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Publications des séminaires de mathématiques et informatique de Rennes, 1979, fascicule 1
« Séminaire de probabilités », , exp. nº 3, p. 1-10
<http://www.numdam.org/item?id=PSMIR_1979___1_A3_0>

ON INTEGRATION WITH RESPECT TO MEASURES WITH VALUES IN ARBITRARY TOPOLOGICAL VECTOR SPACES

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Abstract. Completions of measures with values in arbitrary topological vector spaces are discussed and applied to the integral of Ph. Turpin. By the completion of Z. Lipecki, integrability is reduced to integrability with respect to measures with values in metric vector spaces. Using this result we are able to compare integration theories for Borel and Radon measures, extending analogous results obtained by the author in the locally convex case.

This note concernes integration with respect to measures m with values in topological vector spaces X which are not necessarily locally convex. A Lebesgue-type integration theory in such a general setting is developed by Ph. Turpin [4]. The natural framework for this theory is given by classes of functions which are measurable with respect to complete measures. On the other hand, considering X as a projective limit of metric vector spaces, from every m we obtain a family $(m_i)_{i\in I}$ of metric space valued measures. Now, it turns out that integrability with respect to the Lebesgue completion $\overline{m}$ of m is not equivalent to integrability with respect to $\overline{m}_i$ of measures m_i. We find that in order to obtain an equivalence-theorem it suffice to apply a Lebesgue-type completion which exists for every additive X-valued set function; in case of measures this completion $\overset{\frown}{m}$ coincides with the completion introduced by Z. Lipecki [2]. In Section 1 various types of extensions and of measurability are studied. In Section 2 we prove that a function is integrable with respect to $\overset{\frown}{m}$ if and only if it is integrable with respect to $\overline{m}_i$ for every $i \in I$. In particular, for locally compact spaces this implies that the integration theory of regular "L^∞-bounded" Borel measures completed in such a way is equivalent to the E. Thomas integration theory of "globally extendible" Radon measures [3].

AMS(MOS) subject classifications (1970): 28A45, 46G10.

Key words and phrases: completions of set functions, integration with respect to vector measures, measures with values in arbitrary topological vector spaces.

1. Completions and measurability. Everywhere in the sequel X denotes a Hausdorff topological vector space i.e. a vector space endowed with a Hausdorff topology which makes continuous the operations $(x,y) \to x+y$ $(X \times X \to X)$ and $(r,x) \to rx$ $(\mathbb{R}(\text{or } \mathbb{C}) \times X \to X)$. The topology of X is always determined by a family $|.|_{i \in I}$ of F-seminorms i.e. of applications $|.|_i : X \to \mathbb{R}_+$ such that $|x+y|_i \leq |x|_i + |y|_i$, $|rx|_i \leq |x|_i$ if $|r| \leq 1$, and $|rx|_i \to 0$ if $r \to 0$. $\hat{X}$ denotes the completion of X, X_i denotes $(X/\{x: |x|_i = 0\})^\wedge$, and π_i denotes the quotient map $\hat{X} \to X_i$. $\overline{X}$ denotes the quasi-completion of X.

In all that follows $\mathcal{A}$ denotes an algebra of subsets of some set $T \subset \mathcal{A}$ and m an additive set function $m: \mathcal{A} \to X$ $(m(\emptyset)=0)$. m is called a quasi-measure if it is exhaustive, i.e. if $m(A_n) \to 0$ for any sequence $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets in $\mathcal{A}$. m is a measure if $\mathcal{A}$ is a σ-algebra and m is σ-additive i.e. if $m(\bigcup_1^\infty A_n) = \sum_1^\infty m(A_n)$ for a sequence as above. m_i denotes $\pi_i \circ m$. $\mathcal{N}(m)$ denotes a family of m-null sets i.e. of A such that $m(S)=0$ for every $S \subset A$, $S \in \mathcal{A}$. Via $\mathcal{N}(m)$ we define m-essential boundedness, m-almost everywhere (m-a.e.), etc. $\overline{\mathcal{A}}^m$ denotes the algebra of all $A \subset T$ such that there existe $M,N \in \mathcal{A}$ with properties $M \subset A \subset N$ and $N-M \in \mathcal{N}(m)$; the set function $\overline{m}: \overline{\mathcal{A}}^m \to X$ defined by $\overline{m}(A) = m(M)$ is called the Lebesgue (or null-) completion of m.

