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Publications de l'Institut de recherche mathématiques de Rennes, 1993, fascicule 2
« Fascicule de probabilités », , p. 1-19
<http://www.numdam.org/item?id=PSMIR_1993___2_A2_0>

Consistency of Estimators of Cyclic Functional Parameters for some Nonstationary Processes

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Abstract. We consider a class of zero mean second order stochastic processes whose covariance kernel admits a Fourier series decomposition, $\mathrm{E}[X(s+t)\overline{X(s)}] \sim \sum_{\alpha\in\mathbf{R}} b_\alpha(t)\, \mathrm{e}^{is\alpha}$, and the subclass of these processes for which the coefficient functions b_α are the Fourier transforms of complex measures m_α, $\alpha \in \mathbf{R}$. These classes of processes which contain the almost periodically correlated processes and the strongly harmonizable processes, are frequently applied in signal analysis.

This paper addresses the problem of the asymptotic behavior of the variance, and of the consistency for some natural estimators of $b_\alpha(t)$, and whenever $m_\alpha(d\lambda) = f_\alpha(\lambda)\, d\lambda$, of $f_\alpha(\lambda)$. We deal with this problem under two different types of hypotheses: first, in terms of conditions on the associated stochastic spectral measure whenever the process is strongly harmonizable, next, in terms of a mixing condition which constrains the dependence of remote events.

Key words. almost periodic, harmonizable process, spectral measure, spectral density, consistent estimator, periodogram, mixing property

AMS Subject Classification (1980). 62G05, 62M15

1 Introduction

In signal theory the processes are not always stationary or periodic, however some processes exhibit cyclic components. This paper concerns processes whose covariance kernel admits a Fourier decomposition. An amount of works have been done on this subject (see [13] and references therein). Here we study the consistency and the rate of convergence of estimators of the cyclic components of these processes. First define the processes we are concerned with. From now, we only consider processes which are zero mean.

CCF processes and CSM processes. Let $(\Omega, \mathcal{A}, \mathrm{P})$ be a probability space. In this work, a process $X : \mathbf{R} \to L^2(\mathrm{P})$ is said to be with *cyclic covariance functions*, and we will write X is a *CCF process*, whenever it satisfies the two following conditions

(A1) for any t, the function $u \to K(u+t, u) = \mathrm{cov}[X(u+t), X(u)]$ is in $L^1_{loc}(\mathbf{R})$,

(A2) for all t and α, the following limit exists

$$\lim_{s\to\infty} \frac{1}{s} \int_0^s K(u+t, u)\, \mathrm{e}^{-iu\alpha}\, du = b_\alpha(t).$$

Then the functions b_α are called the *cyclic covariance functions* of X, and for all $\alpha, t \in \mathbf{R}$, we have $|b_\alpha(t)| \le b_0(0)$.

Whenever in addition,

(**A3**) the function $(t, u) \to K(u+t, u)$ is measurable,

(**A4**) and the function b_0 is continuous at $t = 0$,

it is known [23] that for any α there exists a unique complex measure m_α on $\mathbf{R}$, called a *cyclic spectral measure* of X, such that

$$b_\alpha(t) = \int_{\mathbf{R}} e^{it\lambda}\, m_\alpha(d\lambda) \ \text{ Leb} - \text{a.e.},$$

and the process is said to be with *cyclic spectral measures, X* is a *CSM process.* Each measure m_α is absolutely continuous with respect to m_0 which is a finite nonnegative measure [6, 18], and for any α, $m_{-\alpha}(d\lambda) = \overline{m_{\alpha}}(d\lambda + \alpha)$. Moreover, if $m_\alpha(d\lambda) = f_\alpha(\lambda) m_0(d\lambda)$, the function $f_\alpha \in L^1(m_0)$ is called a *cyclic spectral density* of X.

Note that a second order stationary white noise $\varepsilon : \mathbf{R} \to L^2(\mathrm{P})$, $\mathrm{E}[\varepsilon(u)] = 0$ and $\mathrm{cov}[\varepsilon(u+t), \varepsilon(u)] = 1$ if $t = 0$ and 0 if $t \neq 0$, is CCF but not CSM .

PC processes and APC processes. A process X is said to be *periodically correlated (PC)* with period $T > 0$ whenever for any t the function $u \to K(u+t, u)$ is periodic with period T and satisfies the condition

(**A1'**) for any t, $u \to K(u+t, u)$ is continuous.

A process X is said to be *almost periodically correlated (APC)* whenever for any t the function $u \to K(u+t, u)$ is almost periodic in the sense of Bohr [3, 6, 14, 18]. In signal analysis these processes are also said to be respectively wide sense cyclostationary and wide sense almost cyclostationary [11].

Note that the almost periodicity of $u \to K(u+t, u)$ implies its uniform continuity in $\mathbf{R}$. Hence, here, any PC process is APC . However in some papers (see [17] and references therein) the condition of continuity (A1') is removed from the definition of the periodically correlated processes and replaced by the measurability condition (A1). Such periodically correlated processes are not necessarily APC but are CCF . In the following we only consider PC processes with the continuity property (A1').

Harmonizable processes. A process X is *harmonizable* (in the sense of Loève) whenever there exists a measure M on $\mathbf{R}^2$, called the *spectral bimeasure* of X, such that

$$K(s,t) = \iint_{\mathbf{R}^2} e^{i(s\lambda_1 - t\lambda_2)}\, M(d\lambda_1, d\lambda_2).$$

This notion can also be generalized for instance to weakly harmonizable processes [24, 25], σ-finite harmonizable processes [8] and locally harmonizable processes [7].

These different classes of processes are frequently applied in signal analysis [2, 11, 12, 24] as generalizations of the second order stationary processes, pointing out a spectral point of view. It is known that the statistical methods based on periodicity, almost periodicity or spectral decomposition are widely used in modelling real life data e.g. in climatology, signal transmission, oceanography (see [11] and references therein).

From the theory of the periodic or almost periodic functions, we easily deduce that the class of the CCF processes contains the PC or APC processes, thus the second order stationary processes. The class of the CSM processes contains the uniformly $L^2(\mathrm{P})$-continuous PC or APC processes, as well as the harmonizable processes, the σ-finite harmonizable processes and the locally harmonizable processes.

Whenever the process X is strongly harmonizable, for any α the cyclic spectral measure m_α can be seen as the restriction of the spectral bimeasure M of the process to the straight line D_α with equation $\lambda_1 - \lambda_2 = \alpha$ [18]. The set $F(X) = \{\alpha \in \mathbf{R} : b_\alpha(t) \neq 0 \text{ for some } t\}$ is countable, and the Fourier series $\sum_{\alpha \in F(X)} b_\alpha(t)\ \mathrm{e}^{is\lambda}$ is absolutely convergent. Its sum is equal to $K(s+t,s)$ if and only if the strongly harmonizable process X is APC [18]. Furthermore an harmonizable process is not PC or APC whenever its spectral measure M is absolutely continuous. See [6, 18] for more details on the relationships between the strong harmonizability and the almost periodicity.

Aim of the paper. In [17, 19] Hurd and Leskow showed that the usual estimators of the covariance function and of the spectral density of a second order stationary process, formed from a single sample path of the process, can be modified to provide consistent estimators of the cyclic covariance functions and of the cyclic spectral densities of PC or APC processes. In this paper, we are concerned with the asymptotic behavior and the consistency of these estimators in the case of the CCF processes and of the CSM processes.

After the presentation of the estimators in Section 2, we tackle those problems with two quite different points of view.

