M. MÉTIVIER

J. PELLAUMAIL

A Basic Course on General Stochastic Integration

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IRISA INSTITUT DE RECHERCHE EN INFORMATIQUE ET SYSTÈMES ALÉATOIRES

Avenue du Général Leclerc
35042 RENNES CEDEX
FRANCE
Tél. (99) 36.48.15

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* A BASIC COURSE ON GENERAL STOCHASTIC INTEGRATION

by

M. METIVIER and J. PELLAUMAIL

Summary :

This course on the stochastic integral is "self-contained". The reader is only expected to have a knowledge of classical measure theory. The fundamental parts of the course are the following : construction of the stochastic integral and the Ito formula (general hilbertian non continuous case), existence and unicity of a "strong" solution for very general differential stochastic equations, basic properties of martingales and Doléans measures, construction and properties of the Meyer process associated with a Doléans measure, Burkholder inequalities, stochastic integral considered as a group-valued integral. A new inequality for semi-martingales is also established and used in several parts of the course.

The methods used are quite different from those of the Strasbourg school.

* *Pour toute correspondance ou demande de tirés à part, écrire à*

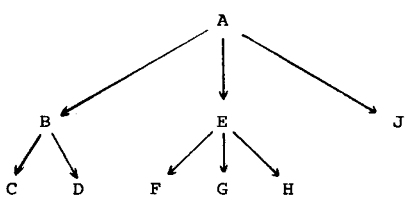
M. METIVIER Ecole Polytechnique, Centre de Mathématiques Appliquées, Route de Saclay 91128 PALAISEAU
ou J. PELLAUMAIL I.N.S.A. B.P. 14 A 35031 RENNES CEDEX

A BASIC COURSE ON GENERAL STOCHASTIC INTEGRATION

INTRODUCTION

This course on the stochastic integral is "self-contained" ; except in paragraph J, the details of all proofs are given. The reader is expected to have a knowledge of classical measure theory and specially the properties of the space L_2, the properties of the conditionnal expectations and some properties of equi-integrable families of random variables. Of course, the understanding is easier for the reader already familiar with the classical study of elementary processes, in particular the brownian motion.

It is often possible to read a paragraph without knowing the previous paragraphs ; more precisely, the planning is as follows :



In other words, knowing the paragraphs A and E, one can read the paragraph G (for example).

An extensive table of contents is given atthe end of this course.

In this course, we are specially concerned with the "hard" parts of theorems ; some easier facts and elementary counterexamples are given in exercises at the end ; some of these exercises are fundamental , specially all the exercises on the brownian motion (C.1, 2, 3, 4) but are not used in the course.

In paragraph A, we give elementary definitions and properties from the theory of stochastic processes as studied in [Del] (stochastic basis, stopping time, predictable set, etc...)

The stochastic integral is defined in paragraph B for a very large class of processes.

The Ito formula is proved in paragraph C for non continuous processes with values in a Hilbert space.

The existence and unicity of a "strong" solution of the stochastic differential equation $dX_t = a(t,X)\ dZ_t$ is proved when a depends on the whole past history of the process X and is lipchitzian ; the proof is based on the fixed point theorem , the process Z being assumed to satisfy an inequality, which, in paragraph G, turns out to be fulfiled for a very large class of processes : actually, all the real semi-martingales.

The definitions and classical properties associated with martingales and Doleans functions are given in paragraph E : in particular, we give conditions for the existence of a Doléans measure, we prove the Doob theorem on the existence of a "cadlag" modification for some processes, the "stopping" theorem for martingales or Doléans measures and the Doob inequality for square integrable martingale ; we also study the stochastic integral with respect to a square integrable martingale.

The Meyer process associated with a Doléans measure is constructed in paragraph F.

A new inequality for semi-martingales is proved in paragraph G.

The Burkholder inequalities are proved in paragraph H and a new inequality is also given.

For convenience of notation, there is no paragraph I.

In the paragraph J, the stochastic integral is defined and studied as a classical integral with respect to a group-valued, or vector-valued, measure.

Some exercises are given and naturally exercises A.1, A.2, etc ... are related to paragraph A B.1, B.2 ... to paragraph B, and so on.