Examples show that $\bigcap_{i \in I} \overline{\mathcal{A}}^{m_i}$ (denoted by $\overleftarrow{\mathcal{A}}^m$) can be larger than $\overline{\mathcal{A}}^m$ (cf. [1, 1.1, p. 257]). The following theorem gives an additive extension $\overleftarrow{m}: \overleftarrow{\mathcal{A}}^m \to \hat{X}$ of m such that $(\pi_i \circ \overleftarrow{m})(A) = (\overline{\pi_i \circ m})(A)$ for every $i \in I$ and $A \in \overleftarrow{\mathcal{A}}^m$. We call $\overleftarrow{m}$ (which is obviously unique) the projective null-completion of m; $\overleftarrow{m}$ is a quasi-measure (resp., a measure) if and only if so is m.

1.1. THEOREM. - For every m there exists the projective null-completion $\overleftarrow{m}$ i.e. for every $A \in \overleftarrow{\mathcal{A}}^m$

$$\overleftarrow{m}(A) = \bigcap_{i \in I} \pi_i^{-1}((\overline{\pi_i \circ m})(A))$$

determines an element of $\hat{X}$ or, more precisely, of $\overleftarrow{X}$.

Proof. The index set I can always be taken such that it is directed under "≤" defined by "$i \le j$ if $|x|_i \le |x|_j$ for every $x \in \hat{X}$ ". Denoting by π_{ij} the canonical map $\pi_{ij}: X_j \to X_i$ $(i \le j)$, $\hat{X}$ can be identified with $\varprojlim(X_i, \pi_{ij})$. Take $i \le j$ and $A \in \overline{\mathcal{A}}^m$. Then $A = B_k \cup C_k$ with $B_k \in \mathcal{A}$ and $C_k \in \mathcal{N}(\overline{m}_k)$ for $k=i,j$. Also, $A = B \cup C$ with $B = B_i \cup B_j \in \mathcal{A}$, $C = C_i \cap C_j \in \mathcal{N}(\overline{m}_k)$ and consequently $\overline{m}_k(A) = m_k(B)$ $(k=i,j)$. But $m_k(B) = \pi_k(m(B))$ and so we have $\overline{m}_i(A) = m_i(B) = \pi_{ij}(m_j(B)) = \pi_{ij}(\overline{m}_j(A))$. Therefore, $(\overline{m}_i(A))_{i \in I}$ can be identified with an element of $\hat{X}$, say x_A. We have $\overline{m}_i(A) = \pi_i(x_A)$, and the required extension is obtained by $\overleftarrow{m}(A) = x_A$. The proof works also with $\tilde{X}$ and $(X/\{x: |x|_i = 0\})^{\frown}$ instead of $\hat{X}$ and X_i.

We relate this extension with two other completions of m.

Denote by $\hat{\mathcal{A}}^m$ the algebra of all $A \subset T$ for which, for every neighborhood V of zero in X there exist $M, N \in \mathcal{A}$, $M \subset A \subset N$, such that $m(S) \in V$ if $N - M \supset S \in \mathcal{A}$. $\hat{\mathcal{A}}^m$ coincides with $\overline{\mathcal{A}}^m$ e.g. in case that the topology of X is discrete [2, Remark 2, p.20], or in case that X is metrizable and $\mathcal{A}$ is a σ-algebra [2, Prop.2, p.21]. m is $\mathcal{F}$-<u>tight</u> on $\mathcal{F} \subset \mathcal{A}$ if for any $A \in \mathcal{A}$ and any zero neighborhood V in X there exists $B \subset A$, $B \in \mathcal{F}$, suchthat $m(S) \in V$ for every $S \in \mathcal{A}$, $S \subset A - B$. If m is a quasi-measure, it has a unique $\mathcal{A}$-tight extension $\hat{m}: \hat{\mathcal{A}}^m \to \hat{X}$ [2, Th.2, p.23] (the <u>Lipecki</u> or <u>topological completion</u> of m).

