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# Infinite Interval Exchange Transformation with Positive Entropy

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# Infinite interval exchange transformation with positive entropy

by

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## ABSTRACT

We construct an interval exchange transformation with infinitely many intervals on [0, 1] which has a positive metrical entropy with respect to the Lebesgue measure.

A transformation $\varphi$ on $[0, 1]$ is called an *interval exchange transformation* if there exist families $I_i$ and $J_i$ of countably many disjoint intervals in $[0, 1]$ such that

(1) $\nu(\bigcup_i I_i)=\nu(\bigcup_i J_i)=1$,

(2) $\nu(I_i)=\nu(J_i)$ for any i, and

(3) $\varphi(\inf I_i + x)=\inf J_i + x$ for any i and x with $0<x<\nu(I_i)$,

where $\nu$ is the Lebesgue measure on $[0, 1]$. It is well known that if $\sum_i -\nu(I_i)\log \nu(I_i)<\infty$, then the metrical entropy $h_\nu(\varphi)=0$. It was asked by Prof. M. Keane (Rennes) whether there exsists an interval exchange transformation with positive entropy or not. Here is an answer to this question to construct such a transformation.

A real number x whose expansion to base 2 is $0.x_0x_1x_2\ldots$ shall be identified with an element $((x_0,x_2,\ldots), (x_1,x_3,\ldots))$ of $\{0,1\}^N \times \{0,1\}^N$. Let $\lambda=\mu\times\mu$ be the measure on $\{0,1\}^N\times\{0,1\}^N$ induced from the Lebesgue measure by this identification.

We select $F_n \subset \{0,1\}^{2^n}$ and $F=\bigcup_{n=1}^{\infty} F_n$ so that

(4) $\{\Gamma_\xi ; \xi\in F\}$ is a family of disjoint subsets of $\{0,1\}^N$ such that $\mu(\bigcup_{\xi\in F} \Gamma_\xi)=1$, where for $\xi=(\xi_0,\xi_1,\ldots,\xi_n)$, $\Gamma_\xi=\{\omega\in\{0,1\}^N;\ \omega(i)=\xi_i$ for $i=0, 1,\ldots,n\}$,

(5) For $\xi=\eta\zeta\in F_n$, where $\eta\zeta$ is the concatenation of $\eta$ and $\zeta$ in $\{0,1\}^{2^{n-1}}$, define $\bar\xi=\zeta\eta$. Then $\{\Gamma_{\bar\xi};\ \xi\in F\}$ is also a family of disjoint sets such that $\mu(\bigcup_{\xi\in F}\Gamma_{\bar\xi})=1$, and

(6) If $\xi\eta\in F_n$, where $\xi, \eta\in\{0,1\}^{2^{n-1}}$, then $\zeta\xi\notin F_n$ for any $\zeta\in\{0,1\}^{2^{n-1}}$.

Such $F$ does exist as we will see later. For $\alpha\in\{0,1\}^N$, let $\tau(\alpha)=n$ if $\alpha\in\Gamma_\xi$ with $\xi\in F_n$. Let $S_k$ be a transformation on $\{0,1\}^N$ such that for any $\alpha\in\{0,1\}^N$,

$$(S_k\alpha)(n)=\begin{cases}\alpha(n+2^{k-1}) & 0\leq n<2^{k-1}\\ \alpha(n-2^{k-1}) & 2^{k-1}\leq n<2^k\\ \alpha(n) & n\geq 2^k\end{cases}$$

Finally, let $T$ be the transformation on $\{0,1\}^N\times\{0,1\}^N$ such that for any $(\alpha,\beta)\in\{0,1\}^N\times\{0,1\}^N$,

$$T(\alpha,\beta)=(S_{\tau(\alpha)}\alpha,\ S_{\tau(\alpha)}\beta).$$

This $T$ can be considered as a transformation on $[0, 1]$ through the above correspondence between $[0, 1]$ and $\{0,1\}^N\times\{0,1\}^N$. In this sense, it is easy to see that $T$ is an interval exchange transformation.

THEOREM $h_\lambda(T)>0$.

(proof) Let $A=\{A_0, A_1\}$ be the partition on $\{0,1\}^N\times\{0,1\}^N$ such that $A_i=\{(\alpha,\beta);\ \beta(0)=i\}$ $(i=0,1)$. Then, it is sufficient to prove that $h_\lambda(T;\ A)=\log 2$. For simplicity, we denote $S\alpha=S_{\tau(\alpha)}\alpha$. By $D_k$, we denote the set of sequences $(n_0,n_1,\ldots,n_{k-1})$ of positive integers such that $n_i\neq\max\{n_{i+1},\ldots,n_{k-1}\}$ for any $i=0,1,\ldots,k-1$. Then, it is easy to see that $\mu\{\alpha;\ (\tau(\alpha),\tau(S\alpha),\ldots,\tau(S^{k-1}\alpha))\in D_k\}=1$. For $k=1,2,\ldots$, define functions $\sigma_k:\{1,2,\ldots\}^k\to\{1,2,\ldots\}$ inductively by

$$\sigma_1(n_0)=2^{n_0-1}$$

$$\sigma_k(n_0,n_1,\ldots,n_{k-1})=\begin{cases}2^{n_0-1}+\sigma_{k-1}(n_1,\ldots,n_{k-1}) & n_0>\max\{n_1,\ldots,n_{k-1}\}\\ \sigma_{k-1}(n_1,\ldots,n_{k-1}) & \text{else.}\end{cases}$$