Some bibliographical notes are given immediately prior to the bibliography and the table of contents.

A - STOCHASTIC BASIS

A-1. STOCHASTIC BASIS : DEFINITION

Let T be a part of the real line. We shall call *stochastic basis* a family $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in T})$ such that $(\Omega,\mathcal{F})$ is a measurable space and $(\mathcal{F}_t)_{t\in T})$ is an increasing family of sub-σ-algebras of $\mathcal{F}$. If $(\Omega,\mathcal{F},P)$ is a probability space, we shall call the family $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\in T})$ a *probabilized stochastic basis* or, only, a *stochastic basis*.

In the following, T is always the unit interval $[0,1]$ of the real line or $\bar{\mathbb{N}}$ the set $\mathbb{N}\cup\{\infty\}$ where $\mathbb{N}$ is the set of the integers. We shall note $T' = T\setminus\{0\}$ and $\Omega' = \Omega\times T'$. Moreover we shall note $T\infty$ the supremum of the elements of T. Intuitively, Ω is the space of all the "possible events" and $\mathcal{F}_t$ is the σ-algebra generated by the events realized before the time t. It is often better to forget this point of view.

In all the paragraph A, we consider a probabilized stochastic basis $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\in T})$.

We shall say that this basis is complete if the space $(\Omega,\mathcal{F},P)$ is a complete probability space and if, for each element A of $\mathcal{F}$ such that $P(A) = 0$ and for each element t of T, A is an element of $\mathcal{F}_t$.

For each element t of T, we note $\mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$ and we shall say that the family $(\mathcal{F}_t)_{t\in T}$ is right continuous if $\mathcal{F}_t = \mathcal{F}_{t+}$ for each element t of T.

If H is a Banach space (with its σ-algebra $\mathcal{H}$ of borelian sets), we note $L_0^H(\Omega,\mathcal{F}_t,P)$ the complete metric space for the convergence in probability which contains all the H-valued $\mathcal{F}_t$-measurable random variables.

A-2. STOPPING TIME AND NOTATION $\mathcal{F}_u$

Let u be a measurable mapping from $(\Omega,\mathcal{F})$ into $(T,\mathcal{C})$, where $\mathcal{C}$ is the σ-algebra of borelian sets. One says that u is a *stopping time* if, for each element t of T, the set $\{\omega:u(\omega)\leq t\}$ belongs to the σ-algebra $\mathcal{F}_t$.

If u and v are two stopping times, it is easily seen that $u\vee v$ and $u\wedge v$ are also stopping times.

If u is a stopping time, one notes $\mathcal{F}_u$ the σ-algebra defined by :

$$\mathcal{F}_u = \{A : A\in\mathcal{F} \text{ and, } \forall t\in T, (A\cap[u\leq t])\in\mathcal{F}_t\}$$

If s belongs to T and if u = s (for each ω), we see that $\mathcal{F}_u = \mathcal{F}_s$ (then there is no possible confusion in the notations).

A-3. STOCHASTIC INTERVAL

Let u and v be two stopping times ; one notes $]u,v]$ the part of $(\Omega\times T)$ defined by :

$(\omega,t)\in]u,v]$ if and only if $u(\omega)<t\leq v(\omega)$

One defines $[u,v[,\ldots$ in the same way. Such sets are called *stochastic intervals*.

Then, there is an ambiguous notation, but, in the general case, there is no possible confusion : if u=s and v=t are two "fixed" stopping times, the set $]u,v] =]s,t]$ can be a part of T or a part of $(\Omega\times T)$ as above.

A-4. PROCESS (definitions) AND FRENCH NOTATION CADLAG

The word "process" has several different meanings in probability theory.

If $(H,\mathcal{H})$ is a measurable space, we shall say that X is an H-valued process if X is an H-valued mapping defined on $(\Omega\times T)$. On the contrary, we shall say that X is a "process defined up to modification" if $X = (X_t)_{t\in T}$ is a mapping from T into $L_0^H(\Omega,\mathcal{F},P)$.

If X and X' are two processes, we shall say that X' is a *modification* of X if, for each element t of T, $X_t = X'_t$ a.e.