The first one, presented in Section 3, and which does not seem to be explored or published to our knowledge, applies whenever the process in consideration is strongly harmonizable. It is founded on the decomposition of such a process X as the Fourier transform of an $L^2(\mathrm{P})$-valued measure $\mu:\mathcal{B}(\mathbf{R}) \to L^2(\mathrm{P})$, called *the spectral stochastic measure* of X [25]

$$X(t) = \int_{\mathbf{R}} \mathrm{e}^{it\lambda}\ \mu(d\lambda).$$

Whenever the product measure $\mu \otimes \overline{\mu}$ exists as an $L^p(\mathrm{P})$-valued measure on $\mathbf{R}^2$, for some $p \geq 1$, the estimator of $b_\alpha(t)$ converges in $L^p(\mathrm{P})$ towards a random variable, and the estimator of $f_\alpha(\lambda)$ converges towards a random transformation (Theorem 3.1). If in addition the process X is Gaussian, $p = 2$ and the spectral stochastic measure μ of X has no point mass, the estimators are consistent in quadratic mean (Theorem 3.2). Simultaneously we point out that the consistency and the rate of convergence of the estimators depend on the repartition of $\mu \otimes \overline{\mu}$ and of M on the straight lines D_α and on their neighborhoods.

In order to control the correlation between time separated random variables in the expressions of the variances of the estimators, and in order to avoid a normality condition, we can consider a mixing condition limiting the dependence of remote events [2]. Thus, in the case of the uniformly mixing APC processes, Hurd and Leskow have established some sufficient conditions for the quadratic mean consistency of the estimators of $b_\alpha(t)$ and $f_\alpha(\lambda)$ [19], next for the almost everywhere consistency and for the asymptotic normality [20, 21].

In Section 4, we introduce a notion of mixing property which is expressed only in terms of moments, and which is sligthly more general that the classical notions of uniform mixing and of strong mixing (Definition 4.1) [1]. Then, this mixing condition and some mild hypotheses provide the consistency in quadratic mean and in almost everywhere convergence for the estimators of $b_\alpha(t)$ for CCF processes, and of $f_\alpha(\lambda)$ for CSM processes (Theorem 4.3 and Theorem 4.6). These results improve those in [5] where the fourth moments of the process X are assumed to be bounded. Furthermore some rates of convergence are obtained for the PC or APC processes, for which the rate of convergence of the bias of the estimators is known (Lemmas 2.1 and 2.2).

In the particular case of the PC or APC processes, another point of view to studying the consistency of the estimators of $b_\alpha(t)$ and $f_\alpha(\lambda)$ has been considered in [6]. It is founded on the Fourier series decomposition of the covariance kernel

$$K(u+t,u) \sim \sum_{\alpha\in\mathbf{R}} b_\alpha(t)\, \mathrm{e}^{is\alpha},$$

and on the convergence of this series. Thus, the results are expressed in terms of the cyclic covariance functions b_α, and in terms of distance between the elements of the set $F(X) = \{\alpha \in \mathbf{R} : b_\alpha(t) \neq 0 \text{ for some } t\}$ which is countable whenever the process X is uniformly $L^2(\mathrm{P})$-continuous. All these points of view provide different sufficient conditions for the consistency of the estimators, but the common problems are:
i) the control of the variance of the estimators, and
ii) the control of the convergences which defined $b_\alpha(t)$ as a time average, and $f_\alpha(\lambda)$ as the inverse Fourier transform of $b_\alpha(t)$.

Throughout the paper, the process X is assumed to be measurable as a function defined from $(\Omega \times \mathbf{R}, \mathcal{A} \otimes \mathcal{B}(\mathbf{R}))$ into $(\mathbf{C}, \mathcal{B}(\mathbf{C}))$.

2 Definition of the estimators

In the following, the hypotheses will always imply that X has moments of sufficiently large order for the following integrals which defined the estimators can be taken in the Lebesgue sense and path by path P-almost everywhere. Moreover we assume that the process X is zero mean.

2.1 Estimators of the cyclic covariance functions

By analogy with the second order stationary case where only the estimation of $b_0(t)$ is considered, Hurd [17] defined the following natural estimators of $b_\alpha(t)$, $\alpha \in \mathbf{R}$,

$$\hat{b}_\alpha(t,s) = \begin{cases} \frac{1}{s}\int_0^{s-t} X(u+t)\overline{X(u)}\,\mathrm{e}^{-iu\alpha}\,du & \text{for } 0 \le t \le s, \\ \frac{1}{s}\int_{-t}^{s} X(u+t)\overline{X(u)}\,\mathrm{e}^{-iu\alpha}\,du & \text{for } -s \le t \le 0, \\ 0 & \text{otherwise,} \end{cases}$$

which is computationable with the data recorded from $t = 0$ to $t = s$. We can easily see that if X is a CCF process then $\hat{b}_\alpha(t,s)$ is an asymptotically unbiased estimator of $b_\alpha(t)$

$$\lim_{s\to\infty} \mathrm{E}[\hat{b}_\alpha(t,s)] = b_\alpha(t).$$

Some rates of convergence towards 0 of the bias of $\hat{b}_\alpha(t,s)$ are established in the following lemma for the PC processes and some APC processes. Remind that for an APC process, the set $F(X,t) = \{\alpha \in \mathbf{R}; b_\alpha(t) \neq 0\}$ is countable for any t. For a PC process or a uniformly $L^2(\mathrm{P})$-continuous APC process, the set $F(X) = \cup_{t\in\mathbf{R}} F(X,t) = \{\alpha \in \mathbf{R}; b_\alpha(t) \neq 0,$ for some $t\}$ is countable [18].

LEMMA 2.1 *Let X be a measurable process, $\alpha_0 \in \mathbf{R}$ and K be a compact subset of $\mathbf{R}$. Assume that*
i) either X is PC,
ii) or X is APC and $\sup_{t\in K}\sum_{\alpha\in F(X,t),\alpha\neq 0} 1/|\alpha|^2 < \infty$,
iii) or X is APC, $\sup_{t\in K}\sum_{\alpha\in F(X,t)} |b_\alpha(t)| < \infty$ and α_0 is not a limit point of $\cup_{t\in K} F(X,t)$,
then for any $\varepsilon < 1$,

$$\lim_{s\to\infty} \sup_{t\in K} s^\varepsilon |\mathrm{E}[\hat{b}_{\alpha_0}(t,s)] - b_{\alpha_0}(t)| = 0.$$

PROOF. The theory of the almost periodic functions [3], implies that

$$|\mathrm{E}[\hat{b}_{\alpha_0}(t,s)] - b_{\alpha_0}(t))| \leq \frac{c}{s}(|tb_{\alpha_0}(t)| + \sum_{\alpha\in F(X,t),\alpha\neq\alpha_0} \frac{|b_\alpha(t)|}{|\alpha-\alpha_0|}),$$

for some $c > 0$. Moreover

$$\sum_{\alpha\in F(X,t)} |b_\alpha(t)|^2 = \lim_{s\to\infty} \frac{1}{s}\int_0^s E[X(u+t)\overline{X(u)}]^2\, du \leq \lim_{s\to\infty}\frac{1}{s}\int_0^s E[X(u)\overline{X(u)}]^2\, du,$$

where the limits exist and are finite since the function $u \to \mathrm{E}[X(u+t)\overline{X(u)}]^2$ almost periodic for any t. Hence we readily deduce the lemma. □

This lemma applies whenever the process is strongly harmonizable since for such a process $\sum_{\alpha\in F(X)} |b_\alpha(t)| \leq |M|(\mathbf{R}\times\mathbf{R})$.

Note that $\sup_{\alpha\in\mathbf{R}} |\mathrm{E}[\hat{b}_\alpha(t,s)] - b_\alpha(t)|$ does not converge towards 0 since for any nonnull APC process the function $\alpha \to b_\alpha(t)$ is not continuous at any point of $F(X,t)$.

2.2 Estimators of the cyclic spectral densities

For the estimation of $f_\alpha(\lambda)$, Hurd [17] introduced the shifted periodogram defined by

$$S_\alpha(\lambda,s) = \frac{1}{2s\pi} I(\lambda,s)\overline{I(\lambda-\alpha,s)} = \frac{1}{2\pi}\int_{-s}^{s} \hat{b}_\alpha(t,s)\,\mathrm{e}^{-it\lambda}\,dt,$$

where $I(\lambda,s) = \int_0^s X(t)\,\mathrm{e}^{-it\lambda}\,dt$.

