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CHAPITRE III

ANALYTIC AND GEVREY HYPOELLIPTICITY

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INTRODUCTION :

We shall begin, for sake of completeness, by recalling samewell knowk definitions and results.

In the following P will denote a linear partial differential operator with analytic coefficients in an open set $\Omega \subset \mathbb{R}^n$ or else, more generally, a classical analytic pseudo differential operator in Ω. Writing a-sing supp u for the analytic singular support of a distribution u, we define P to be analytic hypoelliptic if :

(1) a-singsupp u = a-sing supp Pu for all u $\in \mathcal{E}'(\Omega)$.

We also recall that P is said to be hypoelliptic if :

(2) sing-supp u = sing supp Pu for all u $\in \mathcal{E}'(\Omega)$,

where sing supp u is the singular support of the distribution u and the definition could be obviously extended now to operators with C^∞ coefficients.

Starting from an operator P which is already known to be hypoelliptic (and, posssibly, for which suitable a-priori estimates have been proved), we shall purpose in the following the discussion of the validity of (1). In the case when P will turn out to be hypoelliptic but not analytic hypoelliptic, it will be natural to investigate the Gevrey hypoellipticity of P. We recall that a distribution u is said of class G^s (Gevrey class of order s), $1 \leq s < \infty$, at a point $x_o \subset \Omega$ if u is of class C^∞ at x_o and in a compact neighborhood $K \subset \Omega$ of x_o.

$$(3) \qquad \max_{x \in K} |D^{\alpha}u(x)| \leq C_K (C_K |\alpha|)^{s|\alpha|} \quad ,$$

for a constant C_K independent of α ; $G^s(\Omega)$ is the set all the functions in $C^{\infty}(\Omega)$ which are of class G^s at every $x_o \subset \Omega$. Writing G^s-sing supp u for the singular support of u with respect to the class G^s, we define the G^s-hypoelliptic operators P by means of the identity.

$$(4) \qquad G^s\text{-}\underline{\text{sing supp}}\ u = G^s\text{-}\underline{\text{sing supp}}\ Pu \qquad \underline{\text{for all}}\quad u \in \mathcal{E}'(\Omega).$$

When $s = 1$ the Gevrey class $G^s(\Omega)$ is the set of all the real analytic functions in Ω and (4) comes down to (1). Let us recall that the problem of the analytic and Gevrey hypoellipticity is completely solved for the linear partial differential operators with constant coefficients : $P = p(D)$ is analytic hypoelliptic if and only if P is elliptic, and for the hypoelliptic non elliptic operators necessary and sufficient conditions for the G^s-hypoellipticity can be easily expressed on the polynomial $p(\xi)$. (See for example Hörmander [7]) . Another class of operators which has been studied is the class of the linear partial differential operators $P = p(x,D)$ of principal type, i.e. the operators for which

$$(5) \qquad d_{\xi}\ p_m(x,\xi) \neq 0 \qquad \underline{\text{whenever}} \qquad p_m(x,\xi) = 0$$

(we write p_m for the principal part of p; observe that (5) is trivially satisfied by the elliptic operators). It has been proved by Trèves [15] that an operator with analytic coefficients P which satisfies (5) is analytic hypoelliptic if (and only if) it is hypoellliptic.

Several results are also known for the operators with multiple characteristics, i.e. the operators which are not of principal type. It is nowadays quite clear that the (non) analytic hypoellipticity of an operator with multiple characteristics depends strongly on the geometric properties of the characteristic manifold.

$$(6) \qquad \Sigma = \{(x,\xi) \in \Omega \times (\mathbb{R}^n \setminus 0),\ p_m(x,\xi) = 0\} \quad ;$$

and precisely on the behaviour in a neighborhood of Σ of the sympletic two-form $\omega = \sum_{j=1}^{n} dx_j \wedge d\xi_j$. In the next section 1 we shall begin by considering the operators of principal type ; a cheap proof of the above-mentioned result of Trèves will be given, under un additional hypothesis of regularity on the characteristic manifold. The arguments of Section 1, which are based on L^2 a-priori estimates, will be generalired in Section 2 to a class of operators with multiple characteristics ; the characteristic manifold Σ will be there symplectic (symplectic means that the restriction of ω to Σ is non-degenerate) and also for these operators we shall prove a result of analytic hypoellipticity.

When the characteristic manifold Σ is not symplectic, the analytic hypoellipticity may fail for the hypoelliptic operators, as the theory of the operators with constant coefficients points out clearly. In Section 3 we shall consider classes of hypoelliptic operators whose charcteristic manifold Σ is involutive (that is, noughly speaking, the degeneration of ω on Σ is maximal ; a more precise definition of regular involutive manifold will be given in the sequel). For these operators we shall prove a result of Gevrey hypoellipticity ; of course, we shall be interested in the best, i.e. the least, real number $s \geq 1$ for which (4) is valid.

Let us finally remark that all the results in the following will be stated and proved in a micro-local form, by replacing singular supports with the corresponding wave front sets. We recall that $(x_o, \xi_o) \notin WF\, u$, with $x_o \in \Omega$, $\xi_o \in \mathbb{R}^n$, $\xi_o \neq 0$, $u \in \mathcal{D}'(\Omega)$, means that there exist a function $\varphi \subset C_o^\infty(\Omega)$, $\varphi(x_o) \neq 0$, and a conic neighborhood Γ of ξ_o such that :

$$(7) \qquad |(\varphi u)^\wedge(\xi)| \leq C_M(1-|\xi|)^{-M} \qquad \text{if } \xi \in \Gamma ,$$

for every $M = 1,2,\ldots$, and for suitable constants C_M.

In the analytic-Gevrey category, we have that $(x_o, \xi_o) \notin WF_s\, u$, $1 \leq s < \infty$, if and only if there are a neighborhood $U \subset \Omega$ of x_o, a conic neighborhood Γ of ξ_o and a bounded sequence $U_N \in \mathcal{E}'(\Omega)$. Which is equal to u in U such that :

$$(8) \qquad |\hat{U}_N(\xi)| \leq C^{N+1}N^N(1+|\xi|)^{-N/s} , \ \xi \in \Gamma , \quad N = 1,2,\ldots,$$

where now the constant C does not depend on N. The analytic wave front set $WF_1 u$ will be also denoted by $WF_a u$. We may now introduce the notion of micro-hypoellipticity for P, by replacing (2) with :

(9) $WF\ u = WF\ Pu$ <u>for all</u> $u \in \mathcal{E}'(\Omega)$;

similarly P will be said G^s (or analytic, when s = 1), micro-hypoelliptic if :

(10) $WF_1 u = WF_s\ Pu$ <u>for all</u> $u \in \mathcal{E}'(\Omega)$.

Since the projections of the wave front sets on Ω are the respective singular supports, the (Gevrey or analytic)-micro-hypoellipticity implies the standard (Gevrey or analytic) hypoellipticity. Take also note of the obvious inclusions.

(11) $WF\ u \subset WF_s u \subset WF_a u$, <u>for all</u> $u \in \mathcal{D}'(\Omega)$, $1 \leqslant s < \infty$.

The use of the wave front sets in the study of the hypoellipticity has at least two advantages, with respect to the standard point of view. First, the classical theorem of regularity for the solutions of the elliptic equations can be refined by means of the formula (see for example Hörmander [4], Trèves [16]) :

(12) $WF_a u \subset WF_a\ Pu \cup \Sigma$,

where P is in general a classical analytic pseudo differential operator, Σ is its characteristic manifold and we may replace WF_a with WF or WF_s. So, to prove the hypoellipticity of P, it will be actually sufficient to prove its micro-hypoellipticity at Σ.

