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ON A CLASS OF WEIGHTED SOBOLEV'S SPACES.

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I - CASE OF THE HALF-LINE $\mathbb{R}_+$.

For an integer $m \in \mathbb{N}$, two real numbers α and $\beta \geqslant 0$ and an interval I of $\mathbb{R}_+$, we consider the space :

$$V^m_{\alpha,\beta}(I) = \{u \in \mathcal{D}'(I) \; ; \; t^\alpha u \in L^2(I) \; , \; t^\beta D^m_t u \in L^2(I)\}$$

equipped by the canonical norm.

Proposition I.1 :

If $u \in V^m_{\alpha,\beta}(0,T)$, where T is a real number strictly positive, we have :

(i) $t^{\beta-j} D_t^{m-j} u \in L^2(0,T)$ for $0 \leqslant j \leqslant \mathrm{Min}(j_o,m)$ with $j_o = [\beta+\frac{1}{2}]_-$;

(ii) $t^{\beta-j_o} D_t^{m-j} u \in L^2(0,T)$ for $j_o+1 \leqslant j \leqslant m$ if $j_o + 1 \leqslant m$;

(iii) $u \in H^{m-\beta}(0,T)$ if $\beta-m \neq$ integer $+ \frac{1}{2}$.

The notation $[A]_-$ means the greatest integer $<A$.

Proof : Let φ be an indefinitely differentiable function such that $\varphi(t) = 1$ if $t \leqslant \frac{T}{2}$ and $\varphi(t) = 0$ if $t \geqslant 3\frac{T}{4}$. Put $v = \varphi u$; then $v \in V^m_{\alpha,\beta}(\mathbb{R}_+)$ with bounded support.

Using the Hardy's inequality, we obtain (i).

Again for (ii) : we have $t^{\beta-j_o} D_t^{m-j_o} v \in L^2(\mathbb{R}_+)$, also $t^{\beta-j_o+1} D_t^{m-j_o} v \in L^2(\mathbb{R}_+)$ and by the Hardy's inequality, we get $t^{\beta-j_o} D_t^{m-j_o-1} v \in L^2(\mathbb{R}_+)$; repeating the same argument, we obtain (ii).

If $\beta > m$, it results from (i) that $t^{\beta-m} u \in L^2(\mathbb{R}_+)$ and consequently if β-m $\neq$ integer + $\frac{1}{2}$, we have ([4]) $u \in H^{m-\beta}(\mathbb{R}_+)$.

If $\beta \leq m$, then $j_o \leq m$ and $-\frac{1}{2} < \beta - j_o \leq \frac{1}{2}$. Then, two cases must be distinguish according to $-\frac{1}{2} < \beta - j_o \leq 0$ and $0 < \beta - j_o \leq \frac{1}{2}$.

First case :

$0 < \beta - j_o \leq \frac{1}{2}$. We have : $t^{\beta-j_o} D_t^{m-j_o-1} v$ and $t^{\beta-j_o} D_t^{m-j_o} v \in L^2(\mathbb{R}_+)$ (see that $0 < \beta - j_o$ and $\beta \leq m$ implies $j_o+1 \leq m$). Then, we have $t^{1/2} D_t^{m-j_o-1} v$ and $t^{1/2} D_t^{m-j_o} v \in L^2(\mathbb{R}_+)$, and now we prove that these two conditions imply $D_t^{m-j_o-1} v \in L^2(\mathbb{R}_+)$.

Lemma I-1 :

([1]). If $u \in V^1_{1/2,1/2}(\mathbb{R}_+)$, then $u \in L^2(\mathbb{R}_+)$.

Proof :

If $u \in \mathcal{D}(\mathbb{R}_+)$, we can write :

$$|u(t)|^2 = 2 \text{ Re} \int_t^{+\infty} u(\sigma)\overline{u'(\sigma)}\, d\sigma$$

and using the Fubini's theorem, it comes :

$$\int_o^{+\infty} |u|^2 dt \leq -2 \text{ Re} \int_o^{+\infty} \sigma\, u(\sigma)\overline{u'(\sigma)}\, d\sigma \leq \int_o^{+\infty} t|u(t)|^2 dt + \int_o^{+\infty} t|u'(t)|^2 dt.$$

At last, by the density of $\mathcal{D}(\mathbb{R}_+)$ in the space $V^1_{1/2,1/2}(\mathbb{R}_+)$, we get the lemma I.1.

Now, we prove that $D_t^{m-j_o-1} v \in H^{\varepsilon}(\mathbb{R}_+)$ with $\varepsilon = 1-(\beta-j_o)$. For that, put $D_t^{m-j_o-1} v = f$ and $D_t^{m-j_o} v = F$ and compute :

$$\int_o^{+\infty} \int_o^{+\infty} \frac{|f(x) - f(y)|^2}{|x-y|^{2\varepsilon+1}}\, dx\, dy = \int_o^{+\infty} \int_o^{+\infty} \frac{|f(x+t) - f(x)|^2}{t^{2\varepsilon+1}}\, dx\, dt \ .$$

But,

$$f(x+t) - f(x) = \int_o^t F(x+\sigma)\, d\sigma .$$

Then,

$$\int_o^{+\infty} \frac{|f(x+t) - f(x)|^2}{t^{2\varepsilon+1}}\, dt = \int_o^{+\infty} \frac{1}{t^{2\varepsilon+1}} \left| \int_o^{+\infty} F(x+\sigma)\, d\sigma \right|^2 dt ,$$

and using the Hardy's inequality,

$$\leqslant C \int_o^{+\infty} \frac{1}{t^{2\varepsilon-1}} |F(x+t)|^2\, dt .$$

(C is a constant).

But,

$$\int_o^{+\infty} \frac{1}{t^{2\varepsilon-1}} |F(x+t)|^2\, dt = \int_x^{+\infty} \frac{1}{|y-x|^{2\varepsilon-1}} |F(y)|^2\, dy$$

$$= x^{-2(\varepsilon-1)} \int_1^{+\infty} \frac{1}{|\sigma-1|^{2\varepsilon-1}} |F(\sigma x)|^2\, d\sigma$$

and using the Fubini's theorem, it comes :

$$\int_o^{+\infty} x^{-2(\varepsilon-1)} \left(\int_1^{+\infty} \frac{1}{|\sigma-1|^{2\varepsilon-1}} |F(\sigma x)|^2 d\sigma \right) dx = \int_1^{+\infty} \frac{\sigma^{2(\varepsilon-1)-1}}{|\sigma-1|^{2\varepsilon-1}}\, d\sigma \cdot \int_o^{+\infty} y^{-2(\varepsilon-1)} |F(y)|^2 dy$$

then, $D_t^{m-j_o-1} v \in H^\varepsilon(\mathbb{R}_+)$ and $v \in H^{m-\beta}(\mathbb{R}_+)$.