$S_\alpha(\lambda,s)$ is known to be an asymptotically unbiased but inconsistent estimator of $f_\alpha(\lambda)$ for the periodic situation. A smoothed shifted periodogram can be defined in the following way [17, 19]. Let $\psi : \mathbf{R} \to \mathbf{C}$ be a bounded measurable function, continuous at $t = 0$ with $\psi(0) = 1$ and $\mathrm{supp}(\psi) \subset [-1,1]$ (covariance averaging kernel), and let $(a_s)_{s>0}$ be a nonincreasing family of numbers (window width) such that $s^{-1} < a_s$. Set $\psi_s(t) = \psi(a_s t)$ and

$$\begin{aligned} \hat{f}_\alpha(\lambda,\psi,s) &= \frac{1}{2\pi}\int_{-s}^{s} \hat{b}_\alpha(t,s)\psi(t)\,\mathrm{e}^{-it\lambda}\,dt, \\ f_\alpha(\lambda,\psi) &= \frac{1}{2\pi}\int_{\mathbf{R}} b_\alpha(t)\psi(t)\,\mathrm{e}^{-it\lambda}\,dt. \end{aligned}$$

In the next sections we obtain some sufficient conditions for $\hat{f}_\alpha(\lambda,\psi_s,s)$ is an asymptotically unbiased estimator of f_α. The following result provides a rate of convergence towards 0 of the bias of this estimator of $f_\alpha(\lambda)$ for some PC or APC processes.

LEMMA 2.2 *Let X be a measurable CSM process and $\alpha_0 \in \mathbf{R}$ such that $|b_{\alpha_0}(t)| \leq c\inf\{|t|^{-b}, 1\}$ for some $b > 1$ and $c > 0$, thus $m_{\alpha_0}(d\lambda) = f_{\alpha_0}(\lambda)d\lambda$. Assume that*
i) either X is PC ,
ii) or X is APC and $\sum_{\alpha\in F(X),\alpha\neq 0} 1/|\alpha|^2 < \infty$,
iii) or X is APC , $\sum_{\alpha\in\mathbf{R}} |b_\alpha| \in L^q(\mathbf{R})$ for some q, $1 \leq q \leq \infty$, and α_0 is not a limit point of $F(X)$,
then for $a_s = s^{-\delta}$ with $0 < \delta < 1$, we have

$$\lim_{s\to\infty} \sup_{\lambda\in\mathbf{R}} |\mathrm{E}[\hat{f}_{\alpha_0}(\lambda, \psi_s, s)] - f_{\alpha_0}(\lambda)| = 0.$$

If in addition $\psi(t) = 1 + \mathcal{O}(t^r)$, as $t \to 0$, for some $r > 0$, then for any ε such that $0 \leq \varepsilon < \inf\{1 - \delta, \delta r(b-1)/(b+r)\}$, we have

$$\lim_{s\to\infty} \sup_{\lambda\in\mathbf{R}} s^\varepsilon |\mathrm{E}[\hat{f}_{\alpha_0}(\lambda, \psi_s, s)] - f_{\alpha_0}(\lambda)| = 0.$$

PROOF. From the proof of Lemma 2.1 we have

$$|\mathrm{E}[\hat{f}_{\alpha_0}(\lambda, \psi_s, s)] - f_{\alpha_0}(\lambda)| = |\int \psi_s(t)\mathrm{E}[\hat{b}_{\alpha_0}(t,s)]\, \mathrm{e}^{-it\lambda}dt - \int b_{\alpha_0}(t)\, \mathrm{e}^{-it\lambda}dt|$$

$$\leq c(\int |\psi_s(t) b_{\alpha_0}(t)|\frac{|t|}{s}dt + \frac{1}{s}\int |\psi_s(t)| \sum_{\alpha\in F(X),\alpha\neq\alpha_0} \frac{|b_\alpha(t)|}{|\alpha-\alpha_0|}dt + \int |\psi_s(t) - 1||b_{\alpha_0}(t)|dt).$$

We estimate the three terms in the following ways, c being a constante whose value can change from an expression to another.

$$\int |\psi_s(t) b_{\alpha_0}(t)|\frac{|t|}{s}dt \leq \frac{c}{s}\int_{-a_s^{-1}}^{a_s^{-1}} |t b_{\alpha_0}(t)|\, dt \leq \begin{cases} (c/s)(1 + a_s^{b-2}), & \text{for } b \neq 2, \\ (c/s)(1 + \ln(a_s^{-1})), & \text{for } b = 2. \end{cases}$$

For $1/q + 1/q' = 1$,

$$\begin{aligned} &\frac{1}{s}\int |\psi_s(t)| \sum_{\alpha\in F(X),\alpha\neq\alpha_0} \frac{|b_\alpha(t)|}{|\alpha-\alpha_0|}dt \\ &\leq \frac{a_s^{-1/q'}}{s}(\int_{-1}^{1} |\psi(t)|^{q'}dt)^{1/q'}(\int_{\mathbf{R}}(\sum_{\alpha\in F(X),\alpha\neq\alpha_0} \frac{|b_\alpha(t)|}{|\alpha-\alpha_0|})^q dt)^{1/q}. \end{aligned}$$

As $r > 0$, $b > 1$ and $\varepsilon < \delta r(b-1)/(b+r)$, there exists Δ such that $(\varepsilon + \delta)/(r+1) < \Delta < \delta - \varepsilon/(b-1)$. Set $c_s = s^{-\Delta}$. Then $\Delta < \delta < (r+1)\Delta$, $\varepsilon < (r+1)\Delta - \delta$, $\varepsilon < (\delta - \Delta)(b-1)$, and $a_s^{-1}c_s \geq 1$. Hence

$$\begin{aligned} \int |\psi(a_s t) - 1||b_{\alpha_0}(t)|dt &= a_s^{-1}\int |\psi(t) - 1||b_{\alpha_0}(a_s^{-1}t)|dt \\ &\leq c a_s^{-1}(\int_{|t|\leq c_s} |\psi(t) - 1|dt + \int_{|t|>c_s} |b_{\alpha_0}(a_s^{-1}t)|dt) \\ &\leq c a_s^{-1}(\int_{|t|\leq c_s} |t|^r dt + a_s^b \int_{|t|>c_s} |t|^{-b}dt) \\ &\leq c(a_s^{-1}c_s^{r+1} + a_s^{b-1}c_s^{1-b}), \end{aligned}$$

for s sufficiently large. Thus we can deduce the lemma. □

Note that in the lemma, we can replace conditions (i) and (ii) by
i')either X is PC and $\sum_{\alpha\in\mathbf{R}} |b_\alpha|^k \in L^q(\mathbf{R})$ for some q, $1 \le q \le \infty$ and for $k = 1$ or 2,
ii') or X is APC, $\sum_{\alpha\in\mathbf{R}} |b_\alpha|^2 \in L^q(\mathbf{R})$ for some q, $1 \le q \le \infty$, and $\sum_{\alpha\in F(X),\alpha\neq 0} 1/|\alpha|^2 < \infty$,
and the condition on ε by
$0 \le \varepsilon < \inf\{1 - \delta/q', 1 - \delta(2-b), (\delta - \Delta)(b-1), \Delta(r+1) - \delta\}$, *for some number Δ such that $\Delta < \delta < (r+1)\Delta$.*

We already noted that for a PC or APC process

$$\sum_{\alpha\in F(X,t)} |b_\alpha(t)|^2 \le \lim_{s\to\infty} \frac{1}{s}\int_0^s E[X(u)\overline{X(u)}]^2\, du < \infty.$$

The conditions on the cyclic covariance functions b_α can be interpreted as mixing conditions .

