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ON THE TWO PARAMETER ITO EQUATIONS

by

CONSTANTIN TUDOR

CONTENT

o. Introduction

In the last time a number of papers are devoted to studying the two parameter Ito equation

$$(o.1) \qquad X_z = H_z + \int_0^z F(u,X)\,dw_u + \int_0^z G(u,X)\,du \quad ; \; z \in R_+^2$$

Here (w_z) is a two parameter Wiener process and the first integral is in the Ito sense.

In section 1 some important results concerning the existence and the uniqueness of solutions of (o.1) are formulated.

When the equation (o.1) possesses a weak solution which is pathwise unique, we give in section 2 a condition which guarantees the convergence in maximal quadratic mean of successive approximations to the solution. Also, a convergence theorem in maximal quadratic mean is established under a topology on the set of drift terms which is weaker than the one usually used.

In section 3 we give bounds for $E(\sup_{u\le z} \| X_u - X_u^n \|^2)$, where X^n is an approximant process obtained from the solution X of (o.1) by a determinist discretization of time.

1. Existence and uniqueness theorems

We introduce the following notations:

$C(R_+^2, R^d)$ = the collection of all continuous functions from R_+^2 to R^d endowed with the uniform convergence on compact subsets.

$D(R_+^2, R^d)$ = the collection of all right continuous and with left hand limits functions from R_+^2 to R^d endowed with the Scorohod topology.

$\Gamma = C(R_+^2, R^d)$ or $D(R_+^2, R^d)$.

$\mathcal{R}_z$ = the Borel field on Γ generated by the sets of the form $\{f \in \Gamma \,; f(u) \in C\}$, where $u \le z$ and $C \in \mathcal{B}_{R^d}$.

$\|f\|_z = \sup_{u \leq z} \|f(u)\|$ if $f \in \Gamma$.

$\mathcal{M}a(\Gamma)$=the collection of all real valued functions α defined on $R^2_+ \times \Gamma$ which are $\mathcal{B}_{R^2_+ \times \Gamma}$-measurable and such that for every z,$\alpha(z,.)$ is $\mathcal{R}_z$-measurable.

Let $(\Omega,\mathcal{F},\mathcal{F}_z,P;z\geq o)$ be a filtered probability space and $(w_z)_{z\geq o}=(w^1_z,\ldots,w^d_z)_{z\geq o}$ be a R^d-valued two parameter Wiener process i.e

1) w is null on the axes and has continuous paths,

2) for every z,w_z is $\mathcal{F}_z$-measurable(w is $\mathcal{F}_z$-adapted) and for each rectangle $[(s,t),(\bar{s},\bar{t})]$ the increment $\square_{s,t} w_{\bar{s},\bar{t}} = w_{\bar{s},\bar{t}} - w_{\bar{s},t} - w_{s,\bar{t}} + w_{s,t}$ is independent on the Borel field $\mathcal{B}(\mathcal{F}_{s,v} \cup \mathcal{F}_{u,t}; u\leq s \text{ or } v \leq t) = \bar{\mathcal{F}}_{s,t}$.

3)the processes $(w^1_z),\ldots,(w^d_z)$ are independent,

4)every $w^j_{s,t}$ has a gaussian law with mean o and variance st.

Let $F:R^2_+ \times\Gamma\times\Omega \longrightarrow R^d \otimes R^d$, $G:R^2_+\times\Gamma\times\Omega \longrightarrow R^d$ be $\mathcal{B}_{R^2_+\times\Gamma} \otimes \mathcal{F}$-measurable such that P-a.s ω the mappings $F(.,\omega)$,$G(.,\omega)$ have the components in $\mathcal{M}a(\Gamma)$.

F is called the diffusion coefficient and G the drift coefficient.

Definition 1. A strong solution of (o.1) with initial process H is a process $(X_z)_{z\geq o}$ with paths in Γ (H is supposed having the paths in Γ), $\mathcal{F}_z$-adapted and which satisfies (o.1) with probability one.

Definition 2. A system $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathcal{F}}_z,\bar{P},\bar{H},\bar{F},\bar{G},\bar{w},\bar{X};z\geq o)$ is a weak solution of (o.1) if

1)$(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathcal{F}}_z,\bar{P})$ is a filtered probability space,

2)On $(\bar{\Omega},\bar{\mathcal{F}},\bar{P})$ are given the processes $\bar{H}_z$,$\bar{F}(z,f)$,$\bar{G}(z,f)$,$\bar{w}_z$,$\bar{X}_z$ such that

2.1)The system $(\bar{H}_z,\bar{F}(z,f),\bar{G}(z,f))$;$z\geq o$,$f\in\Gamma$, is stochastic equivalent with the system $(H_z,F(z,f),G(z,f))$;$z\geq o$;$f\in\Gamma$, $\bar{H}$ has the paths in Γ,

2.2)$(\bar{w}_z)$ is a two parameter Wiener process, $(\bar{X}_z)$ is a process with paths in Γ, $\bar{\mathcal{F}}_z$-adapted and which satisfies (o.1) with probability one.

It is clear that every strong solution is a weak solution.

Definition 3. We say that (o.1) has a pathwise unique weak solution if every two weak solutions X,Y on the same filtered probability space and with the same initial processes and the coefficients and having the same two parameter Wiener process are indistinguishable.

Definition 4. We say that (o.1) has a unique in law solution if every two weak solutions X, Y have the same law on Γ.

Remark_1. There is a deep difference between strong and weak solutions. The investigation of strong solutions uses usually the successive approximation and for weak solutions one use in principal the two parameter martingale technique.

Remark_2. The famous result of Watanabe and Yamada from the one dimensional time which states that pathwise uniqueness implies uniqueness in law holds in the two parameter case too.
The study of two parameter Ito equations begins around 1972 with existence and uniqueness theorems for Lipschitz coefficients (without past) (see Cairoli[1], Ponomarenco[7], Tzarenco[10]).
Let us define

$B_2(\mathcal{F}_z,\Gamma)$ = the R^d-valued processes $(f_z)_{z\geq 0}$ having paths in Γ, $\mathcal{F}_z$-adapted and such that for every T>o

$$\int_0^T\int_0^T E(\|f_z\|^2)dz < \infty$$

$B_2(\mathcal{F}_z,\Gamma)$ = the processes $(f_z)_{z\geq 0}$ from $B_2(\mathcal{F}_z,\Gamma)$ such that for every T>o verify

$$\int_0^T\int_0^T E(\|f\|_z^2)dz < \infty$$

The following two theorems give convenient forms of existence and uniqueness theorems in the Lipschitz case.

