega \text{ and } \mu_\omega\text{- a.e. } \mathbf{i}, \tag{4.2}$$

or equvalently

$$\lim_{n\to\infty} \frac{\mu_\omega B(\mathbf{i}|n)}{\psi_{1/\theta}(|B(\mathbf{i}|n)|)} = 0 \quad \text{for } P\text{-a.e. } \omega \text{ and } \mu_\omega\text{- a.e. } \mathbf{i}, \tag{4.2}'$$

where $\psi_{1/\theta}(t) = t^\alpha(\log \frac{1}{t})^{1/\theta}$.

Proof. (0) It is known that $\mathbb{E}W\log^+W < \infty$ if and only if $\mathbb{E}Z\log^2 Z < \infty$ (see for example Athreya and Ney 1972). So it suffices to show that, if $\mathbb{E}W\log^+W < \infty$, then

$$\lim_{n\to\infty} \frac{\log W_{\mathbf{i}|n}}{n} = 0 \quad Q\text{ - a.e.} \tag{4.0}''$$

By the distribution of $W_{\mathbf{i}|n}$ (Lemma 4.0), for all $\varepsilon>0$, we have

$$Q(|\log W_{\mathbf{i}|n}| \geq n\varepsilon) = Q(W_{\mathbf{i}|n} \geq e^{n\varepsilon} \text{ or } W_{\mathbf{i}|n} \leq e^{-n\varepsilon})$$

$$= \mathbb{E}\,[W(1_{\{W\geq e^{n\varepsilon}\}} + 1_{\{W\leq e^{-n\varepsilon}\}})].$$

Thus

$$\sum_{n=1}^{\infty} Q(|\log W_{\mathbf{i}|n}| \geq n\varepsilon) \leq \sum_{n=1}^{\infty} \mathbb{E}\,[W\, 1_{\{W\geq e^{n\varepsilon}\}}] + \sum_{n=1}^{\infty} e^{-n\varepsilon}$$

$$= O\{\mathbb{E} W\log^+ W\} + \sum_{n=1}^{\infty} e^{-n\varepsilon} < \infty.$$

The Borel - Cantelli lemma gives $\limsup_{n\to\infty} \dfrac{|\log W_{\mathbf{i}|n}|}{n} \leq \varepsilon$ Q - a.e. and then the result desired.

(i) It is equivalent to show that, if $\mathbb{E}(e^{tW^{\theta}}) < \infty$ for some $\theta\in(0,\infty)$ and $t\in(0,\infty)$, then

$$\limsup_{n\to\infty} \frac{W_{\mathbf{i}|n}^{\theta}}{\log n} \leq t^{-1} \quad Q.\text{- a.e.} \qquad (4.1)'''$$

For all $0<t'<t$, we have $\mathbb{E}(We^{t'W^{\theta}}) <\infty$ since $\mathbb{E}(e^{tW^{\theta}}) <\infty$. Then $\forall\, \varepsilon > 0$

$$Q(e^{tW_{\mathbf{i}|n}^{\theta}} \geq n^{1+\varepsilon}) = \mathbb{E}\left[W\, 1_{\left\{e^{tW^{\theta}} \geq n^{1+\varepsilon}\right\}}\right]$$

and consequently

$$\sum_{n=1}^{\infty} Q(e^{tW_{\mathbf{i}|n}^{\theta}} \geq n^{1+\varepsilon}) = O\{\mathbb{E}(We^{\frac{tW^{\theta}}{1+\varepsilon}})\} <\infty$$

(since $t/(1+\varepsilon)<t$). The Borel-Cantelli lemma ensures that

$$\limsup_{n\to\infty} \frac{tW_{\mathbf{i}|n}^{\theta}}{\log n} \leq (1+\varepsilon) \quad Q\text{- a.e.}$$

Hence the result follows.

(ii) Again equivalently, we have to prove that, if $\mathbb{E}(W^{\theta+1}) < \infty$ for $\theta\in(0,\infty)$, then

$$\lim_{n\to\infty} \frac{W_{\mathbf{i}|n}^{\theta}}{n} = 0 \quad Q\text{ - a.e.}$$

The approach is almost the same as above by means of the Borel-Cantelli Lemma, noting that, for all $\varepsilon>0$,

$$\sum_{n=1}^{\infty} Q(W_{\mathbf{i}|n}^{\theta} \geq n\varepsilon) = \sum_{n=1}^{\infty} \mathbb{E}(W\, 1_{\{W^{\theta} \geq n\varepsilon\}}) = O(\tfrac{1}{\varepsilon}\mathbb{E}W^{1+\theta}) < \infty. \qquad \square$$

Remark 4.1. (i) If $\mathbb{E}e^{tW^{\theta}} = \infty$, then a simple calcul as above shows

$$\sum_{n=1}^{\infty} Q(e^{tW_{\mathbf{i}|n}^{\theta}} \geq n) = \infty.$$

If the events $(e^{tW_{\mathbf{i}|n}^{\theta}} \geq n)$ satisfied some asymptotic independent properties (it is so in the case that Z is of geometric distribution, see Hawkes 1981), then we could conclude that

$$\limsup_{n\to\infty} \frac{W_{\mathbf{i}|n}^{\theta}}{\log n} \geq t^{-1} \quad Q.\text{- a.e.} \tag{4.1)""$$

and then (4.1)" could be strengthened to

$$\limsup_{n\to\infty} \frac{\mu_{\omega}B(\mathbf{i}|n)}{\phi_{1/\theta}(|B(\mathbf{i}|n)|)} = r_{\theta}^{-1/\theta} \quad \text{for P-a.e. } \omega \text{ and } \mu_{\omega}\text{- a.e.}\mathbf{i}. \tag{4.1)"$$