3 Harmonizability hypotheses

For simplicity, in this section we call *S_p-harmonizable process*, $p \ge 1$, any harmonizable process whose spectral stochastic measure μ is such that the product measure $\mu\otimes\overline{\mu}$ exists as an $L^p(\mathrm{P})$-valued measure on $\mathbf{R}^2$ with $\mu\otimes\overline{\mu}(A,B) = \mu(A)\overline{\mu(B)}$ for all A and B in $\mathcal{B}(\mathbf{R})$. For instance, this condition on μ is satisfied whenever μ has a bounded total $L^{2p}(\mathrm{P})$-variation [9, 10].

Consider an S_p-harmonizable measurable process X. Thanks to [4], for all α, λ and $|t| \le s$, the estimators $\hat{b}_\alpha(t,s)$ and $\hat{f}_\alpha(\lambda,\psi_s,s)$ can be expressed in the following ways

$$\hat{b}_\alpha(t,s) = \iint_{\mathbf{R}^2} (N(s(\lambda_1 - \lambda_2 - \alpha)) + \varepsilon(t,s))\, e^{it\lambda_1}\, \mu\otimes\overline{\mu}(d\lambda_1, \lambda_2),$$

$$\begin{aligned} \hat{f}_\alpha(\lambda,\psi_s,s) &= \frac{1}{2\pi}\iint_{\mathbf{R}^2} [N(s(\lambda_1 - \lambda_2 - \alpha))\hat{\psi}_s(\lambda_1 - \lambda) \\ &\qquad + \int_{\mathbf{R}} \varepsilon(t,s)\psi_s(t)\, e^{it(\lambda_1 - \lambda)}\, dt]\, \mu\otimes\overline{\mu}(d\lambda_1, \lambda_2), \end{aligned}$$

where $N(u) = (\,e^{iu} - 1)/iu$ for $u \neq 0$, and 1 for $u = 0$, and where $|\varepsilon(t,s)| \le |t|/s$. Denote by $\hat{L}^1(\mathbf{R})$ the set of the Fourier transform functions of the elements of $L^1(\mathbf{R})$.

3.1 Convergence of the estimators

Consider the process $\hat{b}_\alpha : \mathbf{R} \to L^p(\mathrm{P})$ defined by

$$\hat{b}_\alpha(t) = \iint_{D_\alpha} \mathrm{e}^{it\lambda_1}\ \mu\otimes\overline{\mu}(d\lambda_1, d\lambda_2) = \int_{\mathbf{R}} \mathrm{e}^{it\lambda}\ \nu_\alpha(d\lambda),$$

where ν_α is the stochastic measure defined by $\nu_\alpha(A) = \mu\otimes\overline{\mu}((A \times \mathbf{R}) \cap D_\alpha)$ [4]. Set

$$\hat{f}_\alpha(\lambda, \psi_s) = \frac{1}{2\pi}\int_{\mathbf{R}} \hat{b}_\alpha(t)\psi_s(t)\ \mathrm{e}^{-it\lambda}\ dt = \frac{1}{2\pi}\int_{\mathbf{R}} \hat{\psi}_s(\lambda_1 - \lambda)\ \nu_\alpha(d\lambda_1).$$

Then, we have $\mathrm{E}[\nu_\alpha(A)] = m_\alpha(A)$, $\mathrm{E}[\hat{b}_\alpha(t)] = b_\alpha(t)$ and

$$\mathrm{E}[\hat{f}_\alpha(\lambda, \psi_s)] = \frac{1}{2\pi}\int_{\mathbf{R}} \hat{\psi}_s(\lambda_1 - \lambda)\ m_\alpha(d\lambda_1) = \frac{2}{\pi}\int_{\mathbf{R}} \hat{\psi}_s(\lambda_1 - \lambda) f_\alpha(\lambda_1)\ d\lambda_1,$$

the last equality being valid only if $m_\alpha(d\lambda) = f_\alpha(\lambda)\ d\lambda$. Furthermore, we know that for any CCF process, $\lim_{s\to\infty} \mathrm{E}[\hat{b}_\alpha(t,s)] = b_\alpha(t)$. In the following theorem we precise the asymptotic behavior in $L^p(\mathrm{P})$ of the estimators $\hat{b}_\alpha$ and $\hat{f}_\alpha$.

THEOREM 3.1 *Let X be an S_p-harmonizable measurable process, for some $p \geq 1$, and let $\alpha \in \mathbf{R}$.*

i) For any compact subset K of $\mathbf{R}$, we have

$$\lim_{s\to\infty} \sup_{t\in K} E[|\hat{b}_\alpha(t,s) - \hat{b}_\alpha(t)|^p] = 0.$$

ii) If $\lim_{s\to\infty} s a_s^2 = \infty$ then we have

$$\lim_{s\to\infty} \sup_{\lambda\in\mathbf{R}} \mathrm{E}[|\hat{f}_\alpha(\lambda, \psi_s, s) - \hat{f}_\alpha(\lambda, \psi_s)|^p] = 0, \tag{1}$$

and for any $g \in L^1(\mathbf{R}) \cap \hat{L}^1(\mathbf{R})$,

$$\lim_{s\to\infty} E[|\int_{\mathbf{R}} \hat{f}_\alpha(\lambda, \psi_s, s) g(\lambda)\ d\lambda - \int_{\mathbf{R}} g(\lambda)\ \nu_\alpha(d\lambda)|^p] = 0. \tag{2}$$

Furthermore, if in addition $b_\alpha \in L^1(\mathbf{R})$ then $m_\alpha(d\lambda) = f_\alpha(\lambda)\ d\lambda$, and

$$\lim_{s\to\infty} \mathrm{E}[\hat{f}_\alpha(\lambda, \psi_s, s)] = \lim_{s\to\infty} \mathrm{E}[\hat{f}_\alpha(\lambda, \psi_s)] = f_\alpha(\lambda).$$

PROOF. The error $\hat{\varepsilon}_\alpha(t,s) = \hat{b}_\alpha(s,t) - \hat{b}_\alpha(t)$ is given by the expression

$$\hat{\varepsilon}_\alpha(t,s) = \iint_{\mathbf{R}^2} (N(s(\lambda_1 - \lambda_2 - \alpha)) + \varepsilon(t,s) - 1_{D_\alpha}(\lambda_1, \lambda_2))\ \mathrm{e}^{it\lambda_1}\ \mu\otimes\overline{\mu}(d\lambda_1, \lambda_2).$$

Since $\lim_{s\to\infty} N(s\lambda) = \delta_0(\lambda)$, $|N(s)| \leq 1$ and $|\varepsilon(t,s)| \leq |t|/s$, the dominated convergence theorem for vector valued measures [10] applies, thus for any compact subset K of $\mathbf{R}$, $\lim_{s\to\infty} \sup_{t\in K} \mathrm{E}[|\hat{\varepsilon}_\alpha(t,s)|^p] = 0$.

In the same way, if $\lim_{s\to\infty} sa_s^2 = \infty$, we easily obtain convergence (1), after noting that

$$|\int_{\mathbf{R}} \varepsilon(t,s)\psi_s(t)\, e^{it(\lambda_1-\lambda)}\, dt| \le s^{-1}a_s^{-2} \int_{\mathbf{R}} |\psi(t)|\, dt \to 0 \text{ as } s \to \infty.$$

For any $g \in L^1(\mathbf{R}) \cap \hat{L}^1(\mathbf{R})$, we have

$$\begin{aligned} &2\pi \int_{\mathbf{R}} \hat{f}_\alpha(\lambda, \psi_s, s) g(\lambda)\, d\lambda \\ &= \iint_{\mathbf{R}^2} N(s(\lambda_1 - \lambda_2 - \alpha))A_s(\lambda_1) + B_s(\lambda_1)\ \mu\otimes\overline{\mu}(d\lambda_1, d\lambda_2), \end{aligned}$$

where

$$\begin{aligned} A_s(\lambda_1) &= \iint_{\mathbf{R}^2} \psi_s(t) g(\lambda)\, e^{it(\lambda_1-\lambda)}\, dt\, d\lambda = \int_{\mathbf{R}} \psi_s(t)\hat{g}(-t)\, e^{it\lambda_1}\, dt, \text{ and} \\ B_s(\lambda_1) &= \iint_{\mathbf{R}^2} \varepsilon(t,s)\psi_s(t) g(\lambda)\, e^{it(\lambda_1-\lambda)}\, dt d\lambda \\ &= \int_{\mathbf{R}} \varepsilon(t,s)\psi_s(t)\hat{g}(-t)\, e^{it\lambda_1}\, dt, \end{aligned}$$