1.2. PROPOSITION. - For every quasi-measure m the projective null-completion $\overleftarrow{m}$ is a restriction of the topological completion $\hat{m}$, and these extensions are generally not equal; if m is defined on a σ-algebra, $\overleftarrow{m}$ and $\hat{m}$ coincide.

Proof. We easily check $\hat{\mathcal{A}}^m = \bigcap_{i \in I} \hat{\mathcal{A}}^{m_i}$ and, by the $\mathcal{A}$-tightness of $\hat{m}$ and $\hat{m}_i$, $\pi_i(\hat{m}(A)) = \hat{m}_i(A)$ for every $A \in \hat{\mathcal{A}}^m$. Therefore,

$$\hat{m}(A) = \bigcap_{i \in I} \pi_i^{-1}((\pi_i \circ m)^{\wedge}(A)),$$

and to end the proof we have to compare $(\pi_i \circ m)^{\wedge}$ and $\overline{(\pi_i \circ m)}$ (the later is clearly also $\mathcal{A}$-tight). $\hat{m}$ and $\overleftarrow{m}$ are generally not equal even in case of scalar m [2, Ex.1, p.21].

The second completion which we relate to $\overleftarrow{m}$ is defined in case of X being a locally convex Hausdorff space. Let X' denote the topological and X^* the algebraic dual of X. Then, by the same argument as in 1.1, $\bigcap_{x'\in X'} x'^{-1}((\overline{x'\circ m})(A))$ determines an element of X'^* for every A from $\tilde{A}^m = \bigcap_{x'\in X'} \overline{A}^{x'\circ m}$. If for all $A \in \tilde{A}^m$ the elements of X'^* determined in such a way are in $\hat{X}$, denote them by $\tilde{m}(A)$. We call $\tilde{m}$ the scalar null-completion of m. By [1, 1.7, p.259], $\tilde{m}$ exists for every quasi-measure m. But if m is a quasi-measure, by [1, 1.11, p.260], $\tilde{A}^m$ just coincides with $\overline{A}^m$ and $\tilde{m}$ coincides with $\overleftarrow{m}$.

In the following m will be an X-valued measure, and therefore $\overleftarrow{m} = \hat{m}$. We take X to be a real vector space, but the statements are valid for the complex case as well.

By f we always denote a function $f:T \to \overline{\mathbb{R}}$. f is m-measurable if $f^{-1}(B) \in A$ for every Borel $B \subset \overline{\mathbb{R}}$. The vector space of m-measurable f is denoted by $L^0(m)$. $L^\infty(m)$ denotes the space of m-essentially bounded f from $L^0(m)$, endowed with the F-seminorm $\|\cdot\|_\infty$ of the essential upper bound. $S(A)$ denotes the space of simple f i.e. the space spanned by f of the form 1_A, $A \in A$ ($S(A)$ is dense in $L^\infty(m)$).

1.3. LEMMA. - A scalar function f is $\overline{m}$-measurable if and only if it differs from an m-measurable function by an $\overline{m}$-null function.

Proof. For f=g+h with $g \in L^0(m)$ and h an $\overline{m}$-null function, f is obviously from $L^0(\overline{m})$. Conversely, $f \in L^0(\overline{m})$ iff $f^{-1}(B) \in \overline{A}^m$ for every $B=[-\infty, a]$, a being rational. But $f^{-1}(B) = A_a \cup N_a$, $A_a \in A$, $N_a \subset M_a \in N(m)$; taking $g(t)=0$ if $t \in \bigcup_a M_a$, and $g(t)=f(t)$ otherwise, we have $g \in L^0(m)$ and f-g is an m- (and $\overline{m}$-) null function.