If additionally there were no μ_{ω}- exceptional sets, then a standard density argument could imply $0<\mathcal{H}^{\phi_{1/\theta}}(K(\omega))<\infty$ if $0<r_{\theta}<\infty$. This explains why $\phi_{1/\theta}$ is the correct gauge function.

(ii) If $\mathbb{E}(W^{\theta+1}) = \infty$, where $\theta\in(0,\infty)$, then $\forall M>0$ $\sum_{n=1}^{\infty} Q(W_{\mathbf{i}|n}^{\theta} \geq nM) = \infty$. Again if the events $(W_{\mathbf{i}|n}^{\theta} \geq nM)$ satisfied some asymptotic independent properties, we could conclude $\lim_{n\to\infty} \frac{W_{\mathbf{i}|n}^{\theta}}{n} = \infty$ Q - a.e. and

$$\lim_{n\to\infty} \frac{\mu_{\omega}B(\mathbf{i}|n)}{\psi_{1/\theta}(|B(\mathbf{i}|n)|)} = \infty \quad \text{for P-a.e. } \omega \text{ and } \mu_{\omega}\text{- a.e. } \mathbf{i}.$$

By the same reson as above, this shows that we might have probably

$$\mathcal{H}^{\psi_{1/\theta}}(K(\omega)) = 0 \text{ a.s. if } \mathbb{E}(W^{\theta+1}) = \infty.$$

Corollary 4.1. *(The dimension α of K and the α-measure associated)*

(i) $\dim K = \alpha$ *a.s.* $K\neq\emptyset$; *(ii)* $\mu^{\alpha}(K)<\infty$ *a.s.*

(iii) $0 < \mu^{\alpha}(K) < \infty$ *a.s. on* $K\neq\emptyset$ *if and only if Z is a.s.a constant.*

Proof. As K can be covered by Z_n balls of radius 2^{-n}, and $Z_n(2^{-n})^{\alpha} = Z_n/\mu^n \to W$ a.s., we have consequently $\mu^{\alpha}(K) \leq W$ a.s. This proves (ii) and $\dim K \leq \alpha$ a.s. To show (i), it then suffices to prove that $\dim K \geq \alpha$ a.s. If $\mathbb{E}Z\log^2 Z < \infty$, this is an immedeate consequence of (4.0)' by a standard argument of density, see for example Taylor (1986, §4), and this appeared explicitly in Hawkes (1981). Otherwise, we truncate the process as follows. Let $0<M\in\mathbb{N}$ and consider the Galton -Watson tree $\mathcal{T}^*$ generated by the defining elements $Z^{*\sigma} := \min(Z^{\sigma},M)$ ($\sigma\in T$). The resulting branching set K^* has a.s. dimension $\alpha^*(M) := \log\mu^*(M)/\log 2$, where $\mu^*(M) := \mathbb{E}\min(Z,M) > 1$ if M is sufficiently large. Hence $\dim K \geq \dim K^* = \alpha^*(M)$ with probability $\geq 1-q^*$, where $q^*=q^*(M) \in [0,1)$ is the extinction probability of the new branching process (Z_n^*), which satisfies

$$p_0 + p_1 q^* + p_2 (q^*)^2 + \dots + p_{M-1}(q^*)^{M-1} + (p_M + p_{M+1} + \dots)(q^*)^M = q^*.$$

Since q^* decreases (remark that Z_n^* increases) when M increases, the limit $q_0 := \lim_{M\to\infty} q^*(M)$ exists. Letting $M\to\infty$, in the above equation gives $p(q_0)=q_0$. As $q_0\in[0,1)$, we have $q_0=q$, the extinction probability of the branching process (Z_n). Thus $\dim K \geq \alpha$ with probability $\geq 1-q$ by letting $M\to\infty$. This establishes (i).

We then turn to (iii). If Z is a.s. a constant, then it is easily verified that $\mu^{\alpha}(K) \equiv 1$ a.s.; if Z is not a.s.a constant, then $\mu^{\alpha}(K)=0$ a.s. by Falconer (1987, Lemma 4.4). □

Proposition 4.2 *(The lower bound) Let* $0 < \theta < +\infty$.

(i) If $\mathbb{E}(e^{rW^{\theta}}) < \infty$ *for some* $r \in (0,\infty)$, *then*

$$\mu^{\phi_{1/\theta}}(K) \geq r^{1/\theta} W \quad a.s. \tag{4.3}$$

where $\phi_{1/\theta}(t) = t^{\alpha}(\log\log \frac{1}{t})^{1/\theta}$.

