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Publications des séminaires de mathématiques et informatique de Rennes, 1978, fascicule 3
« Séminaire d'analyse fonctionnelle », , exp. nº 2, p. 1-21
<http://www.numdam.org/item?id=PSMIR_1978___3_A2_0>

POWERS AND GEVREY'S REGULARITY FOR A SYSTEM OF DIFFERENTIAL OPERATORS

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The purpose of this paper is to give some results about powers and Gevrey regularity in the interior and up to boundary for a system of differential operators, which is, in particular, an extension of those of Kotake-Narashiman [8] and Nelson [11].

I - POWERS AND G_S REGULARITY.

At first, we recall the definition (or characterization) of the analyticity of a function :

Definition I-1 :

A function u, C^∞ in an open set of $\mathbb{R}^n$, is analytic in Ω if, for every compact set K of Ω, there exists a constant $L = L_K > 0$ such that, for every $\alpha \in \mathbb{N}^n$, we have :

$$||D^\alpha u||_{L^2(K)} \leqslant L^{|\alpha|+1}(|\alpha|!)$$

where we have written, for $\alpha = (\alpha_1,\ldots,\alpha_n)$,

$$|\alpha| = \alpha_1+\ldots+\alpha_n \quad \text{and} \quad D^\alpha = i^{-|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}} .$$

We denote by $a(\Omega)$ the space of analytic functions in Ω.

In [8], Kotake-Narashiman characterize the analyticity with the help of the powers of an elliptic operator in the following manner :

THEOREM 0 :

Let P be an elliptic differentiel operator of order $m \geqslant 1$ with analytic coefficients in an open set Ω of $\mathbb{R}^n$, the two following propositions are equivalent :

(i) $u \in a(\Omega)$;

(ii) $u \in C^{\infty}(\Omega)$ *and, for every compact set* K *of* Ω, *there exists a constant* $L = L_K > 0$ *such that, for every* $k \in \mathbb{N}$, *we have :*

$$||P^k u||_{L^2(K)} \leq L^{k+1}((mk)!).$$

In [11], Nelson characterizes the analyticity with the help of the powers of n real vector fields linearly independant in the following manner :

THEOREM 0' :

Let $P_1,\ldots,P_n$ *be some real vector fields, with analytic coefficients and linearly independant in every point of an open set* Ω *of* $\mathbb{R}^n$, *the two following propositions are equivalent :*

(i) $u \in a(\Omega)$;

(ii) $u \in C^{\infty}(\Omega)$ *and, for every compact set* K *of* Ω, *there exists a constant* $L = L_K > 0$ *such that, for every* $1 \leq i_j \leq n$, $1 \leq j \leq k$ *and* $k \geq 1$, *we have :*

$$||P_{i_1}\ldots P_{i_k} u||_{L^2(K)} \leq L^{k+1}(k!).$$

The purpose of this paper is to extend these results for more general operators and in the Gevrey's classes of order $s \geq 1$ in the interior and also up to the boundary.

We recall the definition of the Gevrey's classes :

Definition 2 :

Let K be a compact set of $\mathbb{R}^n$ and S a real number ≥ 1. We mean by Gevrey's class of order s in K the space $G_S(K)$ of the restrictions over K of C^{∞} functions u in a neighbourhood of K such that there exists a constant $L > 0$ such that, for every $\alpha \in \mathbb{N}^n$, we have :

$$||D^{\alpha}u||_{L^2(K)} \leq L^{|\alpha|+1}(|\alpha|!)^S.$$

Let Ω be an open set of $\mathbb{R}^n$; we mean by Gevrey's class of order s in the space $G_S(\Omega)$ of the functions which are in $G_S(K)$ for every compact subset K of Ω.

If K is "smooth enough", we can replace the $L^2(K)$-norm by the $L^\infty(K)$-norm. For s = 1, we get, of course, the analytic functions.

Let Ω be an open set of $\mathbb{R}^n$ with boundary $\partial\Omega$ and $P_j \equiv P_j(x;D)$, $j = 1,\ldots,N$, some differential operators of order $m_j \in \mathbb{N}$. let be denote by $P_j' = P_j'(x;D)$ the principal part of order m_j of P_j ; we introduce the two following conditions :

(A) for every $x \in \Omega$, the polynomial $P_j'(x;\xi)$, for $1 \leq j \leq N$, have no common non trivial real zero ;

(B) for every $x \in \partial\Omega$, the polynomials $P_j'(x;\xi)$, for $1 \leq j \leq N$, have no common non trivial complex zero.

