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An Elasto-Plastic Contact Problem

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Introduction

Duvaut [1] has studied the problem of finding the stresses in an elastic-perfectly plastic body $\mathcal{E}$ in frictionsless contact with a rigid body $\mathcal{B}$ which is pressed against $\mathcal{E}$. In this note we extend the study of Duvaut and look also for the displacements of $\mathcal{E}$ and $\mathcal{B}$. Since the displacements in an elastic-perfectly plastic body may be discontinuous, we shall assume a suitable type of hardening of the elasto-plastic material in which case the displacements are more regular (cf. [4]). We shall consider a stationary case corresponding to Henky's law. The evolution case, corresponding to the Prandtl-Reuss' law, seems to be more difficult and is not treated here.

In Section 1 we prove existence and uniqueness of a solution of the elasto-plastic contact problem. Then, in Section 2 we study a finite element method for finding an approximate solution of the contact problem and we prove a convergence result. Finally, we give an algorithm for solving the discrete problem.

By C we shall denote a positive constant not necessarily the same at each occurence.

1. THE CONTACT PROBLEM

We shall assume that initially the elasto-plastic body $\mathcal{E}$ occupies the bounded region $\Omega \subset \mathbb{R}^3$ with boundary Γ and that Γ contains an open set $\mathring{\Gamma}_1$ in the plane $\{x = (x_1,x_2,x_3) \in \mathbb{R}^3 : x_3 = 0\}$. Moreover, we shall assume that initially the rigid body $\mathcal{B}$ occupies the region $B = \{x \in \mathbb{R}^3 : -x_3 \geq \varphi(x_1,x_2),\ (x_1,x_2) \in \Gamma_0\}$, where Γ_0 is a compact set contained in $\mathring{\Gamma}_1$ with smooth boundary and $\varphi: \Gamma_0 \to \mathbb{R}$ is smooth, non-negative and $\varphi(\bar{x}) = 0$ for some $\bar{x} \in \Gamma_0$ (see Fig. 1.).

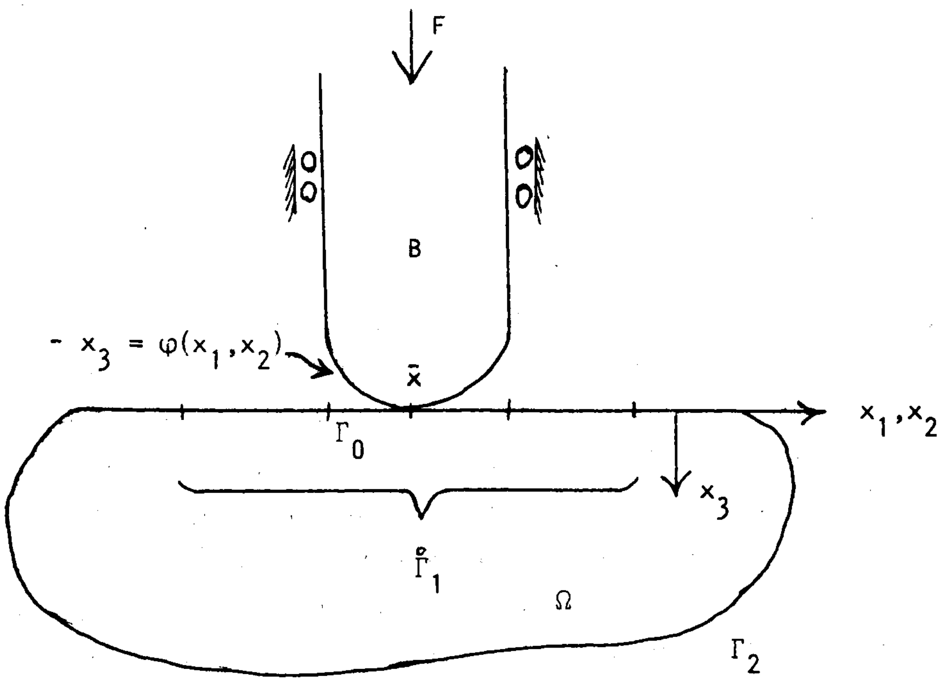

Fig. 1.

