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Publications de l'Institut de recherche mathématiques de Rennes, 1992-1993, fascicule 1
« Fascicule d'équations aux dérivées partielles », , exp. nº 6, p. 1-11
<http://www.numdam.org/item?id=PSMIR_1992-1993___1_A6_0>

BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER OPERATORS IN LIPSCHITZ AND C^1 DOMAINS

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§0 INTRODUCTION.

In this article we discuss some recent progress in the theory of higher order homogeneous elliptic operators. These operators have the general form $L = \sum a_\alpha D^\alpha$, where α is a multi-index. Ellipticity, or strong ellipticity, for L is the requirement that there exists a constant C such that

$$\text{for all } \xi = (\xi_1, \ldots \xi_n) \text{ in } \mathbf{R}^n, \quad C^{-1}|\xi|^{2m} \geq \sum_{|\alpha|=2m} a_\alpha \xi^\alpha \geq C|\xi|^{2m},$$

where α is a multi-index, and m is an integer. The coefficients a_α are assumed to be real and this, together with the ellipticity condition, forces the order $(2m)$ of the operator to be even. If $m \geq 2$ the operator is said to be of higher order.

The behavior of solutions to higher order operators is vastly different from that of solutions to second order operators, even in smooth domains. Solutions need not satisfy a Harnack inequality or a maximum principle; the Green's function need not be of one sign and the fundamental solution may even change sign, all unlike the second order situation. Indeed, the property of unique continuation for such an operator may fail. In 1961, Plis [Pl] constructed an example of a 4^{th} order homogeneous elliptic operator with smooth coefficients (and constant coefficients outside the unit ball) which has a nontrivial solution supported in the unit ball. Thus, as we are interested in the unique solvability of the problem $Lu = 0$ in a domain Ω with Dirichlet conditions on the boundary of Ω, we shall henceforth assume that the coefficients a_α of L are constant. This guarantees that unique continuation holds, but yet gives rise to a theory which is much different from the second order one, exhibiting still all the aforementioned pathology of solutions and of Green's functions.

Such operators arise naturally in physical problems, for instance in the theory of elastostatics. One well known problem involving the biharmonic operator is the clamped plate problem: to solve $\Delta^2 u = f$ in Ω with zero Dirichlet conditions on $\partial\Omega$. (We shall be more specific about the boundary conditions later on.) The function f represents the force acting on a clamped plate and the solution u is the displacement of that plate. Hadamard conjectured that positive f should give rise to positive u. Physically this means that the displacement should take place in one

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Supported in part by an A. P. Sloan Foundation Fellowship and the NSF.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

direction if the force acts in one direction. And it would mean that the Green's function for Δ^2 in Ω would be of one sign. This is true if Ω is a ball. But Duffin [Du] showed that the Green's function for Δ^2 will change sign in an infinite strip and, later, Garabedian [G] showed that a sign change occurs if the domain Ω is a sufficiently eccentric ellipse. Near the vertex of some infinite cones, the Green's function may even change sign infinitely often [O].

We turn now to a discussion of boundary value problems associated with solving $Lu = 0$ where L has constant coefficients. In the late 50's and early 60's a rather complete theory was developed by Agmon, Douglis and Nirenberg and by Browder in [ADN1], [ADN2] and [B] for the upper half space and for domains with smooth boundary, and for very general boundary conditions. On the the upper half space, in [ADN1], explicit Poisson kernels are constructed, L^p estimates up to the boundary and and extensions of the maximum principle are proven, and interior extimates and Schauder estimates are obtained. The techniques and results of the work cited above (see also [A]) lead to solvability of the Dirichlet problem, in the sense of nontangential estimates, when the domain is sufficiently smooth (and the smoothness depends on the order of the operator).