Let H be a topological space. Let f be an H-valued function defined on T. We shall say that f is a *cadlag* function if, for each element t of T, f is right continuous and with left limit (in french, f est continue à droite et admet une limite à gauche).

Let X be an H-valued process ; we shall say that X is a *cadlag* process if, for each element ω of Ω, the *sample function* $t\rightsquigarrow f(t) = X_t(\omega)$ is cadlag as defined above.

We shall use also the notations *caglad* (left continuous and with right limit), *laglad* (with left and right limit), and so on....

In the same way, we shall say that a process X is *continuous* if, for each element ω of Ω, the sample function $t \rightsquigarrow f(t) = X_t(\omega)$ is continuous.

One says that two processes X and X' are indistinguishable if $P(\{\omega : \exists t,\ X_t(\omega) \neq X'_t(\omega)\})=0$.

Actually, in the following, we consider processes defined up to indistinguishability.

If X is a cadlag process, one notes $(X_{t-})_{t\in T} = (Y_t)_{t\in T}$ the caglad process, unique up to an indistinguishability, such that $Y_t = \lim_{s\uparrow t} X_s$ for each element t of T.

Moreover, if X and Y are two cadlag processes such that X is a modification of Y, then X and Y are indistinguishable (we have $X_t(\omega) = Y_t(\omega)$ for each element t of T except if there is a rationnal number q such that $X_q(\omega) \neq Y_q(\omega)$).

A-5. PREDICTABLE SETS ; NOTATIONS $\mathcal{R}$, $\mathcal{A}$ and $\mathcal{P}$

We shall note $\mathcal{R}$ the family of parts A of $\Omega' = \Omega\times T'$ such that $A = F\times]s,t]$ where F belongs to $\mathcal{F}_s$. We shall note $\mathcal{A}$ the algebra generated by $\mathcal{R}$. We shall note $\mathcal{P}$ the σ-algebra generated by $\mathcal{R}$ (or $\mathcal{A}$) : the elements of this σ-algebra are called the *predictable sets.* One says that an $(H,\mathcal{H})$-valued process is *predictable* if this process is measurable relative to $\mathcal{P}$ and $\mathcal{H}$.

A-6. DECOMPOSITION OF EACH ELEMENT OF $\mathcal{A}$ (lemma)

If A is an element of $\mathcal{A}$, there exists a finite family $(A_i)_{i\in I}$ of elements of $\mathcal{R}$, which is a partition of A.

Proof

Let $\mathcal{A}'$ be the class of all the elements A of $\mathcal{A}$ such that there exists a finite family $(A_i)_{i\in I}$ of elements of $\mathcal{R}$ which is a partition of A. To prove that $\mathcal{A} = \mathcal{A}'$, it is sufficient to prove that $\mathcal{A}'$ is an algebra. For that, it is sufficient to prove that, if A and B are elements of $\mathcal{A}'$, it is the same for $A\setminus B$. Then, we suppose that $B = \bigcup_{i=1}^{n} B_i$ where $(B_i)_{1\leq i\leq n}$ is a finite family of elements of $\mathcal{R}$. We define C_i, by recurrence, by $C_1 = A$ and $C_{i+1} = C_i \setminus B_i$. We have $C_{n+1} = A\setminus B$. Reasonning by recurrence, it is sufficient to prove that, if D is an element of $\mathcal{A}'$, it is the same for $D\setminus B_i$; it is sufficient to prove that if D is an element of $\mathcal{R}$ and that is easy to verify.

A-7. $\mathcal{A}$ AND THE STOCHASTIC INTERVALS $]u,v]$ (lemma)

The algebra $\mathcal{A}$ is identical to the algebra $\mathcal{A}'$ generated by the stochastic intervals $]u,v]$ where u and v are simple stopping times (i.e. the number of elements of $u(\Omega)$ and $v(\Omega)$ is finite).

Proof

1°) First, we prove $\mathcal{A} \subset \mathcal{A}'$. It is sufficient to prove that, if $B = F\times]s,t]$ is an element of $\mathcal{R}$, then B is also an element of $\mathcal{A}'$; but $B =]u,v]$, where u are v are the stopping times defined by $v(\omega) = t$ $(\forall\omega)$ and $u(\omega) = t$ if $\omega \in (\Omega\setminus F)$ and $u(\omega)=s$ if $\omega \in F$.