At first, we have the following theorem on powers in the Gevrey's classes $G_S(\Omega)$, which generalizes the Kotaké-Narashiman and Nelson's theorems :

<u>*THEOREM 1* </u>*:*

If the operators P_j, $j = 1,\ldots,N$, have coefficients in $G_S(\Omega)$ and satisfy the condition (A), the two following propositions are equivalent :

(i) $u \in G_S(\Omega)$;

(ii) $u \in C^\infty(\Omega)$ and for every compact subset K of Ω, there exists a constant $L = L_K > 0$ such that, for every $1 \leq i_j \leq N$, $1 \leq j \leq k$ and $k \geq 1$, we have :

$$||P_{i_1}\ldots P_{i_k}u||_{L^2(K)} \leq L^{k+1}\left(\left(\sum_{j=1}^{k} m_{i_j}\right)!\right)^S.$$

Also, we have the following result which is a result on powers in the Gevrey's classes $G_S(\bar{\Omega})$:

THEOREM 2 :

If Ω is a bounded open set of $\mathbb{R}^n$ with Lipschitzian boundary if the operators P_j, for $1 \leq j \leq N$, have coefficients in $G_S(\bar{\Omega})$ and satisfy the conditions (A) and (B), the two following propositions are equivalent :

(i) $u \in G_S(\bar{\Omega})$;

(ii) $u \in C^\infty(\bar{\Omega})$ and there exists a constant $L > 0$ such that, for every $1 \leq i_j \leq N$, $1 \leq j \leq k$ and $k \geq 1$, we have :

$$||P_{i_1}\cdots P_{i_k}||_{L^2(\Omega)} \leq L^{k+1}\left(\left(\sum_{j=1}^{k} m_{i_j}\right)!\right)^S .$$

We recall that an open set Ω of $\mathbb{R}^n$ with Lipschitzian boundary $\partial\Omega$ is an open set such that, for every point $x_o \in \partial\Omega$, there exists a real number $r > 0$, a system of local coordinates $(x_1,\ldots,x_n)$ and a Lipschitzian function $h = h(x_1,\ldots,x_{n-1})$ such that :

$$\Omega \cap B(x_o,r) = \{(x_1,\ldots,x_n) \; ; \; x_n > h(x_1,\ldots,x_{n-1})\} \cap B(x_o,r)$$

where $B(x_o,r)$ is a ball of center x_o and radius r.

The implications (i)$\Longrightarrow$(ii) always are true and are easy to prove.

The method used to prove the theorem 2 (like for the theorem 1) in the implication (ii)$\Longrightarrow$(i) is an adaptation of this of Kotakê-Narashiman [8] using the tools of Morrey-Nirenberg [10].

At first, we can only consider the operators with the same order m. In fact, for $j = 1,\ldots,N$, we put $\hat{m}_j = \prod_{i \neq j} m_i$ and $Q_j = P_j^{\hat{m}_j}$. The operators

$Q_j = Q_j(x;D)$, for $1 \leq j \leq N$, have the order $m = \prod_{j=1}^{N} m_j$ and satisfy the conditions (A) and (B) if and only if the operators P_j for $j = 1,\dots,N$ satisfy the conditions (A) and (B). And more, if $u \in C^\infty(\bar{\Omega})$ and if there exists a constant $L > 0$ such that, for every $1 \leq i_j \leq N$, $1 \leq j \leq k$ and $k \geq 1$, we have :

$$||P_{i_1}\dots P_{i_k} u||_{L^2(\Omega)} \leq L^{k+1}((\sum_{j=1}^{k} m_{i_j})!)^S ,$$

then also we have :

$$||Q_{i_1}\dots Q_{i_k} u||_{L^2(\Omega)} \leq L'^{k+1}((km)!)^S$$

with $L' = (\max(L,1))^m$.

Then, for the following, we assume that all the operators P_j have the same order m.

The point of the begining of the proof is a global a priori estimate which is given in Aronszajn [2], Smith [12] (cf. also Bolley-Camus [3]) :

Proposition I-1 :

Under the assumptions of the theorem 2, for every $k \geq 1$, there exists a constant $L > 0$ such that, for every $u \in C^\infty(\bar{\Omega})$, we have :

$$||u||_{H^k(\Omega)} \leq C. \{\sum_{j=1}^{N} ||P_j u||_{H^{k-m}(\Omega)} + ||u||_{L^2(\Omega)}\} .$$

By localization, we are going to deduce two others a priori estimates. At first :

Proposition I-2 :

Under the assumptions of the theorem 2, for every $x \in \bar{\Omega}$, for every open neighbourhoods W and W' of x in $\bar{\Omega}$, W' being relatively compact in W, there exists a constant $A > 0$ such that, for every $u \in C^\infty(W)$, we have :

$$||u||_{H^m(W')} \leqslant A. \{ \sum_{j=1}^{N} ||P_j u||_{L^2(W)} + ||u||_{L^2(W)} \} .$$

Proof :

From the proposition I-1, then exists a constant $C > 0$ such that, for every $u \in C^\infty(W)$, and $1 \leqslant k \leqslant m$, we have :

$$||u||_{H^k(W)} \leqslant C. \{ \sum_{j=1}^{N} ||P_j u||_{H^{-m+k}(W)} + ||u||_{L^2(W)} \} .$$

