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by

Masahiro SHIOTA

§1. Introduction

Let $0 \leq r \leq \omega$, and let U, V open semialgebraic subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively. We call a C^r map f from U to V a C^r Nash map if the graph of f is semialgebraic in $\mathbb{R}^n \times \mathbb{R}^m$. We note that the composition of C^r Nash maps between open semialgebraic sets is a C^r Nash map and that the inverse map of a C^r Nash diffeomorphism (a C^0 Nash homeomorphism if $r=0$) is a C^r Nash map. Hence we can define C^r Nash manifolds as follows. An abstract C^r Nash manifold of dimension m is a C^r manifold with a finite system of coordinate neighborhoods $\{\psi_i : U_i \to \mathbb{R}^m\}$ such that for each i and j, $\psi_i(U_i \cap U_j)$ is an open semialgebraic subset of $\mathbb{R}^m$ and the map $\psi_j \circ \psi_i^{-1}$: $\psi_i(U_i \cap U_j) \to \psi_j(U_i \cap U_j)$ is a C^r Nash diffeomorphism (a C^0 Nash homeomorphism if $r=0$). We call such coordinate neighborhoods C^r Nash coordinate neighborhoods. We say that a C^r map from an abstract C^r Nash manifold M of dimension m to another N of dimension n is a C^r Nash map if for each C^r Nash coordinate neighborhoods $\psi_i : U_i \to \mathbb{R}^m$ and $\varphi_j : V_j \to \mathbb{R}^n$ of M and N, respectively, $\psi_i(f^{-1}(V_j) \cap U_i)$ is semialgebraic and open in $\mathbb{R}^m$, and the map $\varphi_j \circ f \circ \psi_i^{-1} : \psi_i(f^{-1}(V_j) \cap U_i) \to \mathbb{R}^n$ is of class C^r Nash. If there exists a C^r Nash embedding of an abstract C^r Nash manifold into a Euclidean space we call the manifold affine.

We note that an abstract C^∞ Nash manifold and a C^∞ Nash map are automatically of class C^ω Nash by Proposition 3.11, Chapter VI in Malgrange [9].

Hence we assume $r \neq \infty$. The purpose of this paper is to study the structures of abstract C^r Nash manifolds. All results in this paper were published already in some journals [18],...,[23]. Therefore we shall give only sketches of proofs for most of the theorems, and for some theorems in §6 we shall give brief proofs using the approximation theorem 2.1. In §2 we shall define a C^r topology on the set of C^r Nash functions on an abstract C^r Nash manifold in the same way as the usual topology on the space $\mathcal{S}$ of rapidly decreasing C^∞ functions, and we shall show an approximation theorem of C^r Nash functions by C^ω Nash functions in the C^r topology, which is a useful tool for the study of Nash manifolds and Nash maps. The case $r = 0$ of the theorem was announced by Efroymson [4] and the proof was completed by Pecker [16]. If the manifolds are compact, then the usual polynomial approximation theorem works well in most cases. But the compact-open or uniform C^r topology is too weak in the noncompact case, and we need our C^r topology and our approximation theorem. §3 treats the case $0 < r < \infty$. We shall show that an abstract C^r Nash manifold is affine and admits a unique affine C^ω Nash manifold structure. But there exists a definite difference between a C^r Nash manifold for $0 < r < \infty$ and an abstract C^ω Nash manifold. We shall find in §4 distinct abstract nonaffine C^ω Nash manifold structures of potency of continuum on any compact or compactifiable C^∞ manifold. §5 proves that we can compactify any C^0 Nash manifold by attaching a boundary, such a compactification is unique, and a compact C^0 Nash manifold possibly with boundary admits uniquely a PL manifold structure. The last statement follows from the theorem of uniqueness of subanalytic triangulation of locally compact subanalytic set [23]. Similar results hold true for affine C^ω Nash manifolds (§6). We can compactify uniquely any affine C^ω Nash manifold by attaching a boundary, and any compact C^∞ manifold possibly with boundary admits uniquely an affine C^ω Nash manifold structure. We shall show also examples of two PL C^0 Nash manifolds which are PL homeomorphic but not C^0 Nash homeomorphic (§5) and examples of two affine C^ω Nash manifolds which are C^ω diffeomorphic but not C^ω Nash diffeomorphic (§6).

§2. Approximation theorem

Let M be an abstract C^r Nash manifold for $r < \infty$. If $r > 0$, then we give naturally the tangent space TM an abstract C^{r-1} Nash manifold structure. Hence we can define a C^{r-1} Nash vector field on M for $r > 0$. Let $N^r(M)$ denote the set of all C^r Nash functions on M. We choose a basis of neighborhood system of the zero function in $N^r(M)$ as follows

$$\mathcal{O}_{v_1,\ldots,v_r;h} = \{f \in N^r(M) : |f(x)| \leq h(x),\ |v_1 f| \leq 1, \ldots, |v_1 \ldots v_r f| \leq 1\}$$

for all C^{r-1} Nash vector fields $v_1,\ldots,v_r$ on M and C^0 Nash functions h. Here, if $r = 0$ then we consider the sets $\{f \in N^0(M) : |f(x)| \leq h(x)\}$. We call the topology on $N^r(M)$ defined by this basis of neighborhood system the C^r topology. We define the C^r topology on the set $N^r(M, \mathbb{R}^n)$ of all C^r Nash maps from M to $\mathbb{R}^n$ in the same way, and we generalize the definition onto the set $N^r(M, M')$ of all C^r Nash maps from M to another abstract C^r Nash manifold M' by embedding M' in some euclidean space , (which is possible by Theorem 3.1). We remark that the definition of the C^r topology on $N^r(M, M')$ does not depend on the choice of embedding of M' in a euclidean space by the inequality of Łojasiewicz [8]. If M and M' are abstract C^ω Nash manifolds, the C^ω topology on the set $N^\omega(M, M')$ of all C^ω Nash maps from M to M' is by definition the projective limit of the topological spaces $N^r(M, M')$, $0 \leq r < \infty$, and the natural maps $N^r(M, M') \to N^{r'}(M, M')$ for $r \geq r'$.

Remark. If M is affine and noncompact, then $N^r(M)$ is not a linear topological space, indeed the multiplication $\mathbb{R} \times N^r(M) \to N^r(M)$ is not contінous.

Remark. If $M = \mathbb{R}^n$ then the family of sets

$$\mathcal{O}'_{r',h} = \{f \in N^r(\mathbb{R}^n) : |D^\alpha f(x)| h(x) \leq 1,\ \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \text{ with } |\alpha| \leq r'\},$$

for all $0 \leq r' \leq r$ with $r' < \infty$ and polynomial functions h on $\mathbb{R}^n$, is a basis of neighborhood system of 0 in $N^r(\mathbb{R}^n)$ by the inequality of Łojasiewicz.

Approximation Theorem 2.1,[20]. Let M be an affine C^ω Nash manifold and let f be a C^r Nash function on M for $0 \leq r < \infty$. Then we can approximate f by a C^ω Nash function in the C^r topology.