(ii) *If* $\mathbb{E}(W^{\theta+1}) < \infty$ *then*

$$\mu^{\psi_{1/\theta}}(K) = \infty \quad a.s. \text{ on } W > 0, \tag{4.3'}$$

where $\psi_{1/\theta}(t) = t^{\alpha}(\log \frac{1}{t})^{1/\theta}$.

Proof. (i) We first note that $\mu_\omega(K(\omega)) = W$ a.s. By Proposition 4.1(i), (4.1)', for each $\varepsilon > 0$ we can choose a compact subset $K'=K'(\omega)$ of $K=K(\omega)$ such that $\mu(K') \geq W-\varepsilon$ and

$$\mu_\omega B(\mathbf{i}|n) \leq (1+\varepsilon)r^{-1/\theta}\, \phi_{1/\theta}(|B(\mathbf{i}|n)|)$$

for all $\mathbf{i} \in K'=K'(\omega)$ and all $n \geq N_0=N_0(\omega)$ (proposition 4.1(i) ensures that this can be done almost surely). This means that with probability 1,

$$\mu_\omega(B(\mathbf{i}|n) \cap K') \leq (1+\varepsilon)r^{-1/\theta}\, \phi_{1/\theta}(|B(\mathbf{i}|n)|)$$

when $n \geq N_0(\omega)$. Let (S_j) be any cover of balls of K' with diam $S_j \leq 2^{-N_0}$. Then

$$\mu_\omega(S_j \cap K') \leq (1+\varepsilon)r^{1/\theta}\phi_{1/\theta}(|S_j|) \qquad (|S_j| = \text{diam } S_j).$$

Hence

$$W-\varepsilon \leq \mu_\omega(K') \leq \mu_\omega(\bigcup_j(S_j \cap K')) \leq \sum_j \mu_\omega(S_j \cap K') \leq (1+\varepsilon)r^{-1/\theta} \sum_j \phi_{1/\theta}(|S_j|).$$

This implies that

$$\mu^{\phi_{1/\theta}}(K) \geq \frac{1}{1+\varepsilon}\, r^{1/\theta}(W-\varepsilon)$$

almost surely. Letting $\varepsilon \to 0$, it gives the result desired.

(ii) The same idea as above: $\forall\eta>0$, choose a compact subset K' of K such that $\mu_\omega(K') \geq W-\eta$ and $\mu_\omega(B(\mathbf{i}|n) \cap K') \leq \eta\, \psi_{1/\theta}(|B(\mathbf{i}|n)|)$ for all $\mathbf{i} \in K'$ and $n \geq N_0$ (That this can be done a.s. by Proposition 4.1(ii), (4.2)'). Thus for any cover $\{S_j\}$ of balls of K' with $|S_j| := \text{diam } S_j \leq 2^{-N_0}$,

$$W-\eta \leq \mu_\omega(K') \leq \mu_\omega(\bigcup_j(S_j \cap K')) \leq \sum_j \mu_\omega(S_j \cap K') \leq \eta \sum_j \psi_{1/\theta}(|S_j|)$$

This qives $\mu^{\psi_{1/\theta}}(K) \geq \frac{1}{\eta}\,(W-\eta)$. Letting $\eta \to 0$, we know that a.s. $\mu^{\psi_{1/\theta}}(K) = \infty$ if $W>0$. □

Lemma 4.3. *Suppose that a function $g\colon \mathbb{R} \to [0,1]$ is non-increasing such that $\int_0^\infty g(t)dt = +\infty$ and that $j\colon \mathbb{N} \to \mathbb{R}$ is a function satisfying $\limsup_{k\to\infty} \frac{j(k)}{k} < 1$ then $\forall \varepsilon>0$*

$$\limsup_{k\to\infty} \int_{j(k)^{1/(1+\varepsilon)}}^{k^{1/(1+\varepsilon)}} g(t)t^{\varepsilon}\, dt - k^{\bar{\varepsilon}/(1+\varepsilon)} = +\infty \tag{4.5}$$

for each $\bar{\varepsilon}$ with $0 < \bar{\varepsilon} < \varepsilon$.

Proof. Since

$$\infty = \int_1^\infty g(t)dt = \sum_{k=1}^{\infty} \int_k^{k+1} g(t)dt \le \sum_{k=1}^{\infty} g(k)$$

we can choose an increasing sequence $(k_\nu)_{\nu\in\mathbb{N}}$ of integers such that

$$g(k_\nu) \ge \frac{1}{k_\nu^{1+\varepsilon'}}, \quad (\ \varepsilon' > 0 \text{ given }\ )$$

for all $\nu = 1,2,\ldots$For each k_ν, choose $K_\nu \in \mathbb{N}$ such that $(k_\nu-1)^{1+\varepsilon} < K_\nu \le k_\nu^{1+\varepsilon}$. This is possible since $k_\nu^{1+\varepsilon}-(k_\nu-1)^{1+\varepsilon} \ge 1$. Now

$$\limsup_{k\to\infty,\, k\in\mathbb{N}} \left[\int_{(j(k))^{1/(1+\varepsilon)}}^{k^{1/(1+\varepsilon)}} g(t)t^{\varepsilon}dt - k^{\bar{\varepsilon}/(1+\varepsilon)} \right]$$