Moreover, the use of the wave front set combined with the machinery of the Fourier integral operators may lead to relevant simplifications in the study of P. Precisely, let X be a homogeneous canonical transformation acting from a conic neighborhood of a point $\rho = (x_o,\xi_o)$ to a conic neighborhood of the point $X(\rho) = (y_o,\eta_o)$ (X canonical means that X perserves the two-form ω) ; let us suppose that X is a real analytic diffeomorphism. Then consider any Fourier integral operator F with a phase function corresponding to X and with a classical analytic elliptic symbol as amplitude function. From the evaluation of the singularities of the kernels of F and its inverse F^{-1}, we have that a fixed pseudodifferential operator P is (Gevrey or analytic) micro-hypoelliptic at ρ if and only if the pseudo differential operator $\tilde{P} = F\, P\, F^{-1}$ is (Gevrey or analytic) micro-hypoelliptic at $X(\rho)$.

On the other hand, the principaal symbol of $\tilde{P}$ is given by $\tilde{p}_m(y,\eta) = p_m(X^{-1}(y,\eta))$; if we assume in particular $\rho \in \Sigma$ and denote by $\tilde{\Sigma}$ the characteristic manifold of $\tilde{P}$, then $X(\rho) \in \tilde{\Sigma}$ and in a conic neighborhood of $X(\rho)$ we have $\tilde{\Sigma} = X(\Sigma)$.

In this way, by fixing a suitable canonical transformation X, we may reduce ourselves to the study of operators $\tilde{P}$ of a truly elemental form. Additional simplifications in the expression of $\tilde{P}$ can be obtained by means of an appropriate choice of the amplitude function of F and composition with elliptic factors. For a complete theory of the Fourier integral operators let us refer to Hörmander [5], [6].

Further references to results and problems on analytic and Gevrey regularity will be given in the conclusive Section 4 (without any pretence to completeness ; the literature on the subject is nowadays very extensive).

1 - OPERATORS OF PRINCIPAL TYPE :

We shall begin by considering the model in $\mathbb{R}^2_{x,y}$

(13) $P = D_x + i\, x^{2k} D_y$,

where $k \geqslant 1$ is a fixed integer. We shall give a simple proof of the analytic micro-hypoellipticity of P by using an equivalent definition of WF_a ; namely with the notations of the introduction :

LEMMA 1 :

Let u be in $\mathcal{E}'(\mathbb{R}^n)$. We have $(x_o,\xi_o) \notin WF_a u$ if and only if there are uniformly bounded functions $\Psi_j \subset C^\infty(\mathbb{R}^n)$, $j = 1,2,\ldots$, $\Psi_j(\xi) = 1$ for $|\xi| \geqslant j$ in a conic neighborhood of ξ_o independent of j, such that :

(14) $$\sup_{x\in V} |D_x^\alpha \Psi_j(D) f(x)| \leqslant c(c_j)^{|\alpha|} \quad \underline{for} \quad |\alpha| \leqslant j\,,$$

for a constant c and a neighborhood V of x_o which are also independent of j.

It is easy to see that (14) implies $(x_o,\xi_o) \notin WF_a u$. In the opposite direction, we first observe that if $(x_o,\xi_o) \notin WF_a u$, then there exists a neighborhood V of x_o and a conic neighborhood Γ of ξ_o such that $V \times \Gamma \cap WF_a u = \emptyset$. To obtain (14) it will be sufficient to take functions $\Psi_j \in C^\infty(\mathbb{R}^n)$ with supp $\Psi_j \subset \Gamma$, $\Psi_j(\xi) = 1$ for $|\xi| \geqslant j$ in a smaller conic neighborhood of ξ_o, and such that :

(14) $$|D_\xi^{\alpha+\beta} \Psi_j(\xi)| \leqslant c_\beta (c_j)^{|\alpha|} (1+|\xi|)^{-|\alpha|-|\beta|} \quad \underline{for} \quad |\alpha| \leqslant j\,,$$

for some constants c, c_β independent of j ; (15) means that the functions φ_j are classical C^∞ symbols of order zero, and they are analytic "up to the order j" (the construction of such a sequence is always possible).

The estimates (14) are then a consequence of the following lemma, which be useful also in the sequel :

LEMME 2 :

Let u be in $\mathcal{E}'(\mathbb{R}^n)$. Let V be an open bounded subset of $\mathbb{R}^n$ and let Γ

be an open conic subset of the space of the dual variables such that $(V \times \Gamma) \subset WF_a u = \emptyset$. Let $\Psi_j(\xi)$, $j = 1,2,\ldots$, be a sequence of functions in $C^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \Psi_j \cap \{|\xi| \geq j\} \subset \Gamma$ and satisfying the estimates (15). Then (14) is valid with V as in the hypothesis and for a new suitable constant c.

The proof of lemma 2 is rather standard ; see for example Rodino [12]. Let us return to the operator P in (13) ; x and y are now the variables in $\mathbb{R}^2$, with dual variables ξ and η, respectively.

The application of (12) shows that we may limit ourselves to argne on the points of the characteristic manifold :

$$(16) \qquad \Sigma = \{(x,y;\xi,\eta) \in \mathbb{R}^2 \times (\mathbb{R}^2\backslash 0)\ ,\ x = 0,\ \xi = 0\}.$$

Since P is invariant for translations in the variable y, we may further assume y = 0. Therefor we have simply to prove that $\rho = (x=0,\ y=0,\ \xi=0,\ \eta=\eta_0\neq 0) \notin WF_a$ Pu implies $\rho \notin WF_a u$ for all $u \subset \mathcal{E}'(\mathbb{R}^2)$.

It is now very important to observe that u is already known to be micro-analytic in the region $(V \times \Gamma) \setminus \Sigma$, where V is a small neighborhood of the origin in $\mathbb{R}^2$ and Γ is a small conic neighborhood of $(\xi=0, \eta=\eta_0)$; this is a trivial consequence of the assumption $\rho \notin WF_a$ Pu and the inclusion (12). In fact, our proof will consist in coming back, by means of suitable cut-off functions, to the estimation of the derivatives of u in the elliptic region $(V \times \Gamma) \setminus \Sigma$ where the analyticity is already acquired.

In view of lemma 1, the inequalities which we have to prove are :

$$(17) \qquad \|D_y^h \Psi_j(D)u\|_{L^2(U)} \leq c(c_j)^h \quad \underline{\text{for}} \quad h \leq j\ ,$$

where the functions Ψ_j are taken as in the proof of lemma 1, satisfying (15) with $\operatorname{supp} \Psi_j \subset \Gamma$, $\Psi_j = 1$ for $\sqrt{\xi^2+\eta^2} \geq j$ in a smaller conic neighborhood Γ' of $(\xi=0, \eta=\eta_0)$; U is a neighborhood of the origin, $U \subset V$. Note that in (17) we are indeed allowed to consider only y-derivations if the ξ axes does not intersect Γ, and the sup-norms in (14) can be replaced by L^2-norms in view of the embedding theorem of Sobolev. We shall use the following a-priori inequality, which we assume known (and which is actually well known) for P in (13) :

$$(18) \qquad |||f||| = ||f|| + ||D_x f|| + ||x^{2k} D_y f|| \leqslant c||Pf|| \, ,$$

$$\underline{\text{for all}} \quad f \in C_o^\infty(V) \, ,$$

where V is a sufficiently small neighborhood of the origin which we may assume coïncident with the previously defined V. Observe that the inequality (18) is not the best possible with respect to the Sobolev norms, but it is "maximal" in the sense that both the terms in the expression of Pf are estimated.