Proof. First we reduce the problem to the case $M = \mathbb{R}^n$. We shall use later the same idea as this. So we show the details. Because M is affine, we assume $M \subset \mathbb{R}^n$. Then $\overline{M} - M$ is a closed semialgebraic subset of $\mathbb{R}^n$. Apply Lemma 6 of Mostowski [13] to $\overline{M} - M$. Then we have a C^0 Nash function h on $\mathbb{R}^n$ such that $h^{-1}(0) = \overline{M} - M$ and $h|_{\mathbb{R}^n - (\overline{M} - M)}$ is a C^ω Nash function. Consider the graph of $1/h|_{\mathbb{R}^n - (\overline{M} - M)}$ in place of M. Then we can suppose M is closed in $\mathbb{R}^n$. Next we want to extend f to $\mathbb{R}^n$. Let $\rho : U \to M$ be a C^ω Nash tubular neighborhood of M in $\mathbb{R}^n$, i.e. $\rho : U \to M$ is a tubular neighborhood such that ρ and U are of class C^ω Nash (the existence is easy to see [18]). Let γ be the square of the distance function from M in $\mathbb{R}^n$.

Then, by the inequality of Łojasiewicz, we have a large positive number C and a large integer k such that $U' = \{x \in \mathbb{R}^n : \gamma(x)(C + |x|^{2k}) \leq 1\}$ is contained in U and γ is of class C^ω Nash on U'. Let g be a C^r Nash function on $\mathbb{R}$ such that $g(0) = 1$ and $g(x) = 0$ for $x \geq 1$. Set $\zeta(x) = g(\gamma(x)(C + |x|^{2k}))$ for $x \in U'$ and $= 0$ for $x \notin U'$. Then ζ is a C^r Nash function on $\mathbb{R}^n$, and we can extend $\zeta \cdot f \circ \rho$ to $\mathbb{R}^n$ by setting $= 0$ outside U. Hence we can assume $M = \mathbb{R}^n$.

For the proof we need a C^r Nash partition of unity whose elements satisfy a good estimate and can be approximated by C^ω Nash functions. Let $X \subset \mathbb{R}^n$ be an algebraic set, U a semialgebraic neighborhood of X, Y a C^ω Nash manifold contained in $X - \operatorname{Sing} X$ and of the same dimension as X, V a closed semialgebraic neighborhood of $X - Y$ in $\mathbb{R}^n$, g a C^r Nash function on $\mathbb{R}^n$ r-flat at Y, and W a neighborhood of 0 in $N^r(\mathbb{R}^n - V)$ in the C^r topology. Then we have the following by elementary calculations.

<u>Lemma 2.2</u>. There exists $F \in N^r(\mathbb{R}^n)$ such that $F = 0$ outside U, $F = 1$ in another neighborhood, F can be approximated by a C^ω Nash function on $\mathbb{R}^n$ in the C^r topology and $gF|_{\mathbb{R}^n - V}$ is in W.

By this lemma we obtain a C^r Nash partition of untiy. Let $\{Y_i\}$ be a finite C^ω Nash stratification of $\mathbb{R}^n$, (i.e. each Y_i is a C^ω Nash manifold), for each i let $g_i \in N^r(\mathbb{R}^n)$ be r-flat at Y_i and let W be a neighborhood of 0 in $N^r(\mathbb{R}^n)$ in the C^r topology.

<u>Lemma 2.3</u>. There exist open semialgebraic neighborhoods $V_i' \subset V_i$ of Y_i and $H_i \in N^r(\mathbb{R}^n)$ such that

(2.3.1) $\quad \overline{V_i'} \subset V_i \cup \overline{Y}_i$, (2.3.2) if $V_i \cap V_j \neq \phi$ and $i \neq j$ then $Y_i \subset \overline{Y}_j - Y_j$ or $Y_j \subset \overline{Y}_i - Y_i$,

(2.3.3) $H_i = 1$ on $V_i' - Z_i$ and $= 0$ outside $V_i - Z_i'$

where $Z_i = \bigcup_{Y_j \subset \overline{Y}_i - Y_i} V_j$ and $Z_i' = \bigcup_{Y_j \subset \overline{Y}_i - Y_i} V_j'$,

(2.3.4) H_i can be approximated by C^ω Nash functions on $\mathbb{R}^n$ in the C^r topology,

(2.3.5) $\Sigma H_i = 1$,

(2.3.6) $g_i H_i \in W$.

For any $f \in N^r(\mathbb{R}^n)$ we obtain easily a finite C^ω Nash stratification $\{Y_i\}$ of $\mathbb{R}^n$ such that for each derivative D^α with $|\alpha| \leqq r$ and for each i $D^\alpha f|_{Y_i}$ is of calss C^ω Nash. But in order to apply Lemma 2.2 we need a better stratification.

<u>Lemma 2.4</u>. There exist a finite C^ω Nash stratification $\{Y_i\}$ of $\mathbb{R}^n$ and $f_i \in N^r(\mathbb{R}^n)$ such that f_i can be approximated by C^ω Nash functions on $\mathbb{R}^n$ and $f - f_i$ are r-flat at Y_i.

Now we can prove the theorem. Let W be a neighborhood of 0 in $N^r(\mathbb{R}^n)$ in the C^r topology. Let $\{Y_i\}$ and f_i be the results in Lemma 2.4. Apply Lemma 2.3 to $\{Y_i\}$, $\{f - f_i\}$ and W. Then we obtain $H_i \in N^r(\mathbb{R}^n)$ which satisfy (2.3.4), (2.3.5) and such that $(f - f_i)H_i \in W$. By (2.3.4) and Lemma 2.4 we have $\tilde{H}_i$, $\tilde{f}_i \in N^\omega(\mathbb{R}^n)$ such that $f_i H_i - \tilde{f}_i \tilde{H}_i \in W$. Then we have

$$f - \Sigma \tilde{f}_i \tilde{H}_i = \Sigma(fH_i - f_i H_i) + \Sigma(f_i H_i - \tilde{f}_i \tilde{H}_i) \in 2kW,$$

where k is the number of indexes of $\{Y_i\}$. Hence the theorem is proved.

Remark 2.5, [20]. Consider the following additional condition in Theorem 2.1. Let M_1 be a C^ω Nash manifold contained and closed in M. Assume any C^ω Nash function on M_1 can be extended to M and there exist $h_1, \ldots, h_k \in N^\omega(M)$ such that $h_1^{-1}(0) \cap \ldots \cap h_k^{-1}(0) = M_1$ and $\operatorname{grad} h_1, \ldots, \operatorname{grad} h_k$ span the normal bundle of M_1 in M. Then if the restriction of f to M_1 is of class C^ω Nash then we can choose an approximation $\tilde{f}$ of f so that $\tilde{f} = f$ on M_1.

Proof. Let f_1 be a C^ω Nash extension of $f|_{M_1}$ to M, and consider $f - f_1$ in place of f. Then we can assume $f = 0$ on M_1. Moreover in Lemma 2.4 we can choose $\{Y_i\}$ so that M_1 is a union of some Y_i. Then $f_i = 0$ on Y_i if $Y_i \subset M_1$, and we can choose the C^ω Nash approximations $\tilde{f}_i$ of f_i so that $\tilde{f}_i = 0$ on Y_i if $Y_i \subset M_1$ by the existence of $h_1, \ldots, h_k$. Hence the C^ω Nash approximation $\Sigma \tilde{f}_i \tilde{H}_i$ vanishes on M_1. Thus the remark is proved.