When the domain fails to be smooth, these boundary value problems have been less well understood. Recently, G. Verchota and I have shown ([PV5]) that, in Lipschitz domains in $\mathbf{R}^n$, the Dirichlet and regularity problems with data in L^p, for p near 2, are uniquely solvable with appropriate nontangential estimates, for all higher order operators which are constant coefficient homogeneous and elliptic (CCHE). The main goals of the remainder of this article are to explain the formulation of this problem in non-smooth domains, give the background and the precursors of this result, to describe the difficulty that arises in the higher order case and to sketch the argument that overcomes this difficulty. Briefly, the problem consists of finding the appropriate substitute for the Rellich identity (see [JK1]) which, in the second order case, allows one to control all derivatives of a solution on the boundary by a conormal derivative.

Acknowledgements. The work described here is joint work with G. Verchota and I am grateful for this fruitful collaboration over the past several years. This article is a synopsis of a talk given at the Fourier Analysis and PDE conference held at Miraflores de la Sierra, June 1992, and I am grateful to the organizers for having had the opportunity to participate in this conference and to be a part of this proceedings.

§1 The Dirichlet Problem on non-smooth domains.

If $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$ is a Lipschitz function then $D = \{(x,y) \in \mathbf{R}^{n-1} \times \mathbf{R} : y > \varphi(x)\}$ is an infinite Lipschitz domain in $\mathbf{R}^n$. If φ is C^1, then D is called a C^1 domain. A bounded domain $\Omega \subseteq \mathbf{R}^n$ is Lipschitz if the boundary of D is given, locally and uniformly, by the graph of a Lipschitz function. (For a more precise definition see [JK2].) Alternatively, such a domain satisfies a uniform interior and exterior cone condition. Thus there exists a family of truncated cones $\{\Gamma(Q) : Q \in \partial\Omega\}$ such that $\Gamma(Q)$ is compactly contained in Ω, and these truncated cones are the appropriate nontangential approach regions to a point on the boundary of the domain. For a function v defined in Ω the nontangential maximal function of v is $v^*(Q) = \sup\{v(X) : X \in \Gamma(Q)\}$. The normal vector $N(Q)$ to $Q \in \partial\Omega$ exists almost everywhere. A function v belongs to L_1^p if it has tangential derivatives in $L^p(\partial\Omega, d\sigma)$. Above a graph, $\{y > \varphi(x)\}$, this simply means that $\nabla_x v(x, \varphi(x))$ belongs to $L^p(dx, \mathbf{R}^{n-1})$ and there is a natural localization of this definition to bounded domains. (See [DK], for example.)

The Dirichlet problem in L^p for Laplace's equation in a Lipschitz domain is the problem of solving $\Delta u = 0$ in Ω, $u|_{\partial\Omega} = f \in L^p(d\sigma)$, with the estimate $||u^*||_{L^p(d\sigma)} \leq C||f||_{L^p(d\sigma)}$. Dahlberg([D1]) showed that this problem was uniquely solvable if $p > 2 - \epsilon$, for $\epsilon = \epsilon(\Omega)$. Moreover, for any $p < 2$, there exists a domain, depending on this p, on which this fails to be uniquely solvable. (Note that the range of solvability $2 < p < \infty$ follows from the $p = 2$ case and the maximum principle by interpolation.) The regularity problem for Laplace's equation is that of solving $\Delta u = 0$ in Ω, $u|_{\partial\Omega} = f \in L_1^p(d\sigma)$ with the estimate $||(\nabla u)^*||_{L_1^p(d\sigma)} \leq C||f||_{L_1^p(d\sigma)}$. Jerison and Kenig ([JK1]) solved this problem for $p = 2$, then Verchota [V1] solved this problem for $1 < p < 2$, by the method of layer potentials, and again this range of $p, 1 < p < 2 + \epsilon$, is sharp.