<u>Second case</u> :

$-\frac{1}{2} < \beta - j_o \leqslant 0$. The case $\beta - j_o = 0$ being trivial, we can assume that $-\frac{1}{2} < \beta - j_o < 0$. Then, $\frac{1}{2} < \beta - j_o + 1 < 1$ and we have $D_t^{m-j_o} v \in L^2(\mathbb{R}_+)$ and $t^{\beta-j_o+1} D_t^{m-j_o+1} v \in L^2(\mathbb{R}_+)$. By the same calculus as before we get that $D_t^{m-j_o} v \in H^\varepsilon(\mathbb{R}_+)$ with $\varepsilon = -(\beta-j_o)$ and finally $v \in H^{m-\beta}(\mathbb{R}_+)$.

The proposition I.1 is proved.

Remark I.1 :

We can improve the result of the proposition I.1 when $\beta-\alpha > m$, in fact we have : if $\beta-\alpha > m$ and if u $V^m_{\alpha,\beta}(0,T)$, then $t^{\alpha+\frac{j}{m}(\beta-\alpha)} D^j_t u \in L^2(0,T)$ for $j = 0,\dots,m$. The proof is analogous to that of the following proposition I.2.

Proposition I.2 :

If $\beta-\alpha < m$ and if $u \in V^m_{\alpha,\beta}(T,+\infty)$ where T is a real number >0, then :

$$t^{\alpha+\frac{j}{m}(\beta-\alpha)} D^j_t u \in L^2(T,+\infty) \qquad \text{for } j = 0,\dots,m.$$

Proof :

It will be made in two steps.

First step :

Reduction to the case $\alpha = 0$.

Lemma I.2 :

If $u \in V^m_{\alpha,\beta}(T,+\infty)$, then : $t^{\beta-m+j} D^j_t u \in L^2(T,+\infty)$.

Proof :

If $\beta \leq \frac{1}{2}$, obviously we have $u \in H^m(T,+\infty)$ and then $t^{\beta-j} D^{m-j}_t u \in L^2(T,+\infty)$ for $j = 0,\dots,m$.

If $\beta > \frac{1}{2}$, then, as in the proposition I.1, we get that $t^{\beta-j} D^{m-j}_t u \in L^2(T,+\infty)$ for $0 \leq j \leq \mathrm{Min}(j_o,m)$ with $j_o = [\beta+\frac{1}{2}]_-$. At last, since $D^{m-j}_t u \in L^2(T,+\infty)$ for $j = 0,\dots,m$, we get that $t^{\beta-j} D^{m-j}_t u \in L^2(T,+\infty)$ for $j = j_o+1,\dots,m$ if $j_o+1 \leq m$ ($\beta-j$ is negative).

Lemma I.3 :

The map $u \longrightarrow t^{\alpha}u$ is an isomorphism from $V^m_{\alpha,\beta}(T,+\infty)$ onto $V^m_{o,\beta-\alpha}(T,+\infty)$.

Proof :

Let u be an element of $V^m_{\alpha,\beta}(T,+\infty)$, we put $v = t^{\alpha}u$; then $t^{\beta-\alpha}D^m_t v(t) = \sum_{j=o}^{m} a_j . t^{\beta-j}D^{m-j}_t u(t)$ and by the lemma I.2, it results that $v \in V^m_{o,\beta-\alpha}(T,+\infty)$.

Conversely, let v be an element of $V^m_{o,\beta-\alpha}(T,+\infty)$, we put $u = t^{-\alpha}v$; then $t^{\beta}D^m_t u(t) = \sum_{j=o}^{m} a_j . t^{\beta-\alpha-j}D^{m-j}_t v(t)$ and by the lemma I.2, it results that $u \in V^m_{\alpha,\beta}(T,+\infty)$.

Seconde step :

We assume $\alpha = 0$.

We use the change of variable $y = t^{\frac{m-\beta}{m}}$ and of function $w(y) = y^{\beta/2(m-\beta)}u(t)$.

By induction on p, we show that, for $0 \leqslant p \leqslant m$, we have :

$$D^p_y w(y) = y^{\beta/2(m-\beta)} \sum_{j=o}^{p} a_{jp} . t^{j-p+p\frac{\beta}{m}} D^j_t u(t) .$$

where $a_{pp} \neq 0$. By the lemma I.2, we get $D^m_y w \in L^2(Y,+\infty)$ where $Y = T^{\frac{m-\beta}{m}}$ and consequently $w \in H^m(Y,+\infty)$ since $w \in L^2(Y,+\infty)$. Then, $D^p_y w \in L^2(Y,+\infty)$ for $p = 0,\ldots,m$ and using the precedent formula, we get, by induction on p and since $j-p+p\frac{\beta}{m} < j\frac{\beta}{m}$ for $j < p$, that $t^{j\frac{\beta}{m}}D^j_t u \in L^2(T,+\infty)$ for $j = 0,\ldots,m$.

The proposition I.2 is proved.

We now apply these results to a sub-class of Sobolev spaces with weights which we will be useful for the following : let be $m \in \mathbb{N}$, $-\sigma$ and δ two real numbers >0 such that $\sigma+m \geqslant 0$ and $\sigma+\delta m \geqslant 0$, we consider the space:

$W^m_{\sigma,\delta}(\mathbb{R}_+) = \{u \in H^{-\sigma}(\mathbb{R}_+) \; ; \; t^{\sigma+\delta k+j} D^j_t u \in L^2(\mathbb{R}_+)$ for $\sigma+\delta k+j \geqslant 0$ and $k+j \leqslant m\}$

equipped by the canonical norm.

By the propositions I.1 and I.2, this space coïncide with the space $V^m_{\sigma+\delta m,\sigma+m}(\mathbb{R}_+)$.

We now give the Sobolev's theorem for the spaces $W^m_{\sigma,\delta}(\mathbb{R}_+)$.