For example, assume M and M_1 are contained and closed in $\mathbb{R}^n$ and M_1 contains only nonsingular points of the Zariski closure of M_1 in $\mathbb{R}^n$. Then M and M_1 satisfy the conditions in Remark 2.5 by the separation theorem of Mostowski [13] and by the extension theorem of Efroymson-Pecker [4]-[16]. We call such M_1 a smooth leaf of an algebraic set. We can generalize the above remark as follows.

Theorem 2.1', [22]. Let M_1 be an affine C^ω Nash manifold contained and closed in another affine M and let f be a C^r Nash function on M for $0 \leq r < \infty$ such that $f|_{M_1}$ is of class C^ω Nash. Then f can be approximated by a C^ω Nash function in the C^r topology which agrees with f on M_1.

Proof. By the above statement it is sufficient to embed M in some euclidean space $\mathbb{R}^m$ so that the image of M is closed in $\mathbb{R}^m$ and the image of M_1 is a smooth leaf of an algebraic set in $\mathbb{R}^m$. Assume M is contained and closed in $\mathbb{R}^n$. Now, by the normalization theorem of Zariski there exists a C^ω Nash map φ from M to some $\mathbb{R}^{n'}$ such that graph φ is a smooth leaf of an algebraic set in $\mathbb{R}^n \times \mathbb{R}^{n'}$ [1], [15]. Hence we only need to extend φ to $\Phi : \mathbb{R}^n \to \mathbb{R}^{n'}$. We can extend φ to $\Phi_1 : \mathbb{R}^n \to \mathbb{R}^{n'}$ as a C^1 Nash map, and we can approximate Φ_1 by a C^ω Nash map Φ_2 (Theorem 2.1). Define $\Theta : \mathbb{R}^n \times \mathbb{R}^{n'} \to \mathbb{R}^{n'}$ by $\Theta(x, y) = \Phi_2(x) - \Phi_1(x)$. Then, by Remark 2.5, there exists a C^ω Nash approximation Θ' of Θ such that $\Theta' = \Theta$ on graph φ. Set $T(x, y) = (x, y + \Theta'(x, y))$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^{n'}$, and choose the above approximations so close that T is a diffeomorphism of $\mathbb{R}^n \times \mathbb{R}^{n'}$ (Lemma 3.4). Let S be the inverse diffeomorphism. Then the map $\Phi : \mathbb{R}^n \to \mathbb{R}^{n'}$, defined by $(x, \Phi(x)) = S(x, \Phi_2(x))$, is a C^ω Nash extension of φ to $\mathbb{R}^n$. Hence we have proved the theorem.

Corollary 2.6, [22]. Let M be a C^ω Nash manifold contained and closed in $\mathbb{R}^n$. Then there exist C^ω Nash functions $f_0, \ldots, f_n$ on $\mathbb{R}^n$ such that $f_0^{-1} \cap \ldots \cap f_n^{-1}(0) = M$ and grad $f_0, \ldots,$ grad f_n span the normal bundle of M in $\mathbb{R}^n$. (We can call M C^ω Nash nonsingular.)

Proof. Let g be the square of the distance function from M, and put $g_0 = g$, $g_i = \frac{\partial g}{\partial x_i}$, $i = 1, \ldots, n$, in a semialgebraic neighborhood of M. Then we can extend $g_0, \ldots, g_n$ to $\mathbb{R}^n$ as C^1 Nash functions in the same way as the above proof of Theorem 2.1. Hence, by Theorem 2.1' we obtain the corollary.

We can generalize Theorem 2.1' to the case of map. The proof proceeds using a C^ω Nash tubular neighborhood in the same way as the proof of Theorem 2.1.

Theorem 2.1''. Let M, M_1 be the same as Theorem 2.1', and let f be a C^r Nash map from M to another affine M_2 for $0 \leqq r < \infty$ such that $f|_{M_1}$ is of class C^ω Nash. Then f can be approximated by a C^ω Nash map from M to M_2 in the C^r topology which agrees with f on M_1.

Problem 2.7. In the proof of Theorem 2.1' we used the fact that any affine C^ω Nash manifold can be embedded in some Euclidean space so that the image is a smooth leaf of an algebraic set. Is it possible to embed so that the image is a nonsingular algebraic set ? The compact case is due to Tognoli, and a C^∞ embedding is shown by Akbulut-King.

Remark 2.8,[22]. Theorem 2.1'' is generalized to the case of cross-section of a C^ω Nash fibre bundle. Theorem 2.1' implies the fact that any C^ω Nash function on M_1, a C^ω Nash manifold contained and closed in M, can be extended to M. The map case of this fact does not hold true, but we can prove the following. Let M, M_1, M_2 be the same as Theorem 2.1''. Then a C^ω Nash map from M_1 to M_2 can be extended to $M \to M_2$ if and only if it is extensible as a C^0 map. Of course a generalization of this to the case of cross-section holds true (Oka's principle).

Problem 2.9. The zero set of a C^r Nash function on a Euclidean space is called a C^r Nash set. A C^ω Nash set is clearly a semialgebraic analytic set. The converse is not correct. But the germ of a semialgebraic analyitc set is a C^ω Nash set germ. Let X be the zero set of an analytic function. If X is semialgebraic, then is it a C^ω Nash set ? In Theorems 2.1', 1'', can we replace M_1 by a C^ω Nash set ?

§3. C^r Nash manifolds for $0 < r < \infty$

Theorem 3.1, [21]. For $r < \infty$, any abstract C^r Nash manifold is affine.

Proof. Let $\{\psi_i : U_i \to \mathbb{R}^m\}$ be a finite system of C^r Nash coordinate neighborhoods of an abstract C^r Nash manifold M. At the beginning of Proof of Theorem 2.1 an affine C^ω Nash manifold in $\mathbb{R}^n$ is modified to be closed in $\mathbb{R}^n$. In the same way, for each i, we obtain a C^r Nash embedding $\varphi_i : U_i \to \mathbb{R}^n$ such that $\varphi_i(U_i)$ is closed in $\mathbb{R}^n$. Then, considering the composition of φ_i with the stereographic projection $\mathbb{R}^n \to S^n \subset \mathbb{R}^{n+1}$ we can assume $\varphi_i(U_i)$ is bounded and $\overline{\varphi_i(U_i)} - \varphi_i(U_i)$ consists of one point, say 0. Set $\Phi_i(x) = \varphi_i(x)$ for $x \in U_i$ and $\Phi_i(x) = 0$ for $x \in M - U_i$. Then Φ_i is a C^0 Nash map. Let $q : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $q(x) = |x|^{2k}x$ for a large integer k. Then $q \circ \Phi_i$ is a C^r Nash map by the inequality of Łojasiewicz. Hence the product $\prod_i q \circ \Phi_i : M \to \prod_i \mathbb{R}^n$ is a C^r Nash embedding.

By this theorem we do not need the terminology "affine" for $r < \infty$. Hence we call an abstract C^r Nash manifold simply a C^r Nash manifold if $r < \infty$.

Theorem 3.2, [20]. Let M be a C^r Nash manifold contained in $\mathbb{R}^n$ for $0 < r < \infty$. Then there exists a C^r Nash embedding of M in $\mathbb{R}^n$ which is chosen arbitrarily close to the identity in the C^r topology and whose image is a C^ω Nash manifold.