$\hat{g}$ denoting the Fourier transform function of g. Since $\lim_{s\to\infty} A_s(\lambda_1) = 2\pi g(\lambda_1)$, $|A_s(\lambda_1)| \le \|\psi\|_\infty \|\hat{g}\|_1 < \infty$, and $|B_s| \le s^{-1}a_s^{-2}\ \|\psi\|_\infty\|\hat{g}\|_1$, the dominated convergence theorem for vector-valued mesures applies, providing convergence (2). Thanks to the classical Lebesgue dominated convergence theorem, whenever in addition $b_\alpha(t) \in L^1(\mathbf{R})$ we obtain that $\hat{f}_\alpha(\lambda, \psi_s, s)$ is an asymptotically unbiased estimator of $f_\alpha(\lambda)$. □

From [25] we can deduce a majorization of $\mathrm{E}[|\hat{\varepsilon}_\alpha(t,s)|^p]$ for $|t| \le s$,

$$\mathrm{E}[|\hat{\varepsilon}_\alpha(t,s)|^p]^{1/p} \le 2(\|\mu\otimes\overline{\mu}\|_p(B(\alpha,l) - D_\alpha) + s^{-1}(|t| + 2l^{-1})\|\mu\otimes\overline{\mu}\|_p(\mathbf{R}^2)),$$

for any α and all $s, l > 0$, where $B(\alpha, l)$ is the band parallel to the line D_α with width $2l$, and where $\|\mu\otimes\overline{\mu}\|_p(A)$ denotes the semi-variation of the L^p-valued measure $\mu\otimes\overline{\mu}$ on the Borel subset A of $\mathbf{R}^2$ [10]. As the behavior of the semi-variation of μ does not seem to be easily obtain in practical situation, we do not go deeper in the study of the rate of convergence following this point of view.

Whenever the spectral stochastic measure μ has a bounded total $L^{2p}(\mathrm{P})$-variation measure m, the majorization formula of the error involves

$$\mathrm{E}[|\hat{\varepsilon}_\alpha(t,s)|^p]^{1/p} \le 2(m\otimes m(B(\alpha,l) - D_\alpha) + s^{-1}(|t| + 2l^{-1})m\otimes m(\mathbf{R}^2)).$$

Hence the rate of convergence of $\mathrm{E}[|\hat{\varepsilon}_\alpha(t,s)|^p]$ towards 0 can be estimated with the repartition of $\mu\otimes\overline{\mu}$ in the neighborhood of the line D_α.

3.2 Consistency of the estimators

Whenever $p = 2$, the variances of $\hat{b}_\alpha(t)$ and $\hat{f}_\alpha(\lambda, \psi)$ satisfy

$$\operatorname{var}[\hat{b}_\alpha(t)] = \iiiint_{D_\alpha \times D_\alpha} e^{it(\lambda_1 - \lambda_1')} M_4(d\lambda_1, d\lambda_2, d\lambda_1', d\lambda_2'), \tag{3}$$

$$\operatorname{var}[\hat{f}_\alpha(\lambda, \psi)] = \frac{1}{4\pi^2} \iiiint_{D_\alpha \times D_\alpha} \hat{\psi}(\lambda_1 - \lambda)\overline{\hat{\psi}(\lambda_1' - \lambda)} M_4(d\lambda_1, d\lambda_2, d\lambda_1', d\lambda_2'), \tag{4}$$

where the bimeasure $M_4 : \mathcal{B}(\mathbf{R}^2) \times \mathcal{B}(\mathbf{R}^2) \to \mathbf{C}$ is defined by

$$M_4(A \times B) = \operatorname{cov}(\mu \otimes \overline{\mu}(A), \mu \otimes \overline{\mu}(B)).$$

Whenever M_4 is not a measure on $\mathbf{R}^4$, we consider the integrals in (3) and (4), as integrals with respect to a bimeasure as defined in [25].

If in addition the process X is Gaussian, we can state sufficient conditions for the random variables $\hat{b}_\alpha(t)$ and $\hat{f}_\alpha(\lambda, \psi)$ are equal to constant values P-almost everywhere.

THEOREM 3.2 *Consider a Gaussian S_2-harmonizable centered measurable process X such that the bimeasure $M^*(A \times B) = E[\mu(A)\mu(B)]$ is extendable as a measure on $\mathbf{R}^2$. Assume that the spectral bimeasure M has no point mass (i.e. $\|\mu(\lambda)\|_2^2 = M(\lambda, \lambda) = 0$ for any λ). Let $\alpha \in \mathbf{R}$. Then, for any compact subset K of $\mathbf{R}$, we have*

$$\lim_{s \to \infty} \sup_{t \in K} \mathrm{E}[|\hat{b}_\alpha(t, s) - b_\alpha(t)|^2] = 0.$$

If $\lim_{s \to \infty} s a_s^2 = \infty$, then for any $g \in L^1(\mathbf{R}) \cap \hat{L}^1(\mathbf{R})$,

$$\lim_{s \to \infty} \int_{\mathbf{R}} \hat{f}_\alpha(\lambda, \psi_s, s) g(\lambda)\, d\lambda = \int_{\mathbf{R}} g(\lambda)\, m_\alpha(d\lambda) \text{ in } L^2(P).$$

Furthermore, if in addition $b_\alpha \in L^1(\mathbf{R})$ then $m_\alpha(d\lambda) = f_\alpha(\lambda)\, d\lambda$ and

$$\lim_{s \to \infty} \sup_{\lambda \in \mathbf{R}} \mathrm{E}[|\hat{f}_\alpha(\lambda, \psi_s, s) - f_\alpha(\lambda)|^2] = 0.$$

PROOF. Since the harmonizable process X is centered and Gaussian, its spectral stochastic measure μ is also centered and Gaussian, and from the fourth moment property for jointly Gaussian complex variables, we have for all Borel subsets A, B, C, D in $\mathbf{R}$

$$M_4(A \times B \times C \times D) = M(A \times C)\overline{M(B \times D)} + M^*(A \times D)\overline{M^*(B \times C)}.$$

The bimeasures M and M^* being extendable as complex measures on $\mathbf{R}^2$, the bimeasure M_4 is extendable as a complex measure on $\mathbf{R}^4$.

This expansion of M_4, relations (3) and (4), and Fubini theorem imply

$$\mathrm{var}[\hat{b}_\alpha(t)] = \iint_{\mathbf{R}^2} e^{it(\lambda-\lambda')}\overline{M(\lambda-\alpha,\lambda'-\alpha)}\, M(d\lambda, d\lambda') + \iint_{\mathbf{R}^2} e^{it(\lambda-\lambda'-\alpha)}\overline{M^*(\lambda-\alpha,\lambda'+\alpha)}\, M^*(d\lambda, d\lambda'), \text{ and}$$

$$4\pi^2\mathrm{var}[\hat{f}_\alpha(\lambda,\psi_s)] = \iint_{\mathbf{R}^2} \hat{\psi}_s(\lambda_1-\lambda)\overline{\hat{\psi}_s(\lambda_1'-\lambda)}\,\overline{M(\lambda_1-\alpha,\lambda_1'-\alpha)}\, M(d\lambda_1, d\lambda_1') + \iint_{\mathbf{R}^2} \hat{\psi}_s(\lambda_1-\lambda)\overline{\hat{\psi}_s(\lambda_1'+\alpha-\lambda)}\,\overline{M^*(\lambda_1-\alpha,\lambda_1'+\alpha)}\, M^*(d\lambda_1, d\lambda_1').$$