Proposition I.3 : we have :

i) If $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, u is continuous on $\mathbb{R}_+$ and there exists a constant $C > 0$ such that for every $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, for every $t > 0$, we have :

$$(1.1) \qquad |u(t)| \leqslant C.\, t^{-\frac{\sigma+m}{2m}} \, ||u||^{1/2m}_{W^m_{\sigma,\delta}} \, ||u||^{1-1/2m}_{L^2} \; ;$$

(ii) We assume $-\sigma > \frac{1}{2}$, then : if $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, u is continuous and bounded on $\mathbb{R}_+$ and there exists a constant $C > 0$ such that for every $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, for every $t > 0$, we have :

$$(1.2) \qquad |u(t)| \leqslant C.\, ||u||^{-1/2\sigma}_{W^m_{\sigma,\delta}} \, ||u||^{1+\frac{1}{2\sigma}}_{L^2} \; ;$$

$$(1.3) \qquad |u(t)| \leqslant C.\, t^{-(\sigma+\delta m)+\frac{1}{2}(\delta-1)} \, ||u||_{W^m_{\sigma,\delta}} \, .$$

Proof :

(i) At first, we apply the usual Sobolev's theorem : if $v \in H^m(\mathbb{R}_+)$ with $m \geqslant 1$, then v is continuous on $\overline{\mathbb{R}_+}$ and there exists a constant $C > 0$ such that for every $v \in H^m(\mathbb{R}_+)$, for every $t \geqslant 0$, we have :

$$|v(t)|^2 \leqslant C \left\{ \int_0^{+\infty} |D^m_t v(\tau)|^2 \, d\tau + \int_0^{+\infty} |v(\tau)|^2 \, d\tau \right\}.$$

If $w \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, the function v defined by $v(\tau) = w(\tau+t)$ belongs to $H^m(\mathbb{R}_+)$ for every $t > 0$. Since $-\sigma > 0$ and $\sigma+m \geq 0$, then $m \geq 1$ and for every $w \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, for every $t > 0$, we have :

$$|w(t)|^2 \leq C. \left\{ \int_t^{+\infty} |D_t^m w(\tau)|^2 \, d\tau + \int_t^{+\infty} |w(\tau)|^2 \, d\tau \right\}.$$

Now, let u be an element of $W^m_{\sigma,\delta}(\mathbb{R}_+)$ and we apply the precedent inequality to the function w defined by $w(\tau) = u(\lambda\tau)$ where λ is a positive constant. Then, there exists a constant $C > 0$ such that, for every $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, for every $t > 0$, for every $\lambda > 0$, we have :

$$(1.4) \quad |u(t)|^2 \leq C/\lambda \left\{ \int_t^{+\infty} |\lambda^m D_t^m u(\tau)|^2 \, d\tau + \int_t^{+\infty} |u(\tau)|^2 \, d\tau \right\},$$

and since $t \leq \tau$, we get :

$$|u(t)|^2 \leq C/\lambda \left\{ \int_t^{+\infty} \lambda^{2m} t^{-2(\sigma+m)} |\tau^{\sigma+m} D_t^m u(\tau)|^2 \, d\tau + \int_t^{+\infty} |u(\tau)|^2 \, d\tau \right\}.$$

Choosing $\lambda = t^{\frac{\sigma+m}{m}}$, a fortiori we obtain :

$$|u(t)|^2 \leq C. \, t^{-\frac{\sigma+m}{m}} \left\{ \int_0^{+\infty} |\tau^{\sigma+m} D_t^m u(\tau)|^2 \, d\tau + \int_0^{+\infty} |u(\tau)|^2 \, d\tau \right\}.$$

Now, we apply this inequality to the function v defined by $v(\tau) = u(\lambda\tau)$ where λ is a constant >0 :

$$|u(\lambda t)|^2 \leq C. \, t^{-\frac{\sigma+m}{m}} \Big/ \lambda \left\{ \int_0^{+\infty} \lambda^{-2\sigma} |\tau^{\sigma+m} D_t^m u|^2 \, d\tau + \int_0^{+\infty} |u(\tau)|^2 \, d\tau \right\}.$$

Putting $\lambda = r^{1/2\sigma}$, we get for every $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, for every $t > 0$, for every $r > 0$, we have :

$$|u(t\, r^{1/2\sigma})|^2 \leq C. \, (t\, r^{1/2\sigma})^{-\frac{\sigma+m}{m}} r^{\frac{1}{2m} - 1} \left\{ \int_0^{+\infty} |\tau^{\sigma+m} D_t^m u|^2 d\tau + r \int_0^{+\infty} |u(\tau)|^2 d\tau \right\}.$$

Finally, there exists a constant $C > 0$ such that, for every $t > 0$, for every $r > 0$, for every $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, we have :

$$|u(t)|^2 \leq C.\ t^{-\frac{\sigma+m}{m}}\ r^{\frac{1}{2m}-1}\ \{\ ||u||^2_{W^m_{\sigma,\delta}} + r\ ||u||^2_{L^2}\ \}\ .$$

Taking $r = ||u||^2_{W^m_{\sigma,\delta}} \big/ ||u||^2_{L^2}$, we obtain the inequality (1.1).

(ii) If $-\sigma > \frac{1}{2}$, the Sobolev's theorem imply that if $v \in H^{-\sigma}(\mathbb{R}_+)$, then v is continuous and bounded on $\overline{\mathbb{R}_+}$ and there exists a constant $C > 0$ such that for every $v \in H^{-\sigma}(\mathbb{R}_+)$, for every $t \geq 0$, we have :

$$|v(t)|^2 \leq C.\ ||v||^2_{H^{-\sigma}(\mathbb{R}_+)}\ .$$

But, from the proposition I.1, the space $V^m_{o,\sigma+m}(\mathbb{R}_+)$ is continuously imbedded in $H^{-\sigma}(\mathbb{R}_+)$, then, for every $t \geq 0$, for every $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, we have :

$$|v(t)|^2 \leq C.\ \{\ \int_o^{+\infty} |\tau^{\sigma+m} D^m_t u|^2\ d\tau + \int_o^{+\infty} |u(\tau)|^2\ d\tau\ \}.$$

Using the same change of functions as before, we get that for every $u \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, for every $t > 0$, for every $r > 0$, we have :

$$|u(t)|^2 \leq C.\ r^{-1-\frac{1}{2\sigma}}\ \{\ ||u||^2_{W^m_{\sigma,\delta}} + r\ ||u||^2_{L^2}\ \}.$$

We obtain the inequality (1.2) in taking $r = ||u||^2_{W^m_{\sigma,\delta}} \big/ ||u||^2_{L^2}$.

To have the inequality (1.3), we start from the inequality (1.4) in which we choose $\lambda = (\int_t^{+\infty} |u(\tau)|^2 d\tau)^{1/2m}\ (\int_t^{+\infty} |D^m_t u(\tau)|^2 d\tau)^{-1/2m}$, that gives :

$$|u(t)|^2 \leqslant C.\ \left(\int_t^{+\infty} |D_t^m u|^2\ d\tau\right)^{1/2m} \left(\int_t^{+\infty} |u(\tau)|^2\ d\tau\right)^{1-1/2m} ;$$

after that, we remark that, since $t \leqslant \tau$, we have :

$$\int_t^{+\infty} |D_t^m u|^2\ d\tau \leqslant t^{-2(\sigma+m)} \int_t^{+\infty} \tau^{2(\sigma+m)}\ |D_t^m u|^2\ d\tau \leqslant t^{-2(\sigma+m)} ||u||^2_{W^m_{\sigma,\delta}}$$

and

$$\int_t^{+\infty} |u(\tau)|^2 d\tau \leqslant t^{-2(\sigma+\delta m)} \int_t^{+\infty} \tau^{2(\sigma+\delta m)}\ |u(\tau)|^2\ d\tau \leqslant t^{-2(\sigma+\delta m)} ||u||^2_{W^m_{\sigma,\delta}}$$

hence the inequality (1.3).