$$\ge \limsup_{\nu\to\infty} \left[\int_{(j(K_\nu))^{1/(1+\varepsilon)}}^{K_\nu^{1/(1+\varepsilon)}} g(t)t^{\varepsilon}dt - K_\nu^{\bar{\varepsilon}/(1+\varepsilon)} \right]$$

$$\ge \limsup_{\nu\to\infty} \left[\frac{1}{1+\varepsilon}\, g(k_\nu) \left(K_\nu - j(K_\nu) \right) - K_\nu^{\bar{\varepsilon}/(1+\varepsilon)} \right]$$

$$\ge \limsup_{\nu\to\infty} \left[\frac{1}{1+\varepsilon}\, k_\nu^{\varepsilon-\varepsilon'} \left(\frac{(k_\nu-1)^{1+\varepsilon}}{k_\nu^{1+\varepsilon}} - \frac{j(K_\nu)}{K_\nu}\frac{K_\nu}{k_\nu^{1+\varepsilon}} \right) - k_\nu^{\bar{\varepsilon}} \right] = +\infty$$

if we choose $\varepsilon' > 0$ such that $\bar{\varepsilon} < \varepsilon-\varepsilon'$, noting that

$$\lim_{\nu\to\infty} \frac{(k_\nu-1)^{1+\varepsilon}}{k_\nu^{1+\varepsilon}} = 1 \quad\text{and}\quad \limsup_{\nu\to\infty} \frac{j(K_\nu)}{K_\nu}\frac{K_\nu}{k_\nu^{1+\varepsilon}} = \limsup_{\nu\to\infty} \frac{j(K_\nu)}{K_\nu} < 1.$$

The proof is then completed. □

Proposition 4.4. *(i) Suppose that*

$$\mathbb{E}(e^{rW^{1/\theta}}) = \infty$$

for some $\theta \in (0,\infty)$ *and* $r \in (0,\infty)$. *For* $t > 0$, *write*

$$B_k^* \equiv B_k^*(\theta,t) = \left\{ \sigma \in \mathbb{N}^k \mid W_{(\sigma|\nu)^*}(\omega) < \left(\frac{1}{t} \log\log \frac{1}{2^{-\nu}} \right)^\theta \right. \\ \left. \text{for all } \nu = [\log k], [\log k]+1, \ldots, k. \right\}, \tag{4.6}$$

where $\tau^* = (\tau_1,\tau_2,\ldots,\tau_{n-1},\tau_n+1)$ *if* $\tau = (\tau_1,\tau_2,\ldots,\tau_n)$, *and*

$$I_k^* \equiv I_k^*(\theta,t) = \int_\Omega \sum_{\sigma\in B_k^*} (2^{-k\alpha})\, 1_{(\sigma\in\mathcal{T})}(\omega) \left(\log\log \frac{1}{2^{-k}}\right)^\theta dP . \tag{4.7}$$

Then for all $t > r$ *we have*

$$\liminf_{k\to\infty} I_k^* = 0 . \tag{4.8}$$

(ii) For $\theta \in (0,\infty)$, *write*

$$\bar{B}_k^*(\theta) = \left\{ \sigma\in\mathbb{N}^k \mid W_{(\sigma|\nu)^*}(\omega) < \left(\log \frac{1}{2^{-\nu}}\right)^\theta \right. \\ \left. \text{for all } \nu = [\log k], [\log k]+1, \ldots, k. \right\} \tag{4.6)'$$

and

$$\bar{I}_k^*(\theta) = \int_\Omega \sum_{\sigma\in\bar{B}_k^*} (2^{-k\alpha})\, 1_{(\sigma\in\mathcal{T})}(\omega) \left(\log \frac{1}{2^{-k}}\right)^\theta dP . \tag{4.7)'$$

Then

$$\limsup_{k\to\infty} \left\{ \sum_{\nu=[\log k]}^{k} P\left[W^{1/\theta} \ge \nu \right] - \theta \log k \right\} > -\infty \tag{4.9}$$

implies

$$\liminf_{k\to\infty} \bar{I}_k^*(\theta) < +\infty. \tag{4.8)'$$

In particular we have

$$\liminf_{k\to\infty} \bar{I}_k^*(\theta-\varepsilon) = 0 \ (\forall\varepsilon>0) \tag{4.8)"$$

if $\mathbb{E}(W^{1/\theta}) = \infty$.

Proof. (i) Using the notations introduced in the proceeding, we can write

$$1_\sigma = 2^{-k\alpha}\, 1_{(\sigma\in\mathcal{T})}(\omega),$$

and, since $W_{(\sigma|\nu+1)^*}$ (ν = [logk],...,k.) are independent and, as a family independent of F_σ (the σ-algebra generated by $Z^{(\sigma|i)}$ $(0 \le i \le |\sigma|)$), we have

$$I_k^* = \sum_{\sigma\in\mathbb{N}^k} \int_{\bigcap_{\nu=[\log k]}^{k} \left\{ W_{(\sigma|\nu)*} < \left(\frac{1}{t}\log\log\frac{1}{2^{-\nu}}\right)^{\theta}\right\}} 1_{\sigma}\left(\log\log\frac{1}{2^{-|\sigma|}}\right)^{\theta} dP$$