2°) Now, we prove $\mathcal{A}' \subset \mathcal{A}$. Let u be a simple stopping time. Then, there exists a finite increasing sequence $(t(k))_{1\leq k\leq n}$ of elements of T and an associated sequence $(F(k))_{1\leq k\leq n}$ of elements of $\mathcal{F}$ such that :

a) for each integer k, F(k) belongs to $\mathcal{F}_{t(k)}$

b) $(F(k))_{1\leq k\leq n}$ is a partition of Ω

c) $u = \sum_{k=1}^{n} t(k).1_{F(k)}$

Then we put $B_k = (F(k)\times]t(k),1])$ (for each integer k) and $(B_k)_{1\leq k\leq n}$ is a partition of $]u,1]$. That proves that $]u,1]$ is an element of $\mathcal{A}$ and completes the proof.

A-8. ADAPTED PROCESS

One says that a process X or a process X defined up to a modification is adapted (with respect to the stochastic basis $(\Omega,\mathcal{F},P(\mathcal{F}_t)_{t\in T})$) if, for each element t of T, the random variable $X_t(.)$ is $\mathcal{F}_t$-measurable.

Let u be a T-valued function defined on Ω and measurable relative to $\mathcal{F}$ and $\mathcal{C}$. It is easily seen that the definitions imply that u is a stopping time if and only if the process $X = 1_{[o,u[}$ is adapted.

A-9. AN EXAMPLE OF STOPPING TIME (lemma)

Let H be a Banach space . Let X be an H-valued adapted process, right or left continuous. Let u be a stopping time and a be a real number. For each element ω of Ω, we put :

$$v(\omega) = \inf.\{t : t\in T, t\geq u(\omega), \|X_t(\omega)-X_{u(\omega)}(\omega)\| > a\}$$

and $v(\omega) = T\infty = \underset{t \in T}{Sup.}\ t$ *if the set above is empty. Then v is a stopping time with respect to the family* $(\mathcal{F}_{t+})_{t \in T}$.

Proof

It is sufficient to consider the case where $T = [0,1]$. In this case, let Q' be the set of rational numbers belonging to T and let $(S(n))_{n>o}$ be a sequence of finite parts of Q' increasing to Q'. We put :

$v'(\omega)=\inf.\{ t : t \in Q', t \geq u(\omega), ||X_t(\omega)-X_{u(\omega)}(\omega)|| > a \}$

$v_n(\omega)=\inf.\{ t : t \in S(n), t \geq u(\omega), ||X_t(\omega)-X_{u(\omega)}(\omega)|| > a \}$

(with the convention $v'(\omega) = 1$ or $v_n(\omega) = 1$ if the sets above are empty).

It is easily seen that, for each element ω of Ω, $v'(\omega) = v(\omega)$ and $v'(\omega) = \inf_{n>o} v_n(\omega)$. It is also easily seen that, for each integer n, v_n is a stopping time. Then, we have only to prove that the limit v of a decreasing sequence $(v(n))_{n>o}$ of stopping times is a stopping time for the family $(\mathcal{F}_{t+})_{t \in T}$.

Let t be an element of T ; we put :
$A = \{\omega : v(\omega) > t\}$ and $A(n,k) = \{\omega : v_n(\omega) > t + \frac{1}{k}\}$
We have $A = \bigcup_{k>0} \{\bigcap_{n>0} A(n,k)\}$. Moreover, the set $\bigcap_{n>0} A(n,k)$ belongs to $\mathcal{F}_{t+1/k}$; thus A belongs to $\mathcal{F}_{t+1/k}$ for each integer k ; then A belongs to $\mathcal{F}_{t+}$ and that proves that v is a stopping time with respect to the family $(\mathcal{F}_{t+})_{t \in T}$

A-10. STOPPED PROCESS AND LOCALIZATION (definitions)

Let u be a T-valued random variable defined on $(\Omega, \mathcal{F})$ and X be a process. Let Z be the process defined by :

$Z_t(\omega) = X_t(\omega)$ if $t \leq u(\omega)$

$Z_t(\omega) = X_{u(\omega)}(\omega)$ if $t \geq u(\omega)$

On says that Z is the process stopped at the random variable u. If u is a stopping time and if X is an adapted process, it is easily seen that Z is also an adapted process.