II - CASE OF THE HALF SPACE $\mathbb{R}^n_+$, $n > 1$.

Let m be an integer, $-\sigma$ and δ two real numbers >0 such that $\sigma+m \geqslant 0$ and $\sigma+\delta m \geqslant 0$, we consider the space :

$$W^m_{\sigma,\delta}(\mathbb{R}^n_+) = \{u \in L^2(\mathbb{R}^n_+);\ t^{\sigma+\delta|\alpha|+j} D_t^j D_x^\alpha u \in L^2(\mathbb{R}^n_+) \text{ for } \sigma+\delta|\alpha|+j \geqslant 0 \text{ and } |\alpha|+j \leqslant m\}$$

equipped by the canonical norm.

The space $\mathcal{D}(\overline{\mathbb{R}^n_+})$ is dense in the space $W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ (cf[2]for example) and also we have :

$$W^m_{\sigma,\delta}(\mathbb{R}^n_+) = \{u \in \mathcal{D}'(\mathbb{R}^n_+);\ t^{\mathrm{Max}(o,\sigma+\delta|\alpha|+j)} D_t^j D_x^\alpha u \in L^2(\mathbb{R}^n_+) \text{ for } |\alpha|+j \leqslant m\}.$$

<u>Proposition II.1.</u> we have :

i) if $m > n/2$ and if $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, then u is continuous on $\mathbb{R}^n_+$ and there exists a constant $C > 0$ such that, for every $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, for every $(t,x) \in \mathbb{R}^n_+$, we have :

$$(2.1) \qquad |u(t,x)| \leqslant C.\ t^{-\frac{\sigma+m}{2m} - \frac{n-1}{2m}(\sigma+\delta m)} ||u||^{n/2m}_{W^m_{\sigma,\delta}} ||u||^{1-n/2m}_{L^2} ;$$

(ii) If Min $(-\sigma, -\sigma/_{\delta}) > n/2$ and if $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, then u is continuous and bounded on $\mathbb{R}^n_+$ and there exists a constant $C > 0$ such that for every $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, for every $(t,x) \in \mathbb{R}^n_+$, we have :

(2.2) $$|u(t,x)| \leqslant C. \, ||u||_{W^m_{\sigma,\delta}}^{-\frac{1+\delta(n-1)}{2\sigma}} \, ||u||_{L^2}^{1+\frac{1+\delta(n-1)}{2\sigma}} .$$

Proof :

The proof is analogous to those made in the chapter I. (i) ; at first, we apply the usual Sobolev's theorem : if $v \in H^m(\mathbb{R}^n_+)$ with $m > n/2$ then v is continuous on $\overline{\mathbb{R}^n_+}$ and there exists a constant $C > 0$ such that for every $v \in H^m(\mathbb{R}^n_+)$, for every $(t,x) \in \mathbb{R}^n_+$, we have :

$$|u(t,x)|^2 \leqslant C. \, \{ \sum_{j+|\alpha|=m} \int_{\mathbb{R}^n_+} |D^j_t D_x v(\tau,y)|^2 \, d\tau \, dy + \int_{\mathbb{R}^n_+} |v(\tau,y)|^2 \, d\tau \, dy \}.$$

If $w \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, the function v defined by : $v(\tau,y) = w(\tau+t,y)$ belongs to $H^m(\mathbb{R}^n_+)$ for every $t > 0$. Hence, for every $w \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, for every $(t,x) \in \mathbb{R}^n_+$, we have :

$$|w(t,x)|^2 \leqslant C. \, \{ \sum_{|\alpha|+j=m} \int_t^{+\infty} \int_{\mathbb{R}^{n-1}} |D^j_t D^\alpha_x w(\tau,y)|^2 d\tau dy + \int_t^{+\infty} \int_{\mathbb{R}^{n-1}} |w(\tau,y)|^2 d\tau dy \} .$$

Let now u an element of $W^m_{\sigma,\delta}(\mathbb{R}^n_+)$ and apply the precedent inequality to the function w defined by : $w(\tau,y) = u(\lambda\tau, \mu y)$ where λ and μ are two constants. Hence, there exists a constant $C > 0$ such that, for every $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, for every $(t,x) \in \mathbb{R}^n_+$, for every λ and $\mu > 0$, we have :

$$|u(t,x)|^2 \leqslant C/_{\lambda.\mu^{n-1}} \cdot \{ \sum_{|\alpha|+j=m} \int_t^{+\infty} \int_{\mathbb{R}^{n-1}} \lambda^{2j} \, \mu^{2(m-j)} \, |D^j_t D^\alpha_x u(\tau,y)|^2 \, d\tau \, dy$$

$$+ \int_t^{+\infty} \int_{\mathbb{R}^{n-1}} |u(\tau,y)|^2 \, d\tau \, dy \},$$

and since $t \leqslant \tau$, that gives:

$$|u(t,x)|^2 \leqslant C/_{\lambda.\mu^{n-1}} \times$$

$$\times \{\sum_{|\alpha|+j=m} \int_t^{+\infty} \int_{\mathbb{R}^{n-1}} \lambda^{2j}\mu^{2(m-j)} t^{-2(\sigma+\delta(m-j)+j)} |\tau^{\sigma+\delta|\alpha|+j} D_t^j D_x^\alpha u|^2 d\tau dy$$

$$+ \int_t^{+\infty} \int_{R^{n+1}} |u(\tau,y)|^2 \, d\tau \, dy\}.$$

choosing $\lambda = t^{\frac{\sigma+m}{m}}$ and $\mu = t^{\frac{\sigma+\delta m}{m}}$, a fortioti we get :

$$|u(t,x)|^2 \leqslant C.t^{-\frac{\sigma+m}{m} - \frac{n-1}{m}(\sigma+\delta m)} \{\sum_{|\alpha|+j=m} \int_{\mathbb{R}^n_+} |\tau^{\sigma+\delta|\alpha|+j} D_t^j D_x^\alpha u|^2 \, d\tau \, dy$$