Whenever μ has no point mass, then the complex measures M and M^* has no point mass and $\mathrm{var}[\hat{b}_\alpha(t)] = \mathrm{var}[\hat{f}_\alpha(\lambda,\psi_s)] = 0$. Thus $\hat{b}_\alpha(t) = \mathrm{E}[\hat{b}_\alpha(t)] = b_\alpha(t)$ P-a.e., and

$$\hat{f}_\alpha(\lambda,\psi_s) = \mathrm{E}[\hat{f}_\alpha(\lambda,\psi_s)] = \frac{1}{2\pi}\int_{\mathbf{R}} \hat{\psi}_s(\lambda_1-\lambda) f_\alpha(\lambda)\, d\lambda \text{ P-a.e..}$$

From Theorem 3.1 we can easily conclude. □

In fact, if μ has no point mass then for any bounded measurable function $f : \mathbf{R}^2 \to \mathbf{C}$ we have

$$\iint_{D_\alpha} f(\lambda_1,\lambda_2)\, \mu\otimes\overline{\mu}(d\lambda_1, d\lambda_2) = \iint_{D_\alpha} f(\lambda_1,\lambda_2)\, M(d\lambda_1, d\lambda_2) \text{ in } L^2(\mathrm{P}).$$

4 Mixing hypotheses

To avoid a normality condition and to control the fourth moments that appear in the variances of the estimators, we consider the following mixing property.

DEFINITION 4.1 *Let $p \geq 2$ and $\beta : \mathbf{R}^+ \to \mathbf{R}^+$ measurable and such that* $\lim_{t\to\infty}\beta(t) = 0$. *A process $X \to L^2(\mathrm{P})$ is said to be β-L^p-mixing , if*

$$|\mathrm{cov}[f,g]| \leq \beta(t)\mathrm{E}[|f|^p]^{1/p}\mathrm{E}[|g|^p]^{1/p}$$

for any $t > 0$, any $s \in \mathbf{R}$, any $\sigma(X(u), u \leq s)$-measurable $f \in L^p(\mathrm{P})$ and any $\sigma(X(u), u \geq s+t)$-measurable $g \in L^p(\mathrm{P})$.

This mixing notion contains the more classical uniformly mixing and strong mixing ones. Indeed, if a process is uniformly mixing with coefficient function Φ, then it is β-L^p-mixing with $p = 2$ and $\beta = 2\Phi^{1/2}$, and if a process is strongly mixing with coefficient function α, then it is β-L^p-mixing with $p > 2$ and $\beta = 6\alpha^{1/r}$, $r = p/(p-2)$ [15]. However some less restrictive mixing conditions can be assumed as

$$|\mathrm{cov}[f,g]| \leq \sum_n \beta_n(t)\mathrm{E}[|f|^{p_n}]^{1/p_n}\mathrm{E}[|g|^{p_n}]^{1/p_n}, \tag{5}$$

with $2 \leq p_1 < p_2 < \ldots < p_n < \ldots \leq \infty$. Some authors considered such mixing conditions as the "absolute regularity" conditions and the "restricted mixing" conditions (see [1, 22] and references therein). Here for simplicity we only consider β-L^p-mixing processes. The results can easily be reformulated for the processes satisfying mixing condition (5).

4.1 Estimation of the cyclic covariance functions

From now on let $I(s) = \int_0^s \beta(t)\,dt$, $A_p(s) = \sup_{0\leq t\leq s} \mathrm{E}[|X(t)|^p]^{1/p}$, and $B_p(s) = 1/s \int_0^s \mathrm{E}[|X(t)|^p]^{2/p}\,dt$. Remark $B_p(s) \leq A_p(s)$, and for $1 \leq q \leq p$, $A_q(s) \leq A_p(s)$. Then we can state the asymptotic behavior of the bias $\hat{b}_\alpha(t,s) - \mathrm{E}[\hat{b}_\alpha(t,s)]$.

LEMMA 4.2 *Let X be a β-L^p-mixing measurable process, for some $p \geq 2$ and $\beta \in L^1_{loc}(\mathbf{R}^+)$. Assume that $(A_{2p}(s))^4 I(s) \leq c(1+s^\gamma)$ for some $0 \leq \gamma < 1$ and $c > 0$. Then for any ε, $0 \leq 2\varepsilon < 1-\gamma$ and for any compact subset K of $\mathbf{R}$*

$$\lim_{s\to\infty} \sup_{\alpha\in\mathbf{R}} \sup_{t\in K} s^{2\varepsilon}\mathrm{E}[|\hat{b}_\alpha(t,s) - \mathrm{E}[\hat{b}_\alpha(t,s)]|^2] = 0.$$

If in addition $\gamma < 1/2$ and $0 \leq 4\varepsilon < 1-2\gamma$, *then for all* α and t,

$$\lim_{s\to\infty} s^\varepsilon(\hat{b}_\alpha(t,s) - E[\hat{b}_\alpha(t,s)]) = 0 \text{ P-a.e.}$$

PROOF. First, the β-L^p-mixing property implies the inequality

$$|\mathrm{var}[\hat{b}_\alpha(t,s)]| \leq 2s^{-1}(A_{2p}(s))^4(2|t| + I(s)). \tag{6}$$

In [19] this inequality is proved in the particular case of uniformly mixing and apc processes with bounded fourth moments, and the proof can easily be fitted to our hypotheses.
The quadratic mean convergence follows directly.
Now we prove the almost everywhere convergence. If $4\varepsilon < 1-2\gamma$, then $\gamma + 2\varepsilon < 1-\gamma-2\varepsilon \leq 1$, and there exists $a > 1$ such that $a(2\varepsilon+\gamma) < 1 < a(1-2\varepsilon-\gamma)$. For the obtention of the almost everywhere convergence, Borel Cantelli lemma will be applied, thus we shall prove that any $\eta > 0$, the series $\sum_{n\in\mathbf{N}} \mathrm{P}[n^{a\varepsilon}|Z_\alpha(t,n^a)|] \geq \eta]$ and $\sum_{n\in\mathbf{N}} \mathrm{P}[\sup\{s^\varepsilon|Z_\alpha(t,s) - Z_\alpha(t,n^a)|; n^a \leq s < (n+1)^a\} \geq \eta]$ converge uniformly with respect to $\alpha \in \mathbf{R}$ and to t in each compact subset of $\mathbf{R}$, where $Z_\alpha(t,s) = \hat{b}_\alpha(t,s) - E[\hat{b}_\alpha(t,s)]$.

Indeed, from Tchebychev inequality and inequality (6), we have for some $c > 0$,

$$\mathrm{P}[n^{a\varepsilon}|Z_\alpha(t,n^a)|] \geq \eta] \leq c\eta^{-2}n^{a(-1+\gamma+2\varepsilon)}.$$

For P-almost every ω, if $n^a \leq s < (n+1)^a$ and, for example, $0 < t \leq n^a$, then we have

$$
\begin{aligned}
|Z_\alpha(t,s,\omega) - Z_\alpha(t,n^a,\omega)| \;\leq\; & \frac{(n+1)^a - n^a}{n^{2a}} \int_0^{n^a-t} |Y(u+t,u,\omega)|\, du \\
& + \frac{1}{(n+1)^a} \int_{n^a-t}^{(n+1)^a-t} |Y(u+t,u,\omega)|\, du,
\end{aligned}
$$

where $Y(u,v) = X(u)\overline{X(v)} - E[X(u)\overline{X(v)}]$. In the same way, for the other values of t we can get similar inequalities. Hence, we deduce that for some $c > 0$,

$$
\begin{aligned}
& \mathrm{P}[\sup_{n^a \leq s < (n+1)^a} s^\varepsilon |Z_\alpha(t,s) - Z_\alpha(t,n^a)| \geq \eta] \\
& \qquad \leq c\eta^{-2}(n+1)^{2a\varepsilon}((1+\frac{1}{n})^a - 1)^2 (A_4((n+1)^a))^4.
\end{aligned}
$$

The convergence of the series follows from the choice of a, and we can readily complete the proof. □

The previous lemma provides the consistency of the estimators of $b_\alpha(t)$ for CCF processes. Thanks to Lemma 2.1 we can also deduce the rate of convergence for some PC or APC processes.