Proof. Let $G_{n,m}$ denote the Grassmann manifold of m-linear subspaces in $\mathbb{R}^n$ for $m = \operatorname{codim} M$. Set $E_{n,m} = \{(\lambda, x) \in G_{n,m} \times \mathbb{R}^n : x \in \lambda\}$ and let $\rho : E_{n,m} \to G_{n,m}$ be the projection. Then $\xi = (E_{n,m}, \rho, G_{n,m})$ is an affine C^ω Nash vector bundle (i.e. $E_{n,m}$ and $G_{n,m}$ are affine C^ω Nash manifolds and ρ is a C^ω Nash map) [15]. Let $\pi : M \to G_{n,m}$ denote the C^{r-1} Nash map defined by $\pi(x) =$ the normal vector space of M in $\mathbb{R}^n$ at x; let π' be a close C^r Nash approximation of π in the C^{r-1} topology, which exists because π can be extended to an open semialgebraic neighborhood of M in $\mathbb{R}^n$ as a C^{r-1} map and we can apply Theorem 2.1 to the extension; and let $\pi'^*\xi$ denote the induced bundle. Then $\pi'^*\xi$ is a C^r Nash vector bundle. Let us regard M and $G_{n,m}$ as subsets of $\pi'^*E_{n,m}$ and $E_{n,m}$, respectively, through the zero cross-sections. Define a C^r Nash map $\varphi: \pi'^*E_{n,m} \to \mathbb{R}^n$ by

$$\varphi(x, y, z) = x + z, \ (x, y, z) \in \pi'^*E_{n,m} \subset M \times E_{n,m} \subset M \times G_{n,m} \times \mathbb{R}^n.$$

Then there exists a C^r Nash tubular neighborhood V of M in $\pi'^*E_{n,m}$ such that $\varphi|_V$ is an embedding. Set $W = \varphi(V)$ and $\psi = q \circ \varphi^{-1} : W \to E_{n,m}$ where $q : \pi'^*E_{n,m} \to E_{n,m}$ is the natural bundle map. Then W is an open semialgebraic tubular neighborhood of M in $\mathbb{R}^n$, ψ is transversal to $G_{n,m}$ and $\psi^{-1}(G_{n,m}) = M$. Hence the theorem follows from Theorem 2.1 and the next lemma.

<u>Lemma 3.3</u>. Let M_1, M_2, M_3 be affine C^ω Nash manifolds such that M_2 is contained and closed in M_3, and let $f : M_1 \to M_3$ be a C^r Nash map transversal to M_2 with $0 < r$. Then there eixsts a neighborhood V of f in $N^r(M_1, M_3)$ in the C^r topology such that for any g in V g is transversal to M_2 and there corresponds a C^r Nash diffeomorphism h of M_1 such that $g^{-1}(M_2) = h(f^{-1}(M_2))$. Moreover the correspondence $g \to h$ is continuous, and if $f = g$ then $h = \mathrm{ident}$.

The essence of the proof of this lemma for $0 < r < \infty$ exists in the next lemma. The case $r = \omega$ follows from the case $r < \infty$, Theorem 2.1" and the next lemma.

Lemma 3.4. Let $f : M_1 \to M_2$ be a C^r Nash diffeomorphism of affine C^ω Nash manifolds for $r > 0$. Then if a C^r Nash map $g : M_1 \to M_2$ is close to f in the C^r topology, then g is a diffeomorphism.

Proof. If M_1 is compact, the lemma is trivial. Hence we assume M_1 is noncompact. We can suppose also $r < \infty$. Because $f^{-1} \circ g : M_1 \to M_1$ is close to the identity, we can assume $M_1 = M_2$ and f is the identity map. By Theorem 6.1 there exist a compact affine C^ω Nash manifold L and a C^ω Nash submanifold L' of codimension one such that M_1 is C^ω Nash diffeomorphic to a union of connected components of $L - L'$. Regard M_1 as the union. Then g is extensible to $L \to L$ as a C^r Nash map by setting $g = \text{ident}$ outside M_1, and the extension is close to the identity map. Therefore the problem is reduced to the compact case, whence we prove the lemma.

Remark 3.5. (3.5.1) In Lemmas 3.3,4 the manifolds may be C^r Nash manifolds by Theorem 3.2. The direct proof in this case requires complicated arguments.

(3.5.2) If we consider the lemmas in the usual category of C^∞ manifolds we need the Whitney topology. The Whitney topology is much stronger than our topology, but our topology is sufficient for our purpose.

(3.5.3) Lemma 3.4 with Theorem 2.1 says that two affine C^ω Nash manifolds C^r Nash diffeomorphic, $0 < r$, are C^ω Nash diffeomorphic. Hence, by Theorem 3.2, any C^r Nash manifold, $0 < r < \infty$, admits uniquely an affine C^ω Nash manifold structure.

(3.5.4) Theorem 3.2 holds true for a pair of C^r Nash manifolds, to be precise, let $M' \subset M$ be C^r Nash manifolds contained in $\mathbb{R}^n$ for $0 < r < \infty$. Then there exists a C^r Nash embedding of M in $\mathbb{R}^n$ which is close to the identity in the C^r topology and whose images of M and M' are C^ω Nash manifolds. Here we need not to assume M' is closed in M. See [20] for the proof.

(3.5.5) By Theorem 2.1' and Lemma 3.4 we can prove that any two pairs (M_1, M_1') and (M_2, M_2') of affine C^ω Nash manifolds C^r Nash diffeomorphic, $0 < r < \infty$, are C^ω Nash diffeomorphic if M_1' and M_2' are closed in M_1 and M_2 respectively. Here the assumption of closedness is necessary. Indeed it is easy to construct a counter-example without the assumption. Hence any C^r Nash manifold for $0 < r < \infty$ and any C^r Nash submanifold closed in the manifold admit uniquely a C^ω Nash manifold pair structure.

(3.5.6) In Theorem 3.2, if a C^ω Nash manifold M' is contained and closed in M, then we can choose an approximation whose restriction to M' is the identity. The proof is easy by Theorems 2.1' and 3.2. Here also we need the assumption of closedness.

§4. Nonaffine abstract C^ω Nash manifolds

First we study the structures of abstract C^ω Nash manifolds as C^ω manifolds.

Theorem 4.1, [21]. An abstract C^ω Nash manifold is compact or C^ω diffeomorphic to the interior of some compact C^ω manifold with boundary.

Proof. Let M be a noncompact abstract C^ω Nash manifold. Then, by Theorem 3.1 and its proof, there exists a C^2 Nash embedding $\varphi: M \to \mathbb{R}^n$ such that $\varphi(M)$ is bounded and $\overline{\varphi(M)} - \varphi(M) = 0$. We see easily that $\{\varphi(M), 0\}$ is a C^2 Whitney stratification of $\overline{\varphi(M)}$ because $\varphi(M)$ is semialgebraic in $\mathbb{R}^n$. Hence, for a small positive number ε, $\varphi(M)$ is C^1 diffeomorphic to $\varphi(M) - \{x \in \mathbb{R}^n : |x| \leqq \varepsilon\}$, and $\varphi(M) - \{|x| \leqq \varepsilon\}$ is the interior of the compact manifold with boundary $\varphi(M) - \{|x| < \varepsilon\}$.

Conversely we have the following, which, in the case of a torus, is due to Chillingworth, Hubbard and Mazur [3].

Theorem 4.2, [21]. Any compact C^ω manifold or the interior of any compact C^ω manifold with boundary admits a continuum number of distinct nonaffine abstract C^ω Nash manifold structures.