We are going to deduce the proposition I-2 of this estimate in proving by induction on p, for $1 \leqslant p \leqslant m$, that there exists a constant $C_p > 0$ and a function $\phi_p \in C_o^\infty(W)$ equal to 1 on $\overline{W}'$ such that, for every function $u \in C^\infty(W)$ we have :

$$(p) \qquad ||u||_{H^m(W')} \leqslant C_p . \{ \sum_{j=1}^{N} ||P_j u||_{L^2(W)} + ||u||_{L^2(W)} + ||\phi_p u||_{H^{m-p}(W)} \} .$$

For $p = 1$, we consider a function $\phi_o \in C_o^\infty(W)$, equal to 1 on $\overline{W}'$; then, if $u \in C^\infty(W)$, from the precedent estimate written with $k = m$, we have :

$$||u||_{H^m(W')} \leqslant ||\phi_o u||_{H^m(W)} \leqslant C. \{ \sum_{j=1}^{N} ||P_j(\phi_o u)||_{L^2(W)} + ||\phi_o u||_{L^2(W)} \}.$$

However, $P_j(\phi_o u) = \phi_o P_j u - [P_j,\phi_o]\phi_1 u$ where $\phi_1 \in C_o^\infty(W)$, equal to 1 on the support of ϕ_o and $[P_j,\phi_o]$ means the commutator of P_j and ϕ_o. Hence,

$$||P_j(\phi_o u)||_{L^2(W)} \leqslant C_1' . \{ ||P_j u||_{L^2(W)} + ||\phi_1 u||_{H^{m-1}(W)} \}$$

for $1 \leqslant j \leqslant N$; then we get (1).

Suppose (p) is true and show $(p+1)$ if $p+1 \leq m$.

From the precedent estimate written with $k = m-p$, we get, for every $u \in C^{\infty}(W)$:

$$||\phi_p u||_{H^{m-p}(W)} \leq C. \{ \sum_{j=1}^{N} ||P_j(\phi_p u)||_{H^{-p}(W)} + ||\phi_p u||_{L^2(W)} \}.$$

Writting $P_j(\phi_p u) = \phi_p P_j u + [P_j, \phi_p]\phi_{p+1} u$ where $\phi_{p+1} \in C_0^{\infty}(W)$, equal to 1 on the support of ϕ_p. Hence,

$$||P_j(\phi_p u)||_{H^{-p}(W)} \leq C'_{p+1}. \{ ||P_j u||_{L^2(W)} + ||\phi_{p+1} u||_{H^{m-(p+1)}(W)} \}$$

for $1 \leq j \leq N$, from where we get $(p+1)$.

In particular, the inequality (m) is exactly the inequality of the proposition I-2.

In the second step, we establish an other a priori estimate localized for some paritular open sets W and W'. For that, we need some notations : let x be a point in $\overline{\Omega}$, $0 \leq \rho < R < R_1$;

$$W = \Omega \cap B(x;R_1) \qquad \underline{W} = \overline{\Omega} \cap B(x;R_1)$$

$$W_\rho = \Omega \cap B(x;R-\rho) \qquad \underline{W}_\rho = \overline{\Omega} \cap B(x;R-\rho).$$

Then, we have the following refined a priori estimate :

<u>Proposition I-3</u> :

Under the assumptions of the theorem 2, for every $x \in \overline{\Omega}$ and $0 < R < R_1$, there exists a constant $C > 0$ such that, for every $u \in C^{\infty}(W)$, for every $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$, ρ and $\rho' > 0$ with $\rho+\rho' < R$ and $\rho \leq 1$, we have :

$$\rho^m ||D^\alpha u||_{L^2(W_{\rho+\rho'})} \leqslant C. \{\rho^m \sum_{j=1}^{N} ||P_j u||_{L^2(W_{\rho'})} + \sum_{|\beta|\leqslant m-1} \rho^{|\beta|} ||D^\beta u||_{L^2(W_{\rho'})}\}.$$

Proof :

We consider a function $\psi \in C_0^\infty(W_{\rho'})$ such that $0 \leqslant \psi \leqslant 1$, $\psi = 1$ on $W_{\rho+\rho'}$, $||D^\alpha \psi||_{L^\infty(W_o)} \leqslant C_\alpha \rho^{-|\alpha|}$ where C_α is a constant which depends on α and not on x, ρ and ρ'.

We apply the proposition I-1 to the function ψu for $u \in C^\infty(\underline{W})$:

$$||D^\alpha(\psi u)||_{L^2(W_o)} \leqslant A. \{\sum_{j=1}^{N} ||P_j(\psi u)||_{L^2(W_o)} + ||u||_{L^2(W_o)}\}$$

for $|\alpha| \leqslant m$.