Let X be a process. It is often useful to consider an increasing sequence $(u(n))_{n>o}$ of stopping times such that $\lim_{n\to\infty} P[u(n) < 1] = 0$ and to consider the processes X^n which are the process X stopped at the stopping time u(n). This procedure is called *localization*. In this situation, one says that X is *locally bounded*, locally measurable, etc... if each process X^n (which is the process X stopped at the stopping time u(n)) is bounded, measurable, etc...

A-11. PREDICTABLE SETS ASSOCIATED TO THE FAMILY $(\mathcal{F}_{t+})$ (proposition)

The σ*-algebra* $\mathcal{P}$ *of the predictable sets associated to the family* $(\mathcal{F}_t)_{t \in T}$ *is the same than the* σ*-algebra* $\mathcal{P}^+$ *of the predictable sets associated to the family* $(\mathcal{F}_{t+})_{t \in T}$.

Proof

We have $\mathcal{P}^+ \supset \mathcal{P}$. Moreover, if H is an element of $\mathcal{F}_{s+}$, we have $H\times]s,t] = \bigcup_{k>o} (H\times]s+\frac{1}{k},t])$ where, for each integer k, $H\times]s+\frac{1}{k},t]$ is an element of $\mathcal{P}$. Then $\mathcal{P}^+ \subset \mathcal{P}$

A-12. LEFT CONTINUOUS PROCESS AND PREDICTABLE PROCESS (proposition)

Let H be a Banach space ; let X be an H-valued caglad adapted process ; then X is a predictable process. Specially, if u is a stopping time, the real process $1_{]o,u]}$ *is a predictable process.*

Proof

1°) We can assume that the family $(\mathcal{F}_t)_{t \in T}$ is right continuous (cf. A-11 above). Moreover, it is sufficient to consider the case where $T = [0,1]$.

2°) First, we consider the case where $X = Y.1_{]u,1]}$ u being a stopping time and Y being an $\mathcal{F}_u$-measurable random variable. Then, for each integer n, we put :

$u(n) = \sum_{k>o} k.2^{-n} . 1_{[k\ 2^{-n} \leq u < (k+1).2^{-n}]}$

We have $u(n) \downarrow u$; thus $X = \lim_{n\to\infty} Y.1_{]u(n),1]}$; now, for each integer n, $Y.1_{]u(n),1]}$ is a predictable process (this is easily seen as in A-7 above) ; then X is also a predictable process.

3°) Let u and v be two stopping times and Y be an $\mathcal{F}_u$-measurable random variable. Then the process $X = Y.1_{]u,v]}$ is a predictable process because $X = Y.1_{]u,1]} - Y.1_{]v,1]}$ (cf. 2°) above).

4°) Now we consider the general case.For each integer n, let $(u(n,k))_{k>o}$ be the increasing sequence of stopping times (cf. A-9)) defined by recurrence by $u(n,o) = 0$ and

$u(n,k+1) = \inf.\{t : t \geq u(n,k), ||X_t - X_{u(n,k)}|| > \frac{1}{n}\}$

(and $u(n,k+1) = T_\infty$ if the set above is empty).

For each element ω of Ω, the function $t \rightsquigarrow X_t(\omega)$ is caglad ; thus, it is classical and not too difficult to prove that, for each integer n, there exists an integer $k(n,\omega)$ such that $u(n,k(n,\omega)) = 1$. That means that, for each integer n, the sequence of the sets $(u(n,k) < 1)_{k>o}$ is decreasing to the void set ; then we can put :

$$X^n = \sum_{k>o} X_{u(n,k)+} \cdot 1_{]u(n,k),u(n,k+1)]}$$

Moreover, X^n is a predictable process (see 2°) above) and the sequence $(X^n)_{n>o}$ converges uniformly to the process X ; thus X is a predictable process.