Theorem_1 (Ghihman-Piasetzkaia[2]). Assume that for every T>o there exists $C_T>0$ such that for $z\leq(T,T)$ and $g,h\in\Gamma$

(1) (Lipschitz_condition): $\|F(\cdot,g)-F(\cdot,h)\|_z+\|G(\cdot,g)-G(\cdot,h)\|_z\leq C_T\|g-h\|_z$

(2) (Growth_condition): $\|F(\cdot,g)\|_z+\|G(\cdot,g)\|_z\leq C_T(1+\|g\|_z)$

Then for $H\in B_2(\mathcal{F}_z,\Gamma)$ the equation (o.1) has a strong solution in $B_2(\mathcal{F}_z,\Gamma)$ and the solution is pathwise unique in $B_2(\mathcal{F}_z,\Gamma)$.
Let us define

$S(\mathcal{F}_z,\Gamma)$ = the R^d-valued processes $(f_z)_{z\geq 0}$ with paths in Γ, $\mathcal{F}_z$-adapted which verify for each T>o

$$\sup_{z\leq(T,T)} E(\|f_z\|^2) < \infty$$

Theorem_2 (Yeh[14]): Let $F:R_+^2\times\Gamma\longrightarrow R^d\otimes R^d$, $G:R_+^2\times\Gamma\longrightarrow R^d$ be with the components in $\mathcal{M}a(\Gamma)$. Moreover assume that there exists a Radon measure λ on R_+^2 such that for every $T>o$ there exists $C_T>o$ such that

(1)(Lipschitz condition): $\|F(z,g)-F(z,h)\|^2+\|G(z,g)-G(z,h)\|^2\leq$

$$\leq C_T\left\{\int_o^z\|h(u)-g(u)\|^2 d\lambda(u)+\|g(z)-h(z)\|^2\right\}$$

(2)(Growth condition): $\|F(z,g)\|^2+\|G(z,g)\|^2\leq C_T\left\{\int_o^T\|g(u)\|^2 d\lambda(u)+\|g(z)\|^2+1\right\}$

for $z\leq(T,T)$ and $g,h\in\Gamma$.

Then for $H\in S(\mathcal{F}_z,\Gamma)$ the equation (o.1) possesses a pathwise unique strong solution in $S(\mathcal{F}_z,\Gamma)$.

For more general coefficients the existence of weak solutions can be obtained. The next three results give a answer in this direction.

Theorem 3 (Ghihmam-Piasetzkaia[2]): Let $F:R_+^2 x C(R_+^2,R^d)x\Omega\longrightarrow R^d\otimes R^d$, $G:R_+^2 x C(R_+^2,R^d)x\Omega\longrightarrow R^d$ be measurable functions such that P-as ω the mappings $F(.,\omega)$, $G(.,\omega)$ have the component in $\mathcal{M}a(C(R_+,R^d))$.

Moreover assume that

(1)(Growth condition): For every $T>o$ there exists $L_T>o$ such that

$$\|F(.,g)\|_z+\|G(.,g)\|_z\leq L_T(1+\|g\|_z)$$

for $z\leq(T,T)$ and $g\in C(R_+^2,R^d)$.

(2)The functions $(z,g)\longrightarrow\int_o^z F(u,g)du$, $(z,g)\longrightarrow\int_o^z G(u,g)du$ are continuous on $R_+^2 x C(R_+^2,R^d)$ and H has the paths in $C(R_+^2,R^d)$.

(3)For every $T>o$ there exists $K_T>o$ such that

$$E(\|H\|^6_{(T,T)})\leq K_T\ ;\ E\Big(\sup_{\substack{o\leq z'-z\leq(h,h)\\ z,z'\leq(T,T)}}\|H_z-H_{z'}\|^6\Big)\leq K_T h^3$$

Then the equation (o.1) has a continuous weak solution.

Theorem 4 (Ghihman[3]) Let $F:R_+^2 x R^d\longrightarrow R^d\otimes R^d$, $G:R_+^2 x R^d\longrightarrow R^d$ such that

(1)F is continuous, nonsingular and G is measurable.

(2)(Growth condition): For every $T>o$ there exists $C_T>o$ such that

$$\|F(z,p)\|+\|G(z,p)\|\leq C_T(1+\|p\|)\quad\text{for } z\leq(T,T) \text{ and } p\in R^d.$$

Then for each $x\in R^d$ there exists a continuous weak solution of the following equation

$$(1.1)\qquad X_z=x+\int_o^z F(u,X_u)dw_u+\int_o^z G(u,X_u)du$$

Theorem 5(Tudor[8]).Suppose we are given $F,G:R_+^2 xR \longrightarrow R$ measurable, bounded and moreover assume that $F(z,x)\geq C>o$ for every z,x. Then the equation (1.1) has a continuous weak solution.

Remark 3.Unfortunately there is no satisfactory results concerning the uniqueness in law of weak solutions.

2.Convergence Theorems

Let $F(z,x):R_+^2 xR^d \longrightarrow R^d \otimes R^d$, $G(z,x):R_+^2 xR^d \longrightarrow R^d$ be measurable in (z,x), locally bounded in the first variable and continuous in the second variable.

We introduce the following assumption:

Assumption (A):There exists a function $\omega:R_+^2 xR_+ \longrightarrow R_+$ such that

(i) $\omega(.,o)=o$, ω is continuous in both variables,concave nondecreasing in the second variable.