The formulation of the Dirichlet problem with data in L^p (D_p) for a 4^{th} order CCHE operator is straightforward. We need to specify two pieces of boundary data. The problem is to solve

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = f \in L_1^p(d\sigma) & \text{on } \partial\Omega \\ \partial u/\partial N = g \in L^p(d\sigma) & \text{on } \partial\Omega, \end{cases}$$

with the estimate

$$||(\nabla u)^*||_{L^p(\partial\Omega)} \leq C\{||f||_{L_1^p} + ||g||_{L^p}\}.$$

The constant C should depend only on the Lipschitz character of Ω and the normal derivative $\partial u/\partial N$ is understood in the sense of nontangential limits, viz. $\nabla u(X).N(Q) \to g(Q)$ as $X \to Q, X \in \Gamma(Q)$ for a.e. Q. To formulate the L^p regularity problem (R_p), which involves a condition on two derivatives on the boundary, more care is required since the boundary of our domain is only once differentiable. To specify the boundary conditions for this problem one can stipulate the existence of a $C_0^\infty(\mathbf{R}^n)$ function F such that

$$\begin{cases} Lu = 0 & \text{in } \Omega \\ u = F & \text{on } \partial\Omega, \\ \partial u/\partial N(Q) = \sum_j N^j(Q) D_j F & \text{on } \partial\Omega \end{cases}$$

with the apriori estimates

$$||(\nabla\nabla u)^*||_{L^p(\partial\Omega, d\sigma)} \leq C \sum ||D_j F||_{L_1^p(\partial\Omega, d\sigma)},$$

and where N^j denotes the jth component of the normal vector. The problem also has an intrinsic formulation involving arrays of functions defined on the boundary of the domain satisfying certain compatibility conditions. See [CG1] and [V2]. In terms of these arrays, or by solving the B.V. problem associated with the restriction of such an F and its derivatives to $\partial\Omega$, the problems (D_p) and (R_p) for any $2m$ order operator may be formulated on non-smooth domains so as to give meaning to the data $u, \ldots, \frac{\partial^{m-1}u}{\partial N^{m-1}}$, when restricted to the boundary of Ω.

In 1982, Cohen and Gosselin [CG] solved (D_p) and (R_p), $1 < p < \infty$, for the biharmonic operator on C^1 domains in the plane. In 1984, via a special representation for solutions to Δ^2, Dahlberg, Kenig and Verchota [DKV] solved the problem (D_2) for the biharmonic equation on Lipschitz domains in $\mathbf{R}^n$. Subsequently, Verchota ([V2] and [V3]) was able to generalize this representation to solve (D_p) on C^1 domains in $\mathbf{R}^n$ for any $1 < p < \infty$ and to solve (D_2) and (R_2) for the polyharmonic operators Δ^m in Lipschitz domains. As in the case of the Laplacian, the L^p Dirichlet problems on Lipschitz domains are not uniquely solvable if $p < 2$ ([DKV]). Unlike the second order case, there is no maximum principle (and therefore no automatic solution to the L^p Dirichlet problem for $p = \infty$) and so from solvability of (D_2) one cannot conclude solvability of (D_p) for $p > 2$.

In [PV1], G Verchota and I established that (D_p) was solvable in Lipschitz domains in $\mathbf{R}^3$ if $2 < p < \infty$, but may fail for some $p > 2$ if the dimension is larger than 3. In [PV3], we showed that this positive result is in fact a consequence of a weak maximum principle (the $p = \infty$) case of (D_p) which holds for Δ^2 in dimension 3, but fails in higher dimensions. The positive and the negative results were also extended to include the polyharmonic operators Δ^m, $m \geq 4$, in [PV4]. In certain dimensions, depending on the order of the operator, these counterexamples can be obtained from a construction in [MNP]. Indeed, parallel to the development and progress on general Lipschitz and C^1 domains described above is a series of remarkable papers by Mazya et.al. analyzing the behavior of solutions to Δ^2 (and more general higher order operators and elliptic systems) on conical domains and polyhedra. See, for example, [KoM1], [KoM2], [KoM3], [KrM], [MN], [MNP], [MP], [MNPl] and [MR]. For related work, and additional sources, the following papers are a small, but representative, sample of the available literature: [Da], [Gr], [KO], [Ko1], [Ko2], [S].

We now wish to describe one means of solving the problem (D_2) for Laplace's equation. It will then be apparent how readily it extends to all constant coefficient 2^{nd} order elliptic operators. The heart of this proof, or of any other proof, is a Rellich identity, or boundary Garding inequality. And this is precisely where the difficulty lies in solving (D_2) for higher order operators. For simplicity and convenience, we work above a graph, and we will also ignore the required limiting arguments needed to make this proof rigorous.