A-13. PRELOCALIZATION

Let u be a stopping time and X be a cadlag process ; let X^u be the process defined by :

$X^u_t(\omega) = X_t(\omega)$ if $t < u(\omega)$

$X^u_t(\omega) = X_{u(\omega)-}(\omega)$ if $t \geqslant u(\omega)$

We shall say that X^u is the process X stopped just before the stopping time u. If X is adapted, it is the same for X^u. As in A-10, it is often convenient to consider a sequence $(u(n))_{n>o}$ of stopping times and the sequence $(X^{u(n)})_{n>o}$ of associated processes. We shall call this procedure prelocalization. If, for each integer n, $X^{u(n)}$ is bounded, continuous, etc.... we shall say that X is prelocally bounded, continuous, etc....

A-14. PREDICTABLE STOPPING TIME (definition)

Let u *be a stopping time. One says that* u *is predictable if there exists a sequence* $(u(n))_{n>o}$ *of stopping times increasing to* u *and such that, for each integer* n, *and each element* ω *of* Ω,

$[u(n)](\omega) < u(\omega)$

In this case, $]o,u[= \bigcup_{n>o}]o,u(n)]$ is a predictable set.

B - STOCHASTIC INTEGRAL

B-1. GENERALITIES

In all this paragraph B, we consider a probabilized stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in T})$ (cf. A-1), three Banach spaces H, J and K and a bilinear mapping from $(H\times J)$ into K which, to (y,x) element of $H\times J$, associates $y.x$ element of K. The norms in H,J and K will be noted $||.||_H$, $||.||_J$ and $||.||_K$ respectively. Moreover, for the convenience of notations, we shall suppose that $T = [0,1]$.

What is the problem of the stochastic integral ?

Let Y be an H-valued process (usually Y is a predictable process) and X be a J-valued process defined up to modification ; then, the problem is :

1°) to define, for each element t of T, the random variable $Z_t = \int_0^t Y_s \cdot dX_s = \int 1_{]o,t]}(s).Y_s \cdot dX_s$

2°) to study the process $(Z_t)_{t\in T}$ *thus defined up to modification.*

Actually, one considers processes X which have a cadlag modification ; we shall note also X this cadlag modification, defined up to indistinguishability. Then, it is natural to define $Z_t(\omega)$ as the usual integral of the H-valued sample function $s \rightsquigarrow Y_s(\omega)$ with respect to the "measure" $dX_s(\omega)$ (ω being fixed).

Actually, this building is not possible in the general case ; indeed, for many processes, specially the real brownian motion, for each element ω of Ω, the sample function $t \rightsquigarrow f(t) = X_t(\omega)$ is not with bounded variation ; then, $dX_t(\omega)$ (ω being fixed) does not define a measure.

The building that we give now is not the more general, but it is very elementary.

B-2. $\mathcal{R}$-SIMPLE PROCESSES ; NOTATION $\mathcal{E}(H)$

We shall note $\mathcal{E}(H)$ the vector space of the H-valued and $\mathcal{R}$-simple processes, i.e. the processes Y such that $Y = \sum_{i\in I} a_i \cdot 1_{A(i)}$ where $(a_i)_{i\in I}$ is a finite family of elements of H and $(A(i))_{i\in I}$ is a finite associated family of elements of $\mathcal{R}$.

We can assume that, in the previous writing, the sets $(A(i))_{i\in I}$ are disjoint and belong to $\mathcal{R}$ (cf. A-6).

In this case, we can build the stochastic integral as suggested above ; for each element ω of Ω, we can define

$Z_t(\omega) = \int 1_{]o,t]}(s).Y_s(\omega).dX_s(\omega)$ if X is a cadlag process ; if X is defined up to modification, it is the same for the process Z.

Then, the stochastic integral $\int_{]o,1]} Y.dX$ is the linear mapping defined on $\mathcal{E}(H)$, with values in $L_o^K(\Omega,\mathcal{F},P)$, such that, for each element $A = F\times]s,t]$ of $\mathcal{R}$ and each element a of H, if $Y = a.1_A$, we have :

$$\int Y.dX = \int a.1_A.dX = 1_F .a .(X_t-X_s)$$

The problem is to extend the mapping $Y \rightsquigarrow \int Y.dX$ to a larger class of processes than the class of the $\mathcal{R}$-simple processes.