Elsewhere, if we put :

$$P_j = P_j(x;D) = \sum_{|\lambda| \leqslant m} a_{j\lambda}(x) D^\lambda ,$$

we have :

$$P_j(\psi u) - \psi P_j u = \sum_{\substack{\beta < \lambda \\ |\lambda|\leqslant m}} a_{j\lambda} \binom{\lambda}{\beta} D^{\lambda-\beta} \psi \; D^\beta u.$$

But, there exists some constants $C_{j,\lambda,\beta} > 0$, independant in ρ, such that :

$$||a_{j\lambda}\binom{\lambda}{\beta} D^{\lambda-\beta}\psi||_{L^\infty(W_o)} \leqslant C_{j,\lambda,\beta}\, \rho^{-|\lambda-\beta|}.$$

Then,

$$||D^\alpha(\psi u)||_{L^2(W_o)} \leqslant A'.\{\sum_{j=1}^{N} ||P_j u||_{L^2(W_{\rho'})} + \sum_{\substack{|\beta|<|\lambda| \\ |\lambda|\leqslant m}} \rho^{-|\lambda|+|\beta|} ||D^\alpha u||_{L^2(W_{\rho'})}\}$$

and, since $\rho \leqslant 1$, we have :

$$||D^\alpha(\psi u)||_{L^2(W_o)} \leqslant A'.\{\sum_{j=1}^{N} ||P_j u||_{L^2(W_{\rho'})} + \sum_{\substack{|\beta|<|\lambda| \\ |\lambda|\leqslant m}} \rho^{-m+|\beta|} ||D^\beta u||_{L^2(W_{\rho'})}\}$$

that gives the inequality of the proposition I-3.

We now do an induction on this inequality to obtain an estimate on one derivative of u in terms of some powers of $P_j u$:

Proposition I-4 :

Under the assumptions of theorem 2, for every $x \in \bar{\Omega}$, $0 < R < R_1$ there exists a constant $A \geq 1$ such that, for every ρ with $0 < \rho < \min(1,R)$, every $u \in C^\infty(W)$, every $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq km$ and $k \geq 1$, we have :

$$\rho^{|\alpha|S} ||D^\alpha u||_{L^2(W_{|\alpha|\rho})} \leq A^{|\alpha|+1} . \{ \sum_{\nu=1}^{k} \rho^{(\nu-1)mS} \sum_{\substack{1 \leq i_j \leq N \\ 1 \leq j \leq \nu}} ||P_{i_1} \ldots P_{i_\nu} u||_{L^2(W)} + ||u||_{L^2(W)} \}.$$

Proof :

The coefficients $a_{j\nu}$ of the operators P_j being in the class $G_S(\bar{\Omega})$, there exists a constant $B > 0$ such that, for every $\alpha \in \mathbb{N}^n$, we have :

$$\sum_{j=1}^{N} \sum_{|\lambda| \leq m} ||D^\alpha a_{j\lambda}||_{L^\infty(W_o)} \leq B^{|\alpha|+1} (\alpha!)^S$$

then

$$\sum_{j=1}^{N} \sum_{|\lambda| \leq m} ||D^\alpha a_{j\lambda}||_{L^\infty(W_\rho)} \leq B^{|\alpha|+1} (\alpha!)^S \rho^{-|\alpha|S}.$$

We put :

$$S_k(u) = S_k(u;\rho) =$$

$$= \sum_{\nu=1}^{k} \rho^{(\nu-1)mS} \sum_{\substack{1 \leq i_j \leq N \\ 1 \leq j \leq \nu}} ||P_{i_1} \ldots P_{i_\nu} u||_{L^2(W)} + ||u||_{L^2(W)} .$$

then we have :

$$\sum_{j=1}^{N} \rho^{mS} S_k(P_j u) \leq S_{k+1}(u)$$

and $$S_k(u) \leqslant S_{k+1}(u) .$$

We now prove the inequality of the proposition I-4 by induction on k. At first, the inequality of the proposition I-2 gives :

$$||D^\alpha u||_{L^2(W_o)} \leqslant A. \{ \sum_{j=1}^{N} ||P_j u||_{L^2(W)} + ||u||_{L^2(W)} \}$$

for $|\alpha| \leqslant m$.

We can choose $A \geqslant 1$ and since $\rho \leqslant 1$, we have the inequality of the proposition I-4 for $k = 1$.

Let $\alpha \in \mathbb{N}^n$ such that $km < |\alpha| \leqslant (k+1)m$ and assume proved the inequality of the proposition I-4 for every $\beta \in \mathbb{N}^n$ such that $|\beta| \leqslant |\alpha| -1$. We put $\alpha = \alpha_o + \alpha'$ with $|\alpha_o| = m$. We use the inequality of the proposition I-3 with $(|\alpha|-1)\rho$ insteed of ρ', α_o instead of α and $D^{\alpha'}u$ instead of u, that gives :

$$\rho^{|\alpha|S}||D^\alpha u||_{L^2(W_{|\alpha|\rho})} \leqslant C. \{\rho^{|\alpha|S} \sum_{j=1}^{N} ||P_j(D^{\alpha'}u)||_{L^2(W_{(|\alpha|-1)\rho})}$$

$$+ \sum_{|\beta| \leqslant m-1} \rho^{|\alpha|S-m+|\beta|}||D^{\beta+\alpha'}u||_{L^2(W_{(|\alpha|-1)\rho})} \}.$$