For every nonnegative constants T,C_1,C_2,C_3 the trivial solution is the unique solution of every inequation

$$o\leq u(s)\leq C_1\int_o^s\int_o^T \omega(p+T,q,u(p))dpdq$$

$$o\leq u(t)\leq C_2\int_o^T\int_o^t \omega(p,q+T,u(q))dpdq$$

$$o\leq u(s,t)\leq C_3\int_o^s\int_o^t \omega(p+T,q+T,u(p,q))dpdq$$

(ii) $\|F(z,x)-F(z,y)\|^2 + \|G(z,x)-G(z,y)\|^2 \leq \omega(z,\|x-y\|^2)$

Consider the following Ito equation

(2.1) $X_z=H_z+\int_o^z F(u,X_u)dw_u + \int_o^z G(u,X_u)du$

Remark 4.Under assumption (A) the stochastic equation has a pathwise unique solution.

Theorem 6.Suppose that (2.1) possesses a weak solution X.Define the successive approximations associated to X by

$$X_z^o=H_z$$

$$X_z^{n+1}=H_z +\int_o^z F(u,X_u^n)dw_u +\int_o^z G(u,X_u^n)du$$

If the assumption (A) is satisfied and for every T>o

$$E(\|H\|^2_{(T,T)})<\infty$$

Then $\lim_{n\to\infty} E(\|X^n-X\|^2_{(T,T)})=0$

We need the following two lemmas whose proofs are easily consequences of the Schwartz inequality, of the Doob inequality and of the fact that the assumption (A) implies the existence, for every z, of a constant C(z) such that

(2.2) $\|F(u,x)\|^2 + \|G(u,x)\|^2 \leq C(z)(1+\|x\|^2)$

for every $u \leq z$ and $x \in \mathbb{R}^d$.

We omit the details.

Lemma 1. Suppose the hypotheses of theorem 6 are satisfied. Then for every z there exists a constant $C_1(z)$ such that we have

(2.3) $\sup_n E(\|X^n\|_z^2) \leq C_1(z)$

(2.4) $E(\|X\|_z^2) \leq C_1(z)$

(2.5) $E(\|X-H\|_u^2) \leq C_1(z)\lambda([o,u])$ for $u \leq z$ (here λ is the Lebesgue measure).

Now fix $T>o$. Define $z_o=(T,T)$ and choose $o<T_1\leq T$ such that

(2.6) $\omega_1(z, C_1(z_o)\lambda([o,z])) \leq C_1(z_o)$ for $z \leq z_1=(T_1,T_1)$

where $\omega_1(z,x)=(32+2T^2)\omega(z,x)$.

For $n=o,1,\ldots$ and $z \leq z_1$ define

$$h_o(z)=C_1(z_o)\lambda([o,z])$$

$$h_{n+1}(z)=\int_o^z \omega_1(u,h_n(u))du$$

$$\bar{h}_n(z)=E(\|X^n-X\|_z^2)$$

Lemma 2. Under the hypotheses of theorem 6 and for every $n=1,\ldots$ the following inequalities hold

(2.7) $o \leq \bar{h}_n(z) \leq h_n(z) \leq h_{n-1}(z)$; $z \leq z_1$

Proof of theorem 6. Step 1. We shall show that

(2.8) $\lim_{n\to\infty} E(\|X^n-X\|_{z_1}^2)=0$

Denote $h(z)=\lim_{n\to\infty} h_n(z)$ for $z \leq z_1$. Of course h is continuous, null on the axes and satisfies

$$h(z)=\int_o^z \omega_1(u,h(u))du$$

The assumption (A) tels us that $h=o$ and (2.8) follows by utilising Lemma 2.

Step 2. We define

$$T_2=\sup\{T;\, o\leq T\leq T \text{ and } \lim_{n\to\infty} E(\|X^n-X\|^2_{(T,T)})=o$$

It is clear that $o<T_1\leq T_2\leq T$. In this step we shall show that

(2.9) $$\lim_{n\to\infty} E(\|X^n-X\|^2_{z_2})=o$$

where $z_2=(T_2,T_2)$.

Let $\bar{\varepsilon}=(\varepsilon,\varepsilon)$, $\varepsilon>o$. We have

(2.1o) $$\lim_{n\to\infty} E(\|X^n-X\|^2_{z_2-\bar{\varepsilon}})=o.$$

Denote $D_1=[(o,T_2-\varepsilon),(T_2-\varepsilon,T_2)]$, $D_2=[z_2-\bar{\varepsilon},z_2]$, $D_3=[(T_2-\varepsilon,o),(T_2,T_2-\varepsilon)]$.

All we must to show is

(2.11) $$E(\sup_{z\in D_i}\|X^n_z-X_z\|^2)\leq L(T)\varepsilon$$

for n enough large. We have

$$E(\sup_{z=(s,t)\in D_1}\|X^n_z-X_z\|^2)\leq 3E(\sup_{z\in D_1}\|X^n_z-X^n_{(s,T_2-\varepsilon)}\|^2)+$$

$$+3E(\sup_{z\in D_1}\|\square_z X_{s,T_2-\varepsilon}\|^2)+3E(\|X^n-X\|^2_{z_2-\bar{\varepsilon}})=3I^n_1+3I_2+3I^n_3$$

By using the Schwartz and Doob inequalities, lemma 1 and (2.2) we get

$$I^n_1\leq 2E\Big[\sup_{z\in D_1}\Big\|\int_o^s\int_{T_2-\varepsilon}^t F(u,X^{n-1}_u)\,dw_u\Big\|^2\Big]+$$

$$2E\Big[\sup_{z\in D_1}\Big\|\int_o^s\int_{T_2-\varepsilon}^t G(u,X^{n-1}_u)\,du\Big\|^2\Big]\leq 32\int_o^{T_2-\varepsilon}\int_{T_2-\varepsilon}^{T_2}E(\|F(u,X^{n-1}_u)\|^2du)+$$

$$2(T_2-\varepsilon)\varepsilon\int_o^{T_2-\varepsilon}\int_{T_2-\varepsilon}^{T_2}E(\|G(u,X^{n-1}_u)\|^2)du\leq 32C(z_2)(T_2-\varepsilon)\varepsilon+$$

$$2C(z_2)(T_2-\varepsilon)\varepsilon\leq L_1(T_2)\varepsilon$$

Similarly we obtain $I_2\leq L_1(T_2)\varepsilon$

On the other hand from (2.1o) we arrive to $I^n_3\leq\varepsilon$ for n enough large.