$$+ \int_{\mathbb{R}^n_+} |u(\tau,y)|^2 \, d\tau \, dy \}.$$

We now apply this inequality to the function v defined by : $v(\tau,y) = u(\lambda\tau,\mu x)$ where λ and μ are some constants :

$$|u(\lambda t,\mu x)|^2 \leqslant C \times$$

$$\times \frac{t^{-\frac{\sigma+m}{m} - \frac{n-1}{m}(\sigma+\delta m)}}{\lambda.\mu^{n-1}} \{\sum_{|\alpha|+j=m} \int_{\mathbb{R}^n_+} \lambda^{-2(\sigma+\delta(m-j))} \mu^{2(m-j)} |\tau^{\sigma+\delta|\alpha|+j} D_t^j D_x^\alpha u|^2 d\tau dy$$

$$+ \int_{\mathbb{R}^n_+} |u|^2 \, d\tau \, dy \}.$$

Putting $\lambda = r^{1/2\sigma}$ and $\mu = \lambda^\delta$, we deduce that for every $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, for every $(t,x) \in \mathbb{R}^n_+$, for every $r > 0$, we have :

$$|u(tr^{1/2\sigma},xr^{\delta/2\sigma})|^2 \leqslant C\ x$$

$$\times (tr^{1/2\sigma})^{-\frac{\sigma+m}{m}-\frac{n-1}{m}(\sigma+\delta m)}\ r^{n/2m-1}\{\sum_{|\alpha|+j=m}\int_{\mathbb{R}^n_+}|\tau^{\sigma+\delta|\alpha|+j}D^j_tD^\alpha_x u|^2 d\tau dy$$

$$+ r\int_{\mathbb{R}^n_+}|u|^2 d\tau\ dy\ \}.$$

Finally, there exists a constant $C > 0$ such that, for every $(t,x) \in \mathbb{R}^n_+$, for every $r > 0$, for every $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, we have :

$$|u(t,x)|^2 \leqslant C.\ t^{-\frac{\sigma+m}{m}-\frac{n-1}{m}(\sigma+\delta m)}\ r^{n/2m-1}\ \{\ ||u||^2_{W^m_{\sigma,\delta}} + r\ ||u||^2_{L^2}\}\ .$$

The inequality (2.1) results form this in choosing $r = ||u||^2_{W^m_{\sigma,\delta}} / ||u||^2_{L^2}$.

(ii), we begin to show the

Lemma II-1 :

We have the algebraic and topologic imbedding :

$$W^m_{\sigma,\delta}(\mathbb{R}^n_+) \subset H^{\mathrm{Min}(-\sigma,-\sigma/\delta)}(\mathbb{R}^n_+)\ .$$

Proof :

By the chapter I, we know $V^m_{\sigma+\delta m,\sigma+m}(\mathbb{R}_+) \subset H^{-\sigma}(\mathbb{R}_+)$, hence, there exists a constant $C > 0$ such that, for every $v \subset W^m_{\sigma,\delta}(\mathbb{R}_+)$, we have :

$$\int_{-\infty}^{+\infty}(1+\tau^2)^{-\sigma}|F(Pv)|^2 d\tau \leqslant C.\{\int_0^{+\infty}|t^{\sigma+m}D^m_t v|^2 dt + \int_0^{+\infty}|t^{\sigma+\delta m}v|^2 dt\}\ ,$$

where F means the Fourier transform in the variable t and P a linear and continuous extension operator from $H^{-\sigma}(\mathbb{R})$ (for example, P can be taken as the Babitch extension).

If $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, the function $u(t) = v(t\Lambda^{-1/\delta})$, where Λ is positive constant, belongs to $W^m_{\sigma,\delta}(\mathbb{R}_+)$; for every $\Lambda > 0$, we have :

$$\int_{-\infty}^{+\infty} (\Lambda^{2/\delta}+\tau^2)^{-\sigma} |F(Pv)|^2 d\tau \leqslant C. \{\int_0^{+\infty} |t^{\sigma+m} D_t^m v|^2 dt + \Lambda^{2m} \int_0^{+\infty} |t^{\sigma+\delta m} v|^2 dt\} .$$

Let now u be an element of $\mathcal{D}(\overline{\mathbb{R}^n_+})$ and for every $\xi \in \mathbb{R}^{n-1}\setminus\{0\}$, we consider the function $v(t) = \hat{u}(t,\xi)$, where $\wedge$ means the Fourier transform in the variable $x \in \mathbb{R}^{n-1}$; then $F(Pv)(\tau) = \mathcal{F}Pu(\tau,\xi)$, where $\mathcal{F}$ means the Fourier transform in the variable (t,x) in $\mathbb{R}^n$ and from the precedent inequality, we deduce, taking $\Lambda = |\xi|$ and after integrate in ξ over $\mathbb{R}^{n-1}$, that there exists a constant $C > 0$ such that for all $u \in \mathcal{D}(\overline{\mathbb{R}^n_+})$, we have : putting $\sigma^* = \text{Min}(-\sigma,-\sigma/\delta)$,

$$||Pu||_{H^{-\sigma^*}(\mathbb{R}_n)} \leqslant C.||u||_{W^m_{\sigma,\delta}(\mathbb{R}^n_+)}$$

and then :

$$||u||_{H^{-\sigma^*}(\mathbb{R}^n_+)} \leqslant C.||u||_{W^m_{\sigma,\delta}(\mathbb{R}^n_+)} .$$

The space $\mathcal{D}(\overline{\mathbb{R}^n_+})$ being dense in the space $W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, we have proved the lemma II-1.

Now, if $\text{Min}(-\sigma,-\sigma/\delta) > n/2$ and if $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, then u is continuous and bounded on $\overline{\mathbb{R}^n_+}$ and there exists a constant $C > 0$ such that for every $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, for every $(t,x) \in \mathbb{R}^n_+$, we have :

$$|u(t,x)|^2 \leqslant C. \{ \sum_{|\alpha|+j=m} \int_{\mathbb{R}^n_+} \tau^{2(\sigma+\delta(m-j)+j)} |D_t^j D_x^\alpha u(\tau,y)|^2 \, d\tau \, dy + \int_{\mathbb{R}^n_+} |u(\tau,y)|^2 \, d\tau \, dy \}.$$

Then, we do the change of variable of (i), that gives :

$$|u(t,x)|^2 \leq C/\lambda\mu^{n-1} \times$$

$$\times \{ \sum_{|\alpha|+j=m} \int_{R^n_+} \lambda^{-2(\sigma+\delta(m-j))} \mu^{2(m-j)} |\tau^{2(\sigma+\delta(m-j)+j)} D_t^j D_x^\alpha u(\tau,y)|^2 d\tau dy$$

$$+ \int_{R^n_+} |u(\tau,y)|^2 \, d\tau \, dy \}:$$

we choose $\lambda = r^{1/2\sigma}$ and $\mu = \lambda^\delta$, that gives :

$$|u(t,x)|^2 \leq C.\, r^{-\frac{2\sigma+1+\delta(n-1)}{2\sigma}} \{ ||u||^2_{W^m_{\sigma,\delta}(\mathbb{R}^n_+)} + r\, ||u||^2_{L^2(\mathbb{R}^n_+)} \},$$

and taking $r = ||u||^2_{W^m_{\sigma,\delta}} / ||u||^2_{L^2}$, we get the inequality (2.2).