But, we have :

$$D^{\alpha'}(P_j u) - P_j(D^{\alpha'}u) = \sum_{|\lambda| \leqslant m} \sum_{\gamma \lneq \alpha'} \binom{\alpha'}{\gamma} D^{\alpha'-\gamma}a_{j\lambda} D^{\gamma+\lambda}u ,$$

$$\sum_{j=1}^{N} ||D^{\alpha'-\gamma}a_{j\lambda}||_{L^2(W_{km\rho})} \leqslant B^{|\alpha'-\gamma|+1}((\alpha'-\gamma)!)^S(mk\rho)^{-|\alpha'-\gamma|S}$$

and

$$\binom{\alpha'}{\gamma} \frac{((\alpha'-\gamma)!)^S}{mk^{|\alpha'-\gamma|S}} \leqslant \left(\binom{\alpha'}{\gamma}\frac{(\alpha'-\gamma)!}{(mk)^{|\alpha'-\gamma|}}\right)^S \leqslant \left(\frac{|\alpha'|}{mk}\right)^{|\alpha'-\gamma|S} \leqslant 1$$

since $|\alpha'| = |\alpha|-m \leq km$.

Hence,

$$\| D^{\alpha'}(P_j u) - P_j(D^{\alpha'}u)\|_{L^2(W_{(|\alpha|-1)\rho})} \leq$$

$$\leq \sum_{|\lambda|\leq m}\ \sum_{\gamma<\alpha'} |\alpha'-\gamma|+1\ \rho^{-|\alpha'-\gamma|S}\ \|D^{\gamma+\lambda}u\|_{L^2(W_{(|\alpha|-1)\rho})}$$

then, for $km < |\alpha| \leq (k+1)m$, we have :

$$\rho^{|\alpha|S}\|D^{\alpha}u\|_{L^2(W_{|\alpha|\rho})} \leq C.\ \{\ \rho^{|\alpha|S}\sum_{j=1}\|D^{\alpha'}P_j u\|_{L^2(W_{|\alpha'|\rho})} +$$

$$+ \sum_{|\beta|<m} \rho^{|\alpha|S-m+|\beta|}\ \|D^{\beta+\alpha'}u\|_{L^2(W_{|\beta+\alpha'|\rho})} +$$

$$+ \sum_{|\lambda|\leq m}\ \sum_{\gamma<\alpha'} \rho^{(m+|\gamma|)S}\ B^{|\alpha'-\gamma|+1}\|D^{\gamma+\lambda}u\|_{L^2(W_{(m+|\gamma|)\rho}}\ )\}.$$

We now can apply the assumption of our induction to estimate each term of the member of the right side of this inequality;the first term ie :

$$\leq \rho^{mS}\ A^{|\alpha'|+1}\sum_{j=1}^{N} S_k(P_j u) \leq A^{|\alpha'|+1}S_{k+1}(u)\ ,$$

the second term is :

$$\leq \sum_{|\beta|\leq m} A^{|\beta+\alpha'|+1}\ S_{k+1}(u)\ ,$$

and the third term is :

$$\leq \sum_{|\lambda|\leq m}\ \sum_{\gamma<\alpha'} B^{|\alpha'-\gamma|+1}\ A^{m+|\gamma|+1}\ S_{k+1}(u).$$

then, we have :

$$\rho^{|\alpha|S}||D^{\alpha}u||_{L^2(W_{|\alpha|\rho})} \leqslant A^{|\alpha|+1}S_{k+1}(u)\{CA^{-m}+C\sum_{|\beta|<m}A^{-1}+\sum_{|\lambda|\leqslant m}\sum_{\gamma<\alpha'}B^{|\alpha'-\gamma|+1}A^{-|\alpha'-\gamma|}\}.$$

But,

$$C\sum_{|\lambda|\leqslant m}\sum_{\gamma<\alpha'}B^{|\alpha'-\gamma|+1}A^{-|\alpha'-\gamma|} \leqslant C.m^n B^2A^{-1}\sum_{|\beta|\geqslant 0}(BA^{-1})^{|\beta|} \; ;$$

We can choose A large enough, independent of α and ρ, in order to the term between the brackets be $\leqslant 1$, that achieves the proof of the proposition I-4.