Hence (2.11) is proved for i=1. The cases i=2,3 follow in the same manner.

Step 3. We shall show that $T_2=T$. Contrary assume that $T_2<T$. Observe that the above calculus shows the existence of a sequence $a_n\searrow o$ such that

(2.12) $E(\|X^n-X\|^2_{z_2})\le a_n$

Define $z=(s,t)$, $D_1(s)=[(T_2,o),(T_2+s,T_2)]$, $D_2(t)=[(o,T_2),(T_2,T_2+t)]$, $D(z)=[z_2,z_2+z]$. For $z_2+z\le z_0$ as in step 2 follows

(2.13) $E(\sup_{u\in D_1(s)}\|X^n_u-X_u\|^2)=I^1_n(s)\le 3a_n+L(T)s$

(2.14) $E(\sup_{u\in D_2(t)}\|X^n_u-X_u\|^2)=I^2_n(t)\le 3a_n+L(T)t$

(2.15) $E(\sup_{u\in D(z)}\|\Box_{z_2}(X^n-X)_u\|^2)=I^3_n(s,t)\le L(T)st$

Choose $o<\eta_1\le T-T_2$ and a positive integer p such that

$$T_2\omega^1(s+T_2,t,3a_p+L(T)s)\le L(T) \quad \text{for } s\le\eta_1,\ t\le T_2$$

where $\omega^1=(48+3T\eta_1)\omega$.

For $s\le\eta_1$ we define

$$\varepsilon_p(s)=3a_p+L(T)s$$

$$\varepsilon_{p+n}(s)=3a_{p+n}+\int_o^s\int_o^{T_2}\omega^1(u+T_2,v,\varepsilon_{p+n-1}(v))dudv$$

$$\bar\varepsilon_{p+n}(s)=E(\sup_{u\in D_1(s)}\|X^{n+p}_u-X_u\|^2)$$

By induction on n we obtain (we omit the details)

(2.16) $\bar\varepsilon_{p+n}\le\varepsilon_{p+n}\le\varepsilon_{p+n-1}$

We put $g(s)=\lim_{n\to\infty}\varepsilon_{p+n}(s)$, We have $g(o)=o$, g is continuous and (2.16) implies

$$g(s)=\int_o^s\int_o^{T_2}\omega^1(u+T_2,v,g(v))dudv$$

Hence $g=o$ by assumption (A). In particular we obtain

(2.17) $\lim_{n\to\infty}E(\sup_{u\in D_1(s)}\|X^n_u-X_u\|^2)=o$; $s\le\eta_1$

Similarly follows the existence of $o<\eta_2\le T-T_2$ such that

(2.18) $\lim_{n\to\infty}E(\sup_{u\in D_2(t)}\|X^n_u-X_u\|^2)=o$; $t\le\eta_2$

Denote $\omega^2=4(32+2T^2)\omega$ and choose $o<\eta_3\le\min(\eta_1,\eta_2,T-T_2)$ such that

$$\omega^2(s+T_2,t+T_2,L(T)st)\le L(T) \quad \text{for } 0\le z=(s,t)\le\eta_3$$

Now for $z=(s,t)\le(\eta_3,\eta_3)$ we define

$$\alpha_o^1=\alpha_o^2=\alpha_o^3=o$$

$$\alpha_n^1=\sup_{k\geq n} I_k^1(\eta_3),\alpha_n^2=\sup_{k\geq n} I_3^2(\eta_3)\ \alpha_n^3=\sup_{k\geq n} I_k^3(\eta_3,\eta_3);n\geq 1$$

$$h_o(z)=L(T)st$$

$$h_{n+1}(z)=\int_o^z \omega^2(u+z_2,h_n(u)+\alpha_n^1+\alpha_n^2+\alpha_n^3)du$$

$$\bar{h}_n(z)=E(\sup_{u\in D(z)}\|\square_{z_2}(X^n-X)_u\|^2)$$

By utilising the inequality

$$\sup_{u\in D(z)}\|X_u^n-X_u\|^2\leq 4\|X^n-X\|_{z_2}^2+4\sup_{u\in D_1(s)}\|X_u^n-X_u\|^2+$$

$$4\sup_{u\in D_2(t)}\|X_u^n-X_u\|^2+4\cdot\sup_{u\in D(z)}\|\square_{z_2}(X_u^n-X_u)\|^2$$

we shall obtain $4\bar{h}_n\leq h_n\leq h_{n-1}$ and hence h=o.In particular

(2.19) $$\lim_{n\to\infty} E(\sup_{u\in D(z)}\|X_u^n-X_u\|^2)=o$$

Finally from (2.9),(2.17)-(2.19) we obtain

$$\lim_{n\to\infty} E(\|X^n-X\|^2_{z_2+(\eta_3,\eta_3)})=o$$

which ic contradictory with the definition of T_2.
Theorem is now proved.

Remark 5.In the one oarameter case a similar result as in theorem 6 has been obtained by Yamada[13].

Let $(\Omega,\mathcal{F},\mathcal{F}_z,P,w_z;z\geq o)$ a two parameter Wiener process and $\mathcal{X}$ be a class of continuous processes $(X_z)_{z\geq o}$ defined on $(\Omega,\mathcal{F},P)$ and with values in R^d such that for every T>o there exists a constant K(T) independent of X with properties

(2.2o) $$E(\|X\|^6_{(T,T)})\leq K(T)$$

(2.21) $$E(\|X_z-X_{z'}\|^6)\leq K(T)\{\lambda([o,z'])-\lambda([o,z])\}^3$$

for $o\leq z\leq z'\leq(T,T)$.

Definition.We say that a measurable function $a(z,x,\omega):R_+^2\times R^d\times\Omega\to R^d$ satisfies assumption (B) if for every T>o there exists a constant L(T) such that P-a.s

$$\|a(z,x)\|\leq L(T)(1+\|x\|)$$

$$\|a(z,x)-a(z,y)\|\leq L(T)\|x-y\|$$

for $o \le z \le z' \le (T,T)$ and x,y in R^d.