For the convenience of notations, we write $\int Y.dX$ instead of $\int_{\Omega'} Y.dX = \int_{]o,1]} Y.dX$.

B-3. A FIRST EXTENSION

Let X be a J-valued process, defined up to a modification, which satisfies the following property :

(i) there exists a positive measure a defined on the σ-algebra of predictable sets and such that, for each H-valued and $\mathcal{R}$-simple process Y, we have :

$$E(||\int Y.dX||_K^2) \leqslant \int ||Y||_H^2 . da$$

In this case, the mapping $Y \rightarrow \int Y.dX$ defined on $\mathcal{E}(H)$ and with values in $L_2^K(\Omega,\mathcal{F},P)$ is uniformly continuous if we consider $\mathcal{E}(H)$ as a subspace of $L_2^H(\Omega',\mathcal{P},a)$; then, there is an unique extension of this mapping in a linear continuous mapping from $L_2^H(\Omega,\mathcal{P},a)$ into $L_2^K(\Omega,\mathcal{F},P)$ (the space $\mathcal{E}(H)$ being dense in $L_2^H(\Omega',\mathcal{P},a)$). The image of a process Y belonging to $L_2^H(\Omega',\mathcal{P},a)$ by this mapping will be noted $\int Y.dX$ and will be called *the stochastic integral of the process Y with respect to the process X.*

B-4. THE STOCHASTIC INTEGRAL PROCESS

Let X be a process which satisfies the condition B-3-(i). Let Y be a process which belongs to $L_2^K(\Omega',\mathcal{P},a)$. For each element t of T, we can define the random variable Z_t by :

$$Z_t = \int 1_{]o,t]}.Y.dX$$

Then, the process Z is defined up to modification, and is called the stochastic integral process of Y with respect to X.

B-5. DOMINATED CONVERGENCE THEOREM

We consider the hypothesis and notations given in B-1, B-2 and B-3. Moreover, we suppose that the family $(\mathcal{F}_t)_{t\in T}$ is right continuous, the basis $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\in T})$ is complete and that X is a cadlag adapted process. Let $(Y_n)_{n>o}$ a sequence of $\mathcal{R}$ simple processes such that, for each integer n,

$$\int ||Y-Y_n||_H^2 . da \leqslant 8^{-n}$$

For each integer n, let Z^n be the cadlag process defined by $Z_t^n = \int 1_{]o,t]}.Y_n.dX$; Z^n can be select cadlag because X is a cadlag process and Y^n is an $\mathcal{R}$-simple process ; thus Z^n is unique up to an indistinguishability. For each integer n, we put :

$$u(n) = \inf.\{ t : ||Z_t^n-Z_t^{n+1}||_K > 2^{-n} \}$$

and $u(n) = 1$ if the set above is void.
Let G(n) be the set defined by

$$G(n) = \{\omega : [u(n)](\omega) < 1\}$$

For each simple stopping time v, we have :

$$E(||Z_v^n-Z_v^{n+1}||_K^2) = E(||\int 1_{]o,v]}.(Y_v^n-Y_v^{n+1}).dX||^2)$$

$$\leqslant \int 1_{]o,v]}.||Y_v^n-Y_v^{n+1}||_H^2 . da \leqslant 2.8^{-n}$$

Then we have the same inequality for a general stopping time (such a stopping time being the decreasing limit of a sequence of simple stopping times : cf. the end of the proof of A-9). Thus, this inequality is satisfied for $v = u(n)$ and we have :

$$2.8^{-n} \geqslant E(||Z_{u(n)}^n-Z_{u(n)}^{n+1}||_K^2) \geqslant 4^{-n}.P[G(n)]$$

Then $P[G(n)] \leqslant 2.2^{-n}$ and $P(G) = 0$ if $G = \bigcap_{k>o} \{ \bigcup_{n\geqslant k} G(n)\}$. Thus, if $\omega \notin G$, there exists an integer k such that, for each integer $n \geqslant k$, $\sup_t ||Z_t^n-Z_t^{n+1}|| \leqslant 2^{-n}$. This means that, for each element ω of $\Omega\setminus G$, the sequence $(Z_t^n(\omega))_{n>o}$ is a

Cauchy sequence which converges uniformly to a function $\hat{Z}_t(\omega)$; the process $\hat{Z}$ is a modification of the process Z. Then, we have proved :

If X has a modification which is a cadlag adapted process, it is the same for the process Z.