It follows that if $(n_0,n_1,\ldots,n_{m-1})\in D_m$, then $\sigma_1(n_0)$, $\sigma_2(n_0,n_1),\ldots,$ $\sigma_m(n_0,n_1,\ldots,n_{m-1})$ are different from each other. Therefore, we have

$$\lambda(A_{i_0}\cap T^{-1}A_{i_1}\cap\cdots\cap T^{-m}A_{i_m})$$

$$=\sum_{(n_0,n_1,\ldots,n_{m-1})\in D_m}\lambda(A_{i_0}\cap T^{-1}A_{i_1}\cap\cdots\cap T^{-m}A_{i_m}\mid \tau(\alpha)=n_0,\tau(S\alpha)=n_1,\ldots,$$
$$\tau(S^{m-1}\alpha)=n_{m-1})\cdot\lambda(\tau(\alpha)=n_0,\tau(S\alpha)=n_1,\ldots,\tau(S^{m-1}\alpha)=n_{m-1})$$

$$=\sum_{(n_0,n_1,\ldots,n_{m-1})\in D_m}\lambda(\beta(0)=i_0,\beta(\sigma_1(n_0))=i_1,\ldots,\beta(\sigma_m(n_0,n_1,\ldots,n_{m-1})$$
$$=i_m\mid\tau(\alpha)=n_0,\tau(S\alpha)=n_1,\ldots,\tau(S^{m-1}\alpha)=n_{m-1})$$
$$\times\lambda(\tau(\alpha)=n_0,\tau(S\alpha)=n_1,\ldots,\tau(S^{m-1}\alpha)=n_{m-1})$$

$$=\sum_{(n_0,n_1,\ldots,n_{m-1})\in D_m}\lambda(\beta(0)=i_0,\beta(\sigma_1(n_0))=i_1,\ldots,\beta(\sigma_m(n_0,n_1,\ldots,n_{m-1}))$$
$$=i_m)\cdot\lambda(\tau(\alpha)=n_0,\tau(S\alpha)=n_1,\ldots,\tau(S^{m-1}\alpha)=n_{m-1})$$

$$=\sum_{(n_0,n_1,\ldots,n_{m-1})\in D_m}2^{-m-1}\cdot\lambda(\tau(\alpha)=n_0,\tau(S\alpha)=n_1,\ldots,\tau(S^{m-1}\alpha)=n_{m-1})$$

$$=2^{-m-1}.$$

Thus $h_\lambda(T;\ A)=\log 2$. (Q.E.D.)

Now, we show how to construct $F$ satisfying (4), (5) and (6). Let $F_1=\{(0,1)\}$. Suppose that $F_1, F_2,\ldots, F_n$ have been selected so that

(7) $\Gamma_\xi\cap\Gamma_\zeta=\emptyset$ for any $\xi\neq\zeta$ in $\bigcup_{i=1}^{n}F_i$,

(8) $\Gamma_{\bar\xi}\cap\Gamma_{\bar\zeta}=\emptyset$ for any $\xi\neq\zeta$ in $\bigcup_{i=1}^{n}F_i$, and

(9) if $\xi\zeta\in F_i$ $(\xi,\zeta\in\{0,1\}^{2^{i-1}}$, $i=1,2,\ldots,n)$, then $\zeta\xi\notin F_i$ for any $\xi\in\{0,1\}^{2^{i-1}}$.

Let

$$\mu\Big(\bigcup_{i=1}^{n}\bigcup_{\xi\in F_i}\Gamma_\xi\Big)=\sum_{i=1}^{n}|F_i|2^{-2^i}=1-\delta_n.$$

Then, we can select subsets $G_n$ and $H_n$ of $\{0,1\}^{2^n}$ satisfying that

(10) $|G_n|=|H_n|$ and $|G_n|2^{-2^n}\geq\frac{\delta_n}{3}$,

(11) $G_n\cap H_n=\phi$,

(12) $\Gamma_\xi\cap\Gamma_\zeta=\phi$ for any $\xi\in G_n$ and $\zeta\in\bigcup_{i=1}^{n}F_n$, and

(13) $\Gamma_\xi\cap\Gamma_{\bar\zeta}=\phi$ for any $\xi\in H_n$ and $\zeta\in\bigcup_{i=1}^{n}F_n$.

Let $F_{n+1}=G_nH_n=\{\xi\zeta;\ \xi\in G_n,\ \zeta\in H_n\}$. Then, it is easy to check that $F_1$, $F_2$, ..., $F_{n+1}$ satisfy the conditions (7), (8) and (9) with n+1 for n. Moreover, since

$$\delta_{n+1}\leqq\delta_n-\Big(\frac{\delta_n}{3}\Big)^2,$$

$\delta_n\to 0$ as $n\to\infty$, so that

$$\mu\Big(\bigcup_{\xi\in F}\Gamma_\xi\Big)=\lim_{n\to\infty}\mu\Big(\bigcup_{i=1}^{n}\bigcup_{\xi\in F_i}\Gamma_\xi\Big)=1.$$

Thus, finally we get F satisfying (4), (5) and (6).