Then, we can give the property about the powers "locally up to the boundary" :

Proposition I-5 :

Under the assumptions of theorem 2, if $x \in \bar{\Omega}$, $u \in C^{\infty}(\bar{\Omega}) \cap B(x;R_2)$ and such that, for every open neighbourhood $\mathcal{U}$ of x in $\bar{\Omega}$ with $\mathcal{U}$ relatively compact in $\bar{\Omega} \cap B(x;R_2)$, there exists a constant $L = L_{\mathcal{U}} > 0$ such that, for every $1 \leqslant i_j \leqslant N$, $1 \leqslant j \leqslant k$ and $k \geqslant 1$, we have :

$$||P_{i_1}\ldots P_{i_k}u||_{L^2(U)} \leqslant L^{k+1}(km!)^S$$

then $u \in G_S(\bar{\Omega} \cap B(x;R_2))$.

Proof :

We fixe $R' < R_2$ and put $\mathcal{U}' = \Omega \cap B(x;R_2)$. We want to show that $u \in G_S(\overline{\mathcal{U}'})$. We choose R_1 and R such that $R' < R < R_1 < R_2$ and with the notations used in the proposition I-4, we have :

$$||P_{i_1}\ldots P_{i_k}u||_{L^2(W)} \leqslant L^{k+1}(km!)^S$$

hence,

$$S_k(u) \leqslant \sum_{\nu=1}^{k}\rho^{(\nu-1)mS}N^{\nu}L^{\nu+1}((\nu m)!)^S + L$$

for every ρ such that $0 < \rho < \mathrm{Min}(1,R)$.

We choose $\rho = \frac{R-R'}{km}$, $R-R'$ being small enough ; then we get :

$$((\nu m)!)^S \rho^{(\nu-1)mS} \leqslant (km)^{mS}$$

for $\nu \leqslant k$.

Therefore, there exists a constant $B_1 > 0$ such that :

$$S_k(u) \leqslant \sum_{\nu=1}^{k} N^\nu L^{\nu+1} (km)^{mS} + L \leqslant B_1^{k+1}$$

for $k \geqslant 1$.

And with the proposition I-4, there exists a constant $B_2 > 0$ such that, for $\alpha \in \mathbb{N}^n$ with $|\alpha| \leqslant km$ and $k \geqslant 1$, we have :

$$||D^\alpha u||_{L^2(W_{R-R'})} \leqslant B_2^{k+1} k^{kS} .$$

In particular, if we apply this formula for $|\alpha| = k$, we get, for every $\alpha \in \mathbb{N}^n$:

$$||D^\alpha u||_{L^2(U')} \leqslant B_2^{|\alpha|+1} |\alpha|^{|\alpha|S} ,$$

that gives $u \in G_S(\overline{U'})$.

The theorem 2, for the assertion (ii)$\Longrightarrow$(i), is proved.

<u>Remark I-1</u> :

In the case where $\overline{\Omega}$ is a C^∞ compact manifold with boundary, the condition (B) can be replaced, in the theorem 2, by the following condition :

(B') for every $x \in \partial\Omega$, the polynomials $P_j'(x;\xi)$ for $1 \leqslant j \leqslant N$ have no common non trivial complex zero with imaginary part orthogonal to $\partial\Omega$ in x.

Remark I-2 :

By the same method, the inequalities of coercivness given in Agmon [1] allow to give some similar results about powers in the classes $G_S(\bar{\Omega})$ for boundary value problems associated to some systems $(P_1,\dots,P_N ; B_1,\dots,B_p)$ where the P_j are differential operators and B_j are differential operators at the boundary ; the case where the system of P_j is reduced to a single operator is the case which was studied by Lions-Magenes 9 and the case where the system of B_j is empty is the case that we have studied here.

II - G_S - REGULARITY.

It comes from the teorem 1 the following corollary about the $G_S(\Omega)$-regularity :

Corollary II-1 :

Under the assumptions of theorem 1, the two following propositions are equivalent :

(i) $u \in G_S(\Omega)$

(ii) $u \in C^\infty(\Omega)$ and $P_j u \in G_S(\Omega)$ for $1 \leqslant j \leqslant N$.

and from the theorem 2, we get the following corollary about the $G_S(\bar{\Omega})$-regularity :

Corollary II-2 :

Under the assumptions of theorem 2, the two following propositions are equivalent :

(i) $u \in G_S(\bar{\Omega})$;

(ii) $u \in C^\infty(\bar{\Omega})$ and $P_j u \in G_S(\bar{\Omega})$ for $1 \leqslant j \leqslant N$.

Remark II-1 :

Using the results of regularity given by Smith [11] (cf. also Bolley-Camus [3]), we can replace $u \in C^\infty(\bar{\Omega})$ by $u \in \mathcal{D}'(\Omega)$ in the corollary II-2. In the same way, we can replace $u \in C^\infty(\Omega)$ by $u \in \mathcal{D}'(\Omega)$ in the corollary I-1, using for that, the ellipticity of the operator $\sum_{j=1}^{N} P_j^* P_j$ in Ω.

It is easy to see that the condition (A) for the corollary II-1 and the conditions (A) and (B) (or (B')) for the corollary II-2 are not necessary.

When the operators $P_j = P_j(D)$ have constant coefficients, we introduce the following condition:

(C) The set of the complex common roots ξ of the polynomails $P_j(\xi)$, for $1 \leq j \leq N$, is finite.