Proof. First we give $\mathbb{R}$ a nonaffine abstract C^ω Nash manifold structure. Put $M_1 = (-\infty, 1)$, $M_2 = (0, \infty)$ and $M_3 = (0, 1)$. Let C^ω Nash embeddings $h_1 : M_3 \to M_1$ and $h_2 : M_3 \to M_1$ be defined by

$$h_1(t) = t^2, \quad h_2(t) = 2t - t^2.$$

Consider the disjoint union of M_1, M_2 and M_3 and an equivalence relation generated by $x \sim h_1(x)$ and $x \sim h_2(x)$ for $x \in M_3$ on the union. Then the quotient topological space is homeomorphic to $\mathbb{R}$ and admits an abstract C^ω Nash manifold structure induced by those of M_1, M_2 and M_3. Let M denote the abstract C^ω Nash manifold. We shall see that M is not affine. Let $\varphi_1 : U_1 \to M_1 \subset \mathbb{R}$, $\varphi_2 : U_2 \to M_2 \subset \mathbb{R}$ and $\varphi_3 : U_3 \to M_3 \subset \mathbb{R}$ be the C^ω Nash coordinate neighborhoods of M naturally defined by M_1, M_2 and M_3, respectively, and let $f : (0, 1) \to M$ be the inverse map of φ_3. We want to extend f to $\mathbb{R}$ as widely as possible as an analytic map. As $\lim_{t \to 1} f(t) \in U_2$ and $\operatorname{Im} f \subset U_2$, it is sufficient to consider $\varphi_2 \circ f$ for the extension $\tilde{f}$ of f to $[1, 1+\varepsilon]$ for small positive ε. Now $\varphi_2 \circ f(t) = 2t - t^2$. Hence $\varphi_2 \circ \tilde{f}(t) = 2t - t^2$ on $[1, 2)$, which implies $\tilde{f}((1, 2)) \subset U_3$, $\varphi_3 \circ \tilde{f}(t) = -t + 2$ on $(1, 2)$. By the same argument at $\varphi_1^{-1}(0)$ we obtain the analytic extension $\tilde{f} : \mathbb{R} \to M$ of f such that

$$\tilde{f}(\mathbb{R} - \mathbb{Z}) \subset U_3, \quad \tilde{f}(2\mathbb{Z}) = \varphi_1^{-1}(0) \quad \text{and} \quad \tilde{f}(2\mathbb{Z}+1) = \varphi_2^{-1}(1),$$

and $\tilde{f}$ oscillates infinitely. It is clear $\tilde{f}$ is locally semialgebraic. Hence, if M were affine, then $\tilde{f}$ would be a C^ω Nash map, because any locally semialgebraic C^ω map between affine C^ω Nash manifolds is a C^ω Nash map. But $\tilde{f}$ oscillates infinitely. This is a contradiction. Therefore M is not affine.

If we vary h_1, then we obtain distinct abstract nonaffine C^ω Nash manifold structures on $\mathbb{R}$. In the general case of manifold we embed the manifold M in a Euclidean space so that M is an affine C^ω Nash manifold (Remark 6.6), we remove a closed ball B, and then we attach a little bigger open ball to $M-B$ like the attachment of M_3 to $(M_1-[1/3,1)) \cup (M_2-(0,2/3])$ in the case $M \cong \mathbb{R}$.

Remark 4.3. Let M be an abstract nonaffine C^ω Nash manifold constructed in the proof of Theorem 4.2. Then there does not exist a nonconstant C^ω Nash function on M. Indeed, let g be a C^ω Nash function on M and let $f:\mathbb{R}\to M$ be the locally semialgebraic C^ω map constructed in the proof which oscillates infinitely. Then $g\circ f : \mathbb{R}\to\mathbb{R}$ is a locally semialgebraic C^ω map and hence a C^ω Nash map. If g were not constant then $g\circ f$ would oscillate infinitely. Hence g must be constant.

Remark 4.4. In Proof of Theorem 4.2, consider $M' =$ the closure of U_3 in M. Then M' is clearly a compact abstract C^ω Nash manifold with boundary, and by the existence of locally semialgebraic C^ω map $\tilde{f}:\mathbb{R}\to M'$ M' is not affine. But $\operatorname{Int} M' = U_3$ is affine. Moreover, changing h_1 we obtain distinct abstract C^ω Nash manifold structures on $[0,1]$.

We know many properties of the ring of C^ω Nash functions on an affine C^ω Nash manifold. For example, the ring is Noetherian (Risler [17]); two affine C^ω Nash manifolds are C^ω Nash diffeomorphic if their rings are isomorphic. But we know nothing about the ring for an abstract nonaffine C^ω Nash manifold. Is this Noetherian ?

§5. C^0 Nash manifolds

The C^0 case is very different to other cases.

Remark 5.1. Let M be a compact PL manifold possibly with boundary contained in $\mathbb{R}^n$. Then the interior of M is a C^0 Nash manifold. Let M' be another one contained in $\mathbb{R}^{n'}$, and let $f : M \to M'$ be a PL map with $f(\mathrm{Int}\ M) \subset \mathrm{Int}\ M'$. Then $f|_{\mathrm{Int}\ M} : \mathrm{Int}\ M \to \mathrm{Int}\ M'$ is a C^0 Nash map. In particular the C^0 Nash manifold structure on $\mathrm{Int}\, M$ does not depend on the choice of PL embedding of M in $\mathbb{R}^n$.

Conversely we have

Theorem 5.2. Let M be a C^0 Nash manifold. Then there exists uniquely a compact PL manifold L possibly with boundary such that M is C^0 Nash homeomorphic to $\mathrm{Int}\, L$. Here the uniqueness means that if there is another L' then L and L' are PL homeomorphic.

Remark 5.3. By Remark 5.1 and Theorem 5.2 the correspondence from the quotient space {compact PL manifolds possibly with boundary} /PL homeomorphisms to the quotient space $\{C^0$ Nash manifolds$\} / C^0$ Nash homeomorphisms, defined by $M \to \mathrm{Int}\ M$, is bijective.

Proof of Theorem 5.2. By the proof of Theorem 3.1 we can embed M in a Euclidean space $\mathbb{R}^n$ so that M is bounded and $\overline{M}-M=a\in\mathbb{R}^n$ and moreover $\overline{M}$ is a polyhedron by the theorem of semialgebraic triangulation of a semialgebraic set [6],[7]. Let K be a simplicial complex such that $|K|=\overline{M}$ and a is a vertex of K. Let K' denote the barycentric subdivision of K, and $\overline{St}(K',a)$, $St(K',a)$ denote the closed star, the open star of a in K'. Then $M-\overline{St}(K',a)$ is PL C^0 Nash homeomorphic to M. On the other hand, by the uniqueness theorem of subanalytic triangulation of a locally compact subanalytic set [23], $M-St(K',a)$ is a PL manifold with boundary.

It rests to prove the uniqueness. Let L and L' be compact PL manifolds possibly with boundary such that M is C^0 Nash homeomorphic to the interiors of L and L'. Let X, X' be polyhedrons obtained from L, L' by adjoining cones over the boundaries respectively, and let x, x' be the vertexes of X, X' respectively. Then $X-x$, $X'-x'$ are C^0 Nash homeomorphic to $\operatorname{Int} L$, $\operatorname{Int} L'$ respectively. Hence $X-x$ and $X'-x'$ are C^0 Nash homeomorphic. Consequently X and X' are C^0 Nash homeomorphic and hence they are PL homeomorphic by the uniqueness theorem of subanalytic triangulation. Moreover we can choose the PL homeomorphism so that x is mapped to x' by 4.10 Remark in [23]. Hence in the same way as the first half we see that L and L' are PL homeomorphic.