To solve $\Delta u = 0$ in Ω with $u|_{\partial\Omega} = f \in L^2(d\sigma)$, we assume f continuous, obtain a solution, and need only derive the apriori estimate $||u^*||_{L^2(\partial\Omega)} \leq C||f||_{L^2(\partial\Omega)}$. We assume that $\partial\Omega = \{(x,y) : y > \varphi(x)\}$. By Green's identity, with $\Gamma(X,Y) = c_n|X-Y|^{2-n}$ the fundamental solution of Δ,

$$\begin{aligned} u(X) &= \iint \Delta_Y \Gamma(X,Y) u(Y) dY \\ &= \int_{\partial\Omega} \frac{\partial\Gamma}{\partial N}(X,Q) f(Q) d\sigma - \int_{\partial\Omega} \Gamma(X,Q) \frac{\partial u}{\partial N}(Q) d\sigma(Q) \\ &= \mathbf{A} + \mathbf{B} \end{aligned}$$

Term A has the desired nontangential estimate in virtue of the theorem of Coifman, McIntosh and Meyer on the Cauchy integral on Lipschitz curves, [CMM]. That is, $||\mathbf{A}^*||_{L^2(d\sigma)} \leq C||f||_{L^2(d\sigma)}$. But term **B** involves an extra derivative on u (and not enough derivatives on Γ). Define a harmonic function v by $u = D_n v$, where $D_n = \partial/\partial y$. Then, if N^j denotes the jth component of the normal vector, on the boundary we have

$$\begin{aligned}\partial u/\partial N &= \sum_j N^j D_j u \\ &= \sum_j N^j D_j D_n v \\ &= \sum_j (N^j D_n - N^n D_j) D_j v\end{aligned}$$

where we have made use of the fact that $\sum_j D_j D_j v = 0$. But $N^j D_n - N^n D_j$ is a tangential derivative, which we now denote T_j. That is, $\partial u/\partial N = \sum_j \frac{\partial}{\partial T_j} D_j v$ and so term **B** becomes, after an integration by parts,

$$\mathbf{B}(X) = \int_{\partial\Omega} \partial\Gamma/\partial T_j(X,Q) D_j v(Q) d\sigma(Q)$$

and again by [CMM],

$$||(\mathbf{B})^*||_{L^2(d\sigma)} \leq C||D_j v||_{L^2(d\sigma)}.$$

To finish the proof, one needs the Riesz transform inequality:

$$\sum_j ||D_j v||_{L^2(\partial\Omega,d\sigma)} \leq C||D_n v||_{L^2(\partial\Omega,d\sigma)} = C||u||_{L^2(\partial\Omega,d\sigma)}$$

and this is the Rellich identity alluded to earlier. The proof is as follows. First, since Ω is Lipschitz, N^n is bounded from below. Thus, dropping the summation, we have,

$$\begin{aligned}\int_{\partial\Omega} |D_j v|^2 d\sigma &\leq c_0 \int_{\partial\Omega} |D_j v|^2 N^n d\sigma \\ &= c_0 \iint_\Omega D_n (D_j v)^2 dX \\ &= 2c_0 \iint_\Omega D_n D_j v D_j v dX \\ &= 2c_0 \iint_\Omega D_j (D_n v D_j v) dX.\end{aligned}$$

The last inequality uses the equation for v. Another integration by parts gives

$$\begin{aligned}\iint_\Omega D_j (D_n v D_j v) dX &= \int_{\partial\Omega} D_n v D_j v d\sigma \\ &\leq \left(\int_{\partial\Omega} |D_n v|^2 d\sigma\right)^{1/2} \left(\int_{\partial\Omega} |D_j v|^2 d\sigma\right)^{1/2},\end{aligned}$$

by Cauchy-Schwarz. Thus we have the inequality $\int |D_j v|^2 d\sigma \leq \int |D_n v|^2 d\sigma$.