Then, we have the following necessary and sufficient condition of $G_S(\Omega)$-regularity :

THEOREM II-1 :

Let Ω be a bounded open set of $\mathbb{R}^n$ with Lipschitzian boundary, and P_j be some operators with constants coefficients, $1 \leq j \leq N$; the two following propositions are equivalent :

(i) The space $\{u \in \mathcal{D}'(\Omega)\ ;\ P_j u \in G_S(\bar{\Omega}),\ 1 \leq j \leq N\}$ is the space $G_S(\bar{\Omega})$;

(ii) The operators P_j, $1 \leq j \leq N$, satisfy the condition (C).

The proof made in the case of the space $C^\infty(\Omega)$ in Bolley-Camus [3] can be applied for the space $G_S(\bar{\Omega})$. We recall it here.

Proof :

We assume that (i) is true. We introduce the space :

$$Y(\Omega) = \{u \in \mathcal{D}'(\Omega) \ ; \ P_j u = 0 \ , \ 1 \leqslant j \leqslant N\}.$$

We denote by $Y^o(\Omega)$ (resp. $Y^1(\Omega)$) the space $Y(\Omega)$ equipped with the $L^2(\Omega)$-norm (resp. $H^1(\Omega)$-norm). The identity map from $Y^1(\Omega)$ into $Y^o(\Omega)$ being continuous and these spaces being Banach spaces, the two norms $L^2(\Omega)$-norm and $H^1(\Omega)$-norm are equivalent on $Y(\Omega)$. Then, there exists a constant $C > 0$ such that, for every $u \in Y(\Omega)$, we have :

$$||u||_{H^1(\Omega)} \leqslant C.||u||_{L^2(\Omega)} .$$

The unit ball of $Y^o(\Omega)$ is then compact and therefore $Y(\Omega)$ is of finite dimension.

But, if $\xi \in \mathbb{C}^n$ satisfies $P_j(\xi) = 0$ for $1 \leqslant j \leqslant N$, the function $u(x) = e^{i<x,\xi>}$ satisfies $P_j u = 0$ for $1 \leqslant j \leqslant N$. Then, necessarily, the set of complex common roots of the polynomials in finite.

We now assume that (ii) is true. Let $\xi^1,\dots,\xi^\nu$ be the complex common roots of the polynomials P_j for $1 \leqslant j \leqslant N$. For each $1 \leqslant j \leqslant n$, we consider the polynomial :

$$Q_j(\xi) = \prod_{i=1}^{\nu} (\xi_j - \xi_j^i)$$

where we have put $\xi = (\xi_1,\dots,\xi_n)$.

Then, we have $Q_j(\xi^i) = 0$ for $1 \leqslant i \leqslant \nu$; that is, the polynomials Q_j, $1 \leqslant j \leqslant n$, vanish on the set of the complex common roots of the polynomials P_j, $1 \leqslant j \leqslant N$. From the Nullstellensatz's theorem (cf. Van der Warden [13] for example), there exists an integer $\rho \geqslant 1$ such that the polynomials Q_j^ρ for $1 \leqslant j \leqslant n$ belong to the ideal spanned by the polynomials P_ℓ, $1 \leqslant \ell \leqslant N$; that is, there exists some polynomials $A_{j\ell}$ such that :

$$Q_j^{\rho}(\xi) = \sum_{\ell=1}^{N} A_{j\ell}(\xi)\, P_{\ell}(\xi) \quad , \quad 1 \leqslant j \leqslant n .$$

The polynomials Q_j^{ρ} are polynomials of order $\nu\rho$ of which the principal part is equal to $\xi_j^{\nu\rho}$: these principal parts have only 0 like complex common root, that is, they satisfy the conditions (A) and (B). Hence, if $u \in \mathcal{D}'(\Omega)$ and satisfy $P_j u \in G_S(\bar{\Omega})$ for $1 \leqslant j \leqslant N$, then $Q_j^{\rho} u \in G_S(\bar{\Omega})$ for $1 \leqslant j \leqslant n$. And from Smith [12], Bolley-Camus [3], $u \in C^{\infty}(\bar{\Omega})$ and the corollary II-2 gives $u \in G_S(\bar{\Omega})$.

From the theorem II-1, in particular, we deduce the following sufficient condition of $G_S(\Omega)$-regularity :

Corollary II-3 :

Let P_j be some differential operators, $1 \leqslant j \leqslant N$, with constant coefficients and satisfying the condition (C) ; the two following propositions are equivalent :

(i) $u \in G_S(\Omega)$;

(ii) $u \in C^{\infty}(\Omega)$ and $P_j u \in G_S(\Omega)$ for $1 \leqslant j \leqslant N$.