Remark 5.4. There exist two PL C^0 Nash manifolds which are PL homeomorphic but not C^0 Nash homeomorphic. For example [11], let M_1 and M_2 denote the PL 3-dimensional lens spaces of type (7, 1) and (7.2) and put $L_1=M_1\times\sigma$, $L_2=M_2\times\sigma$, where σ is a 4-simplex. Then it is known that L_1 and L_2 are not PL homeomorphic but $\operatorname{Int} L_1$ and $\operatorname{Int} L_2$ are PL homeomorphic. If $\operatorname{Int} L_1$ and $\operatorname{Int} L_2$ were C^0 Nash homeomorphic, then L_1 and L_2 would be PL homeomorphic by Theorem 5.2. Hence $\operatorname{Int} L_1$ and $\operatorname{Int} L_2$ are not C^0 Nash homeomorphic.

Remark 5.5. Let M_1, M_2 be C^0 Nash PL manifolds which are PL homeomorphic. Each of the following is a sufficient condition for M_1, M_2 to be C^0 Nash homeomorphic. (i) $\dim M_1 \leqq 3$. (ii) $\dim M_1 \geqq 6$ and M_1 is simply connected at infinity (i.e. for each compact subset K of M_1 there exists a compact subset K' of M_1 such that $M_1 - K'$ is simply connected and $K' \supset K$). Indeed the compactification of such M_1 by an attachment of boundary is unique by the h-cobordism theorem [12].

Corollary 5.6. Let M_1, M_2 be compact PL manifolds such that $M_1 \times \mathbb{R}$ and $M_2 \times \mathbb{R}$ are C^0 Nash homeomorphic. Then M_1 and M_2 are PL homeomorphic.

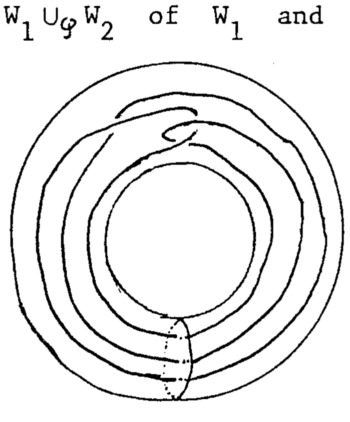
Proof. $M_1 \times \mathbb{R}$ and $M_2 \times \mathbb{R}$ are C^0 Nash homeomorphic to $M_1 \times (0, 1)$ and $M_2 \times (0, 1)$ respectively. Clearly $M_1 \times (0, 1)$ and $M_2 \times (0, 1)$ are the interiors of the compact C^0 Nash manifolds with boundary $M_1 \times [0, 1]$ and $M_2 \times [0, 1]$ respectively. Hence, by Theorem 5.2, $M_1 \times [0, 1]$ and $M_2 \times [0, 1]$ are PL homeomorphic, which implies $\partial(M_1 \times [0,1]) = M_1 \times 0 \cup M_1 \times 1$ and $\partial(M_2 \times [0,1]) = M_2 \times 0 \cup M_2 \times 1$ are PL homeomorphic. Therefore M_1 and M_2 are PL homeomorphic.

Remark 5.7. In this corollary we can not replace the condition "C^0 Nash homeomorphic" by "PL homeomorphic". For example, assume two compact PL manifolds M_1 and M_2 are PL distinct and PL h-cobordant (i.e. there exists a compact PL manifold with boundary M such that ∂M is the disjoint union of M_1 and M_2 and the inclusions $M_1 \to M$ and $M_2 \to M$ are homotopy equivalent). Then it is known [12] that $M_1 \times \mathbb{R}$ and $M_2 \times \mathbb{R}$ are PL homeomorphic.

The difference, which causes the above phenomenon, between a C^0 Nash homeomorphism and a PL homeomorphism is that a C^0 Nash map is "finite" and a PL map may be "infinite". But it is not easy to explain the following fact.

We can not eliminate the compact assumption in Corollary 5.6, namely there exist two C^0 Nash PL manifolds Y_1, Y_2 such that $Y_1 \times \mathbb{R}$ and $Y_2 \times \mathbb{R}$ are C^0 Nash PL homeomorphic but Y_1 and Y_2 are not C^0 Nash homeomorphic nor PL homeomorphic. By Mazur [10] there exists a compact contractible PL manifold with boundary X_1 of dimension 4 such that ∂X_1 is not simply connected and $X_1 \times [0, 1]$ is PL and hence C^0 Nash homeomorphic to $X_2 \times [0, 1]$, where X_2 is a 4-simplex. Here "contractible" means that X_1 is homotopy equivalent to a point. Let Y_1, Y_2 be the interiors of X_1, X_2 respectively. Then $Y_1 \times (0, 1)$ ($= \mathrm{Int}(X_1 \times [0, 1])$) and $Y_2 \times (0, 1)$ ($= \mathrm{Int}(X_2 \times [0, 1])$) are C^0 Nash PL homeomorphic. But Y_1 and Y_2 are not homeomorphic because Y_1 is not simply connected at infinity.

Mazur constructed X_1 as follows. Put $W_1 = B^3 \times S^1$, $W_2 = B^2 \times B^2$, where B^i and S^1 mean a PL i-ball and a PL 1-sphere respectively. Let $B^2_+ \times S^1$ be the upper half of the boundary $\partial W_1 = S^2 \times S^1$, choose a PL Jordan curve C in $B^2_+ \times S^1$ like the figure, and let U be a PL closed tubular neighborhood of C in $B^2_+ \times S^1$. Then we obtain naturally a PL homeomorphism $\varphi : B^2 \times S^1 \to U$ such that $\varphi(0 \times S^1) = C$. Regard $B^2 \times S^1$ as the front of $\partial W_2 = B^2 \times S^1 \cup S^1 \times B^2$. Then the attaching space $W_1 \cup_\varphi W_2$ of W_1 and W_2 by φ is X_1.

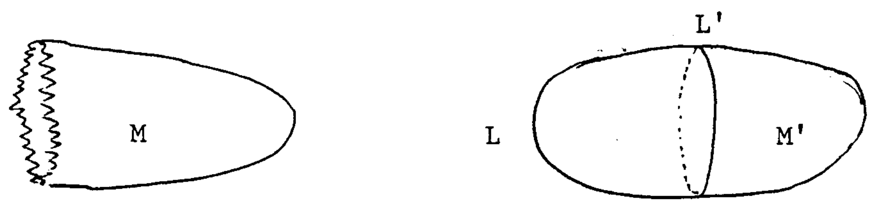


Remark 5.8,[23]. We supply the following in Theorem 5.2. Let M_1, M_2 be C^0 Nash manifolds, let L_2 be a compact C^0 Nash manifold with boundary such that $\operatorname{Int} L_2 = M_2$, and let $f: M_1 \to M_2$ be a C^0 Nash map. Then there exist a compact C^0 Nash manifold with boundary L_1 and a C^0 Nash homeomorphism $\pi: \operatorname{Int} L_1 \to M_1$ such that $f \circ \pi : \operatorname{Int} L_1 \to M_2$ is extensible to $L_1 \to L_2$. Of course we can choose as L_1 a PL manifold with boundary.