The method works just as well for a general constant coefficient 2^{nd} order operator, which we may write as $L = div A\nabla$ where $A = (a_{ij})$ is elliptic, i.e. $A\xi.\xi > C|\xi|^2$. That is, to solve (D_2) for L, one begins by expressing the solution u in terms of a potential involving the fundamental solution of L. An integration by parts yields two boundary integrals, one of which contains the data $u|_{\partial\Omega}$, and is thus readily estimated. Finally, it is only the Rellich identity which is needed to finish the argument. The essential element needed for the Rellich identity is to introduce a form on the boundary which enables one to make use of the equation satisfied by v, where $u = D_n v$. In the second order situation, ellipticity is a very strong condition, for we may apply it to the vector ∇v. Hence

$$(*) \qquad \int_{\partial\Omega} |D_j v|^2 N^n d\sigma \leq C' \int_{\partial\Omega} A\nabla v.\nabla v N^n d\sigma$$

holds because $A\nabla v.\nabla v \geq C|\nabla v|^2$ pointwise. The rest of the argument goes through just as in the case of the Laplacian, for in the solid integral $\iint D_n(A\nabla v.\nabla v)dX$ one will be able to use the equation $Lv = 0$ as before. Now, it is exactly this pointwise estimate, $A\nabla v.\nabla v \geq C|\nabla v|^2$, which has no analog in the case of higher order elliptic equations and the desired version of inequality (*) need not be true. There is a substitute, however, which makes this method work, and, in what follows, we shall describe the method used in [PV5] to obtain these Riesz transform type inequalities and so to solve the Dirichlet and regularity problems for any CCHE operator in such domains.

Briefly, the set-up is as follows. I shall describe only the 4^{th} order case, although the necessary boundary Garding identity is valid in all dimensions. Let $L = \sum_{|\alpha|=4} a_\alpha D^\alpha$ be constant coefficient and let $\Gamma(X,Y)$ denote the fundamental solution, which has size $|X-Y|^{4-n}$ in dimensions $n = 3$ and $n \geq 5$. The solution $u(X)$ is given by

$$u(X) = \iint_\Omega L_Y \Gamma(X,Y) u(Y) dY,$$

and the Dirichlet conditions on the boundary mean that $|\nabla u| \in L^2(\partial\Omega, d\sigma)$. The solid integral gives rise to four boundary integrals, one of which has the form

$$\mathbf{A} = \int_{\partial\Omega} D^2\Gamma(X,Q) Du(Q) d\sigma$$

and D denotes some derivative in Q which is explicit from the integration by parts. We recall now that the desired estimate involves the nontangential maximal function of the gradient of the solution in the fourth order situation. Again by the theory of singular integrals and the theorem of Coifman, McIntosh and Meyer [CMM] we have the estimate $||(\nabla \mathbf{A})^*||_{L^2(\partial\Omega)} \leq C||\nabla u||_{L^2(d\sigma)}$. (Note that it is three derivatives of Γ which satisfies the estimates for which the theory of [CMM] applies.) There are three other boundary integrals involving too few or too many derivatives on u. To handle such terms, we introduce v by setting $u = D_n D_n v$, so that $Lv = 0$. (The number of D_n's introduced here is connected with the order of the operator.) The claim is that the following boundary inequality, the analog of the Riesz transform inequality for solutions of second order operators, is the key element in the proof of the L^2 estimate:

$$(**)\qquad ||\nabla\nabla\nabla v||_{L^2(\partial\Omega,d\sigma)} \leq C||\nabla D_n D_n v||_{L^2(\partial\Omega,d\sigma)}.$$

The expression $|\nabla\nabla\nabla v|^2$ abbreviates the sum over all $j,k,l \leq n$ of $|D_jD_kD_lv|^2$.