Remark II-2 :

It comes from the precedent theorems that, if the polynomials $P_j \equiv P_j(\xi)$, $1 \leqslant j \leqslant N$, (with constant coefficients), have principal parts without complex common root different from 0, that is the condition (B), then, they have only a finite number of complex common roots, that is satisfy the condition (C) : it is a "classical" result in algeabraic geometry.

III - "REDUCED POWERS" AND G_S-REGULARITY.

In [5], Damlakhi gives a refinement about the Nelson's theorem (theorem 0') in the following sense :

THEOREM [5] :

Let $P_1, \ldots, P_n$ be some real vectors fields, with analytic coefficients and linearly independant in each point of an open set Ω; the two following propositions are equivalent :

(i) $u \in a(\Omega)$;

(ii) $u \in C^\infty(\Omega)$ and, for every compact subset K of Ω, there exists a constant $L = L_K > 0$ such that, for every $k \geq 1$ and $1 \leq i \leq n$, we have :

$$||P_i^k u||_{L^2(K)} \leq L^{k+1}(k!) .$$

In a similar way and, according to the precedent chapters I and II, we are going to put the two following conjectures :

Conjecture 1 :

Under the assumption of theorem 1, the two following propositions are equivalent :

(i) $u \in G_S(\Omega)$;

(ii) $u \in C^\infty(\Omega)$ and, for every compact subset K of Ω, there exists a constant $L = L_K > 0$ such that, for every $k \geq 1$ and $1 \leq i \leq N$, we have :

$$||P_i^k u||_{L^2(K)} \leq L^{k+1}((km_i)!)^S.$$

Conjecture 2 :

Under the assumptions of theorem 2, the two following propositions are equivalent :

(i) $u \in G_S(\bar{\Omega})$;

(ii) $u \in C^\infty(\Omega)$ and there exists a constant $L > 0$ such that, for every $k \geq 1$ and $1 \leq i \leq N$, we have :

$$||P_i^k u||_{L^2(\Omega)} \leq L^{k+1}((km_i)!)^S.$$

Then, a positive answer is given in a particular case by Damlakhi [5] who uses for that the notion of analytic wave front set of an hyperfunction and the fundamental theorem of Sato, and also the idea to add an other variable t (in $\mathbb{R}$) and to consider the evolution operators $P_j = \frac{\partial}{\partial t} - i\, P_j$, $1 \leqslant j \leqslant N$.

Also, the conjecture 1 is true in the case of operators P_j of order 1, with complex and constant coefficients. The proof of this result is based on the following proposition :

<u>Proposition III-1</u> :

Let $P_j = P_j(\xi)$ be some polynomials, $j = 1,\dots,N$, of order 1 with complex and constant coefficients ; we assume that their principal parts have no real common root different from 0. Then, for every compact sets K_1 and K_2 of R^n, K_1 being included in the interior $\overset{o}{K_2}$ of K_2, there exists a constant $C > 0$ such that, for every $u \in C^\infty(K_2)$ and $\alpha \in \mathbb{N}^n$, we have :

$$||D^\alpha u||_{L^2(K_1)} \leqslant C^{|\alpha|+1} \sum_{i=1}^{N} \sum_{|\beta|\leqslant|\alpha|} \sum_{j=o}^{|\alpha|-|\beta|} C^{|\beta|} |\alpha|^{|\beta|} \frac{|\alpha|!}{(|\alpha|-|\beta|-j)!\,j!\,\beta!} ||P_i^{|\alpha|-|\beta|-j} u||_{L^2(K_2)} .$$

This proposition is obtained in using, in particular, the special function of truncation given in Hormander [7].

Another positive answer to the conjecture 2 has been given for $s = 1$, $\Omega =]-1,+1[^n$ and for the canonical system of the first partial derivatives by Damlakhi [5] who uses for that the spectral theory of the Legendre's operator in n-variables.

The conjecture 2 is also true "locally" in the half-space $\mathbb{R}^n_+ = \{(x,t);t \geqslant 0\}$ for the case of a transversal operator P_1 of order 1 with constant and real coefficients and some tangential operators $P_2,\dots,P_N$ with complex and constant coefficients. The proof is based on the following a priori estimate : there exists a constant $C > 0$ such that, for all $u \in C_o^\infty(\overline{R^n_+})$, $u(x,t) = 0$ for $t \geqslant 1$, $k \geqslant 1$ and $\alpha \in \mathbb{N}^{n-1}$, we have :

$$||D_x^\alpha P_1^k u||_{L^2(\mathbb{R}^n_+)} \leqslant C^{|\alpha|+k+1} \{ ||P_1^{|\alpha|+k+1} u||_{L^2(\mathbb{R}^n_+)} + \sum_{j=2}^{N} \sum_{\ell=o}^{|\alpha|+k+1} \binom{\ell}{|\alpha|+k+1} ||P_j^{|\alpha|+k+1-\ell} u||_{L^2(\mathbb{R}^n_+)} \} .$$

We prove such an inequality in using the inequalities given in Cartan [4] and Hardy-Littlewood-Polya [6].

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