<u>Proposition II.2</u> :

Let ℓ be an integer, $o \leq \ell < -\sigma - \frac{1}{2}$; then the map $u \longrightarrow \gamma_\ell u = D_t^\ell u(t=o)$: $\mathcal{D}(\overline{R^n_+}) \longrightarrow (R^{n-1})$ can be extended in a linear and continuous map from $W^m_{\sigma,\delta}(R^n_+)$ into $H^{-\frac{2(\sigma+\ell)+1}{2\delta}}(R^{n-1})$.

<u>Proof</u> :

It comes, by the chapter I, that there exists a constant $C > 0$ such that, for every $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, we have :

$$|D_t^\ell v(o)|^2 \leq C.\, \{ \int_o^{+\infty} |t^{\sigma+m} D_t^m v|^2 dt + \int_o^{+\infty} |t^{\sigma+\delta m} v|^2 dt \}.$$

If $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, the function $u(t) = v(t\,\Lambda^{-\frac{1}{\sigma}})$, where Λ is a positive constant, belongs to $W^m_{\sigma,\delta}(\mathbb{R}_+)$; hence here exists a constant $C>0$ such that for every

$v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, for every $\Lambda > 0$, we have :

$$\Lambda^{-\frac{2(\sigma+\ell)+1}{2\delta}} |D_t^\ell v(o)|^2 \leqslant C. \{ \int_o^{+\infty} |t^{\sigma+m} D_t^m v|^2 dt + \Lambda^{2m} \int_o^{+\infty} |t^{\sigma+\delta m} v|^2 dt \}.$$

Let now u be an element of $\mathcal{D}(\overline{R^n_+})$, and for every $\xi \in \mathbb{R}^{n-1} \setminus \{o\}$, we consider the function $v(t) = \hat{u}(t,\xi)$, where $\wedge$ is the Fourier transform in the variable $x \in \mathbb{R}^{n-1}$; as in lemma II-1, we deduce that :

$$||\gamma_\ell u||_{H^{-\frac{2(\sigma+\ell)+1}{2\delta}}} \leqslant C. ||u||_{W^m_{\sigma,\delta}} .$$

It will be very useful for the following to have an inequality of type "compacity" for the spaces $W^m_{\sigma,\delta}$:

Proposition II.3.

Let m be an integer $\geqslant 1$ and put $\delta_1 = \text{Min}(1,\delta)$. There exists a constant $C > 0$ such that, for every $\varepsilon > 0$, for every $u \in W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, with supp $u \subset \{|t| \leqslant 1\}$, we have :

(2.3) $$||u||_{W^{m-1}_{\sigma+\delta_1,\delta}} \leqslant C. \{ \varepsilon . ||u||_{W^m_{\sigma,\delta}} + \varepsilon^{-(m-1)} ||u||_{L^2} \} .$$

Proof :

We begin to establish a lemma :

Lemma II-2 :

The map $u \longrightarrow \{ ||t^{\sigma+m} D_t^m u||^2_{L^2} + \sum_{|\alpha|=m} ||t^{\sigma+\delta m} D_x^\alpha u||^2_{L^2} + ||u||^2_{L^2} \}^{1/2}$ is an equivalent norm for the space $W^m_{\sigma,\delta}(\mathbb{R}^n_+)$.

Proof :

Let k and j be some integers such that $\sigma+\delta k+j \geq 0$ and $k+j \leq m$. From the chapter I, it results that if $v(t) \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, then $t^{\sigma+\delta k+j}D^j_t v \in L^2(\mathbb{R}_+)$ and :

$$\int_0^{+\infty} |t^{\sigma+\delta k+j}\, D^j_t v|^2 dt \leq C. \left\{ \int_0^{+\infty} |t^{\sigma+m} D^m_t v|^2 dt + \int_0^{+\infty} |t^{\sigma+\delta m} v|^2 dt \right\}$$

where C is a constant >0 which does not depend on v.

If $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, the function $u(t) = v(t\,\Lambda^{-1/\sigma})$, where Λ is a positive constant belongs to $W^m_{\sigma,\delta}(\mathbb{R}_+)$; hence, there exists a constant $C > 0$ such that for every $v \in W^m_{\sigma,\delta}(\mathbb{R}_+)$, for every Λ, we have :

$$(2.4)\quad \Lambda^{2k}\int_0^{+\infty} |t^{\sigma+\delta k+j} D^j_t v|^2 dt \leq C.\left\{\int_0^{+\infty} |t^{\sigma+m} D^m_t v|^2 dt + \Lambda^{2m}\int_0^{+\infty} |t^{\sigma+\delta m} v|^2 dt\right\}.$$

Let now u be an element of $\mathcal{D}(\overline{R^n_+})$ and for every $\xi \in \mathbb{R}^{n-1}\setminus\{o\}$, we consider the function $v(t) = \hat{u}(t,\xi)$, where $\wedge$ means the Fourier transform in the variable $x \in \mathbb{R}^{n-1}$, and from the precedent inequality, we deduce, taking $\Lambda = |\xi|$ and after integration in ξ over $\mathbb{R}^{n-1}$, that there exists a constant $C > 0$ such that for every $u \in \mathcal{D}(\overline{R^n_+})$, we have :

$$||u||^2_{W^m_{\sigma,\delta}} \leq C.\ \left\{||t^{\sigma+m} D^m_t u||^2_{L^2} + \sum_{|\alpha|=m} ||t^{\sigma+\delta m} D^\alpha_x u||^2_{L^2} + ||u||^2_{L^2}\right\}.$$

The space $\mathcal{D}(\overline{R^n_+})$ being dense in the space $W^m_{\sigma,\delta}(\mathbb{R}^n_+)$, the lemma II-2 is a consequence of this inequality and the Banach's theorem.