THEOREM 4.3 *Let X be a β-L^p-mixing measurable CCF process for some $p \geq 2$ and $\beta \in L^1_{loc}(\mathbf{R}^+)$, and let $\alpha \in \mathbf{R}$. Assume that $(A_{2p}(s))^4 I(s) \leq c(1+s^\gamma)$ for some $c > 0$. Then, we have the following convergence in $L^2(\mathrm{P})$ if $0 \leq \gamma < 1$, and P-a.e. if $0 \leq 2\gamma < 1$,*

$$\lim_{s\to\infty} \hat{b}_\alpha(t,s) = b_\alpha(t).$$

In addition, assume that X, α_0 and K satisfy the hypotheses of Lemma 2.1. Then we have the following convergence in $L^2(\mathrm{P})$ if $0 \leq 2\varepsilon < 1-\gamma$,

$$\lim_{s\to\infty} \sup_{t\in K} s^\varepsilon |\hat{b}_{\alpha_0}(t,s) - b_{\alpha_0}(t)| = 0,$$

and the following convergence P-a.e. for any $t \in K$ if $0 \leq 4\varepsilon < 1 - 2\gamma$,

$$\lim_{s\to\infty} s^\varepsilon |\hat{b}_{\alpha_0}(t,s) - b_{\alpha_0}(t)| = 0.$$

4.2 Estimation of the cyclic spectral densities

The β-L^p-mixing property provides the existence of the cyclic spectral densities, and that $\hat{f}_\alpha(\lambda,\psi_s,s)$ is an asymptotically unbiased of estimator of $f_\alpha(\lambda)$ [5].

LEMMA 4.4 *Let X be a β-L^p-mixing CSM process, for some $p \geq 2$ and $\beta \in L^1(\mathbf{R}^+)$, and such that $B_p(s) \leq c$ for some $c > 0$. Then, for any α, the cyclic covariance function b_α belongs to $L^1(\mathbf{R})$, and the cyclic spectral density f_α exists, $m_\alpha(d\lambda) = f_\alpha(\lambda)d\lambda$. Moreover,*

$$\lim_{s\to\infty} \sup_{\lambda\in\mathbf{R}} |E[\hat{f}_\alpha(\lambda, \psi_s, s)] - f_\alpha(\lambda)| = 0.$$

PROOF. The mixing property and Schwarz inequality imply that for all $s > 0$, and $t \neq 0$,

$$\int_0^s |K(u+t, u)|\, du \leq \beta(|t|)(s(s+|t|)B_p(s)B_p(s+t))^{1/2}. \tag{7}$$

Hence $|b_\alpha(t)| \leq c\beta(|t|)$ for some $c > 0$, and $b_\alpha \in L^1(\mathbf{R})$. The spectral density function f_α exists and is the inverse Fourier transform of b_α.

Furthermore, from inequality (7) we deduce that $|\mathrm{E}[\hat{b}_\alpha(t,s)]| \leq \beta(|t|)B_p(s)$, and thanks to Lebesgue dominated convergence theorem we can easily prove that the estimator $\hat{f}_\alpha(\lambda, \psi_s, s)$ is asymptotically unbiased uniformly with respect to $\lambda \in \mathbf{R}$. □

LEMMA 4.5 *Let X be a β-L^p-mixing measurable process, for some $p \geq 2$ and $\beta \in L^1_{loc}(\mathbf{R}^+)$ such that $A_{2p}(s) \leq c(1+s^{\gamma_1})$, $I(s) \leq c(1+s^{\gamma_2})$ with $0 \leq \gamma_1, \gamma_2$* and *$4\gamma_1 + 3\gamma_2 < 1$, $c > 0$. Assume that $a_s = s^{-\delta}$, with $0 < \delta$, $\gamma_2 \leq \delta < (1-4\gamma_1)/3$. Then for any ε, such that $0 \leq 2\varepsilon < 1 - 4\gamma_1 - 3\delta$,*

$$\lim_{s\to\infty} \sup_{\alpha\in\mathbf{R}} \sup_{\lambda\in\mathbf{R}} s^{2\varepsilon}\mathrm{E}[|\hat{f}_\alpha(\lambda, \psi_s, s) - \mathrm{E}[\hat{f}_\alpha(\lambda, \psi_s, s)]|^2] = 0.$$

In addition, assume that $8\gamma_1 + 5\gamma_2 < 1$, $\gamma_2 \leq \delta < (1-8\gamma_1)/5$, and the function ψ is even and nonincreasing on $\mathbf{R}^+$. Then for any ε such that $0 \leq 4\varepsilon < 1 - 8\gamma_1 - 5\delta$, and for any $\lambda \in \mathbf{R}$ we have

$$\lim_{s\to\infty} s^\varepsilon(\hat{f}_\alpha(\lambda, \psi_s, s) - E[\hat{f}_\alpha(\lambda, \psi_s, s)]) = 0 \quad \mathrm{P-a.e.}$$

PROOF. The convergence in quadratic mean is a consequence of the following inequality (8) which is implied by inequality (6)

$$\begin{aligned} \mathrm{var}[\hat{f}_\alpha(\lambda, \psi_s, s)] &\leq \frac{(A_{2p}(s))^4}{2\pi^2 s a_s^3}\left(\int_{\mathbf{R}} \psi(t)(2|t| + a_s I(s))^{1/2}\, dt\right)^2 \\ &\leq c s^{-1+4\gamma_1+3\delta}(1 + s^{-\delta+\gamma_2}). \end{aligned} \tag{8}$$

Assume that ψ is even and nonincreasing on $\mathbf{R}^+$. If $8\gamma_1 + 5\delta + 4\varepsilon < 1$ then $4\gamma_1 + 2\delta + 2\varepsilon < 1 - 4\gamma_1 - 3\delta - 2\varepsilon \leq 1$, and there exists $a > 1$ such that $a(4\gamma_1 + 2\delta + 2\varepsilon) < 1 < a(1 - 4\gamma_1 - 3\delta - 2\varepsilon)$. As in the proof of the almost everywhere convergence in Lemma 4.2, Borel Cantelli lemma will be applied and we shall prove that for any $\eta > 0$ that the series $\sum_{n\in\mathbf{N}} \mathrm{P}[n^{a\varepsilon}|F_\alpha(\lambda, n^a)| \geq \eta]$ and $\sum_{n\in\mathbf{N}} \mathrm{P}[\sup\{s^\varepsilon|F_\alpha(\lambda, s) - F_\alpha(\lambda, n^a)|;\ n^a \leq s < (n+1)^a\} \geq \eta]$ converge uniformly with respect to α and λ in $\mathbf{R}$, where $F_\alpha(\lambda, s) =$

$\hat{f}_\alpha(\lambda,\psi_s,s) - E[\hat{f}_\alpha(\lambda,\psi_s,s)]$. Indeed, from Tchebychev inequality and inequality (8), for some $c > 0$

$$P[n^{a\epsilon}|F_\alpha(\lambda,n^a)| \geq \eta] \;\leq\; c\eta^{-2}n^{a(-1+4\gamma_1+3\delta+2\epsilon)}.$$

For P-almost every ω if $n^a \leq s < (n+1)^a$, we have

$$\begin{aligned}
&2\pi|F_\alpha(\lambda,s,\omega) - F_\alpha(\lambda,n^a,\omega)| \\
&\quad\leq \int_{\mathbf{R}} |\psi(a_s t) - \psi(n^{-a\delta}t)||Z_\alpha(t,n^a,\omega)|\,dt \\
&\qquad + \int_{\mathbf{R}} \psi(a_s t)|Z_\alpha(t,s,\omega) - Z_\alpha(t,n^a,\omega)|\,dt.
\end{aligned}$$