Similarly, $f \in L^\infty(\overline{m})$ iff it differs from an $L^\infty(m)$-function by an $\overline{m}$-null function; therefore, $S(A)$ is also dense in $L^\infty(\overline{m})$. Because m and m_i are defined on A, $L^0(m) = L^0(m_i)$ for any $i \in I$. On the other hand, by 1.1, for the completions we have

1.4. PROPOSITION. - For any X-valued measure m,

$$L^0(\widehat{m}) = \bigcap_{i\in I} L^0(\overline{m}_i).$$

The semivariations $\|m_i\|$ of measures m_i (or i-semivariations $\|m\|_i$ of m) on sets 'rom $\mathcal{A}$ are defined by

$$\|m_i\|(A) \ (=\|m\|_i(A)) = \sup\{|\sum_{k=0}^{n} r_k m_i(A_k)| : n\geq 0,\ \sup_k |r_k|\leq 1,\ A\supset A_k\in\mathcal{A},\ k\neq l \Rightarrow A_k\cap A_l\neq\emptyset\}.$$

ı case of an m on a Hausdorff topological space T and with $\mathcal{A}$ containing all compact ıbsets of T, by $\|m_i\|$ we define the so called "Luzin measurability" (cf. [3, 1.8, p.13]). is Luzin m-measurable if, for every $i\in I$, compact $K\subset T$, and $\varepsilon>0$, there exists a compact ', $K'\subset K$, such that $\|m_i\|(K-K')<\varepsilon$ and $f|K'$ is continuous. On the other hand, for such we define regularity again in terms of $\|m_i\|$: m is innerly regular if, for every $i\in I$, $\in\mathcal{A}$, and $\varepsilon>0$, there exists a compact $K\subset A$ such that $\|m_i\|(A-K)<\varepsilon$. Both notions apply ɔ the Borel measures (i.e. measures defined on the algebra of Borel subsets of T) and ɔ their extensions.

1.5. COROLLARY. - Let m be an innerly regular X-valued Borel measure. Then the space of Luzin m-measurable functions coincides with $L^0(\widehat{m})$.

Proof. By 1.4, we have to prove that $\overline{m}_i$-measurability and Luzin m_i-measurability coincide. For a Luzin m_i-measurable f there exists a partition $T=N+\sum_1^\infty K_n$ with $N\in\mathcal{N}(m_i)$ and K_h a compact set such that $f|K_n$ is continuous [3, 2.7(proof), p.24]. With $f^o(t) = f(t)$ for $t\in K_n$, and $f^o(t) = 0$ for $t\in N$, we have $f^o\in L^0(m_i)$ and $f\in L^0(\overline{m}_i)$ (1.3). Conversely, for $f\in L^0(\overline{m}_i)$, there exists an $\mathcal{S}(\mathcal{A})$-sequence m_i-a.e.-converging to f. By inner regularity, $\mathcal{S}(\mathcal{A})$ are Lusin m_i-measurable functions, and the property is preserved by an m_i-a.e. limit (cf. the proofs of [3, 2.8, p.25] and [3, 2.6, p.23]).