Proposition 1. Let $(a_n)_{n=o,1,..}$ be a sequence of functions which satisfy the assumption (B) with the same constant $L(T)$.
Define for $n=o,1,\ldots$ and $T,N>o$

$$\alpha_n(N,T)=E(\sup_{\|x\|\le N}\ \sup_{o\le z\le z'\le(T,T)}\left|\int_z^{z'}[a_n(u,x)-a_o(u,x)]du\right|^2)$$

Assume that for every n,N,T we have

(2.22) $$\lim_{n\to\infty}\alpha_n(N,T)=o$$

Then

~~(2.23) $$\lim_{n\to\infty}\sup_{X\in\mathcal{X}}E(\sup_{o\le z\le z'\le(T,T)}\left|\int_z^{z'}[a_n(u,X)-a_o(u,X)]du\right|^2)=o$$~~

Proof. Let D be a partition of the rectangle $[o,T]^2$ of the form $o=s_o<\ldots<s_{r+1}=T$, $o=t_o<\ldots<t_{s+1}=T$ and denote $z_{i,j}=(s_i,t_j)$, $D_{i,j}=[z_{i,j}, z_{i+1,j+1})$, $b_n=a_n-a_o$.
Computation gives

(2.24) $$E(\sup_{o\le z\le z'\le(T,T)}\left|\int_z^{z'}b_n(u,X_{z_{i,j}})du\right|^2)=I_n^{i,j}(X,T)\le 2\alpha_n(N,T)+$$

$$L_1(T)N^{-4}\longrightarrow o$$

as $n\to\infty$, $N\to\infty$ (in this order) uniformly in $X\in\mathcal{X}$
Let η be arbitrarily fixed and choose the partition D such that

$$\max_{i,j}\lambda(D_{i,j})\le\eta;\ \max_{i,j}[(s_{i+1}-s_i)+(t_{j+1}-t_j)]^{1/2}\le\eta$$

Then we shall obtain

$$\{E(\sup_{o\le z\le z'\le(T,T)}\left|\int_z^{z'}b_n(u,X)du\right|^2\}^{1/2}\le$$

$$\sum_{i=o}^{r}\sum_{j=o}^{s}\{E(\left|\int_{D_{i,j}}b_n(u,X)du)\right|^2\}^{1/2}+$$

$$4(r+1)(s+1)\{E[\sup_{i,j}(\int_{D_{i,j}}|b_n(u,X)|du)^2]\}^{1/2}=$$

$$J_1+4(r+1)(s+1)J_2$$

Next we have

$$J_1\le\sum_{i=o}^{r}\sum_{j=o}^{s}\{E(\left|\int_{D_{i,j}}b_n(u,X_{z_{i,j}})du\right|^2)\}^{1/2}+$$

$$\sum_{i=o}^{r}\sum_{j=o}^{s}E(\left|\int_{D_{i,j}}[b_n(u,X)-b_n(u,X_{z_{i,j}})]du)\right|^2\}^{1/2}\le$$

$$(r+1)(s+1)\sup_{i,j}\left[I_n^{i,j}(X,T)\right]^{1/2}+$$

$$\sum_{i=o}^{r}\sum_{j=o}^{s}\left\{E\left(\left|\int_{D_{i,j}}\left[b_n(u,X)-b_n(u,X_{z_{i,j}})\right]du\right|^2\right\}^{1/2}=$$

$$(r+1)(s+1)\sup_{i,j}\left[I_n^{i,j}(X,T)\right]^{1/2}+\sum_{i=o}^{r}\sum_{j=o}^{s}L_{i,j}$$

A computation gives

$$L_{i,j}\le L_2(T)\,\lambda(D_{i,j})\left[(s_{i+1}-s_i)+(t_{j+1}-t_j)\right]^{1/2}$$

If we choose n_o enough large such that

$$(r+1)(s+1)\sup_{i,j}\sup_{X\in\mathfrak{X}}\left[I_n^{i,j}(X,T)\right]^{1/2}\le\eta \text{ for } n\ge n_o \text{ (see (2.24))}$$

then we get

$$(2.25)\qquad J_1\le\eta+L_2(T)\sum_{i,j}\lambda(D_{i,j})\eta=(1+L_2(T)T^2)\eta=L_3(T)\eta$$

for $n\ge n_o$.

Also a simple computation gives

$$J_2\le\max_{i,j}\left[\lambda(D_{i,j})\right]^{1/2}\left\{\int_{D_{i,j}}E(|b_n(u,X)|^2)du\right\}^{1/2}\le L_4(T)\eta$$

Finally we deduce

$$\left\{E\left(\sup_{o\le z\le z'\le(T,T)}\left|\int_z^{z'}b_n(u,X)du\right|^2\right)\le L_3(T)\eta+\right.$$

$$4(r+1)(s+1)L_4(T)\eta=L_5(T)\eta$$

for $n\ge n_o$, uniformly in $X\in\mathfrak{X}$.

The proof is now complete.

<u>Theorem 7</u>. Suppose we are given for $n=o,1,..$ the functions

$$F_n:R_+^2\times R^d\times\Omega\longrightarrow R^d\otimes R^d\ ;\ G_n:R_+^2\times R^d\times\Omega\longrightarrow R^d$$

which satisfy assumption (B) with the same constant $L(T)$.

Let $(H_z^n)_{z\ge o}$; $n=o,1,...,$ be a sequence of continuous and $\mathcal{F}_z$-adapted R^d-valued processes such that for every $T>o$ there exists a constant $M(T)$ satisfying

$$\sup_n E(\|H^n\|^6_{(T,T)})\le M(T)$$

$$\sup_n E(\|H_z^n-H_{z'}^n\|^6)\le M(T)\left\{\lambda([o,z'])-\lambda([o,z])\right\}^3$$

for $o\le z\le z'\le(T,T)$.

Moreover assume that for every $N,T>o$

(i_1) $\lim_{n\to\infty} E(\sup_{\|x\|\le N} \int_0^T\int_0^T \| F_n(u,x)-F_o(u,x)\|^2 du)=o$

(i_2) $\lim_{n\to\infty} E(\sup_{\|x\|\le N} \sup_{o\le z\le z'\le(T,T)} \left| \int_z^{z'} \left[G_n(u,x)-G_o(u,x)\right] du\right|^2)=o$

(i_3) $\lim_{n\to\infty} E(\|H^n-H^o\|^2_{(T,T)})=o$

Let (X^n_z) be the solution of the Ito equation

$$X^n_z=H^n_z+\int_o^z F_n(u,X^n_u)dw_u+\int_o^z G_n(u,X^n_u)du$$

Then

$$\lim_{n\to\infty} E(\|X^n-X^o\|^2_{(T,T)})=o$$

Proof. The existence and tha pathwise uniqueness of X^n follow from [2] and [14].