Proof of the proposition II-3 :

From the inequality (2.4) in which we take $j = m-1$, $k = 1$ and $\Lambda^{-1} = \varepsilon > 0$, we deduce that :

$$\int_0^{+\infty} |t^{\sigma+\delta+m-1} D_t^{m-1} v|^2 \, dt \leqslant C \times$$

$$\times \; \varepsilon^2 \int_0^{+\infty} |t^{\sigma+m} D_t^m v|^2 \, dt + \varepsilon^{-2(m-1)} \int_0^{+\infty} |t^{\sigma+\delta m} v|^2 \, dt \}.$$

We apply this inequality to the function $v(t) = \hat{u}(t,\xi)$ for $u \in \mathcal{D}(\overline{R^n_+})$ and $\xi \in \mathbb{R}^{n-1} \setminus \{o\}$, we integrate in ξ over R^{n-1}, that gives :

$$(2.5) \quad ||t^{\sigma+\delta+m-1} D_t^{m-1} u||^2_{L^2(R^n_+)} \leqslant C. \{\varepsilon^2 ||t^{\sigma+m} D_t^m u||^2_{L^2(R^n_+)} + \varepsilon^{-2(m-1)} ||u||^2_{L^2(R^n_+)} \}$$

if supp $u \subset \{|t| \leqslant 1\}$.

Besides, we know that there exists a constant $C > 0$ such that for every $\varepsilon > 0$, for every $v(x) \in H^m(R^{n-1})$, we have :

$$(2.6) \quad \sum_{|\alpha|=m-1} \int_{R^{n-1}} |D_x^\alpha v|^2 dx \leqslant C. \{\varepsilon^2 \sum_{|\alpha|=m} \int_{R^{n-1}} |D_x^\alpha v|^2 dx + \varepsilon^{-2(m-1)} \int_{R^{n-1}} |v|^2 dx \}.$$

Then, we use this inequality to the function $v(x) = u(t,x)$, $t > 0$, where $u \in \mathcal{D}(\overline{R^n_+})$; we multiply by $t^{\sigma+\delta m}$, and we integrate in $t > 0$ over R_+, that gives :

$$(2.7) \quad \sum_{|\alpha|=m-1} ||t^{\sigma+\delta m} D_x^\alpha u||^2_{L^2(R^n_+)} \leqslant C. \{\varepsilon^2 \sum_{|\alpha|=m} ||t^{\sigma+\delta m} D_x^\alpha u||^2_{L^2(R^n_+)}$$

$$+ \varepsilon^{-2(m-1)} ||u||^2_{L^2(R^n_+)} \} .$$

if supp $u \subset \{ |t| \leqslant 1 \}$.

The inequality (2.3), for $\delta \leqslant 1$, is a consequence of (2.5) and (2.7). For $\delta \geqslant 1$, we replace the inequality (2.5) by the inequality :

$$(2.8) \quad ||t^{\sigma+m} D_t^{m-1} u||^2_{L^2(R^n_+)} \leqslant C. \{\varepsilon^2 ||t^{\sigma+m} D_t^m u||^2_{L^2(R^n_+)} + \varepsilon^{-2(m-1)} ||u||^2_{L^2(R^n_+)} \}.$$

if supp u $\subset$ $\{|t| \leq 1\}$. This inequality is easy to prove like for (2.5).

After, in (2.7), we multiply by $t^{2(\sigma+1+\delta(m-1))}$ and we choose $\varepsilon = \eta t^{\delta-1}$, $\eta > 0$, and we achieve as before.

III - CASE OF A BOUNDED OPEN SET Ω OF $\mathbb{R}^n$, $n > 1$.

Let Ω be a bounded open set of $\mathbb{R}^n$, with boundary Γ. We assume that Ω is a compact C^∞ manifold. We give $\varphi: \mathbb{R}^n \longrightarrow \mathbb{R}$ a C^∞ function such that :

$$(3.1) \quad \begin{cases} \Omega = \{x \in \mathbb{R}^n \ ; \ \varphi(x) > 0\} , \\ \Gamma = \{x \in \mathbb{R}^n \ ; \ \varphi(x) = 0\} , \\ \text{grad } \varphi(x) \neq 0 \text{ for } x \in \Gamma , \end{cases}$$

Where grad $\varphi(x) = \left(\frac{\partial \varphi}{\partial x_1}(x), \ldots, \frac{\partial \varphi}{\partial x_n}(x) \right.$ is the gradient vector associated to φ.

Let $(X_i)_{o \leq i \leq q}$ be some vector fields with C^∞ coefficients on $\mathbb{R}^n$ such that :

(3.2) X_o is transversal to Γ on Γ, ie : $(X_o \varphi)(x) \neq 0$ for $x \in \Gamma$;

(3.3) X_i is tangent to Γ on Γ for $i = 1,\ldots,q$, ie : $(X_i \varphi)(x) = 0$ for $x \in \Gamma$;

(3.4) for every $x \in \overline{\Omega}$, the rank of the system $(X_i(x))_{o \leq i \leq q}$ is equal to n.

Let m be an integer, $-\sigma$ and δ two real numbers >0 such that $\sigma+m \geqslant 0$ and $\sigma+\delta m \geqslant 0$, we consider the space:

$$W^m_{\sigma,\delta}(\Omega) = \{u \in L^2(\Omega) \ ; \ \varphi^{\mathrm{Max}(o,\sigma+<\delta,\alpha>)} X^\alpha u \in L^2(\Omega) \text{ for } |\alpha| \leq m \}$$

equipped by the canonical norm. We have used the notation $X^\alpha = X_o^{\alpha_o} \dots X_q^{\alpha_q}$ for $\alpha = (\alpha_o, \dots, \alpha_q) \in \mathbb{N}^{q+1}$ and $<\delta,\alpha> = \delta \sum_{i=1}^{q} \alpha_i + \alpha_o$.

Proposition III-1.

With the precedent assumptions, we have :

(i) $W^m_{\sigma,\delta}(\Omega) \subset H^m_{loc}(\Omega)$;

(ii) for every $\phi \in C^\infty(\bar\Omega)$ and for every $u \in W^m_{\sigma,\delta}(\Omega)$, we have : $\phi\, u \in W^m_{\sigma,\delta}(\Omega)$.

Proof :

(i) With the assumption (3.4), for every $x_o \in \Omega$, there exists a neighbourhood $V(x_o)$ of x_o in Ω in which we can write :

$$\frac{\partial}{\partial x_k} = \sum_{i=o}^{q} \beta_i^k(x)\; X_i$$

for $k = 1,\dots,n$ with some convenient functions β_i^k which are C^∞ in $V(x_o)$ and we can easily get (i).