We shall also assume as already proved the C^∞ micro-hypoellipticity of P ; this will allow us to suppose from the beginning $\Psi_j(D)u\big|_V \in C^\infty(V)$.

Using (18) and arguing by induction on $h \leqslant j$, we shall prove the estimates :

$$(19) \qquad |||D_y^h \;{}_j(D)u|||_{L^2(U)} = ||D_y^h \Psi_j(D)u||_{L^2(U)} +$$

$$||D_x \Psi_j(D)u||_{L^2(U)} + ||x^{2k} D_y D_y^h \Psi_j(D)u||_{L^2(U)}$$

$$\leqslant c(c\, j)^h \qquad \underline{\text{for}} \quad h \leqslant j \, ,$$

with will imply trivially (17).

In order to apply (18), we introduce a cut-off function $\varphi \in C_o^\infty(V)$ with $\varphi = 1$ in U. Let us define φ as a product : $\varphi(x,y) = X(x)X(y)$, with $X(t) \in C_o^\infty(\mathbb{R})$, $0 \leqslant X(t) \leqslant 1$, $X(t) = 1$ for $|t| \leqslant \varepsilon$, $X(t) = 0$ for $|t| \geqslant 2\varepsilon$, assuming, as we can, $U = \{|x| < \varepsilon,\ |y| < \varepsilon\}$, $V = \{|x| < 2\varepsilon,\ |y| < 2\varepsilon\}$; we have in this way :

(20) $\quad \text{supp } D_x \varphi \subset V \setminus \{|x| \leqslant \varepsilon\}$.

The important future of (20) is that :

(21) $\quad ||(D_x\varphi) D_y^h \Psi_j(D)U|| \leqslant c(c\ j)^h$.

In fact, we know that $(V \setminus \{|x| \leqslant \varepsilon\}) \times \Gamma \cap WF_a u = \emptyset$ and we may apply lemma 2 ; (21) follows the from (14) where we replace V with supp $D_x \varphi$.

Applying (18) we have now :

(22) $$|||D_y^h \Psi_j(D)U|||_{L^2(U)} \leqslant ||| \varphi D_y^h \Psi_j(D)U||| \leqslant$$

$$\leqslant c||P(\varphi D_y^h \Psi_j(D)U||.$$

Noting that $[P,D_y^k] = 0$, we may further write :

(23) $$||P(\varphi D_y^h \Psi_j(D)U)|| \leqslant ||\varphi D_y^h \Psi_j(D)Pu||$$

$$+ ||[P,\varphi] D_y^h \Psi_y(D)U|| + || \varphi[P,\Psi_j(D)]D_y^h U||.$$

It is easy to estimate the first term in the rihgt-hand side of (23) by using lemma 1 and the assumption $\rho \notin WF_a Pu$. The crucial point of the proof is then the analysis of the other two terms containing the commutators.

We first consider :

(24) $$[P,\Psi_j(D)] = i[X^{2k},\Psi_j(D)]D_y ,$$

which is the pseudo-differential operator with symbol

(25) $$- i \sum_{o<\gamma\leqslant 2k} \binom{2k}{\gamma} X^{2k-\gamma} \Phi_{\gamma,j}(\xi,\eta) ,$$

where

$$(26)\qquad \Phi_{\gamma,j}(\xi,\eta) = \eta(D_\xi^\gamma \Psi_j)(\xi,\eta).$$

The sequences $\Phi_{\gamma,j}$, $j = 1,2,\dots$, satisfy the hypotheses of lemma 2 with supp $\Phi_{\gamma,j} \cap \{\sqrt{\xi^2+\eta^2} \geqslant j\} \subset \Gamma\backslash\Gamma'$ (recall that $\Psi_j = 1$ in $\Gamma' \cap \{\sqrt{\xi^2+\eta^2} \geqslant j\}$). On the other hand we have $V \times (\Gamma\backslash\Gamma') \cap WF_a\, u = \emptyset$, so that applying lemma 2 we may conclude

$$(27)\qquad ||\varphi[P,\Psi_j(D)]D_y^h u|| \leqslant c \max_\gamma \{\sup_{x\in V} |\Phi_{\gamma,j}(D)D_j^h\, u(x)|\} \leqslant c(cj)^h.$$

Finally we consider the term in (23) containing the commutator

$$(28)\qquad [P,\varphi] = D_x\varphi + i\, x^{2k}\, D_y\varphi.$$

The contribution coming from the first term in the right-hand side has been already estimated in (21) ; to handle the remaring part we may write

$$(29)\qquad ||(D_y\varphi)\, x^{2k}\, D_y^h\, \Psi_j(D)u|| \leqslant \max|D_y\varphi|\ |||D_y^{h-1}\, \Psi_j(D)u|||_{L^2(V)},$$

and then apply the inductive hypothesis. There is however an evident trouble; namely : we started in (19), (22) with the neighborhood U, where as in (29) we ask the induction for bounds in the larger neighborhood V. To overcome the difficulty we nest j+1 neughborhoods, $V = U_{j,o} \supset U_{j,1} \supset \dots \supset U_{j,j} = U$ and we take j cut-off functions : $\varphi_{j,1},\dots,\varphi_{j,j}$, with $\varphi_{j,h} \in C_o^\infty(U_{j,h-1})$, $\varphi_{j,h}=1$ in $U_{j,h}$. We may construct $U_{j,h}$ and $\varphi_{j,h}$ in such a way that

$$(30)\qquad |\varphi_{j,h}| \leqslant 1,\ |D_x\, \varphi_{j,h}| \leqslant c\, j,\ |D_y\, \varphi_{j,h}| \leqslant c\, j,$$

for a constant C which does not depend on j and h ; we may also assume that the inclusion (20) is still valid for $\varphi_{j,h}$.

The proceeding is then standard : what we want to prove by induction is that $|||D_y^h \Psi_j(D)u|||_{L^2(U_{j,h})} \leqslant K(Kj)^h$, for a constant K independent of j and h, $h \leqslant j$, and we get it by replacing in (22) and subsequent formulas φ with $\varphi_{j,h}$. In fact, if K has been chosen sufficiently large, we obtain for (29) exactly the required bounds, in view of (30) and the inductive hypothesis. Let us also observe that replacing in (21) φ with $\varphi_{j,h}$ we have no change in the rihgt hand side, as we see using (30) and arguing on the derivations of the function $D_y u$ whose analytic wave front set has olviously empty intersection with $(V\backslash\{|x|\leqslant\varepsilon\}) \times \Gamma$. The analytic micro-hypoellipoticity of P in (13) is therefore proved.

Arguing on a real analytic manifold Ω and using Fourier integral operators, we whall now generalire the preceding result to a wide class of equations of principal type. Let P be a linear partial differential operator with analytic coefficients in Ω or, more generally, a classical analytic pseudo-differential operator in Ω. Write p_m for its principal symbol. Let $\Sigma \subset T^*\Omega\backslash 0$ be a closed conic submanifold of codimentison 2. We suppose Σ is the characteristic manifold of P, and we make the following assumption.

(31) For every fixed $\rho \in \Sigma$, there exist a conic neighborhood Λ of ρ and a complex number > such that in Λ we have > $p_m = a + ib$, with a and b real valued symbols, $d_\xi a \neq 0$ and $H_a^j b = 0$ on $\Sigma \cap \Lambda$ if $j < 2k$, but $H_a^{2k} b \neq 0$ for some fixed integer $k \geqslant 1$, which we assume independent of .