2. Integration. The integration theory in [4] is developed for the class of L^{∞}-bounded measures, that is for measures m having bounded a convex ballanced hull of the range $m(\mathcal{A})$. The integral, as an element of $\hat{X}$, is defined in an obvious way for simple functions, and is extended by continuity from $\mathcal{S}(\mathcal{A})$ to $\mathcal{L}^{\infty}(m)$. In the general case, f is integrable ($f \in \mathcal{L}^1(m)$) if there exists a sequence $(f_n \in \mathcal{L}^{\infty}(m))_{n\in\mathbb{N}}$ such that $(f_n)_{n\in\mathbb{N}}$ converges m-a.e. to f and $(\int f_n h \, dm)_{n\in\mathbb{N}}$ converges for every $h \in \mathcal{L}^{\infty}(m)$; $\int f \, dm$ is defined as $\lim_{n\in\mathbb{N}} \int f_n \, dm$. Following the lines of the proof of 2.1 below it turns out that $\mathcal{L}^1(m) = \bigcap_{i\in I} \mathcal{L}^1(m_i)$. As a matter of fact, for any m, integrability and measurability in [4] are defined with respect to $\overline{m}$, and in that setting the next theorem holds true (notice that $(\overset{\frown}{m})^{-} = \overset{\frown}{m}$; also, instead of $\hat{X}$ we can take $\overset{\frown}{X}$):

2.1. THEOREM. - Let $m:\mathcal{A} \to X$ be an L^{∞}-bounded measure. Then f is $\overset{\frown}{m}$-integrable if and only if it is $\overline{m}_i$-integrable for every $i \in I$, and for such f we have

$$\pi_i \int f \, d\overset{\frown}{m} = \int f \, d\overline{m}_i .$$

Proof. First, notice that $\pi_i \circ \overset{\frown}{m}$ and $\overline{m}_i$ are consecutive extensions of m_i, and that $\mathcal{S}(\mathcal{A})$ is dense in $\mathcal{L}^{\infty}(\pi_i \circ \overset{\frown}{m})$ as well as in $\mathcal{L}^{\infty}(\overline{m}_i)$. Therefore, $\mathcal{L}^{\infty}(\pi_i \circ \overset{\frown}{m}) \subset \mathcal{L}^{\infty}(\overline{m}_i)$, and we have $\int g \, d(\pi_i \circ \overset{\frown}{m}) = \int g \, d\overline{m}_i$ for every $g \in \mathcal{L}^{\infty}(\pi_i \circ \overset{\frown}{m})$. On the other hand, $\mathcal{L}^{\infty}(\overset{\frown}{m}) \subset \mathcal{L}^{\infty}(\pi_i \circ \overset{\frown}{m})$, and by continuity, $\pi_i \int g \, d\overset{\frown}{m} = \int g \, d(\pi_i \circ \overset{\frown}{m})$ for every $g \in \mathcal{L}^{\infty}(\overset{\frown}{m})$. Therefore we have $\pi_i \int g \, d\overset{\frown}{m} = \int g \, d\overline{m}_i$ as fahr as the class $\mathcal{L}^{\infty}(\overset{\frown}{m})$ is concerned.

Next, for a given f, define f_n, $n \in \mathbb{N}$, by $f(t) \geq n \Rightarrow f_n(t) = n$, $-n < f(t) < n \Rightarrow f_n(t) = f(t)$, $f(t) \leq -n \Rightarrow f_n(t) = -n$. By [4, 2.12, p.8.10], $f \in \mathcal{L}^1(\overline{m})$ if and only if $f_n \in \mathcal{L}^0(\overline{m})$, $n \in \mathbb{N}$, and $\int f_n h \, d\overline{m}$ converges for all $h \in \mathcal{L}^{\infty}(\overline{m})$.

Let now $f \in \mathcal{L}^1(\overline{m}_i)$ for every $i \in I$. Then $f_n \in \mathcal{L}^0(\overline{m}_i)$ for every $i \in I$, which implies $f_n \in \mathcal{L}^0(\overset{\frown}{m})$, and therefore also $f_n \in \mathcal{L}^{\infty}(\overset{\frown}{m})$. If h is an arbitrary function from $\mathcal{L}^{\infty}(\overset{\frown}{m})$, we have $\pi_i \int f_n h \, d\overset{\frown}{m} = \int f_n h \, d\overline{m}_i$, and when $n \to \infty$ the right-hand-side converges in X_i for every $i \in I$. Therefore $\int f_n h \, d\overset{\frown}{m}$ converges in $\hat{X}$, i.e. $f \in \mathcal{L}^1(\overset{\frown}{m})$; also, $\int f \, d\overline{m}_i = \pi_i \int f \, d\overset{\frown}{m}$.