(conditioned on F_σ firstly)

$$= \sum_{\sigma\in\mathbb{N}^k} \int 1_{\sigma}\left(\log\log\frac{1}{2^{-|\sigma|}}\right)^{\theta} dP \prod_{\nu=[\log k]}^{k} P\left\{ W < \left(\frac{1}{t}\log\log\frac{1}{2^{-\nu}}\right)^{\theta}\right\}$$

$$= \left(\log\log\frac{1}{2^{-k}}\right)^{\theta} \mathbb{E}\left[\sum_{\sigma\in\mathbb{N}^k} 1_{\sigma}\right] \prod_{\nu=[\log k]}^{k} P\left\{ W < \left(\frac{1}{t}\log\log\frac{1}{2^{-\nu}}\right)^{\theta}\right\}$$

$$= \left(\log\log\frac{1}{2^{-k}}\right)^{\theta} \prod_{\nu=[\log k]}^{k} P\left\{ W < \left(\frac{1}{t}\log\log\frac{1}{2^{-\nu}}\right)^{\theta}\right\}$$

$$\le \left(\log\log\frac{1}{2^{-k}}\right)^{\theta} \exp\left\{ -\sum_{\nu=[\log k]}^{k} P\left\{ W \ge \left(\frac{1}{t}\log\log\frac{1}{2^{-\nu}}\right)^{\theta}\right\}\right\}$$

$$\le (\log k)^{\theta} \exp\left\{ -\sum_{\nu=[\log k]}^{k} P\left[W \ge \left(\frac{1}{t}\log \nu\right)^{\theta}\right]\right\}.$$

Hence

$$I_k^* \le \exp\left\{ -\sum_{\nu=[\log k]}^{k} P\left[e^{tW^{1/\theta}} \ge \nu\right] + \theta \log\log k\right\}. \tag{4.10}$$

It then suffices to prove that

$$\limsup_{k\to\infty} \left\{ \sum_{\nu=[\log k]}^{k} P\left[e^{tW^{1/\theta}} \ge \nu\right] - \theta\log\log k\right\} = +\infty. \tag{4.11}$$

To show this we note that

$$\sum_{\nu=[\log k]}^{k} P\left[e^{tW^{1/\theta}} \ge \nu\right] \ge \sum_{\nu=[\log k]}^{k-1} \int_{\nu}^{\nu+1} P\left[e^{tW^{1/\theta}} \ge x\right] dx$$

$$= \int_{[\log k]}^{k} P\left[e^{tW^{1/\theta}} \ge x\right] dx = \frac{1}{1+\varepsilon}\int_{[\log k]^{1/(1+\varepsilon)}}^{k^{1/(1+\varepsilon)}} P\left[e^{\frac{t}{1+\varepsilon}W^{1/\theta}} \ge y\right] y^{\varepsilon} dy$$

($x=y^{1+\varepsilon}$), where $\varepsilon > 0$ is chosen such that $t/(1+\varepsilon) > r$. Writing

$$f(y) = P\left[e^{\frac{t}{1+\varepsilon}W^{1/\theta}} \ge y\right],$$

then

$$\int_0^\infty f(y)dy = \mathbb{E}\left[e^{\frac{t}{1+\varepsilon} W^{1/\theta}} \right] = +\infty.$$

By use of Lemma 4.3, we have eventually

$$\limsup_{k\to\infty} \left\{ \sum_{\nu=[\log k]}^{k} P\left[e^{tW^{1/\theta}} \ge \nu \right] - \theta \log \log k \right\}$$

$$\ge \limsup_{k\to\infty} \frac{1}{1+\varepsilon} \left\{ \int_{[\log k]^{1/(1+\varepsilon)}}^{k^{1/(1+\varepsilon)}} P\left[e^{\frac{t}{1+\varepsilon} W^{1/\theta}} \ge y\right] y^{\varepsilon} dy - k^{\bar{\varepsilon}/(1+\varepsilon)} \right\} = +\infty,$$

which ends the proof of (i).

(ii) The same argument as above shows that

$$\bar{I}_k^*(\theta) \le \exp\left\{ -\sum_{\nu=[\log k]}^{k} P\left[W^{1/\theta} \ge \nu \right] + \theta \log k \right\}, \qquad (4.10)'$$

from which the first conclusion follows. To prove the second it suffices to show that $\forall \varepsilon > 0$

$$\limsup_{k\to\infty} \left\{ \sum_{\nu=[\log k]}^{k} P\left[W^{1/(\theta-\varepsilon)} \ge \nu \right] - (\theta-\varepsilon) \log k \right\} = +\infty. \qquad (4.12)$$