THEOREM 3 **:**

Under the preceding assumptions, and in particular under the hypothesis (31), P is analytic hypoelliptic in Ω.

Note that condition (31) implies (5), i.e. P is of principal type. As we wrote in the introduction, our result is a particular case of the result of Trèves [15], where the integer k for which $H_a^{2k} b \neq 0$ was allowed to vary with $\rho \in \Sigma$. To prove theorem 3 we first observe that in view of (12) it will be sufficient to prove analytic micro-hypoellipticity at every $\rho \in \Sigma$. Moreover, it is known (see for example Hörmander [6]) that under the assumption (31) there exists an elliptic Fourier integral operator F defined in a neighborhood of ρ such that $\tilde{P} = F\, P\, F^{-1}$ has, modulo a multiplicative elliptic factor, exactly the form (13), or more precisely :

$$(32) \qquad \tilde{P} = D_x + i\, x^{2k}\, D_{y_1} \ ,$$

where theother n-2 variables $y_2, \ldots, y_{n-1}$ do not appear explicity. According to the arguments of the end of the intorduction, we have that P in theorem 3 is analytic micro-hypoelliptic at $\rho \in \Sigma$ if and only if $\tilde{P}$ is analytic micro-hypoelliptic at $\tilde{\rho} = (x = 0,\ y_1 = y_1^o, \ldots, y_{n-1} = y_{n-1}^o \ ;\ \xi = 0,\ \eta_1 = \eta_1^o, \ldots, \eta_{n-1} = \eta_{n-1}^o)$, $\eta_1^o \neq 0$, where one may assume $y_1 = 0, \ldots, y_{n-1}, \eta_2 = 0, \ldots, \eta_{n-1} = 0$. We proced now as in the proof of the analytic hypoellipticity of P in (13). Since in a conic neighborhood of $\tilde{\rho}$ we may limit ourselves to estimate y_1-derivatives, nothing has to be changed in the preceding computations, but the definition of the cut-off functions which now depend on n variables. As for φ, $\varphi_{j,k}$, the construction is trivial and no comment is needed. As for the construction of Ψ_j, we shall refer to the following.

***LEMMA 4* :**

Let the dual variables be split into two groups of variables, ξ and η. Fix $\tau \subset \{\xi = o\}$. There exists a sequence $\Psi_j(\xi,\eta)$ of classical symbols of order zero, analytic "up to the order j" (i.e. (15) is satisfied, with respect to all the dual variables), such that supp Ψ_j is included in a fixed conic neighborhood of τ, $\Psi_j = 1$ in a smaller conic neighborhood for $\sqrt{|\xi|^2+|\eta|^2} \geqslant j$, and such that :

$$(33) \qquad D_\xi^\alpha \Psi_j(\xi,\eta) = O$$

in a conic neighborhood of the manifold $\{\xi = o\}$, if $|\alpha| \neq O$.

As in the proof of lemma 1 we take symbols of order zero, which are "analytic up to the order j" ; to satisfy the additional condition (33) we set $\Psi_j(\xi,\eta) = \Psi_j^o(\eta)\ \Psi_j^1(\xi,\eta)$, where Ψ_j^o, is a suitable symbol depending only on η, where as the support of Ψ_j^1 is included in a small conic neighborhood of the manifold $\{\xi = o\}$ with $\Psi_j^1 = 1$ in a smaller neighborhood. Choosing $\Psi_j(\xi,\eta)$ as in lemma 4 (let now ξ be one real variable) and using (33), we may again conclude for the functions $\Phi_{\gamma,j}$ in (26) that supp $\Phi_{\gamma,j} \cap \{\sqrt{\xi^2+\eta^2} \geqslant j\}$ is included in the elliptic region $\{\xi \neq 0\}$ and the inequalities (27) are still valid in view of lemma 2. Nothing has to be added for the proof of theorem 3.

2 - OPERATORS WITH MULTIPLE SYMPLECTIC CHARACTERISTICS :

We want to show how the method which we have used in the preceding section for the study of the operators of principal type can be generalired to some operators with multiple characteristics.

We shall limit ourselves to argue on a class of models. As in the preceding section the result will be stated and proved in a microlocal form.

Let x and y be two groups of real variables and write ξ and η, respectively, for the dual variables. We consider an operator of the form :

$$(34) \qquad P = \sum_{(\alpha,\beta,\gamma)\in \mathfrak{m}} c_{\alpha\beta\gamma}\, x^{\alpha} D_x^{\beta} D_y^{\gamma} ,$$

where $\mathfrak{m}$ is a finite set of indices and $c_{\alpha\beta\gamma} \in \mathbb{C}$ are given constants. The following hypotheses, represent exactly what we shall need to apply our method of proof :

(35) *P is C^{∞} micro-hypoelliptic, and for a small neighborhood V of the origin*

$$|||f||| = ||f|| + \sum_{(\alpha,\beta,\gamma)\in \mathfrak{m}} ||x^{\alpha} D_x^{\beta} D_y^{\gamma} f|| \leqslant c\, ||Pf||$$

for all $f \in C_o^{\infty}(V)$.

(36) *P is (microlocally) elliptic outside the manifold $\{x = O, \xi = O\}$.*

(37) *m depends only on $|\gamma|$, i.e. : $(\alpha,\beta,\gamma) \in \mathfrak{m}$ implies $(\alpha,\beta,\gamma') \in \mathfrak{m}$ for every γ' with $|\gamma'| = |\gamma|$.*

Observe that (37) is always satisfied if there is only one variable y. If we denote by Σ the characteristic manifold of P, condition (36) reads $\Sigma \subset \{x=o, \xi=o\}$; for some relevant classes of examples at the end of this section it will come out exactly $\Sigma = \{x=o, \xi=o\}$, which represents prototype of symplectic characteristic manifold (every symplectic manifold can be reduced to this form by a canonical transformation). Observe also that the C^{∞} hypoellipticity of P could be regarded as a consequence of the "maximal estimates" in (35).

THEOREM 5 :

If (35), (36), (37) are satisfied, the operator P in (34) is analytic micro-hypoelliptic.

Proof :

In view of (36) and of the form of the operator, we may limit ourselves to prove that $\rho = (x=o,\ y=o,\ \xi=o,\ \eta=\eta_o) \notin WF_a\ P\ u$ implies $\rho \notin WF_a\ u$. We then observe that, shrinking V in (35) and fixing a sufficiently small conic neighborhood Γ of $(\xi=o,\ \eta=\eta_o)$, we have

$$(38) \qquad WF_a\ u \cap (V \times \Gamma \setminus \{x=o,\ \ =o\}) = \emptyset\ ;$$

this is again a consequence of (36), if we now assume $\rho \notin WF_a\ P\ u$.

To prove $\rho \notin WF_a\ u$ it will be sufficient, in view of lemma 1, to prove the estimates

$$(39) \qquad |||D_y^\delta\ \Psi_y(D)u|||_{L^2(U)} \leqslant c(cj)^{|\delta|} \qquad \text{for } |\delta| \leqslant j.$$

The operators $\Psi_j(D) = \Psi_j(D_x,D_y)$ are defined according to lemma 1 and lemma 4 ; in particular $\Psi_j(\xi,\eta)$ satisfies (33). Observe that $\Psi_j(D)u|_V \in C^\infty(V)$ since in (35) P is supposed to be C^∞ micro-hypoelliptic ; U in (39) is a neighborhood of the origin, $U \subset V$. The formal norm $|||.|||_{L^2(U)}$ is defined as in (35), where we consider now the norms in $L^2(U)$ of the terms of the sum.