Note also that from [2;p.79,8o] it follows that (2.2o),(2.21) are satisfied for $\mathcal{X}=\{X^n;n\ge o\}$.

Now the theorem follows by an application of the standard method for the Ito integral and af the previous proposition for the Lebesgue integral.

Remark 6. Usually the assumption (i_2) is given with the norm inverted with the integral.

Remark 7. In the one parameter case a similar result as in theorem 7 has been obtained by Vârsan[11].

3. A priori estimate of errors in the numerical resolution

Let S be a real separable Hilbert space with norm $|.|$ and inner product $\langle .,. \rangle$ and let $(\Omega,\mathcal{F},\mathcal{F}_z,P;z\in[o,1]^2)$ be a equiped probability space.

We denote

$$I=[o,1]^2 : I_o=\{o,1\}\times\{o\}\cup\{o\}\times[o;1]$$

$\mathcal{L}_2(S)$=the Hilbert space of all Hilbert-Schmidt operators on S with the norm $\|\cdot\|_2$.

$L_2(\Omega,\mathcal{F},P,S)$=the Hilbert space of all S-valued functions on $(\Omega,\mathcal{F},P)$ which are square integrable in the Bochner sense.

$D(I,S)=\{f:I\to S;$ f is right continuous and with left limits$\}$

$\mathcal{P}$=the Borel field on $I\times\Omega$ generated by the sets $(z,z']\times A, A\in\mathcal{F}_z$.

(a S-valued process which is $(\mathcal{P},\mathcal{B}_S)$-measurable is called predictable)

Definition 6. A S-valued process $(M_z)_{z\in I}$ is called square integrable strong martingale (or difference martingale) if

(a) for every z, $M_z\in L_2(\Omega,\mathcal{F}_z,P,S)$ and M vanishes on I_o

(b) $(M_{s,o},\mathcal{F}_{s,1})$, $(M_{o,t},\mathcal{F}_{1,t})$ are S-valued martingales with one parameter

(c) $E(\square_{s',t'}M_{s,t}|\mathcal{B}(\mathcal{F}_{s,1}\cup\mathcal{F}_{1,t}))=o$ if $s<s'$, $t<t'$.

Definition 7. If M_z is a square integrable strong martingale then the process denoted by $\langle M\rangle_z$ with properties

(i) $\langle M\rangle$ has continuous paths.

~~(j) $\langle M\rangle$ vanishes on I_o and is increasing.~~

(k) $E(|\square_{s',t'}M_{s,t}|^2|\mathcal{B}(\mathcal{F}_{s,1}\cup\mathcal{F}_{1,t}))=E(\square_{s',t'}\langle M\rangle_{s,t}|\mathcal{F}_{s,1}\cup\mathcal{F}_{1,t})$

is called the quadratic variation of M ($\langle M\rangle$ exists and is uniquely determined).

Remark 8. If H is a predictable process which satisfies

$$E\left(\int_I\|H\|_2^2 d\langle M\rangle\right)<\infty$$

then the Ito integral $\int_I HdM$ as usual. We note the validity of the Doob inequality

$$E\left(\sup_{u\leq z}\left|\int_o^u HdM\right|^2\right)\leq 16E\left(\int_o^z\|H\|_2^2 d\langle M\rangle\right)$$

We have the following version of theorem 1 (The proof is similar).

Theorem 8. Suppose we are given:

(i_1) $(M_z)_{z\in I}$ a continuous S-valued process which is a square integrable strong martingale with quadratic variation M satisfying

$$\square_{s',t'}\langle M\rangle_{s,t}\leq C_1(s'-s)(t'-t)$$

(i_2) $(V_z)_{z\in I}$ a continuous S-valued process with finite variation

$$A_z=\int_o^z d|V_u|$$

that satisfies

$$\square_{s',t'}A_{s,t}\leq C_2(s'-s)(t'-t)$$

(i_3) $(H_z)_{z\in I}$ an adapted S-valued process with paths in $D(I,S)$ such that $E(\|H\|^2_{(1,1)})=C_3<\infty$.

(i_4) $F,G: I\times D(I,S)\times\Omega\longrightarrow\mathcal{L}_2(S)$ $\mathcal{R}_z\otimes\mathcal{F}_z$-predictable processes such that

$$\|F(z,h)-F(z,g)\|_2^2\leq K_1\|h-g\|_z^2\ ;\ \|G(z,h)-G(z,g)\|_2^2\leq K_2\|g-h\|_z^2$$

$$\|F(z,h)\|_2^2\leq K_3+K_4\|h\|_z^2\ ;\ \|G(z,h)\|_2^2\leq K_5+K_6\|h\|_z^2$$

Then there exists a pathwise unique S-valued process X_z adapted and with paths in $D(I,S)$ (if H is continuous so is X) and such that

(2.26) $X_z=H_z+\int_0^z F(u,X)dM_u + \int_0^z G(u,X)dV_u$ a.s

Next we suppose the hypothesee of theorem 8 hold.

For simplicity we denote

$t_i=i/2^n, D_{i,j}=[t_i,t_{i+1}]\times[t_j,t_{j+1}]$ i, j=o,...,2^n-1

Proposition 2. Assume that

$$|H_z-H_{z'}|^2 \le C_4\|z-z'\|^2 \quad ; \quad 48C_1K_4+3C_2^2K_6\le 2^n$$

Then we have

$$E(\sup_{u\in D_{i,j}} |X_u-X_{t_i,t_j}|^2)\le d_n=\alpha_1 2^{-4n}+\alpha_2 2^{-2n}+\alpha_3 2^{-n}$$

where

$$\alpha_1=7C_2^2K_5+38C_2^2K_6(3C_3+48C_1K_3+3C_2^2K_5)\exp(48C_1K_4+3C_2^2K_6)$$

$$\alpha_2=14C_4+14C_2^2K_5+112C_1K_3+(3C_3+48C_1K_3+3C_2^2K_5)(448C_1K_4+28C_2K_6)\exp(48C_1K_4+3C_2^2K_6)$$

$$\alpha_3=56C_1K_3+112C_1K_4(3C_3+48C_1K_3+3C_2^2K_5)\exp(48C_1K_4+3C_2^2K_6)$$

A proof may be found in [9].