Let the boundary of $\mathcal{E}$ be fixed on the portion $\Gamma_2 = \Gamma \setminus \mathring{\Gamma}_1$ and free on $\mathring{\Gamma}_1$, where the surface measure of Γ_2 is positive. Let $\mathcal{B}$ be acted upon by the vertical force F and assume that $\mathcal{B}$ is free to move vertically, whereas rotation and horisontal displacement are prevented. We want to find the vertical displacement U of $\mathcal{B}$ and the stress $\sigma = \{\sigma_{ij}\}$, i,j, = 1,2,3, and displacement $u = \{u_i\}$ of $\mathcal{E}$, where u_i, i = 1,2,3, is the displacement in the x_i-direction. The reference configuration is the one in fig. 1. We shall assume that the displacements are small, i.e., the nonlinearity due to change in geometry is neglible.

support of σ_{33} is contained in $\Gamma_0 \cup \Gamma_2$. In this case it follows from (1.4 b) that also

(1.7) $$- \int_{\overset{0}{\Gamma}_1} \sigma_{33} ds = F,$$

(1.8) $$\int_{\overset{0}{\Gamma}_1} \sigma_{33}(u_3 + \varphi - U) ds = 0.$$

Since

(1.9) $$u_3 + \varphi - U \geq 0 \qquad \text{on } \overset{0}{\Gamma}_1 ,$$

the relation (1.8) implies that formally

(1.10) $$\sigma_{33} = 0 \qquad \text{if } u_3 + \varphi - U > 0.$$

We note that (1.7), (1.9) and (1.10) is the intuitive way of formulating the contact conditions. Conversely, if (1.5) - (1.9) hold then (1.4 b) follows. Finally, (1.4 a) is one way of formulating the constitutive law for the elasto-plastic material relation $\hat{\sigma}$ and $\varepsilon(u)$.

Remark 2. Let us observe that $(\hat{\sigma},(u,U))$ can equivalently be characterized as a saddle point for the Lagrangian $L : P \times K \to \mathbb{R}$ defined by

(1.11) $$L(\hat{\tau},(v,V)) = \frac{1}{2} \| \hat{\tau} \|^2 - (\varepsilon(v),\tau) + FV.$$

We shall make the following assumption:

(1.12) There is a constant C such that for all $\hat{\sigma} \in P$ and $\chi \in H$ there is $\xi \in [L^2(\Omega)]^m$ with $\| \zeta \| \leq C \| \chi \|$ such that $\hat{\sigma} + (\chi,\zeta) \in P$.

Lemma 1 If (1.12) holds and $(\hat{\bar{\sigma}},\bar{u}) = ((\bar{\sigma},\bar{\xi}),\bar{u}) \in P \times W$ satisfies

(1.13) $$[\hat{\bar{\sigma}},\hat{\tau} - \hat{\sigma}] - (\varepsilon(\bar{u}), \tau - \bar{\sigma}) \geq 0, \ \forall \hat{\tau} \in P,$$

then $\bar{u}$ is uniquely determined by $\hat{\bar{\sigma}}$ and

$$\| \varepsilon(\bar{u}) \| \leq C \| \dot{\sigma} \| .$$

Proof For arbitrary $\chi \in H$ we shoose $\zeta \in [L^2(\Omega)]^m$ according to the above assumption and we take $\hat{\tau} = \hat{\sigma} + (\chi,\zeta)$ in (1.13). If then follows that

$$(\varepsilon(\bar{u}),\chi) \leq (\hat{\bar{\sigma}},(\chi,\zeta)) \leq C\|\hat{\bar{\sigma}}\| \; \|\chi\|,$$

which proves the lemma.

Exampel With m = 1 and

$$D = \{(\sigma,\xi)\in \mathbb{R}^{10} : \qquad |s| \leq 1 + \xi\},$$

where $s = \{s_{ij}\}, s_{ij} = \sigma_{ij} - (\sigma_{kk}/3)\delta_{ij}$, one can easily verify that (1.12) holds. This choice corresponds to von Mise's yield condition and isotropic hardening (c.f. [4]).