Conversely, let $f \in \mathcal{L}^1(\overline{m})$, let $i \in I$, and let $h \in \mathcal{L}^\infty(\overline{m}_i)$. Then $f_n \in \mathcal{L}^\infty(\overline{m}_i)$, $h = h_1 + h_2$ with h_1 from $\mathcal{L}^0(m_i)$ and bounded (and therefore from $\mathcal{L}^\infty(\overline{m})$), and h_2 a $\overline{m}_i$-null-function. Hence, $\int f_n h \, d\overline{m}_i = \int f_n h_1 \, d(\overline{\pi_i \circ m}) = \pi_i \int f_n h_1 \, d\overline{m}$. By the convergence of $\int f_n h_1 \, d\overline{m}$, $f \in \mathcal{L}^1(\overline{m}_i)$ follows.

We consider $\mathcal{L}^1(m)$ endowed with the topology determined by the family of F-seminorms

$$\|f\|_{1,i} = \sup\{ \, |\int f \, h \, dm|_i \, : h \in \mathcal{L}^\infty(m), \, \|h\|_\infty \leq 1\} , \quad i \in I$$

[4, 2.9(5), p. 8.8]; in case of $f = 1_A$, $A \in \mathcal{A}$, this reduces to $\|m_i\|(A)$.

2.2. COROLLARY. - For an L^∞-bounded measure $m: \mathcal{A} \to X$ and $f \in \mathcal{L}^1(\overline{m})$,

$$\|f\|_{1,i} = \sup \{ |\int f \, h \, d\overline{m}_i| : h \in S(\mathcal{A}), \, \max|h(t)| \leq 1 \}.$$

Proof. By [4, 2,9, p.8.8] and the denseness of $S(\mathcal{A})$ in $\mathcal{L}^\infty(\overline{m}_i)$, for $f \in \mathcal{L}^1(\overline{m}_i)$, $\sup |\int f \, h \, d\overline{m}_i|$ taken with respect to $h \in \mathcal{L}^\infty(\overline{m}_i)$, $\|h\|_\infty \leq 1$, is the same as taken with respect to $h \in S(\mathcal{A})$, $\max|h(t)| \leq 1$. For $f \in \mathcal{L}^1(\overline{m})$ and $h \in \mathcal{L}^\infty(\overline{m})$ we have $fh \in \mathcal{L}^1(\overline{m})$ and $|\int fh \, d\overline{m}|_i = |\int fh \, d\overline{m}_i|$. Now we can take the supremum with respect to $h \in \mathcal{L}^\infty(\overline{m})$, resp. $h \in \mathcal{L}^\infty(\overline{m}_i)$ ($\|h\|_\infty \leq 1$), or for both integrals with respect to $h \in S(\mathcal{A})$, $\max|h(t)| \leq 1$.

By 2.2, for $f \in \mathcal{L}^1(\overline{m})$ (or $f \in \mathcal{L}^1(m)$), $\|f\|_{1,i}$ with respect to $\overline{m}$ coincides with $\|f\|_{1,i}$ with respect to $\overline{m}$.