Let $w = D_jv$, and consider one of the terms arising in (**). We first want to obtain the inequality

$$||\nabla\nabla w||_{L^2(d\sigma)} \leq C||\nabla D_n w||_{L^2(d\sigma)}.$$

Iteration of this step yields (**). The problem here is the introduction of a bilinear form on the boundary which permits one to make use of the equation satisfied by w (or v) in the solid integral. The substitute is the following Boundary Garding Inequality ([PV5]) stated here in the 4^{th} order case only, but valid as well, with appropriate modifications, for operators of any order.

$$\int_{\partial\Omega} |\nabla\nabla w|^2 d\sigma \leq C\Big(\int_{\partial\Omega} |\nabla D_n w|^2 d\sigma + \sum_{|\alpha|=|\beta|=2,\alpha_n=0=\beta_n} \int_{\partial\Omega} D^\alpha w a_{\alpha\beta} D^\beta w N^n d\sigma\Big)$$

where $Lw = 0$ and $L = \sum_{|\alpha|=|\beta|=2} a_{\alpha\beta} D^\alpha D^\beta$.

Before sketching a proof of this inequality, we describe the new algebraic identities that underlie this in the general situation. The idea is to make use of the Fourier transform by passing from an integral on the boundary of our Lipschitz domain to an integral over $\mathbf{R}^{n-1}$. Toward this end, we define a quadratic form

$$Q(m,\xi,\eta) = \frac{1}{2}\sum_{i,j=1}^{r}\sum_{|\alpha|=m-2} |\xi_i\eta(\alpha+e_j) - \xi_j\eta(\alpha+e_i)|^2$$

where η is complex valued. In applications, r is the dimension $(n-1)$ and $2m$ is the order of the operator. Given a positive definite form on $\mathbf{R}^r$, let us write the constants as $a^{ij}_{\alpha\beta}$; that is, we are assuming the existence of a constant C such that

$$C^{-1}|\xi|^{2r} \geq \sum_{|\alpha|=|\beta|=m-1}\sum_{i,j=1}^{r} \xi_i\xi^\alpha a^{ij}_{\alpha\beta}\xi_j\xi^\beta \geq C|\xi|^{2m}$$

We then claim

(1) $Q(m,\xi,\eta) = 0$ iff there exists a constant $c \in \mathbf{C}$ such that $\eta(\beta) = c\xi^\beta$

(2) There are constants E and E' such that

$$EQ(m,\xi,\eta) + \operatorname{Re}\Big(\sum_{|\alpha|=|\beta|=m-1}\sum_{i,j=1}^{r} \xi_i a^{ij}_{\alpha\beta}\xi_j\eta(\alpha)\overline{\eta(\beta)}\Big) \geq E'|\xi|^2|\eta|^2_{m-1}$$

where we define

$$|\eta|^2_{m-1} = \sum_{|\alpha|=m-1} |\eta(\alpha)|^2.$$

Statement (1) is proved by induction and reduces to knowing when equality holds in the Cauchy-Schwarz inequality. Statement (2) is a quantitative version of (1) combined with the ellipticity condition. See [PV5] for details.

Consider now the Boundary Garding Inequality in the 4^{th} order case. Take $r = n-1$ above. Then, if a_{ijkl} are the coefficients of the positive definite bilinear form associated to L, we have

$$\sum_{l,k=1}^{n} \sum_{i,j=1}^{n-1} \int_{\partial\Omega} D_i D_k w a_{ijkl} D_j D_l w N^n d\sigma$$

N^n

$$= \int_{\partial\Omega} (\frac{\partial}{\partial x_i} D_k w - \frac{\partial \varphi}{\partial x_i} D_n D_k w) a_{ijkl} (\frac{\partial}{\partial x_j} D_l w - \frac{\partial \varphi}{\partial x_j} D_n D_l w) d\sigma,$$

and all terms with a DD_n component are good terms for the purposes of our inequality. It therefore suffices to estimate the integral