From the behavior of ψ, we deduce that

$$\begin{aligned}
&P[\sup_{n^a\leq s<(n+1)^a} s^\epsilon \int_{\mathbf{R}} |\psi(a_s t) - \psi(n^{-a\delta}t)||Z_\alpha(t,n^a)|\,dt > \eta] \\
&\leq c\eta^{-2}(n+1)^{2a\epsilon}(A_4((n+1)^a))^4\Big(\int_{\mathbf{R}} \psi((n+1)^{-a\delta}t) - \psi(n^{-a\delta}t)\,dt\Big)^2 \\
&\leq c\eta^{-2}(n+1)^{2a(\epsilon+2\gamma_1)}((n+1)^{a\delta} - n^{a\delta})^2\Big(\int_{\mathbf{R}} \psi(t)\,dt\Big)^2.
\end{aligned}$$

Moreover, for P-almost every ω,

$$\begin{aligned}
&\int_{\mathbf{R}+} \psi(a_s t)|Z_\alpha(t,s,\omega) - Z_\alpha(t,n^a,\omega)|\,dt \\
&\leq \int_0^{n^a} \psi((n+1)^{-a\delta}t)(n^{-a} - (n+1)^{-a}) \int_0^{n^a-t} |Y(u+t,u,\omega)|\,du\,dt \\
&\quad + \int_0^{(n+1)^a} \psi((n+1)^{-a\delta}t)n^{-a} \int_{n^a-t}^{(n+1)^a-t} |Y(u+t,u,\omega)|\,du\,dt,
\end{aligned}$$

Hence we get for some $c > 0$,

$$\begin{aligned}
&P[\sup_{n^a\leq s<(n+1)^a} s^\epsilon \int_{\mathbf{R}+} \psi(a_s t)|Z_\alpha(t,s) - Z_\alpha(t,n^a)|\,dt > \eta] \\
&\leq c\eta^{-2}(n+1)^{2a\epsilon}((n+1)^a - n^a)^2 n^{-2a}(A_4((n+1)^a)^4 \\
&\leq c\eta^{-2}n^{2a(\epsilon+\delta+2\gamma_1)-2}.
\end{aligned}$$

We can complete the proof from the choice of a. □

Finally, we deduce the consistency of $\hat{f}_\alpha(\lambda,\psi_s,s)$, and thanks to Lemma 2.2 we obtain the rate of convergence for some PC or APC processes.

THEOREM 4.6 *Let X be a β-L^p-mixing measurable CSM process, for some $p \geq 2$ and $\beta \in L^1(\mathbf{R})$ such that $A_{2p}(s) \leq c(1+s^\gamma)$ with $0 \leq \gamma$* and *$0 < c$. Assume that $a_s = s^{-\delta}$ for some $0 < \delta < 1$, and that the function ψ is even and nonincreasing on $\mathbf{R}^+$. Then for any*

α, the cyclic spectral density f_α exists and we have the following convergence in $L^2(\mathrm{P})$ if $0 < 3\delta < 1 - 4\gamma$, and P*-a.e. if $0 < 5\delta \leq 1 - 8\gamma$*

$$\lim_{s\to\infty} \hat{f}_\alpha(\lambda, \psi_s, s) = f_\alpha(\lambda).$$

In addition, assume that X, α_0 and b satisfy the hypotheses of Lemma 2.2, and $\psi(t) = 1 + O(t^r)$ as $t \to 0$ for some $r > 0$. Then we have the following convergence in $L^2(\mathrm{P})$ if $0 \leq \varepsilon < \inf\{(1 - 4\gamma - 3\delta)/2, 1 - \delta, \delta r(b-1)/(b+r)\}$,

$$\lim_{s\to\infty} \sup_{\lambda\in\mathbf{R}} s^\varepsilon |\hat{f}_{\alpha_0}(\lambda, \psi_s, s) - f_{\alpha_0}(\lambda)| = 0,$$

and the following convergence P*-a.e. for any $\alpha \in \mathbf{R}$ if $0 \leq \varepsilon < \inf\{(1 - 8\gamma - 5\delta)/4, 1 - \delta, \delta r(b-1)/(b+r)\}$,*

$$\lim_{s\to\infty} s^\varepsilon (\hat{f}_{\alpha_0}(\lambda, \psi_s, s) - f_{\alpha_0}(\lambda)) = 0.$$

References

[1] R. C. Bradley and M. Peligrad (1986), *Invariance principle under a two part mixing assumption*, Stochastic Process. Appl. 22, 271-289.

[2] R. A. Boyles and W. A. Gardner (1983), *Cycloergodic properties of discrete-parameter nonstationary stochastic processes*, IEEE Transactions on Information Theory IT-29 (1), 105-114.

[3] C. Corduneanu (1961), *Almost periodic functions*, Wiley (New York).

[4] D. Dehay (1991), *On the product of two harmonizable processes*, Stochastic Process. Appl. 39, 347-358.

[5] D. Dehay (1992), *Estimation de paramètres fonctionnels spectraux de certains processus non-nécessairement stationnaires*, Comptes Rendus de l'Académie des Sciences de Paris, 314 (4), 313-316.

[6] D. Dehay (1994), *Spectral analysis of the covariance of the almost periodically correlated processes*, to appear in Stochastic Process. Appl.

[7] D. Dehay and A. Loughani (1994), *Locally harmonizable covariances: spectral analysis*, to appear in Kybernetika.

[8] D. Dehay and R. Moché (1992), *Trace measures of a positive definite bimeasure*, J. Multivariate Anal. 40, 115-131.

[9] R. M. Dudley and L. Pakula (1972), *A counter example of the inner product of measures*, Indiana Univ. Math. J. 21, 843-845.

[10] N. Dunford and J. T. Schwartz (1957), *Linear operators, parts I and II: general theory*, Interscience Pub. (New York).

[11] W. A. Gardner (1985), *Introduction to random processes with applications to signals and systems*, Macmillan (New York), 2nd ed. 1989 McGraw-Hill.

[12] W. A. Gardner (1988), *Correlation estimation and time series modeling for nonstationary processes*, Signal Processing 15, 31-41.

[13] W. A. Gardner (1994), *Cyclostationarity in communications and signal processing*, IEEE Press (New York).

[14] E. Gladyshev (1963), *Periodically and almost periodically correlated random processes with continuous time parameter*, Th. Probability Appl. 8, 173-177.

[15] C. Hipp (1979), *Convergence rates of the strong law for stationary mixing sequences*, Z. Wahrscheinlichkeitstheorie verw. Gebiete 49, 49-62.

[16] J. E. Huneycutt (1972), *Products and convolutions of vector valued set functions*, Studia Math. 41, 119-129.

[17] H. L. Hurd (1989), *Nonparametric time series analysis for periodically correlated processes*, IEEE Transactions on Information Theory IT-35 (2), 350-359.

[18] H. L. Hurd (1991), *Correlation theory for the almost periodically correlated processes with continuous time parameter*, J. Multivariate Anal. 37 (1), 24-45.

[19] H. L. Hurd and Leskow, J. (1992), *Estimation of the Fourier coefficient functions and their spectral densities for ϕ-mixing almost periodically correlated processes*, Statistics and Probability Letters 14 (4), 299-306.

[20] H. L. Hurd and J. Leskow (1992), *Strongly consistent and asymptotically normal estimation of the covariance for almost periodically correlated processes*, Statistics and Decisions 10, 201-225.

[21] J. Leskow (1992), *An asymptotic normality of the spectral density estimators for almost periodically correlated stochatic processes*, preprint.

[22] M. Peligrad (1992), *On the central limit theorem for weakly dependent sequences with a decomposed strong mixing coefficient*, Stochastic Process. Appl. 42, 181-193.

[23] R. S. Phillips (1950), *On Fourier Stieltjes integrals*, Trans. Amer. Math. Soc. 69, 312-323.

[24] M. M. Rao (1985), *Harmonizable, Cramér, and Karhunen classes of processes*, Handbook of Statistics 5, 279-310, Elsevier Science Publ.

[25] Yu. A. Rozanov (1959), *Spectral analysis of abstract function*, Th. Probability Appl. 4, 271-287.

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mercredi 17 février 1994