Note that

$$\sum_{\nu=[\log k]}^{k} P\left[W^{1/(\theta-\varepsilon)} \ge \nu \right] \ge \sum_{\nu=[\log k]}^{k-1} \int_{\nu}^{\nu+1} P\left[W^{1/(\theta-\varepsilon)} \ge x \right] dx$$

$$= \int_{[\log k]}^{k} P\left[W^{1/(\theta-\varepsilon)} \ge x\right] dx = \int_{[\log k]}^{k} P\left[W^{1/\theta} \ge x^{1-\varepsilon/\theta}\right] dx$$

$$= \frac{1}{1-\varepsilon/\theta} \int_{[\log k]^{1-\varepsilon/\theta}}^{k^{1-\varepsilon/\theta}} P(W^{1/\theta} \ge t)\, t^{\varepsilon/(\theta-\varepsilon)} dt \qquad (x^{1-\varepsilon/\theta} = t).$$

$$= \frac{1}{1+\varepsilon'} \int_{[\log k]^{1/(1+\varepsilon')}}^{k^{1/(1+\varepsilon')}} P(W^{1/\theta} \ge t)\, t^{\varepsilon'} dt \qquad (4.13)$$

where $\varepsilon' = \varepsilon/(\theta-\varepsilon)$, using Lemma 4.3 with $j(x) = [\log x]$, $g(t) = P(W^{1/\theta} \ge t)$ (remark that $\int_0^\infty g(t)dt = \mathbb{E}(W^{1/\theta}) = \infty$) and $\varepsilon' = \varepsilon/(\theta-\varepsilon)$ gives then

$$\limsup_{k\to\infty} \left\{ \sum_{\nu=[\log k]}^{k} P\left[W^{1/(\theta-\varepsilon)} \ge \nu \right] - (\theta-\varepsilon) \log k \right\}$$

$$\ge \limsup_{k\to\infty} \frac{1}{1+\varepsilon'} \left\{ \int_{[\log k]^{1/(1+\varepsilon')}}^{k^{1/(1+\varepsilon')}} P(W^{1/\theta} \ge t)\, t^{\varepsilon'} dt - k^{\bar{\varepsilon}/(1+\varepsilon')} \right\} = +\infty,$$

where $\bar{\varepsilon}\in(0,\varepsilon')$. This shows (4.12), which ends the proof of the proposition. □

Proposition 4.5 (*the upper bound*)

(i) If

$$\mathbb{E}\,[e^{rW^{1/\theta}}] = +\infty$$

for some $\theta \in (0,+\infty)$ *and* $r \in (0,+\infty)$, *then*

$$\mu^{\phi_\theta}(K) < +\infty \quad a.s. \tag{4.14}$$

where $\phi_\theta(t) = t^\alpha(\log\log \frac{1}{t})^\theta$. *Moreover*

$$\mathbb{E}[\mu^{\phi_\theta}(K)] \le r^\theta. \tag{4.15}$$

(ii) If $\mathbb{E}(W^{1/\theta}) = +\infty$ *for some* $\theta \in (0,\infty)$, *then*

$$\mu^{\psi_{\theta-\varepsilon}}(K) = 0 \text{ a.s. } \forall\varepsilon>0, \tag{4.16}$$

where $\psi_{\theta-\varepsilon}(t) = t^\alpha(\log\frac{1}{t})^{\theta-\varepsilon}$. *Moreover, (4.9) implies* $\mathbb{E}\,[\mu^{\psi_\theta}(K)] <\infty$ *(thus* $\mu^{\psi_\theta}(K) <\infty$ *a.s.).*

Proof. (i) Let $t > 0$, $\varepsilon > 0$, $k_0\in \mathbb{N}$ and $\delta = 2^{-[\log k_0]}$, where $[x]$ denotes the integral part of x. Define

$$B^* = \left\{ \sigma \in \mathbb{N}^{\mathbb{N}} : W_{(\sigma|k)^*}(\omega) < \left(\frac{1}{t} \log\log \frac{1}{2^{-k}} \right)^\theta \right.$$
$$\left. \text{for all } k = [\log k_0], [\log k_0]+1, \dots, k_0 \right\}. \tag{4.17}$$

For $\sigma \in \mathbb{N}^{\mathbb{N}} - B^*$, let $k(\sigma)$ be the smallest $k \ge [\log k_0]$ such that

$$W_{(\sigma|k)^*}(\omega) \ge \left(\frac{1}{t} \log\log \frac{1}{2^{-k}} \right)^\theta .$$

Then $k(\sigma) \le k_0$. Set $\tilde{\Gamma} = \left\{ \sigma|k(\sigma) : \sigma \in \mathbb{N}^{\mathbb{N}}-B^* \right\}$ ($\tilde{\Gamma} = \tilde{\Gamma}(k_0)$). It is easy to check that $\tilde{\Gamma}$ is an antichain. Hence there exists a maximal antichain $\Gamma = \Gamma(k_0)$ with $\tilde{\Gamma} \subseteq \Gamma$. For every $\sigma \in B^*$, we have $\sigma|k_0 \in B^*_{k_0}(\omega)$, where $B^*_{k_0}(\omega)$ is defined in proposition 4.4(i). Thus

$$K(\omega) \subseteq \left(\bigcup_{\sigma\in\tilde{\Gamma}} D_\sigma(\omega) \right) \cup \left(\bigcup_{\sigma\in B^*_{k_0}(\omega)} D_\sigma(\omega) \right), \tag{4.18}$$

where $D_\sigma(\omega)=\{\tau\in K(\omega):\ \tau>\sigma\}\equiv B(\sigma)\cap K(\omega)$ denotes the closed descendants of σ. By Proposition 4.4(i), we can choose a sequence (k_i) of integers increasing to $+\infty$ such that

$$\lim_{i\to\infty}\mathbb{E}\left[\sum_{\sigma\in B^*_{k_i}(\omega)} 1_\sigma\left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta\right]=0\ . \tag{4.19}$$