We then take a cut-off function $\varphi(x,y) = \chi(x)\ \Psi(y) \in C_o^\infty(V)$, $\varphi = 1$ in U, in such a way that

(40) $\operatorname{supp} D_x^{\beta} \varphi \subset V \setminus \{|x| \leqslant \varepsilon\} \quad \text{for} \quad |\beta| \neq 0.$

Applying (35) we have

(41) $|||D_y^{\delta} \Psi_j(D)u|||_{L^2(U)} \leqslant |||\varphi D_y^{\delta} \Psi_y(D)u||| \leqslant$

$c\, ||P(\varphi D_y^{\delta} \Psi_j(D)u)|| \leqslant c\, \{||\varphi D_y^{\delta} \Psi_j(D)Pu|| +$

$||[P,\varphi]D_y^{\delta} \Psi_j(D)u|| + ||\varphi[P,\Psi_j(D)]D_y^{\delta}u||\}.$

It is easy to estimate the first term in the right hand side by using lemma 1 and the assumption $\rho \notin WF_a\, P\, u$. As for the term containing the commutator $[P,\Psi_j(D)]$, there is almost no change with respect to the arguments in the preceding section ; in fact, applying (33) we are reduced to a micro-local estimate of the derivatives of u in the elliptic region $\xi \neq 0$, where analyticity is trivial in view of (38). Let us fix attention on the ramaining term ; we shall begin by writing the commutator $[P,\varphi]$ as a linear combination of operators of the form :

(42) $x^{\alpha}(D_x^{\beta_1} D_y^{\gamma_1} \varphi) D_x^{\beta_2} D_y^{\gamma_2}$,

with $(\alpha,\ \beta=\beta_1+\beta_2,\ \gamma=\gamma_1+\gamma_2) \in \mathfrak{M}$, $|\beta_1|+|\gamma_1| \neq 0$. Assume first $|\beta_1| \neq 0$; then $\operatorname{supp} D_x^{\beta_1} D_y^{\gamma_1} \varphi$ satisfies the inclusion (40), and the corresponding contribution in (41) can be easily estimated by applying (38).

We are therefore reduced to the study of the terms

(43) $x^{\alpha}(D_y^{\gamma_1} \varphi) D_x^{\beta} D_y^{\gamma_2} D_y^{\delta} \Psi_j(D)u.$

Let us fix δ_1, $\delta_1 \leqslant \delta$, in such a way that $|\delta_1| = |\gamma_1|$ and re-write in (43)

$$(44) \qquad D_y^{\gamma_2} D_y^{\delta} = D_y^{\gamma_2+\delta_1} D_y^{\delta-\delta_1}$$

(observe that we may always suppose $|\delta| \geqslant$ order of the operator $P \geqslant |\gamma_1|$, since (39) is trivial for bounded values of $|\delta|$). We shall now apply the hypothesis (37) : if $(\alpha,\beta,\gamma) \in \mathfrak{m}$, then also $(\alpha,\beta,\gamma_2+\gamma_1) \in \mathfrak{m}$, since $|\gamma_2+\gamma_1| = |\gamma_2|+|\gamma_1| = |\gamma|$.

Therefore the L^2-norm of (43) is estimated by

$$(45) \qquad \max |D_y^{\gamma_1} \varphi| \; |||D_y^{\delta-\delta_1} \Psi_j(D)u|||_{L^2(V)} .$$

As in section 2, a strict conclusion can be obtained by nesting j+1 neighborhoods, $V = U_{j,o} \supset U_{j,1} \supset \ldots \supset U_{j,j} = U$, and constructing related cut-off functions $\varphi_{j,1},\ldots,\varphi_{j,j}$, for which we assume now

$$(46) \qquad |D_x^{\beta} D_y^{\gamma} \varphi_{j,h}| \leqslant c(cj)^{|\beta|+|\gamma|}$$

Replacing in (41) φ with $\varphi_{j,h}$, $h = |\delta|$, from (46) and from the inductive hypothesis $|||D_y^{\delta} \Psi_j(D)u|||_{L^2(U_{j,h})} \leqslant K(Kj)^{|\delta|}$ we get for (45) the required bounds.

$$(47) \qquad c(cj)^{|\gamma_1|} K(Kj)^{|\delta|-|\delta_1|} \leqslant K(Kj)^{|\delta|} ,$$

if the constant K has been chosen sufficiently large. Theorem 5 is therefore proved.

Let us observe that the hypothesis and the conclusion of theorem 5 can be microlocalired at a given point of the manifold $\{\xi = 0\}$. Precisely, fixing for example attention on $\rho = (x=o,\ y=o,\ \xi=o,\ \eta_1=\eta_1^o,\ \eta_2=o,\dots,\eta_\nu=o)$, we shall introduce the condition

$$(37)' \qquad (\alpha,\beta,\gamma = (\gamma_1,\dots,\gamma_\nu)) \in \mathfrak{m} \qquad \underline{\text{implies}}$$

$$(\alpha,\beta,\gamma_1+\gamma_2'+\dots+\gamma_\nu',\gamma_2-\gamma_2',\dots,\gamma_\nu-\gamma') \in \mathfrak{m} \ \underline{\text{for all}}\ \gamma_2'\leqslant\gamma_2,\dots,\gamma_\nu'\leqslant\gamma_\nu .$$

A repetition of the preceding arguments shows then that (35), (36) and (37)' imply the analytic micro-hypoellipticity of P and ρ (i.e. : $\rho \notin WF_a\, P\, u \rightarrow \rho \notin WF_a u$ for every distribution u). We end this section with some application of theorem 5.

Example 6 :

We consider the operators P of the form (34) for which :

$$(48) \qquad \mathfrak{m} = \{(\alpha,\beta,\gamma),\ |\beta|+|\gamma| \leqslant m,\ |\alpha| = |\beta|+|\gamma|(1+h)-m\},$$

where m and h are fixed positive integers.

We assume P is elliptic for $x \neq 0$ and we introduce an auxiliary operator L_p acting on $\mathcal{S}$, the Schwartz space with respect to the x-variables :

$$(49) \qquad L_p = \sum_{(\alpha,\beta,\gamma)\in\mathfrak{m}} c_{\alpha\beta\gamma}\, \eta^\gamma\, x^\alpha\, D_x^\beta .$$

From a well-known result of Grushin we have that P is C^∞ micro-hypoelliptic (and the estimates in (35) are satisfied) if and only if :

$$(50) \qquad \mathrm{Ker}\, L_p \cap \mathcal{S} = \emptyset .$$

Since (36) and (37) are obviously true, from theorem 5 we have that P is also analytic micro-hypoelliptic (such result was already proved in Rodino [12] by means of the same technique of a-priori estimates). Note that the operator in (13) is of the preceding form with m = 1, h = 2k. An example with double characteristics in $\mathbb{R}^2_{x,y}$ is given by

$$(51) \qquad P = (D_x - r_1 x^h D_y)(D_x - r_2 x^h D_y) + \lambda x^{h-1} D_y ,$$

where $\operatorname{Im} r_1 \cdot \operatorname{Im} r_2 < 0$ and (50) is satisfied if avoids a certain discrete set of values in $\mathbb{C}$.