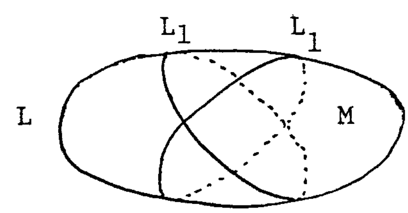
Problem 5. 9(cf.[19]). Let M be a C^0 Nash manifold and f be a C^0 Nash function on M. Do there exist a PL C^0 Nash manifold M_1 and a C^0 Nash homeomorphism $\pi: M_1 \to M$ such that $f \circ \pi$ is PL ?

§6. Affine C^ω Nash manifolds

Theorem 6.1,[18]. Let M be an affine C^ω Nash manifold. Then there exist a compact affine nonsingular algebraic set L, a nonsingular algebraic subset L' of codimension 1, which is empty if M is comapct, and a union M' of some connected components of $L - L'$ such that M is C^ω Nash diffeomorphic to M' and $\overline{M'}$ is a compact affine C^ω Nash manifold with boundary $= L'$.



Proof. If M is compact, the theorem is due to Nash [14] and is proved easily in the same way as Theorem 3.2. Hence we consider the noncompact case. We can assume also M is connected because it is sufficient to prove the theorem for each connected component of M. We shall reduce the problem to the case of the figure below. First embed M in $\mathbb{R}^n$ in the same way as the proof of Theorem 3.1 so that M is contained in a unit sphere and $\overline{M} - M = a \in \mathbb{R}^n$. Next apply the normalization theorem of Zariski [1], [15] to the Zariski closure of M. Then we have a compact affine algebraic set X and an algebraic subset X' of X such that a connected component of $X - X'$ is C^ω Nash diffeomorphic to M and contains only nonsingular points of X. Last apply the desingularization theorem of Hironaka [5] to X and X'. Then there exist a compact nosingular algebraic set L in $\mathbb{R}^m$ and an algebraic subset L_1 of L which has only normal crossings in L such that a connected component of $L - L_1$ is C^ω Nash diffeomorphic to M. We regard the component as M. Here "normal crossing" means that for every point $a \in L_1$ there exist a smooth rational map $\zeta : L \to \mathbb{R}^{n''}$ for some integer $n'' \leqq n' = \dim L$ and a neighborhood U of $\underline{a}$ in L such that $\zeta(a) = 0$, $L_1 \cap U = \zeta^{-1}\{(x_1, \ldots, x_{n''}) \in \mathbb{R}^{n''} : x_1 \ldots x_{n''} = 0\} \cap U$ and ζ is a submersion on U.



Let h be the restriction to L of the square sum of finite generators of the ideal in $\mathbb{R}[x_1, \ldots, x_m]$ defined by L_1. Then for every $a \in L_1$ we have a C^ω Nash local coordinate system $(x_1, \ldots, x_{n'})$ of L around $\underline{a}$ such that $a = 0$ and $h = x_1^2 \ldots x_{n''}^2$. By this property we see easily M is C^∞ diffeomorphic to $M_\varepsilon = \{x \in M : h(x) > \varepsilon\}$ for a small positive number $\varepsilon > 0$. Moreover it is possible to prove directly after a long sequence of arguments that they are C^ω Nash diffeomorphic [18]. But it is much easier to see that they are C^1 Nash diffeomorphic as follows. We construct first a C^1 Nash flow near L_1 which is transversal to each irreducible component of L_1, we transfer $M-M_{2\varepsilon}$ to $M_\varepsilon - M_{2\varepsilon}$ along the flow, we extend the transfer to $M \to M_\varepsilon$ by the identity map of $M_{2\varepsilon}$, we smoothen the extension at $\partial\overline{M_\varepsilon}$, and then we obtain a C^1 Nash diffeomorphism $f: M \to M_\varepsilon$. Hence, by Lemma 3.3 and Theorem 2.1", we have a C^ω Nash diffeomorphism $\tilde{f}: M \to M_\varepsilon$. Here we have to be careful to apply Lemma 3.3 because we used already Theorem 6.1 in the proof of Lemma 3.3. But we needed in the proof only the fact that at least one of M_1 and M_2 in Lemma 3.3 can be compactified by an attachment of its boundary, and now $\overline{M_\varepsilon}$ is a compact C^ω Nash manifold with boundary and with interior $=M_\varepsilon$, hence Lemma 3.3 can be applied.

For the proof of the theorem it rests to modify $\partial\overline{M_\varepsilon} = \{x \in M : h(x) = \varepsilon\}$ to be a nonsingular algebraic set. Let $\alpha: L \to \mathbb{R}$ be a C^∞ function such that $\alpha^{-1}(0) = \partial\overline{M_\varepsilon}$ and α is C^∞ regular at $\alpha^{-1}(0)$, and let $\beta: L \to \mathbb{R}$ be a polynomial approximation of α in the C^1 topology. Then $L' = \beta^{-1}(0)$ is a nonsingular algebraic subset of L of codimension one and we prove in the same way as above that $\{x \in M : \beta(x) \geqq 0\}$ is a compact C^ω Nash manifold with boundary $=L'$ whose interior is C^ω Nash diffeomorphic to M. Thus we complete the proof.

<u>Theorem 6.2, [18]</u>. Let L_1, L_2 be compact affine C^ω Nash manifolds possibly with boundary, and let M_1, M_2 denote their interiors. Then the following conditions are equivalent.

(i) L_1 and L_2 are C^1 diffeomorphic.

(ii) L_1 and L_2 are C^ω Nash diffeomorphic.

(iii) M_1 and M_2 are C^ω Nash diffeomorphic.

<u>Proof</u>. (i) $\Rightarrow$(ii). Let $\tau: L_1 \to L_2$ be a C^1 diffeomorphism and let $L_2 \subset \mathbb{R}^{n_2}$. Applying the polynomial approximation theorem to $\tau|_{\partial L_1}$ and using

a C^ω Nash tubular neighborhood of ∂L_2 in $\mathbb{R}^{n_2}$ in the same way as the proof of Theorem 2.1, we modify τ so that $\tau|_{\partial L_1}$ is of class C^ω Nash. Moreover we reduce the problem to the case in which τ is globally of class C^1 Nash by C^ω Nash collars of L_1 and L_2 and by a C^1 Nash partition of unity. Let W_1, W_2 be the doubles of L_1, L_2 respectively. It is easy to give W_1, W_2 affine C^ω Nash manifold structures so that L_1, L_2 are C^ω Nash submanifolds respectively (see Remark 6.6). Then we have a C^1 Nash diffeomorphism extension of τ to $W_1 \to W_2$. Apply Theorem 2.1" to the extension. Then we obtain a C^ω Nash diffeomorphism from L_1 to L_2.