Now

$$\mu_\delta^{\phi_\theta}(K(\omega))\le\sum_{\sigma\in\tilde\Gamma,\sigma\in\mathcal{T}}\phi_\theta(\operatorname{diam}D_\sigma(\omega))+\sum_{\sigma\in B^*_{k_0}(\omega),\sigma\in\mathcal{T}}\phi_\theta(\operatorname{diam}D_\sigma(\omega))$$

$$=\sum_{\sigma\in\tilde\Gamma}2^{-|\sigma|\alpha}\,1_{(\sigma\in\mathcal{T})}\left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta+\sum_{\sigma\in B^*_{k_0}(\omega)}2^{-|\sigma|\alpha}\,1_{(\sigma\in\mathcal{T})}\left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta.$$

$$=\sum_{\sigma\in\tilde\Gamma}1_\sigma\left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta+\sum_{\sigma\in B^*_{k_0}(\omega)}1_\sigma\left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta.$$

Let k_0 run through (k_i) and $\delta=\delta_i=2^{-[\operatorname{Log}k_i]}$, this gives

$$\liminf_{i\to\infty}\mathbb{E}[\mu_{\delta_i}^{\phi_\theta}(K(\omega))]\le\liminf_{i\to\infty}\mathbb{E}\left[\sum_{\sigma\in\tilde\Gamma(k_i)}1_\sigma\left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta\right]$$

$$+\lim_{i\to\infty}\mathbb{E}\left[\sum_{\sigma\in B^*_{k_i}(\omega)}1_\sigma\left(\log\log\frac{1}{2^{-|\sigma|}}\right)^\theta\right]$$

$$\le\liminf_{i\to\infty}\mathbb{E}\left[\sum_{\sigma\in\Gamma(k_i)}t^\theta\,1_\sigma W_{\sigma*}\right]. \tag{4.20}$$

First conditioned on $\mathbb{F}_{k_i}$, the σ-algebra generated by $Z^\sigma(|\sigma|\le k_i)$, we obtain that

$$\liminf_{i\to\infty}\mathbb{E}[\mu_{\delta_i}^{\phi_\theta}(K(\omega))]\le t^\theta\liminf_{i\to\infty}\mathbb{E}\left[\sum_{\sigma\in\Gamma(k_i)}1_\sigma\right].$$

Since $\mathbb{E}(W_\sigma)=1$, the same reason shows

$$\mathbb{E}\left[\sum_{\sigma\in\Gamma(k_i)}1_\sigma W_\sigma\right]=\mathbb{E}\left[\sum_{\sigma\in\Gamma(k_i)}1_\sigma\right].$$

With this at hands, and noting that $W = \sum_{\sigma\in\Gamma} 1_\sigma W_\sigma$ for any maximal antichain Γ, we have eventually

$$\liminf_{i\to\infty} \mathbb{E}[\mu^{\phi}_{\delta_i}(K(\omega))] \le t^{\theta}. \tag{4.21}$$

Since $\mu^{\phi_\theta}_{\delta_i}(K(\omega))$ increases as δ_i decreases, this implies that $\mathbb{E}[\mu^{\phi_\theta}(K(\omega))] \le t^{\theta}$ for any $t > r$, which ends the proof of the part (i) of the proposition.

(ii) To obtain the second conclusion, by Proposition 4.4(ii), it suffices to show that, if (4.8)' holds then $\mathbb{E}[\mu^{\phi_\theta}(K(\omega))]<\infty$. The argument is the same as above, using (4.8)' instead of (4.8): write

$$I \equiv I(\theta) := \liminf_{k\to\infty} \bar{I}^*_k(\theta) \ (<+\infty),$$

and define

$$\bar{B}^* = \left\{ \sigma \in \mathbb{N}^{\mathbb{N}} : W_{(\sigma|k)^*}(\omega) < \left(\log \frac{1}{2^{-k}} \right)^{\theta} \right.$$
$$\left. \text{for all } k = [\log k_0], [\log k_0]+1, \ldots, k_0 \right\}. \tag{4.17)'}$$

For $\sigma \in \mathbb{N}^{\mathbb{N}} - \bar{B}^*$, let $\bar{k}(\sigma)$ be the smallest $k \ge [\log k_0]$ such that

$$W_{(\sigma|k)^*}(\omega) \ge \left(\log \frac{1}{2^{-k}} \right)^{\theta}.$$

Then $\bar{k}(\sigma) \le k_0$. Setting $\tilde{\Gamma}' \equiv \tilde{\Gamma}'(k_0) = \left\{ \sigma|\bar{k}(\sigma) : \sigma \in \mathbb{N}^{\mathbb{N}}-\bar{B}^* \right\}$, then $\tilde{\Gamma}'$ is an antichain and there exists a maximal antichain $\bar{\Gamma} = \bar{\Gamma}(k_0)$ with $\tilde{\Gamma}' \subseteq \bar{\Gamma}$. For every $\sigma \in \bar{B}^*$, we have then $\sigma|k_0 \in \bar{B}^*_{k_0}(\omega)$, where $\bar{B}^*_{k_0}(\omega)$ is defined in Proposition 4.4(ii), and, instead of (4.18), (4.19), (4.20) and (4.21) we have respectively