As an application, we compare integration theories for bounded measures. In the following T will be a locally compact (Hausdorff) space, $K(T)$ the linear space of continuous functions with compact support, $C_0(T)$ the space of continuous functions tending to zero at infinity. A bounded Radon map on T is a linear map $\mu: K(T) \to X$ which is continuous if we endow $K(T)$ with the topology of uniform convergence. If μ, extended to an $\overline{X}$-valued map on $C_0(T)$ (cf. [3, p.37]), transforms weakly compact subsets of $C_0(T)$ into relatively compact sets of $\overline{X}$, it will be called a globally extendible Radon measure on T [3. 3.4, p.40]. $\mu^\bullet$, $\mathcal{L}^1(\mu)$, $\mathcal{L}^0(\mu)$, etc. are as in [3]; $\mu_i = \pi_i \circ \mu$. Firstly, we formulate a Riesz-type proposition (by regularity arguments, μ and m bellow are uniquelly determined by $\mu(L) = \int 1_L d\mu$ and $m(L)$ with L crossing all compact subsets of T).

2.3. PROPOSITION. - Let T be locally compact and let X be quasi-complete. Then there exists a linear bijection W between all X-valued globally extendible Radon measures μ and all X-valued L^∞-bounded innerly regular Borel measures m on T such that μ and $W(\mu)$ coincide on compact subsets of T; $m = W(\mu)$ is obtained from μ via

$$m(A) = \int 1_A d\mu$$

for Borel sets $A \subset T$, and with this m, $\mu: K(T) \to X$ is represented by

$$\mu(\phi) = \int \phi \, dm, \quad \phi \in K(T).$$

Proof. Let μ be globally extendible and let m be defined by $m(A) = \int 1_A d\mu$ as above. m is countably additive [3, 4.3(4), p.53], and therefore a Borel measure. Let $r = \sum_{k=0}^{n} r_k 1_{A_k}$ with pairwise disjoint $A_k \subset A$ be a simple function on T (with respect to the Borel field). By the estimate

$$\left| \sum_{k=0}^{n} r_k \, m(A_k) \right|_i = \left| \int r \, d\mu \right|_i \leq \mu_i^{\bullet}(|r|)$$

we have $\|m_i\|(A) \leq \mu_i^{\bullet}(A)$ and $\|\varepsilon \, m_i\|(T) \leq \mu_i^{\bullet}(\varepsilon 1_T)$. By [3, 2.5, p.23], we have that m is innerly regular. On the other hand, by [4, 2.2(ii), p.8.5], we have that L^∞-boundedness is equivalent to $\lim_{\varepsilon \to 0} \|\varepsilon m_i\|(T) = 0$ for every $i \in I$, and therefore, by [3, 1.2(d), p.7], m is L^∞-bounded. Notice, that m and μ coincide on compact subsets of T.

Conversely, let m be an innerly regular L^∞-bounded Borel measure on T. Define μ by $\mu(\phi) = \int \phi \, dm$, $\phi \in K(T)$. $\{\mu(\phi) : \|\phi\|_\infty < 1\}$ is a bounded set, being in the closure of the unit ball in $S(A)$, and therefore μ is a bounded map. To show that μ is globally extendible we have to prove $1_L \in L^1(\mu)$ for every compact $L \subset T$ [3, 2.1, p.20] and $1_T \in L^1(\mu)$ [3, 3.3(2), p.38]. Now, for every $i \in I$ and $\varepsilon > 0$ there exists a relatively compact open $\omega \supset L$ such that $\|m_i\|(\omega - L) \leq \varepsilon$ (regularity) and a function $\phi \in K(T)$ such that $\phi(t) = 1$, $t \in L$, and $\phi(t) = 0$, $t \in T - \omega$. $\phi - 1_L$ is lower semi-continuous, and therefore there exists $\psi \in K(T)$, $\|\psi\|_\infty \leq 1$, Supp $\psi \subset \omega - L$, such that

$$\mu_i^{\bullet}(\phi - 1_L) \leq |\mu(\psi)|_i + \varepsilon = \left| \int \psi \, dm \right|_i + \varepsilon \leq 2\varepsilon$$

[3, 1.3, p.7], which proves the integrability of 1_L. $1_T \in L^1(\mu)$ is obtained analogously taking a compact $K \subset T$ such that $\|m_i\|(T-K) < \varepsilon$, and noticing that $1_T - 1_K$ is lower semi-continuous.