(ii) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (i). Let $\tau : M_1 \to M_2$ be a C^ω Nash diffeomorphism, and let $h_1 : L_1 \to \mathbb{R}$, $h_2 : L_2 \to \mathbb{R}$ be nonnegative C^2 Nash functions such that $h_1^{-1}(0) = \partial L_1$, $h_2^{-1}(0) = \partial L_2$ and h_1, h_2 are C^2 regular at ∂L_1, ∂L_2 respectively (the existence is clear). Then L_1, L_2 are C^2 Nash diffeomorphic to $\{x \in N_1 : h_1(x) \geqq \varepsilon\}$, $\{x \in N_2 : h_2(x) \geqq \varepsilon\}$ for small $\varepsilon > 0$ respectively. Consider two C^2 Nash functions h_1 and $h_2 \circ \tau$ on L_1. Then, by the next lemma, $\{x \in N_1 : h_1(x) \geqq \varepsilon\}$ and $\{x \in N_1 : h_2 \circ \tau(x) \geqq \varepsilon\}$ are C^1 diffeomorphic. Hence, by the equality

$$\{x \in N_1 ; h_2 \circ \tau(x) \geqq \varepsilon\} = \tau^{-1}\{x \in N_2 : h_2(x) \geqq \varepsilon\},$$

L_1 and L_2 are C^1 diffeomorphic.

<u>Lemma 6.3</u>. Let L be a compact affine C^r Nash manifold with boundary, $1 < r \leqq \omega$. Let f_1, f_2 be nonnegative C^r Nash functions on L such that $f_1^{-1}(0) = f_2^{-1}(0) = \partial L$. Then for each small $\varepsilon > 0$ there exists a C^{r-1} diffeomorphism from $\{x \in L : f_1(x) \geqq \varepsilon\}$ to $\{x \in L : f_2(x) \geqq \varepsilon\}$. Here if $r = \omega$ then $r - 1$ means ∞.

By considering $1/f_1$ and $1/f_2$, we obtain this by the next lemma.

Lemma 6.4, [18]. Let M be an affine C^r Nash manifold, $1 < r \leqq \omega$, and let g_1, g_2 be proper positive C^r Nash functions on M. Then there exists a C^{r-1} diffeomorphism π of M such that $g_1 \circ \pi = g_2$ outside some compact subset of M. If $r = \omega$ then $r-1 = \infty$ like the above.

We construct π by the integration of the vector field $\operatorname{grad} f_1/|\operatorname{grad} f_1| + \operatorname{grad} f_2/|\operatorname{grad} f_2|$ defined outside a compact subset of M.

Remark 6.5. Let M be an affine C^ω Nash manifold contained in $\mathbb{R}^n$ such that M is bounded and $\overline{M} - M = 0$. Then for any small positive number ε, $M_\varepsilon = \{x \in M : |x| > \varepsilon\}$ is C^ω Nash diffeomorphic to M. Indeed, let L, L' and M' be sets satisfying the properties in Theorem 6.1, and let $\varphi : M \to M'$ be a C^ω Nash diffeomorphism. Let h_1 be a C^2 Nash function on L such that $h_1 > 0$ on M', $h_1^{-1}(0) = L'$ and h_1 is C^2 regular at L'. Put $h_2 = d \circ \varphi^{-1}$ where $d(x) = |x|$ on M. Then we only need to prove that $\{x \in M' : h_1(x) \geqq \varepsilon\}$ is C^1 diffeomorphic to $\{x \in M' : h_2(x) \geqq \varepsilon\}$ for small $\varepsilon > 0$ by Theorem 6.2. But this is an immediate consequence of Lemma 6.4.

Remark 6.6. Let $\mathcal{CN}$, $\mathcal{C}$, $\mathcal{N}$ denote the quotient spaces {compact affine C^ω Nash manifolds possibly with boundary} $/C^\omega$ Nash diffeomorphisms, {compact C^1 manifolds possibly with boundary} $/C^1$ diffeomorphisms, {affine C^ω Nash manifolds}$/C^\omega$ Nash diffeomorphisms respectively. Then, by Theorems 6.1,2, the correspondences $\mathcal{CN} \to \mathcal{C}$ and $\mathcal{CN} \to \mathcal{N}$ are bijective. It rests to prove only that $\mathcal{CN} \to \mathcal{C}$ is surjective. Let L be a compact C^1 manifold possibly with boundary and let W be the C^1 double of L. Apply a theorem of Benedetti-Tognoli [2] to $(W, \partial L)$. Then $(W, \partial L)$ admits an affine

C^ω Nash manifold pair structure. Hence L is given an affine C^ω Nash manifold structure. By this we can choose L, L' and M' in Theorem 6.1 so that L is diffeomorphic to the double of $\overline{M}'$.

Remark 6.7. The C^ω cases of Remarks 5.4,5 hold true. There exist two affine C^ω Nash manifolds which are C^ω diffeomorphic but not C^ω Nash diffeomorphic. Moreover we can choose affine nonsingular algebraic sets as the examples. Let M_1, M_2 be affine C^ω Nash manifolds which are C^ω diffeomorphic. Then each of the following is a sufficient condition for M_1, M_2 to be C^ω Nash diffeomorphic. (i) $\dim M_1 \leqq 3$. (ii) $\dim M_1 \geqq 6$ and M_1 is simply connected at infinity.

The proofs are the same as Remarks 5.4,5.

Problem 6.8 (the nonaffine case of Theorem 6.1). Let M be a noncompact abstract nonaffine C^ω Nash manifold. Then, does there exist a compact abstract nonaffine C^ω Nash manifold with boundary whose interior is C^ω Nash diffeomorphic to M? Even if such a compactification is possible, it is not unique. Namely there exist, by Remark 4.4, two compact abstract nonaffine C^ω Nash manifolds with boundary which are not C^ω Nash diffeomorphic but whose interiors are C^ω Nash diffeomorphic.

Remark 6.9. Remark 5.8 is not correct in the C^ω case. For example, set

$$M_1 = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 1\},\quad L_2 = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leqq 2\},$$

$$M_2 = \operatorname{Int} L_2 \quad \text{and} \quad f(x, y) = (xy, (1-x)(1-y)).$$

Then $\overline{f(M_1)} - f(M_1) = \{(x, 0) : 0 \leqq x \leqq 1\} \cup \{(0, y) : 0 \leqq y \leqq 1\}$. Assume the existence of a compact affine C^ω Nash manifold with boundary L_1 and a C^ω Nash diffeomorphism $\pi : \mathrm{Int}\, L_1 \to M_1$ such that $f \circ \pi : \mathrm{Int}\, L_1 \to M_2$ is extended to $g : L_1 \to L_2$. Then

$$g(L_1) - f(M_1) = g(\partial L_1) = \overline{f(M_1)} - f(M_1).$$

Hence ∂L_1 consists of two analytic sets $g^{-1}\{(x, 0) : x \in \mathbb{R}\}$ and $g^{-1}\{(0, y) : y \in \mathbb{R}\}$. Therefore $\partial L_1 = g^{-1}\{(x, 0) : x \in \mathbb{R}\}$ or $\partial L_1 = g^{-1}\{(0, y) : y \in \mathbb{R}\}$ because ∂L_1 is analytically irreducible, which implies $g(\partial L_1) = \{(x, 0) : 0 \leqq x \leqq 1\}$ or $g(\partial L_1) = \{(0, y) : 0 \leqq y \leqq 1\}$. This is a contradiction.

Problem 6.10. Let M be an affine C^ω Nash manifold and let $f \in N^\omega(M)$ be bounded. Then, do there exist a compact affine C^ω Nash manifold with boundary L and a C^ω Nash diffeomorphism $\pi : \mathrm{Int}\, L \to M$ such that $f \circ \pi$ is extensible to L ?

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