$$K(\omega) \subseteq \left(\bigcup_{\sigma\in\tilde{\Gamma}'} D_\sigma(\omega) \right) \cup \left(\bigcup_{\sigma\in\bar{B}^*_{k_0}(\omega)} D_\sigma(\omega) \right), \tag{4.18)'}$$

$$I(\theta) = \lim_{i\to\infty} \mathbb{E}\left[\sum_{\sigma\in\bar{B}^*_{k_i}(\omega)} 1_\sigma \left(\log \frac{1}{2^{-|\sigma|}} \right)^{\theta} \right] < +\infty, \tag{4.19)'}$$

$$\liminf_{i\to\infty} \mathbb{E}[\mu_{\delta_i}^{\psi_\theta}(K(\omega))] \le \liminf_{i\to\infty} \mathbb{E}\left[\sum_{\sigma\in\bar{\Gamma}(k_i)} 1_\sigma W_{\sigma^*} \right] + I, \tag{4.20'}$$

$$\liminf_{i\to\infty} \mathbb{E}[\mu_{\delta_i}^{\psi_\theta}(K(\omega))] \le 1 + I \tag{4.21'}$$

where k_i is a sequences of integers increasing to $+\infty$ and $\delta_i = 2^{-[\mathrm{Log} k_i]}$. Hence

$$\mathbb{E}(\ \mu^{\psi_\theta}(K(\omega))\ ) \le 1 + I \text{ and } \mu^{\psi_\theta}(K(\omega)) < +\infty \text{ a.s.}$$

This establishes the second assertion. To see the first, we note that $I(\theta-\varepsilon)=0$ for all $\varepsilon>0$ if $\mathbb{E}(W^{1/\theta})=\infty$. Thus by the preceding argument, $\mu^{\psi_{\theta-\varepsilon}}(K(\omega)) < +\infty$ a.s. Since $\mu^{\psi_{\theta-\varepsilon/2}}(K(\omega)) < +\infty$ also, we have $\mu^{\psi_{\theta-\varepsilon}}(K(\omega)) = 0$ a.s. The proof is then completed. □

Proposition 4.6. *(The fundamental theorem) Let* $0 < \theta < +\infty$, $r_\theta = r(W^\theta)$ *be the radius of convergence of the moment generating function* $\mathbb{E}(e^{tW^\theta})$ *of* W^θ *and* $\phi_\theta(t) = t^\alpha(\log\log \frac{1}{t})^\theta$. *Then*

$$\mu^{\phi_\theta}(K) = (r_{1/\theta})^\theta W \quad a.s.$$

In particular

$\mu^{\phi_\theta}(K)$ *is zero, positive and finite, or infinite*

almost surely on $K \neq \emptyset$ *if and only if*

$r(W^{1/\theta})$ *is zero, positive and finite, or infinite*

respectively.

Proof. This is a mere combination of propositions 4.2(i) and 4.5(i): from proposition 4.2(i) we have $\mu^{\phi_\theta}(K) \ge (r_{1/\theta})^\theta W$ a.s. and from proposition 4.5(i) we have $\mathbb{E}[\mu^{\phi_\theta}(K)] \le (r_{1/\theta})^\theta = \mathbb{E}[(r_{1/\theta})^\theta W]$. □

Our results have nearly been proved till now. **Theorem 1** comes directly from Corollary 4.1, and **Theorem 2** from Proposition 4.6.

Proof of Theorem 3. We first note that $0 \le \beta < 1$. If $0 < \beta < 1$, the result follows immediately from Lemma 3.2 and Theorem 2; if $\beta = 0$, then $Z_1 \equiv \mu$

a.s. and the result comes from Corollary 4.1 and its proof. This ends the proof of the theorem. □

Proof of Theorem 4. (i) If $\theta < 1$, then $r(W^{1/\theta}) = 0$ by Lemma 3.3 and hence $\mu^{\phi_\theta}(K) = 0$ a.s. by Theorem 2. (ii) If $\mathbb{E}(e^{tZ}) < \infty$ for some $t > 0$, then $r(Z) > 0$ and consequently $r(W) > 0$ by Lemma 3.1. Thus $\mu^{\phi_1}(K) > 0$ a.s. on $K \neq \emptyset$, again by Theorem 2. (iii) Finally if $\mathbb{E}(e^{tZ}) = \infty$ for some $t > 0$, then $r(Z) < \infty$ and Lemma 3.1 and Theorem 2 imply again $r(W) < \infty$ and $\mu^{\phi_1}(K) < \infty$ a.s. □

Theorem 5 comes directly from Theorem 4. □

Proof of Theorem 6. This is a combination of propositions 4.2(ii) and 4.5(ii). In the case where $\gamma=\infty$, we use also the fact that $\mu^{\alpha}(K) < \infty$ a.s., which implies $\mu^{\psi_\theta}(K) =0$ a.s. □

Remark 7 is seen by the proofs. □

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