The following three theorems are proved in [9].

Theorem 9. For each n let X^n be the process defined by

$$X^n_{t_i,o}=H_{t_i,o} \;;\; X^n_{o,t_j}=H_{o,t_j}$$

$$X^n_{t_i,t_j}=H_{t_i,t_j}+\int_0^{t_i}\int_0^{t_j} F(u,X^n)dM_u+\int_0^{t_i}\int_0^{t_j} G(u,X^n)dV_u$$

$$X^n_{s,t}=X^n_{t_i,t_j} \quad \text{if } t_i\le s<t_{i+1}, t_j\le t<t_{j+1}$$

Then we have

(2.27) $$E(\|X-X^n\|^2_{t_i,t_j})\le d_{n,i,j}=\beta_1 2^{-4n}+\beta_2 2^{-2n}+(96C_1K_1+24C_2^2K_2).$$

$$(d_n+\beta_1 2^{-4n}+\beta_2 2^{-2n})t_i t_j\exp\{(96C_1K_1+24C_2^2K_2)t_i t_j\}$$

where

$$\beta_1=12C_2^2K_5+48C_2^2K_6(3C_3+48C_1K_3+3C_2^2K_5)\exp(48C_1K_4+3C_2^2K_6)$$

$$\beta_2=6C_4+24C_2^2K_5+48C_2^2K_6(3C_3+48C_1K_3+3C_2^2K_5)\exp(48C_1K_4+3C_2^2K_6)$$

Theorem 10. We add to $(i_1)-(i_4)$ and the hopotheses of proposition 2 the assumptions

$$\|F(z,g)-F(z',g)\|_2^2\le L_1 \sup_{z\le u\le z'}|g_u-g_z|^2$$

$$\|G(z,g)-G(z',g)\|_2^2\le L_2 \sup_{z\le u\le z'}|g_u-g_z|^2$$

We define for every n

$$X^n_{t_i,o}=H_{t_i,o} \;:\; X^n_{o,t_j}=H_{o,t_j}$$

$$X^n_{t_{i+1},t_{j+1}}=H_{t_{i+1},t_{j+1}}+\sum_{r=0}^{i}\sum_{s=0}^{j}[F(t_r,t_s,X^n)(\square_{t_{r+1},t_{s+1}}M_{t_r,t_s})+$$

$$G(t_r,t_s,X^n)(\square_{t_{r+1},t_{s+1}}V_{t_r,t_s})$$

$$X^n_{s,t}=X^n_{t_i,t_j} \quad \text{if} \quad t_i\leq s<t_{i+1},\ t_j\leq t<t_{j+1}$$

Then

$$(2.28)\ E(\|X-X^n\|^2_{t_i,t_j})\leq\beta_1 2^{-4n}+\beta_2 2^{-2n}+(96C_1L_1+24C_2^2L_2)(d_n+$$

$$\beta_1 2^{-4n}+\beta_2 2^{-2n})t_i t_j\exp\{(96C_1L_1+24C_2^2L_2)t_i t_j\}$$

Theorem 11. Assume that H is continuous, $F,G: I\times S\times\Omega\longrightarrow\mathcal{L}_2(S)$ are $\mathcal{F}_z$-adapted and continuous such that there exists constants $k_i,\bar{k}_i$, $i=1,\ldots,6$ with properties:

$$\|F(z,h)-F(z,g)\|_2^2\leq k_1|h-g|^2 : \|G(z,h)-G(z,g)\|_2^2\leq\bar{k}_1|h-g|^2$$

$$\left\|\frac{\partial F}{\partial x}(z,h)\right\|\leq k_3 \ :\ \left\|\frac{\partial G}{\partial x}(z,h)\right\|\leq\bar{k}_3 \ ;\ \|F(z,h)\|_2^2\leq k_2 \ ;\ \|G(z,h)\|_2^2\leq\bar{k}_2$$

$$\left\|\frac{\partial^2 F}{\partial x^2}(z,h)\right\|\leq k_4 \ ;\left\|\frac{\partial^2 G}{\partial x^2}(z,h)\right\|\leq\bar{k}_4$$

$$\left\|\frac{\partial F}{\partial s}(z,h)\right\|\leq k_5 \ :\ \left\|\frac{\partial F}{\partial s}(z,h)\right\|\leq\bar{k}_5$$

$$\left\|\frac{\partial F}{\partial t}(z,h)\right\|\leq k_6 \ :\ \left\|\frac{\partial G}{\partial t}(z,h)\right\|\leq\bar{k}_6$$

For each n let X^n be the process defined by

$$X^n_{s,t}=H_{s,t}+\sum_{r=0}^{i-1}\sum_{s=0}^{j-1}\Big[F(t_r,t_s,X^n_{t_r,t_s})(\square_{t_{r+1},t_{s+1}}M_{t_r,t_s})+$$

$$G(t_r,t_s,X^n_{t_r,t_s})(\square_{t_{r+1},t_{s+1}}V_{t_r,t_s})+$$

$$\int_{t_r}^{t_{r+1}}\int_{t_s}^{t_{s+1}}(F.\frac{\partial F}{\partial x})(t_r,t_s,X^n_{t_r,t_s})(\square_{t_r,t_s}M_{u,v})dM_{u,v}+$$

$$\int_{t_r}^{t_{r+1}}\int_{t_s}^{t_{s+1}}(F.\frac{\partial G}{\partial x})(t_r,t_s,X^n_{t_r,t_s})(\square_{t_r,t_s}M_{u,v})dV_{u,v}\Big]+$$

$$F(t_i,t_j,X^n_{t_i,t_j})(\square_{s,t}M_{t_i,t_j})+G(t_i,t_j,X^n_{t_i,t_j})(\square_{s,t}V_{t_i,t_j})+$$

$$\int_{t_i}^{s}\int_{t_j}^{t}(F.\frac{\partial G}{\partial x})(t_i,t_j,X^n_{t_i,t_j})(\square_{u_1,u_2}M_{t_i,t_j})dM_{u_1,u_2}+$$

$$+\int_{t_i}^{s}\int_{t_j}^{t}(F.\frac{\partial G}{\partial x})(t_i,t_j,X^n_{t_i,t_j})(\square_{u_1,u_2}M_{t_i,t_j})dV_{u_1,u_2}$$

if $(s,t)\in[t_i,t_{i+1})\times[t_j,t_{j+1})$.