(ii) Let ϕ be a C^∞ function on $\bar\Omega$ and $u \in W^m_{\sigma,\delta}(\Omega)$. Then $\phi\, u \in L^2(\Omega)$ and for $|\alpha| \leq m$, we have :

$$X^\alpha(\phi u) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta}\; (X^\beta\phi)\; (X^{\alpha-\beta}u)$$

it results that $\varphi^{\mathrm{Max}(o,\sigma+<\delta,\alpha>)} X^\alpha(\phi u) \in L^2(\Omega)$, that is to say $\phi\, u \in W^m_{\sigma,\delta}(\Omega)$.

Remark III-1 :

It is easy to prove that the space $W^m_{\sigma,\delta}(\Omega)$ does not depend of the choice of the vector fields $(X_i)_{o\leq i\leq q}$ satisfying the conditions (3.2), (3.3), (3.4).

Proposition III-2 :

We have :

(i) If $m > n/2$ and if $u \in W^m_{\sigma,\delta}(\Omega)$, then u is continuous on Ω and there exists a constant $C > 0$ such that, for every $u \in W^m_{\sigma,\delta}(\Omega)$, for every $x \in \Omega$, we have :

$$(3.5) \qquad |u(x)| \leq C.\ \varphi(x)^{-\frac{\sigma+m}{2m} - \frac{n-1}{2m}(\sigma+\delta m)} \, ||u||^{n/2m}_{W^m_{\sigma,\delta}} \, ||u||^{1-n/2m}_{L^2} \ ;$$

(ii) if $\mathrm{Min}(-\sigma,-\sigma/\delta) > n/2$ and if $u \in W^m_{\sigma,\delta}(\Omega)$, then u is continuous and bounded on Ω there exists a constant $C > 0$ such that for every $u \in W^m_{\sigma,\delta}(\Omega)$, for every $x \in \Omega$, we have :

$$(3.6) \qquad |u(x)| \leq C.\ ||u||^{-\frac{1+\delta(n-1)}{2\sigma}}_{W^m_{\sigma,\delta}} \, ||u||^{1+\frac{1+\delta(n-1)}{2\sigma}}_{L^2} \ .$$

Proof :

(i) With the proposition III-1 and by a partition of unity the inequality (3.5) can be only obtained for functions $u \in W^m_{\sigma,\delta}(\Omega)$ with support in a neighbourhood of the boundary Γ of Ω.

Let x_o be a point of Γ; from the properties (3.1), we see that there exists a neighbourhood $V(x_o)$ of x_o in R^n and a diffeomorphism $\Theta = (\theta_1 \ldots,\theta_n)$ with $\theta_n = \varphi$ from $V(x_o)$ on to the unit ball of $\mathbb{R}^n$ such that :

$$(3.7) \begin{cases} \Theta\ (V \cap \Omega) = B_+ = \{y \in R^n \ ; \ |y| \leq 1,\ y_n > 0\} \ ; \\ \Theta\ (V \cap \Gamma) = B_o = \{y \in \mathbb{R}^n \ ; \ |y| \leq 1,\ y_n = 0\} \ ; \\ X_o(\theta_k) = 0 \text{ in } V \text{ for } k = 1,\ldots,n-1. \end{cases}$$

In these conditions, if $u \in W^m_{\sigma,\delta}(\Omega)$ with supp $u \subset V$ and if $v = u \circ \Theta^{-1}$, then $v \in W^m_{\sigma,\delta}(R^n_+)$ with supp $v \subset \overline{B_+}$. In fact, it suffices for that to remark that by the diffeomorphism Θ, the vector fields $(X_i)_{o\leq i\leq q}$ are become the vector fields $(I_i)_{o\leq i\leq q}$ with :

(3.8) $\quad I_o = \alpha \dfrac{\partial}{\partial y_n}$, $\alpha(y) \neq 0$ for $y \in B = \{y \in \mathbb{R}^n \; ; \; |y| \leq 1\}$;

(3.9) $\quad I_i = I_i^t + \left[(X_i\varphi) \cdot \Theta^{-1}\right] \dfrac{\partial}{\partial y_n}$ for $i = 1,\dots,q$,

where I_i^t means an homogeneous differential operator of order 1, with C^∞ coefficients in the variables $y_1,\dots,y_{n-1}$;

(3.10) for every $y \in B = \{y \in \mathbb{R}^n \; ; \; |y| \leq 1\}$, the rank of the system $(I_i)_{o\leq i\leq q}$ is equal to n.

Hence, the inequality (3.5) comes from the inequality (2.1) and the proposition II-1.

(ii) In the same way, the inequality (3.6), at the boundary comes from the inequality (2.2) of the proposition II-1.

In the interior, it comes from the fact that if $u \in W^m_{\sigma,\delta}(\Omega)$, then $u \in H^m_{loc}(\Omega)$ and then too belongs to $H^{m'}_{loc}(\Omega)$ where $m' = - \dfrac{\sigma n}{1+\delta(n-1)}$; in fact, since $\sigma+m\geq 0$ and $\sigma+\delta m \geq 0$, we have $m' \leq m$. Then, the inequality (3.6), in the interior, is a consequence of the classical inequality :

$$|u(x)| \leq C. \; ||u||^{n/2m'}_{H^{m'}} \; ||u||^{1-n/2m'}_{L^2} .$$

Proposition III-3 :

Let ℓ be an integer, $0 \leq \ell < -\sigma - \frac{1}{2}$; then, the map $u \longrightarrow \gamma_\ell u = \dfrac{\partial^\ell u}{\partial n^\ell}\Big|_\Gamma : \mathcal{D}(\bar\Omega) \longrightarrow \mathcal{D}(\Gamma)$ can be extended in a linear and continuous map from $W^m_{\sigma,\delta}(\Omega)$ into $H^{-\frac{2(\sigma+\ell)+1}{2\delta}}(\Gamma)$.

($\frac{\partial}{\partial n}$ means the derivative along that unit normal vector to Γ, interior in Ω).

This proposition comes from the proposition II-2.

Proposition III-4 :

Let m be an integer $\geqslant 1$ and $\delta_1 = \mathrm{Min}(1,\delta)$. There exists a constant $C > 0$ such that, for every $\varepsilon > 0$, for every $u \in W^m_{\sigma,\delta}(\Omega)$, we have :

$$||u||_{W^{m-1}_{\sigma+\delta_1,\delta}} \leqslant C.\ \{\ \varepsilon\ ||u||_{W^m_{\sigma,\delta}} + \varepsilon^{-(m-1)}||u||_{L^2}\}. \tag{3.11}$$

Proof :

As before, we see that the inequality (3.11) at the boundary comes from the inequality (2.3) and, in the interior, from the classical inequality for the usual Sobolev spaces :

$$||u||_{H^{m-1}} \leqslant C.\ \{\ \varepsilon\ ||u||_{H^m} + \varepsilon^{-(m-1)}||u||_{L^2}\}.$$

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