The result of this section is the following:

Theorem 1 If (1.12) holds, then there exists a unique element $(\hat{\sigma},(u,U)) \in P\times K$ satisfying (1.4).

Proof We introduce the regularized Lagrangian L_μ: $P\times(W\times\mathbb{R})$ defined by

$$L_\mu(\hat{\tau},(v,V)) = \frac{1}{2}\|\hat{\tau}\|^2 - (\varepsilon(v),\tau) - \frac{1}{2\mu}\int_{\Gamma_0}[(v_3 + \varphi - V)^-]^2 ds$$

$$- \frac{\mu}{2}\|\varepsilon(v)\|^2 - \frac{\mu}{2}V^2 + FV,$$

where

$$w^- = \begin{cases} w & \text{if } w \leq 0, \\ 0 & \text{if } w > 0, \end{cases}$$

and μ is a positive constant. By [3, Theorem] we obtain existence of a saddle point $(\hat{\sigma}_\mu, (u_\mu,U_\mu))$. The extremality relations are

$$(1.14) \qquad [\hat{\sigma}_\mu,\hat{\tau} - \hat{\sigma}_\mu] - (\varepsilon(u_\mu),\tau - \sigma_\mu) \geq 0, \quad \forall \hat{\tau}\in P,$$

$$(1.15) \qquad (\sigma_\mu,\varepsilon(w)) = -\int_{\Gamma_0}\frac{1}{\mu}(u_{\mu 3} + \varphi - U_\mu)^- w_3 ds - \mu(\varepsilon(u_\mu),\varepsilon(w)), \quad \forall w\in W,$$

$$(1.16) \qquad -\int_{\Gamma_0}\frac{1}{\mu}(u_{\mu s} + \varphi - U_\mu)^- = F - \mu U_\mu .$$

We shall now derive an priori estimate for $(\hat{\sigma}_\mu, (u_\mu, U_\mu))$ and then pass to the limit as μ tends to zero. Taking $\hat{\hat{\tau}} = 0$ in (1.14) and using (1.15) with $w = u_\mu$, we get

$$\| \hat{\sigma}_\mu \|^2 \leq (\varepsilon(u_\mu), \sigma_\mu)$$

$$\leq - \int_{\Gamma_0} \frac{1}{\mu} (u_{\mu 3} + \varphi - U_\mu)^- u_\mu ds$$

$$= \int_{\Gamma_0} - \frac{1}{\mu} (u_{\mu 3} + \varphi - U_\mu)^- (u_{\mu 3} + \varphi - U_\mu)^- ds$$

$$+ \int_{\Gamma_0} - \frac{1}{\mu} (u_{\mu 3} + \varphi - U_\mu)^- U_\mu + \int_{\Gamma_0} \frac{1}{\mu} (u_{\mu 3} + \varphi - U_\mu)^- \varphi \, ds$$

$$\leq \int_{\Gamma_0} - \frac{1}{\mu} (u_{\mu 3} + \varphi - U_\mu)^- U_\mu,$$

where the last inequality follows from the fact that φ is nonnegative. Thus using also (1.16), we have

$$\| \hat{\sigma}_\mu \|^2 \leq F\, U_\mu - \mu U_\mu^2 \leq F\, U_\mu. \tag{1.17}$$

To proceed we now need the following estimate:

<u>Lemma 2</u> There is a constant C independent of μ, $0 < \mu \leq 1$, such that

$$0 \leq U_\mu \leq C(1 + \| \varepsilon(u_\mu) \|).$$

<u>Proof</u> The fact that U_μ is nonnegative follows immedeately from (1.17), since $F > 0$. To prove the other inequality, we define

$$\Gamma_0^- = \{x \in \Gamma_0 : u_{\mu 3} + \varphi - U_\mu \leq 0\},$$

$$\Gamma_0^+ = \{x \in \Gamma_0 : u_{\mu 3} + \varphi - U_\mu > 0\},$$

where we suppress the dependence on μ. By (1.16) we have

$$\int_{\Gamma_0^-} (u_{\mu 3} + \varphi - U_\mu) ds = \int_{\Gamma_0} (u_{\mu 3} + \varphi - U_\mu)^- ds$$

$$= \mu^2 U_\mu - \mu F \geq -\mu F.$$

Let now $\delta = m(\Gamma_0)/2$, where m denotes twodimensional Lebesque measure. Then $\delta > 0$ and either $m(\Gamma_0^-) \geq \delta$ or $m(\Gamma_0^+) \geq \delta$. If $m(\Gamma_0^-) \geq \delta$, then we have by the above inequality

$$\delta U_\mu \leq \int_{\Gamma_0^-} U_\mu \, ds \leq \mu F + \int_{\Gamma_0} |u_{\mu 3}| ds + \int_{\Gamma_0} \varphi \, ds.$$

Thus, using the trace inequality (see [2]),

$$(1.18) \qquad \int_{\Gamma_0} |v| ds \leq C \| \varepsilon(v) \| , \quad v \in \mathcal{W},$$

we have

$$U_\mu \leq C(F + \int_{\Gamma_0} \varphi + \| \varepsilon(u_\mu) \|),$$

if $m(\Gamma_0^-) \geq \delta$. On the other hand, if $m(\Gamma_0^+) \geq \delta$, then we rely on the following result: There exists a constant C such that if $(v,V) \in \mathcal{W} \times \mathbb{R}$, $E \subset \Gamma_0$ and $V \leq v_3 + \varphi$ on E, $m(E) \geq \delta$, then

$$V \leq C(1 + \| \varepsilon(v) \|).$$

To prove this we argue by contradiction. Thus, we assume that there are sequences $\{C_i\}$, $\{(v_i,V_i\}$ and $\{E_i\}$ such that $(v_i,V_i) \in \mathcal{W} \times \mathbb{R}$, $E_i \subset \Gamma_0$ and

$$(1.19) \qquad V_i \leq v_i + \varphi \text{ on } E_i,$$

$$(1.20) \qquad m(E_i) \geq \delta$$

$$(1.21) \qquad C_i \to \infty,$$

$$(1.22) \qquad V_i > C_i(1 + \| \varepsilon(v_i) \|).$$

Setting $\tilde{v}_i = v_i/V_i$, we see using (1.22) that

$$1 > C_i \| \varepsilon(\tilde{v}_i) \| ,$$

so that by (1.21),

$$\| \varepsilon(\tilde{v}_i) \| \to 0.$$

It follows by the trace inequality (1.18) that

$$\int_{E_i} |\check{v}_i| ds \to 0.$$

But (1.21) and (1.22) imply that $V_i \to \infty$ and thus we obtain from (1.19) and (1.20),

$$\liminf \int_{E_i} \check{v}_i ds \geq \liminf \int_{E_i} \left(1 - \frac{\varphi}{V_i}\right) ds \geq \delta,$$

which leads to a contradiction. This completes the proof of the lemma.

<u>End of proof of Theorem 1:</u>

By Lemma 1, the hardening assumption (1.12) and (1.17), we now obtain the following a priori estimates:

$$\| \hat{\sigma}_\mu \| \leq C,$$

$$\| \varepsilon(u_\mu) \| \leq C,$$

$$0 \leq U_\mu \leq C.$$

We also have by (1.16),

$$\int_{\Gamma_0} \chi_\mu \, ds \leq F,$$

where

$$\chi_\mu = -\frac{1}{\mu} (u_{\mu 3} + \varphi - U_\mu)^- \geq 0.$$

It follows from these estimates that there exists $(\hat{\sigma},(u,U)) \in P \times K$ and $\chi \in \mathcal{M}$, where $\mathcal{M}$ is the set of positive measures on Γ_0, such that for some sequence $\{\mu\}$ tending to zero, (1.23) $\hat{\sigma}_\mu \to \hat{\sigma}$ weakly in $\hat{H}$,

(1.24) $u_\mu \to u$ weakly in $\mathcal{W}$,

(1.25) $U_\mu \to U$,

(1.26) $\chi_\mu \to \chi$ weak star in $\mathcal{M}$.