To end the proof we have to show that for μ constructed in such a way from m we have $W(\mu) = m$. This would be the case if $\int 1_L\, d\mu$ and $m(L)$ coincide on compact $L \subset T$. By the above construction of ϕ, $|\mu(\phi) - \int 1_L d\mu|_i \leq \mu_i^{\bullet}(\phi - 1_L) \leq 2\varepsilon$, $|\int \phi dm - m(L)|_i \leq \|m_i\|(\omega - L) \leq \varepsilon$, and the result follows.

Notice, that the conditions on Borel measures in 2.3 are independent. In fact, the non-atomic Borel measure in [4. 4.1, p.8.12] is innerly regular, but it is not L^{∞}-bounded; conversely, locally convex space-valued measures are L^{∞}-bounded, regardless regularity.

Two measures μ and m from 2.3, such that $m = W(\mu)$, we call corresponding (to each other); cf. [1, 3.1, p.266].

2.4. THEOREM. - Let μ and m be corresponding quasi-complete space valued measures on a locally compact T. Then $\mathcal{L}^0(\mu) = \mathcal{L}^0(\overline{m})$, $\mathcal{L}^1(\mu) = \mathcal{L}^1(\overline{m})$ and, for integrable f,

$$\int f \, d\mu = \int f \, d\overline{m}$$

and $\mu_i^{\bullet}(|f|) = \|f\|_{1,i}$ $(i \in I)$ (cf. [1, 3.5, p.269]).

Proof. By 2.3 and [3, 2.16(Cor.), p.33], for every $i \in I$, $\|m_i\|$ and $\mu_i^{\bullet}$ coincide on Borel sets. Therefore the space of Luzin m_i-measurable functions coincide with $\mathcal{L}^0(\mu_i)$ [3, 1.8, p.13]; by 1.5 we have $\mathcal{L}^0(\mu_i) = \mathcal{L}^0(\overline{m}_i)$ and $\mathcal{L}^0(\mu) = \mathcal{L}^0(\overline{m})$ ([3, p.49] and 1.4). The class of μ_i- (resp. μ-) negligible sets consists of measurable members, and coincides with $\mathcal{N}(\overline{m}_i)$ (resp. with $\mathcal{N}(\overline{m})$).

To prove $\mathcal{L}^1(\mu) = \mathcal{L}^1(\overline{m})$, by 2.1 and [3, p.49], it suffice to show $\mathcal{L}^1(\mu_i) = \mathcal{L}^1(\overline{m}_i)$, $i \in I$. A bounded μ_i-a.e. defined function is in $\mathcal{L}^1(\mu_i)$ iff it is in $\mathcal{L}^0(\mu_i)$ [3, 1.9, p.14], and therefore iff it is in $\mathcal{L}^0(\overline{m}_i)$. In the general case, f is in $\mathcal{L}^1(\mu_i)$ iff $\lim_{n\to\infty} \int g f_n d\mu$ exists for every bounded Borel g, where f_n are defined as in the proof of 2.1 [3, 2.14, p.31]; instead of the set of $g \in \mathcal{L}^{\infty}(m_i)$ we can take the set $\mathcal{L}^{\infty}(\overline{m}_i)$ as well. But $\int g f_n d\mu_i = \int g f_n d\overline{m}_i$, and the convergence for $n\to\infty$ is necessary and sufficient for $f \in \mathcal{L}^1(\overline{m}_i)$. By an argument similar to that of the proof of [3, 2.16(Cor.), p.33], $\mu_i^{\bullet}(|f|)$ in [3, 1.4(d), p.10] writes as $\sup |\int hf \, d\mu_i|$, $h \in \mathcal{S}(A)$ and $\max|h(t)| \leq 1$, which gives $\mu_i^{\bullet}(|f|) = \|f\|_{1,i}$ by 2.2.

The author is indebted to Ph. Turpin for fruitfull conversations and suggestions on the subject.

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