Métivier [10] has studied the case when h = 1 in (48) ; supposing $c_{\alpha\beta\gamma}$ in (34) are classical analytic pseudo differential operators of order zero, he proves a result of analytic micro-hypoellipticity which can be stated in a geometric invariant form.

<u>Example 7</u> :

Theorem 5 applies to some other operators which are not in the class defined in example 6. Consider for example the "sum of squares"

$$(52) \qquad P = \Delta_x + (\sum_j x_j^{2k_j})\Delta_y ,$$

where k_j are fixed positive integers ; the assumptions (35), (36), (37) are satisfied and P is therefore analytic micro-hypoelliptic.

We understand in (52) that <u>all</u> the x-variables actually appear in the sum $\sum_j x_j^{2k_j}$; this would be unessential for the C^∞ hypoellipticity, but it is nesessary for the simultaneous validity of (36), (37) and the analytic hypoellipticity of P (cf. Baouendi-Goulaouic [1]).

3 - OPERATORS WITH INVOLUTIVE CHARACTERISTICS :

Let Ω be a n-dimensional real analytic manifold. Let Σ be an analytic conic submanifold of $T^*\Omega\setminus 0$ having codimension ν, $1 \leq \nu < n$; Σ is said regular involutive when the following condition is satisfied.

(53) *There exist local homogeneous analytic equations $u_1(x,\xi)=\ldots=u_\nu(x,\xi)=0$ for Σ such that $du_1,\ldots,du_\nu$ and $\sum_{j=1}^{n} \xi_j\, dx_j$ are independent on Σ and the Poisson brackets $\{u_j,u_k\}$, $1 \leq j, h \leq \nu$, vanish on Σ.*

We shall now consider a classical analytic pseudo differential operator P in Ω, with symbol $p(x,\xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x,\xi)$, where the p_{m-j} are positively homogeneous of degree m-j, and we shall suppose that p_m vanishes exactly of order M, $M \geq 2$, on $\Sigma = p_m^{-1}(0)$, i.e. for every $K \subset\subset \Omega$ there is a constant C such that :

$$C^{-1}\, d_\Sigma(x,\xi)^M \leq |p_m(x,\xi)|/|\xi|^m \leq C\, d_\Sigma(x,\xi)^M \tag{54}$$

for all $(x,\xi) \in K \times \mathbb{R}^n$, $|\xi| \geq 1$ ($d_\Sigma(x,\xi)$ is the distance from $(x,\xi/|\xi|)$ to Σ).

For any $\rho \subset \Sigma$ and for any C^∞ vector field Y defined in a neighborhood of ρ we set :

$$I_p(\rho,Y) = (M!)^{-1}(Y^M p_m)(\rho) + p'_{m-1}(\rho) \ , \tag{55}$$

where p'_{m-1} denotes the subprincipal symbol of P : $p'_{m-1} = p_{m-1} + (i/2) \sum_{j=1}^{n} \partial^2 p_m/\partial x_j \partial\xi_j$ (observe that p'_{m-1} if $k \geq 3$) ; $I_p(\rho,Y$ depends only on the equivalence class of $Y(\rho)$ in $T_\rho(T^*\Omega\setminus 0)/T_\rho(\Sigma)$. Our first result is the following (cf. Liess-Rodino [8], [9]) :

THEOREM 8 :

Under the assumptions (53), (54) for the operator P and its characteristic manifold Σ, *suppose further* $I_p(\rho,Y) \neq 0$ *for every* ρ *and* Y. *Then we have that P is* G^s *micro-hypoelliptic for* $s \geq s_0 = M/(M-1)$.

Observe that the condition $I_p \neq 0$ is necessary and sufficient for the C^∞ hypoellipticity of P with lass of one derivative.

Proof :

We shall limit ourselves to a sketch. As in the preceding proofs, we may fix attention on an arbitrary $\rho \in \Sigma$. It is a well known result of Duistermaat-Hörmander and Sato-Kawai-Kashiwara that under the assumption (53) there exists a (real analytic) homogeneous canonical transformation χ, defined in a neighborhood of ρ, squch that in a conic neighborhood of $\chi(\rho)$ we have :

$$(56) \qquad \chi(\Sigma) = \{(z,t;\zeta,\tau),\ \zeta = (\zeta_1,\dots,\zeta_\nu) = 0\}$$

Considering the corresponding Fourier integral operator F we have that the symbol of $\tilde{P} = F\,P\,F^{-1}$ satisfies the same hypotheses of $p(x,\xi)$; in particular, it is easily seen that $I_{\tilde{p}}(\chi(\rho),\ d\chi(\rho)Y) = I_p(\rho,Y) \neq 0$.

We also observe that there is no lass generality in assuming $m = M$. Summing up, we may limit ourselves to argue on $p(x,\xi) \sim \sum_{j=0}^{\infty} p_{M-j}(x,\xi)$ defined in $\Omega \times \Gamma$, where now Ω is an open neighborhood of $x_0 = (z_0,t_0) \in \mathbb{R}^n_x = \mathbb{R}^\nu_z \times \mathbb{R}^{n-\nu}_t$, and Γ is a conic neighborhood of $\xi_0 = (\zeta_0,\tau_0) \in \mathbb{R}^n_\xi = \mathbb{R}^\nu_\zeta \times \mathbb{R}^{n-\nu}_\tau$, with $\rho = (x_0,\xi_0) \in \Sigma = p_M^{-1}(0)$ with equations $\zeta = (\zeta_1,\dots,\zeta_\nu) = 0$. In view of the second estimate in (54) we may write :

$$(57) \qquad p_M(x,\xi) = \sum_{|\alpha|=M} q_\alpha(x,\xi)\zeta^\alpha$$

where the q_α are analytic homogeneous of degree zero ; moreover

$$(58) \qquad |p_M(x,\xi)| \geqslant c|\zeta|^M$$

in view of the first estimate in (54). The condition $I_P \neq 0$ reads now

$$(59) \qquad p^*(x,\xi) = \sum_{|\alpha|=M} p_\alpha(z,t;o,\tau)\zeta^\alpha + p_{M-1}(z,t;o,\tau) \neq 0.$$

From (57), (58), (59) one deduces easily the following estimates:

$$(60) \qquad |p(x,\xi)| \geqslant c(|\zeta|^M + |\tau|^{M-1}) ,$$

$$(61) \qquad |D_x^\alpha D_\xi^\beta p(x,\xi)| \leqslant C^{|\alpha|+|\beta|+1} \alpha! \beta! (1+|\zeta|^M+|\tau|^{M-1}) (1+|\xi|)^{-|\beta|s_o} ,$$

with $s_o = M/(M-1)$; the inequalities are valid for large $|\xi|$, after a shrinking of Γ.

The conclusion in theorem 8 is then given by the following

LEMMA 9 :

Let Ω be a enighborhood of $x_o \in \mathbb{R}^n$ and let Γ be a conic neighborhood of $\xi_o \in \mathbb{R}^n\backslash\{0\}$. Let $p(x,\xi)$ be a classical analytic symbol in $\Omega \times \Gamma$ and assume there exist C, $c > 0$ and r such that for large $|\xi|$

$$(62) \qquad |p(x,\xi)| \geqslant c|\xi|^r ,$$

$$(63) \qquad |D_x^\alpha D_\xi^\beta p(x,\xi)| \leqslant C^{|\alpha|+|\beta|+1} \alpha!\beta! |p(x,\xi)| (1+|\xi|)^{-|\beta|/s} ,$$

for some s, $1 \leqslant s < \infty$. Let $p(x,D)$ be the (micro) pseudo differential operator associated to $p(x,\xi)$; then $(x_o,\xi_o) \notin WF_s\, p(x,D)u$ implies $(x_o,\xi_o) \notin WF_s u$ for all $u \in \mathcal{E}'(\Omega)$.