We shall now prove that $(\hat{\sigma},(u,U))$ satisfies (1.4) by passing to the limit in (1.14) - (1.16). Note that since we have only weak convergence, we cannot guarantee that $(\varepsilon(u_\mu),\ \sigma_\mu)$ tends to $(\varepsilon(u),\sigma)$. First, for any $\hat{\tau} \in P$, we have by (1.14), (1.23) and (1.24),

$$0 \leq \limsup\ \{[\hat{\sigma}_\mu,\hat{\tau}-\hat{\sigma}_\mu] - (\varepsilon(u_\mu),\ \tau - \sigma_\mu)\}$$

$$\leq [\hat{\sigma},\hat{\tau} - \hat{\sigma}] - (\varepsilon(u),\tau) + \limsup\ (\varepsilon(u_\mu),\sigma_\mu).$$

By by (1.15), (1.25) and (1.26),

$$\limsup\ (\varepsilon(u_\mu),\sigma_\mu)$$

$$= \limsup\ \{ \int_{\Gamma_0} \chi_\mu u_{\mu 3} - \mu\|\ \varepsilon(u_\mu)\|^2 \}$$

$$\leq \limsup\ \{\int_{\Gamma_0} \chi_\mu(u_{\mu 3} + \varphi - U_\mu)ds + \int_{\Gamma_0} \chi_\mu(U_\mu - \varphi)ds\}$$

$$\leq \limsup\ \int_{\Gamma_0} \chi_\mu(U_\mu - \varphi) = \langle\chi,U - \varphi\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between $C(\Gamma_0)$, the set of continuous functions on Γ_0, and $\mathcal{M}$. Thus,

(1.27) $$[\hat{\sigma},\hat{\tau} - \hat{\sigma}] - (\varepsilon(u),\tau) + \langle\chi,U - \varphi\rangle \geq 0, \qquad \forall\hat{\tau}\in P.$$

Next, passing to the limit in (1.15), we get

(1.28) $$(\sigma,\varepsilon(v) = \langle\chi,v_3\rangle, \qquad \forall v\in\mathcal{W} \text{ with } v_3 \in C\,\Gamma_0).$$

Since $\sigma \in H$, this relation implies that we may identify the measure χ with an element in $H^{-1/2}(\Gamma)$ also denoted by χ. Thus, it follows from (1.28) that

$$(1.29) \qquad (\sigma,\varepsilon(v)) = \int_\Gamma \chi\, v_3 ds \qquad \forall v \in \mathcal{W},$$

where the integral on the right hand side is to be interpreted as the duality pairing between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$. Moreover, $\chi \geq 0$ and $\mathrm{supp}(\chi) \subset \Gamma_0$. (Note that in order to verify the above properties of χ, we rely heavily on the fact that Γ_0 is compactly contained in $\overset{0}{\Gamma}_1$).

Extending $U - \varphi$ to a function in $H^{1/2}(\Gamma)$ again denoted by $U - \varphi$, we also have

$$(1.30) \qquad \langle \chi, U - \varphi \rangle = \int_\Gamma \chi(U - \varphi) ds.$$

Now, taking $v=u$ in (1.29), adding to (1.27) and using also (1.30), we find

$$(1.31) \qquad [\hat{\sigma},\hat{\tau} - \hat{\sigma}] - (\varepsilon(u),\tau - \sigma - \int_\Gamma \chi(u_3 + \varphi - U) ds \geq 0, \qquad \forall \hat{\tau} \in P.$$

Taking here $\hat{\tau} = 0$, we get

$$\int_\Gamma \chi(u_3 + \varphi - U) ds \leq 0.$$

But $u_3 + \varphi - U$ is nonnegative on $\mathrm{supp}(\chi)$ and $\chi \geq 0$, so that

$$\int_\Gamma \chi(u_3 + \varphi - U) ds \geq 0,$$

and therefore

$$(1.32) \qquad \int_\Gamma \chi(u_3 + \varphi - U) ds = 0.$$

Thus, it follows from (1.31) that $(\hat{\sigma},u)$ satisfies (1.4 a).