N^n

$$\int_{\partial\Omega} \frac{\partial}{\partial x_i} D_k w a_{ijkl} \frac{\partial}{\partial x_j} D_l w d\sigma$$

which, by Plancherel, equals

$$\int_{\xi \in \mathbf{R}^{n-1}} \xi_i \eta(k) a_{ijkl} \xi_j \overline{\eta(l)} d\xi$$

with $\eta(k) = \widehat{D_k w}$. To this integral we add and subtract the quantity

$$\int_{\mathbf{R}^{n-1}} Q d\xi = \int_{\mathbf{R}^{n-1}} |\frac{\partial}{\partial x_i} D_k w - \frac{\partial}{\partial x_k} D_i w|^2 dx$$
$$= \int_{\mathbf{R}^{n-1}} |D_n D_k w \frac{\partial \varphi}{\partial x_i} - \frac{\partial \varphi}{\partial x_k} D_n D_i w|^2 dx$$

which is again a good term for the purposes of our inequality, since it contains terms involving $D_n w$. Hence an application of (2) and Parseval's theorem yields the inequality. We conclude then, with a precise statememt of the main results of [PV5]. (See [V2] for a precise definition of the boundary array space $WA^p_{m-1}(\partial\Omega)$.)

Theorem 1. *Let $\Omega \subset \mathbf{R}^n$ be a bounded Lipschitz domain with nontangential approach regions $\Gamma^\alpha(Q)$ for all $Q \in \partial\Omega$ for α large enough depending on the Lipschitz character of Ω. Let L be a homogeneous real constant coefficient elliptic partial differential operator of order $2m$ in $\mathbf{R}^n$ with ellipticity constant E. Then there is an $\epsilon > 0$ depending on n, on the Lipschitz character of Ω and on E so that if $2-\epsilon < p < 2+\epsilon$, $g \in L^p(\partial\Omega)$ and $\dot{f} \in WA^p_{m-1}(\partial\Omega)$ there is a unique real analytic* *solution u to $Lu = 0$ in Ω so that*

(i) $(\nabla^{m-1}u)^* \in L^p(\partial\Omega)$

(ii) $\lim \partial^{m-1}u(X)/\partial N_Q^{m-1} = g(Q)$ a.e. as $X \to Q, X \in \Gamma^\alpha(Q)$

(iii) $\lim D^\gamma u(X) = f_\gamma(Q)$ a.e. as $X \to Q, X \in \Gamma^\alpha(Q)$ *for* $0 \le |\gamma| \le m-2$

In addition

(iv) $$||(\nabla^{m-1}u)^*||_{L^p(\partial\Omega)} \le ||g||_{L^p(\partial\Omega)} + C \sum_{|\gamma|=m-2} ||\nabla_T f_\gamma||_{L^p(\partial\Omega)}$$

and

(v) *The nontangential limit of $D^\gamma u(X)$ exists a.e. for $|\gamma| = m-1$, so that* $\nabla_T D^\gamma u(X) \to \nabla_T f_\gamma(Q)$ a.e. as $X \to Q, X \in \Gamma^\alpha(Q)$, *for $|\gamma| = m-2$ where C depends only on n, m, E, p and the Lipschitz character of Ω.*

Theorem 2. *With the same hypotheses as Theorem 1 there is an $\epsilon > 0$ depending on n, E and the Lipschitz character of Ω so that if $2-\epsilon < p < 2+\epsilon$ and $\dot{f} \in WA_m^p(\partial\Omega)$ then there is a unique real analytic solution u to $Lu = 0$ in Ω so that*

(i) $(\nabla^m u)^* \in L^p(\partial\Omega)$

(ii) $\lim D^\gamma u(X) = f_\gamma(Q)$ as $X \to Q, X \in \Gamma^\alpha(Q)$ a.e. *for* $0 \le |\gamma| \le m-1$.

In addition

(iii) $$||\nabla^m u)^*|| \le C \sum_{|\gamma|=m-1} ||\nabla_T f_\gamma||_{L^p(partial\Omega)},$$

(iv) $$||\nabla^{m-1}u)^*||_{L^p(\partial\Omega)} \le C \sum_{|\gamma|=m-1} ||f_\gamma||_{L^p(\partial\Omega)}$$

and

(v) *the nontangential limit of $D^\gamma u(X)$ exists a.e. for $|\gamma| = m$ so that* $\lim \nabla_T D^\gamma u(X) = \nabla_T f_\gamma(Q)$ as $X \to Q, X \in \Gamma^\alpha(Q)$ *for $|\gamma| = m-1$ where C depends only on n, m, E, p and the Lipschitz character of Ω.*

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