For the proof of lemma 9 let us refer to Liess-Rodino [8] (see also Bolley-Camus-Métivier [2]).

Since (60), (61) imply (62), (63) with $r = M-1$, $s \geqslant s_o$, theorem 8 is proved.

<u>Example 10</u> :

Let us consider in $\mathbb{R}^\nu_z \times \mathbb{R}_t$ the operator

$$P = D^2_{z_1} + \ldots + D^2_{z_\nu} + \lambda\, D_t . \tag{64}$$

The principal symbol has involutive characteristic manifold $\Sigma = \{\zeta_1 = \ldots = \zeta_\nu = 0\}$ and it satisfies there (54). We have $I_P \neq 0$ if and only if

$$\zeta_1^2 + \ldots + \zeta_\nu^2 + \lambda\, \tau \neq 0 , \tag{65}$$

that is, if and only if $\mathrm{Im}\,\lambda \neq 0$; under this assumption P is G^2 micro-hypoelliptic, according to theorem 8.

Let us observe that the result is the best possible in the classes G^s ; in fact, independently of the value of λ, P is never G^s-hypoelliptic for $1 \leqslant s < 2$ (see for example Hörmander [7]) and actually a theorem of propagation for $WF_s\, u$, $1 \leqslant s < 2$, is valid on the manifolds

$$\Lambda_{t_o,\tau_o} = \{\zeta_1 = \ldots = \zeta_\nu = 0,\ \tau = \tau_o,\ t = t_o\}. \tag{66}$$

When λ is real, we have no hypoellipticity ; in particular, when $\lambda = 0$ we have propagation along Λ_{t_o,τ_o} also for the wave front sets $WF_s\, u$, $s \geqslant 2$, and $WF\, u$, whereas when λ is real $\lambda \neq 0$ the propagation of the wave front sets $WF_s u$, $s \geq 2$ and WF u may be limited to straight lines on Λ_{t_o,τ_o}.

When the condition $I_p \neq 0$ is not satisfied we may still have Gevrey hypoellipticity under suitable additional assumptions. Let us for example consider the class of the operators for which (54) is joined with the following estimates on the lower order terms ; $k \geq 2$ is a fixed real number and it will be convenient to argue on a symbol with asymptotic expansion $p(x,\xi) \sim \sum_{j=o}^{\infty} p_{m-j/k}(x,\xi)$, where $p_{m-j/k}$ is positively homogeneous of degree $m-j/k$. We set

$$(67) \qquad |p_{m-j/k}(x,\xi)|/|\xi|^{m-j/k} \leq C\, d_{\Sigma}(x,\xi)^{M-j} \qquad \text{for } 0 < j < M.$$

More generally, if a term $p_{m-m'}(x,\xi)$ positively homogeneous of degree m-m' occurs in the asymptotic expansion of $p(x,\xi)$, it will be assumed to vanish on Σ of order m'k (at least) ; whenever $m' = j/k$, $j = 0,1,2,\ldots,M$, the presence of such terms will not affect the result of Gevrey hypoellipticity we are going to state. The case when $k = 2$ in (67) is studied in Liess-Rodino [9], where it is proved that the hypellipticity with lass of M/2 derivatives implies for $P = p(x,D)$ the G^2 micro-hypoellipticity. Here we shall concentrate on the case $k > 2$. Let us begin by defining

$$(68) \qquad J_P(\rho,Y) = \sum_{j=o}^{M} \frac{1}{(M-j)!} \; (Y^{M-j}\; p_{m-j/k})(\rho) \;,$$

where as in (55) $\rho \in \Sigma$ and Y is a vector field defined in a neighborhood of ρ.

THEOREM 11 :

Let P be an analytic pseudo differential operator, whose symbol satisfies (54) and (67), with regular involutive characteristic manifold Σ. *Assume* $J_P(\rho,Y) \neq 0$ *for every* ρ *and Y. Then P is* G^s*-micro hypoelliptic for* $s \geq s_o = k/(k-1)$.

Theorem 8 is recaptured for $k \geqslant 3$ by observing that $J_P = I_P$ if $M = k$ and $p_{m-j/k} \equiv 0$ for $j = 1,\dots,M-1$.

The proof of theorem 11 is similar to the proof of theorem 8. We first observe that the estimates (67) have an invariant meaning ; $J_P(\rho,Y)$ is also invariant under changes of variables and conjugation by Fourier integral operators if $k > 2$.

Arguing as in the proof of theorem 8 we are then reduced to consider an operator with symbol

$$(69) \qquad p(x,\xi) = \sum_{0\leqslant|\alpha|\leqslant M} q_\alpha(x,\zeta)\zeta^\alpha + r(x,\xi)$$

where q_α is analytic homogeneous of degree $(M-|\alpha|)(k-1)/k$ and $r(x,\xi)$ is a calssical analytic symbol of order $< M(k-1)/k$. Under the hypotheses of theorem 11 we have for $s_o = k/(k-1)$

$$(70) \qquad |p(x,\xi)| \geqslant c\ (|\zeta|^M + |\tau|^{M/s_o})$$

$$(71) \qquad |D_x^\alpha D_\xi^\beta p(x,\xi)| \leqslant C^{|\alpha|+|\beta|+1}\ \alpha!\ \beta!(1+|\zeta|^M + |\tau|^{M/s_o})(1+|\xi|)^{-|\beta|/s_o}$$

and the application of lemma 9 gives therefore the conclusion.

4 - FURTHER RESULTS AND OPEN PROBLEMS :

We shall begin with a further analysis of the operators with involutive characteristics considered in the preceding section ; we shall actually argue on a wider class, which we define in the following way. Let P be a classical analytic pseudo differential operator in $\Omega \subset \mathbb{R}^n$ with symbol $p(x,\xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x,\xi)$ and assume its characteristic manifold Σ is regular involutive. Let us observe that the functions $u_1(x,\xi),\ldots,u_\nu(x,\xi)$ in (53) can be assumed positively homogeneous of degree 1 ; we shall also suppose, as it is always possible, that $\{u_j,u_k\} = 0$ in a whole neighborhood of Σ. Let (54) be valid, and set.

DEFINITION 12 :

We say that P satisfies the s-levi condition, $1 < s < \infty$, if it can be written in the form (microlocally, in a conic neighborhood of every fixed $\rho \in \Sigma$) :

$$(72) \qquad P = \sum_{0 \leqslant |\alpha| \leqslant M} A_\alpha U^\alpha \ ;$$

where $U_1,\ldots,U_\nu$ are classical analytic pseudo differential operators with principal symbols $u_1(x\xi),\ldots,u_\nu(x\xi)$ and A_α are classical analytic pseudo differential operators of order $\leqslant m-M+(M-|\alpha|)/s$.

The definition 12 has an invariant meaning, and it can be generalired to symbols with lower order trms of different degree of homogenety (as for example the operators in theorem 11).