It remains to prove that (1.4b) holds. Passing to the limit in (1.16), we see that

$$\int_\Gamma \varkappa \, ds = F.$$

Using also (1.29) and (1.32) it follows that if $(v,V) \in K$, then

$$(\sigma,\varepsilon(v-u)) - F(V - U)$$

$$= \int_\Gamma \varkappa \, (v_3 - u_3) ds - (V - U) \int_\Gamma \varkappa \, ds$$

$$= \int_\Gamma \varkappa \, (v_3 + \varphi - V) ds - \int_\Gamma \varkappa (u_3 + \varphi - U) ds$$

$$= \int_\Gamma \varkappa \, (v_3 + \varphi - U) ds \geq 0,$$

since $v_3 + \varphi - U \geq 0$ on $\mathring{\Gamma}_1$.

Finally, to prove uniqueness assume that $(\hat{\sigma}',(u',U')) \in P \times K$ also satisfies (1.4). Then it follows easily that

$$\| \hat{\sigma} - \hat{\sigma}' \|^2 \leq 0,$$

so that $\hat{\sigma} = \hat{\sigma}'$. Thus, by Lemma 1, we have $u = u'$ and therefore also $U = U'$. This completes the proof of the theorem.

2. A FINITE ELEMENT METHOD FOR THE CONTACT PROBLEM

We shall use the same finite element approximation as in [4] and [6], i.e. stresses and hardening parameters will be approximated by piecewise constant functions and the displacements by piecewise linear continuous functions.

Let us now assume for simplicity that Ω is a polyhedral domain so that Ω can be "triangulated", i.e., Ω can be written as the union of a family $\mathcal{T}_h$ of disjoint tetrahedrons T:

$$\Omega = \bigcup_{T\in\mathcal{T}_h} T \quad .$$

Let h denote the maximum of the diameters of the triangles T.
We define

$$\hat{H}_h = \{\hat{\tau}\in\hat{H} : \hat{\tau} \text{ is constant on each } T\in\mathcal{T}_h\} ,$$

$$W_h = \{w\in W : w \text{ is linear on each } T\in\mathcal{T}_h\} ,$$

$$\mathcal{W}_h = \{v\in W : v_i\in W_h, \ i = 1,2,3\} ,$$

$$P_h = \hat{H}_h \cap P,$$

$$K_h = \mathcal{W}_h \cap K.$$

We can now formulate the discrete problem: Find $(\hat{\sigma}_h,(u_h,U_h))\in P_h\cap K_h$ such that

$$\begin{cases} [\hat{\sigma}_h,\hat{\tau} - \hat{\sigma}_h] - (\varepsilon(u_h),\tau - \sigma_h) \geq 0, & \forall\hat{\tau}\in P_h, \quad (2.1\text{ a}) \\ (\sigma_h,\varepsilon(v - u_h)) \geq F(V-U_h), & \forall(v,V)\in K_h \quad (2.1\text{ b}) \end{cases}$$

Existence and uniqueness of a solution of this problem can be demonstrated along the lines of the existence proof for the continuous problem above. In particular we then obtain the following a priori estimate:

$$(2.2) \qquad \| \sigma_h \| \leq C.$$

Moreover, the passage to the limit is easier in this case, since P_h and K_h are finite dimensional.