Then

$$(2.29)\quad E(\|X-X^n\|^2_{t_i,t_j})\leq 6\left[(\bar\alpha_1+\bar\beta_1)+(\bar\alpha_2+\bar\beta_2)2^{-n}+(\bar\alpha_3+\bar\beta_3)2^{-2n}+(\bar\alpha_4+\bar\beta_4)2^{-4n}\right]t_it_j\exp\{(96C_1k_1+6C_2^2\bar k_1)+12.16^2C_1^2(k_2k_4^2+k_3^2k_1)+192C_1C_2^2(k_2\bar k_4^2+\bar k_3^2k_1)\}$$

where

$$\bar\alpha_1=4oC_1k_4^2E(\sup_{u\in D_{i,j}}|X_u-X_{t_i,t_j}|^4)$$

$$\bar\alpha_2=16o(32k_2k_3^2C_1+2C_2^2k_3^2)$$

$$\bar\alpha_3=16o\left[k_5^2+k_6^2+2k_3^2+32C_1k_1E(\sup_{u\in D_{i,j}}|X_u-X_{t_i,t_j}|^2)+C_2^2k_3^2\bar k_2\right]$$

$$\bar\alpha_4=16o(64C_1k_3^2k_5^2+64C_1k_6^2k_3^2+k_3^2C_2^2\bar k_2)$$

$$\bar\beta_1=1o/4C_2^2\bar k_4^2E(\sup_{u\in D_{i,j}}|X_u-X_{t_i,t_j}|^4)$$

$$\bar\beta_2=1oC_2^2(16k_2\bar k_3^2C_1+16k_2k_3^2C_1+\bar k_3^2C_2^2+k_3^2C_2^2)$$

$$\bar\beta_3=1oC_2^2\left[k_5^2+\bar k_6^2+2\bar k_3^2+32C_1\bar k_3^2k_1E(\sup_{u\in D_{i,j}}|X_u-X_{t_i,t_j}|^2)+\bar k_3^2C_2^2\bar k_2\right]$$

$$\bar\beta_4=1oC_2^2(\bar k_3^2C_2^2\bar k_2+64C_1k_5^2\bar k_3^2+64C_1\bar k_3^2k_6^2)$$

Proposition 3. Under the hypotheses of theorem 11 and moreover if M_z is a two parameter Wiener process i.e

(1) M vanishes on I_o and has continuous paths,

(2) $E(M_z)=o$ and $M_z\in L_2(\Omega,\mathcal{F}_z,P,S)$ for each z,

(3) $Cov(\square_{s',t'}M_{s,t})=(s'-s)(t'-t)W$, where W is a positive, trace class operator on S,

(4) the increments over disjoint rectangles are independent,

then the following inequality holds:

$$(2.3o)\quad E(\sup_{u\in D_{i,j}}|X_u-X_{t_i,t_j}|^4)\leq 7^3C_2^4\bar k_2^2 2^{-8n}+7^3\left[C_2^4\bar k_2^2(t_i^4+t_j^4)+36(3/4)^4k_2^2.(1+\|W\|^2)^2+4C_4^2\right]2^{-4n}+36.7^3(4/3)^4k_2^2(1+\|W\|^2)^2(t_i^2+t_j^2)2^{-2n}$$

A proof may be found in[9].

Remark 9. Approximation theorems for 1-parameter has been obtained in [4] . [6]

References

1. CAIROLI R., Sur une équation differentielle stochastique. Compte Rendus Acad. Sc. Paris, 274, Ser. A, 1972.
2. GHIHMAN I.I., PIASETZKAIA T.E., A class of stochastic differential equations containing the two parameter white noise. In: Limit theorems for random processes, Kiev, 1977 (in Russian).
3. GHIHMAN I.I., The existence of weak solutions for a class of hyperbolic systems containing the two parameter white noise. Theory of Random processes, 6, 1978 (in Russian).
4. GLORENNEC P., Estimation a priori des erreurs dand la résolution numérique d'équations différentielles stochastiques. Sem. de Prob. 1 (Rennes), 1977.
5. LEFORT P., Une propriété markovienne pour les processus à deux indices. Thèse de Doctorat de 3^{ième} cycle, Université de Paris 6, 1980.
6. PLATEN E., An approximation method for a class of Ito processes. Lit. Mat. Sb, 1, 1981.
7. PONOMARENCO L., Stochastic integral with respect to the multiparameter brownian motion and attached stochastic equations. Teor. Veroiatn. i Mat. Stat. 7, 1972 (in Russian).
8. TUDOR C., Stochastic integral equations in the plane. Rev. Roum. Math. Pures et Appl. 3, 1981.
9. TUDOR C., TUDOR M., On approximation in quadratic mean for the solutions of two parameter stochastic differential equations in Hilbert spaces. Analele Univ. Bucureşti, 1983.
10. TZARENCO T.T., Oh the existence and the uniqueness of solutions to stochastic equations of Darboux type. Kibernetika, 4, 1972 (in Russian).
11. VÀRSAN C., Some properties of Ito differential equations (to appear in Rev. Roum. Math. Pures et Appl.).
12. YAMADA T., WATANABE S., On the uniqueness of solutions of stochastic differential equations. J. of Math. of Kyoto Univ. 11, 1971
13. YAMADA T., On the successive approximation of solutions of stochastic differential equations. J. of Math. of Kyoto Univ. 3, 1981.
14. YEH J., Existence of strong solutions for stochastic differential equations in the plane. Pacific J. of Math. 1, 1981.

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