If P satisfies the s-Levi condition for some s, $1 < s \leqslant 2$, then its symbol satisfies the inequalities (67) with $k = s/(s-1) \geqslant 2$, and the opposite is also true. On the contrary, if $k < 2$ the inequalities (67) have no longer an invariant meaning, since after changes of variables or conjugation by Fourier integral operators a homogeneous term p_m vanishing of order M on Σ may give rise to terms of degree m-1, m-2,... vanishing exactly of order M-2, M-4,... on Σ. Theorem 11 can be easily generalired to the operators in Definition 12 ; in fact, modulo conjugation by elliptic Fourier integral operators and multiplication by elliptic factors, we may assume $u_1 = \zeta_1, \ldots, u_n = \zeta_n$, $m = M$, nd write :

$$(73) \qquad P = \sum_{o \leqslant |\alpha| \leqslant M} A_\alpha D_z^\alpha .$$

Denoting by $a_\alpha(x,\xi) = q_\alpha(z,t;\zeta,\tau)$ the principal symbol of A_α, which is a homogeneous function of degree $(M-|\alpha|)/s$, and using the arguments in the proofs of section 3, we obtain that P is G^s-micro-hypoelliptic if

$$(74) \qquad \sum_{o \leqslant |\alpha| \leqslant M} a_\alpha(z,t;o,\tau)\zeta^\alpha \neq 0 \quad \underline{\text{for}} \quad \tau \neq 0.$$

The condition (74) is equivalent to the condition $J_P \neq 0$ in theorem 11 when $s < 2$ (i.e. $k > 2$). For $s \geqslant 2$ there is not a simple geometric-invariant formulation of (74) ; see nevertheless the preceding theorem 8 for the case $k = M = 2$ and Helffer [3], where the case $M \geqslant k = 2$ was studied in the C^∞ framework.

We shall now generalire to the operators in Definition 12 the results of non-Gevrey hypoellipticity and WF_s propagation of the Exeample 10. Since we have not precise references for the result (and we are not stable to prove it) we shall state it as.

CONJECTURE 13 :

Assume P satisfies the s_o-Levi condition, $1 < s_o < \infty$. Let ρ be a point of the characteristic manifold Σ and let Λ_ρ be the canonical leaf of Σ for ρ (after a canonical transformation we may write $\Sigma = \{(z,t;\zeta,\tau), \zeta = o\}$; if $\rho = (z_o,t_o,o,\tau_o)$, then $\Lambda_\rho = \{(z,t;\zeta,\tau),\ t = t_o,\ \zeta = o,\ \tau = \tau_o\}$). There exists a conic *neighborhood V of* ρ *such that for every* $s < s_o$ *:*

i) Let u be in $\mathcal{E}'(\Omega)$ *and assume* $WF_s\, Pu \cap V = \emptyset$ *; if* $\rho \in WF_s u$ *, then* $\Lambda_\rho \cap V \subset WF_s u$.

ii) There exists $u \in \mathcal{E}'(\Omega)$ *suche that* $WF_s\, Pu \cap V = \emptyset$ *and* $WF_s u \cap V = \Lambda_\rho \cap V$.

Conjecture 13 interpolates two well known results ; namely when $s = 1$, and (72) is a consequence of (54), the analytic wave front set propagates along Λ_ρ ; when $s = +\infty$, $\frac{1}{j} = o$, and (72) is the standard Levi condition, we have propagation also for the C^∞ wave front set.

In the second part of this section we shall try to give a more complete picture for the operators with multiple characteristics, recalling what we have proved in sections 2 and 3, adding references to other results and pointing out what is missing to a general theory of the analytic and Gevrey-hypoellipticity. Actually, we shall limit ourselves to consider and analytic pseudo differential operator P with double characteristics ; precisely, we shall assume that the principal symbol p_m satisfies (54) with $M = 2$ (and without any hypothesis on the characteristic manifold Σ , which we now simply suppose to be a C^∞ submanifold of $T^*\Omega\backslash 0$). We shall also assume P is already known to be C^∞ hypoelliptic with lass of one derivative.

When Σ is symplectic, P is analytic hypoelliptic. In fact, after conjugation by Fourier integral operators and multiplication by elliptic factors, we may suppose $\Sigma = \{(x,y;\xi,\eta),\ x = o,\ \xi = o\}$ and P has then the form :

$$(75) \qquad P = \sum_{|\alpha|+|\beta|=2} c_{\alpha\beta}\, x^\alpha\, D_x^\beta\, D_{y_1}^{2-|\beta|} + c_o\, D_{y_1} ,$$

here we argue in a conic neighborhood of the point $\rho = (o,o;o,\ \eta = \eta_1^o,\ \eta_2 = o,\dots,\eta_\nu = o)$, as in the remark the end of the proof of theorem 5, with $\eta_1^o > 0$. When $c_{\alpha\beta}$, c_o are constants, from the same remark and from example 6 we have that P is micro-hypoelliptic and analytic micro-hypoelliptic at ρ if

$$(76) \qquad \mathrm{Ker}\ (\sum_{|\alpha|+|\beta|=2} c_{\alpha\beta}\ x^\alpha\ D_x^\beta + c_o) \cap \quad = \emptyset\ ,$$

which can be translated into an explicit algebraic condition on $c_{\alpha\beta}$, c_o. In general when $c_{\alpha\beta}$ and c_o are analytic pseudo differential operators of order zero, we shall deduce the analytic hypoellipticity from the ready mentioned result of Métivier [10].

The condition Σ symplectic is nearly necessary for the analytic hypoellipticity. Precisely, let us call canonical leaf of Σ a connectec analytic manifold $V \subset \Sigma$ (of dimension $\geqslant 1$) such that $T_\rho V = T_\rho\ \Sigma \cap (T_\rho\ \Sigma)^{\perp}$ for every $\rho \in \Sigma$, where $(T_\rho\ \Sigma)^{\perp}$ is the orthogonal of $T_\rho\ \Sigma$ with respect to the symplectic two form ω. Suppose for examlple the rank of the restriction of ω to Σ is constant, and Σ is non-symplectic ; then after a cononical transformation we have $\Sigma = \{(x,y,t;\xi,\eta,\tau),\ x = o,\ \xi = o,\ \eta = o\}$ and the canonical leaves are given by $V = \{x = o,\ t = t_o\ ,\ \xi = o\ ,\ \eta = o\ ,\ \tau = \tau_o\}$; in particular, if the variables x and ξ are missing, Σ is involutive and we recapture the canonical leaves of example 10 and conjecture 13. The result is now following : if a canonical leaf V exists, then P is not analytic hypoelliptic (Métivier [11]) and there is a propagation of the analytic wave front set along V (see Sjöstrand [13]).

In particular, if the rank of the restriction of ω to Σ is constant and Σ is non-symplectic, we have not analytic hypoellipticity ; a representative example is given by

$$(77) \qquad P = \Delta_x + \Delta_y + |x|^2 \Delta_t$$

(cf. the remark at the end of example 7 and Bouendi-Goulaouic [1]).

When the restriction of ω to Σ has not constant rank the ciscussion of the analytic hypoellipticity us more delicate ; let us mention that, according to Sjöstrand [14], there are examples of analytic hypoelliptic operators with double characteristics whose characteristic manifold Σ is not symplectic (but canonical leaves do not exist on Σ , of course).

We conclude with some remarks concerning the Gevrey hypoelipticity of an operator P with double characteristics ; we shall suppose as before P is already known to be C^∞ hypoelliptic with lass of one derivative. Under the additional assumption "Σ regular involutive" we have from theorem 8 that P is G^s-hypoelliptic for $s \geqslant 2$, whereas the wave front ser WF_s , $1 < s < 2$, propagates along the canonical leaves of Σ, according to conjecture 13. A further natural conjecture suggested by the example (77) is that these two statements hold in general, without any assumption on the geometry of Σ.

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