We have the following error estimate:

<u>Theorem 2</u> There is a constant C independent of h such that

(2.3) $$\| u - u_h \|_{\mathcal{W}} \leq C(\| \hat{\sigma} - \hat{\sigma}_h \| + \| \varepsilon(u) - \varepsilon(v) \|), \quad \forall v \in \mathcal{W},$$

(2.4) $$\| \hat{\sigma} - \hat{\sigma}_h \| \leq C(\| \varepsilon(u - v) \| + |U - V|)^{1/2}, \quad \forall (v,V) \in K_h.$$

<u>Proof</u> Let $\hat{\chi}_h$ be the L^2 - projection of $\hat{\sigma}$ onto of $\hat{H}_h$, i.e.,

(2.5) $$\hat{\chi}_h|_T = \frac{1}{\text{area}(T)} \int_T \hat{\sigma}(x)dx, \qquad T \in \mathcal{T}_h.$$

Since $\hat{\sigma}(x) \in D$ a.e. in Ω and D is convex, it follows that $\hat{\chi}_h|_T \in D$, $T \in \mathcal{T}_h$, so that $\hat{\chi}_h \in P_h$. Taking $\hat{\tau} = \hat{\chi}$ in (2.1 a) and $\hat{\tau} = \hat{\sigma}_h$ in (1.4 a), we find that

$$[\hat{\sigma} - \hat{\sigma}_h, \hat{\sigma} - \hat{\sigma}_h]$$

$$= [\hat{\sigma}, \hat{\sigma} - \hat{\sigma}_h] - [\hat{\sigma}_h, \chi_h - \hat{\sigma}_h] - [\hat{\sigma}_h, \hat{\sigma} - \hat{\chi}_h]$$

$$\leq (\varepsilon(u), \sigma - \sigma_h) - (\varepsilon(u_h), \chi_h - \sigma_h) - [\hat{\sigma}_h, \hat{\sigma} - \hat{\chi}_h]$$

$$= (\varepsilon(u - u_h), \sigma - \sigma_h),$$

since by (2.5), $[\hat{\sigma}_h, \hat{\sigma} - \hat{\chi}_h] = 0$ and $(\varepsilon(u_h), \sigma - \chi_h) = 0$ (recall that $\varepsilon(u)|_T$ is constant if $u_h \in \mathcal{W}_h$). But, taking $(v,V) = (u_h, U_h)$ in (1.4 b) and adding (2.1 b), we see that

$$[\hat{\sigma} - \hat{\sigma}_h, \hat{\sigma} - \hat{\sigma}_h] \leq (\sigma - \sigma_h, \varepsilon(u - u_h)) \leq F(U - V) + (\sigma_h, \varepsilon(v - u)),$$

which together with (2.2) proves (2.4). Finally, (2.3) follows easily from our assumption (1.12).

<u>Corollary</u> $\|\hat{\sigma} - \hat{\sigma}_h\| + \|u - u_h\|_{\mathcal{W}} \to 0$ as $h \to 0$.

We shall now give an algorithm (Uzawa's method) for solving the discrete problem (2.1). We note that the solution $(\hat{\sigma}_h,(u_h,U_h))$ can equivalenty be characterized as a saddle point for the Lagrangian $L:P_h \times K_h \to R$, where L is given by (1.11). It is then natural to consider the following <u>iterative method</u> (cf. [3], [5]): For n = 0,1,..., do the following:

(i) given $u_h^n \in \mathcal{W}_h$, find $\hat{\sigma}_h^{n+1} \in P_h$ such that

$$[\hat{\sigma}_h^{n+1}, \hat{\tau} - \hat{\sigma}_h^{n+1}] - (\varepsilon u_h^n), \tau - \sigma_h^{n+1}) \geq 0, \qquad \forall \tau \in P_h,$$

(ii) find $(u_h^{n+1}, U_h^{n+1}) \in K_h$ such that

$$(\varepsilon(u_h^{n+1}), \varepsilon(v - u_h^{n+1})) + (U_h^{n+1}, V - U_h^{n+1})$$

$$\geq \rho(\sigma_h^{n+1}, \varepsilon(v - u_h^{n+1})) + \rho F(V - U_h^{n+1}),$$

where $(u_h^0, U_h^0) \in K_h$ is given and ρ is a positive constant. One can easily prove (see [3], [5]) that

$$\hat{\sigma}_h^n \to \hat{\sigma}_h, \quad u_h^n \to u_h \quad \text{as} \quad n \to \infty.$$

if ρ is sufficiently small.

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