Actually, we have proved more than that : let X be a cadlag process which satisfies the property B-3.(i). Let $(Y_n)_{n>o}$ be a sequence of H-valued processes which converges to Y in the following sense : for each integer n, $\int ||Y-Y_n||_H^2.da \leq 8^{-n}$ For each integer n, let Z^n be a cadlag process which is a modification of the stochastic integral process $\int Y_n.dX$; we can prove as above that :

the sequence $(Z_n)_{n>o}$ converges almost uniformly to a cadlag process which is a modification of the stochastic integral process $Z = \int Y.dX$.

We note that, if $(Y_n)_{n>o}$ is a sequence which converges to Y as in the dominated convergence theorem, there exists a sub-sequence $(Y_{n(k)})_{k>o}$ which converges as above ; thus the sub-sequence $(Z_{n(k)})_{k>o}$ of the cadlag stochastic integral processes associated *converges almost uniformly to the cadlag stochastic integral process* $Z = \int Y.dX$.

This theorem is very useful to prove many properties. We give some examples :

If X has a modification which is a continuous (or predictable, etc...) process, it is the same for the stochastic integral process $\int Y.dX$.

If u is a T-valued random variable, the stochastic integral process stopped at u is the same as the process stochastic integral of Y with respect to the process X stopped at u.

If u is a T-valued random variable, we have : $Z_u - Z_{u-} = Y_{u-}.(X_u - X_{u-})$ if Z is the cadlag stochastic integral process $\int Y.dX$ and if X is a cadlag adapted process.

All these properties are obvious if Y is an $\mathcal{R}$-simple process ; they are true in the general case by the dominated convergence theorem above.

B-6. A SECOND EXTENSION

We consider the hypothesis and notations given in B-1 and B-2. Moreover, we suppose that there exists a real positive finite increasing adapted cadlag process A such that the following property is fulfiled :

(i) for each H-valued $\mathcal{R}$-simple process Y and for each stopping time u,

$$E\left\{ \left|\left| \int_{]o,u]} Y.dX - Y_{u-}.(X_u - X_{u-}) \right|\right|_K^2 \right\} \leq E\left(A_{u-} \left\{ \int_{]o,u[} ||Y_t||_H^2 .dA_t \right\}\right)$$

Then, we can define the stochastic integral $\int Y.dX$ for each H-valued predictable bounded process Y in the following way :

Let u be a stopping time such that $\sup_{\omega\in\Omega} .(A_{u-}) < +\infty$. We note X^u the process defined by :

$$X^u = X \,.\, 1_{[o,u[}$$

For each H-valued $\mathcal{R}$-simple process Y, we have :

$$\int Y.dX^u = \int_{]o,u]} Y.dX - Y_{u-}.(X_u - X_{u-})$$

Then, we can define the stochastic integral $Z_t^u = \int_{]o,u]} Y.dX^u$ and the cadlag stochastic integral process $Z^u = \int Y.dX^u$ as in B.4 and B.5 for each H-valued bounded predictable process Y. Moreover, if u and v are two stopping times such that $\sup_{\omega\in\Omega} (A_{u-}) + \sup_{\omega\in\Omega} (A_{v-}) < +\infty$, the cadlag stochastic integral processes Z^u and Z^v are indistinguishable on the stochastic interval $[o, u\wedge v[$ (this is obvious if Y is an $\mathcal{R}$-simple process and is true in the general case by the dominated convergence theorem above).

Then, we consider a fixed H-valued bounded predictable process Y and the sequence $(u(n))_{n>o}$ of stopping times defined by :

$$u(n) = \inf.\{ t : A_t > n \}$$

(and $u(n) = 1$ if the set above is void).

We have $\lim_{n\to\infty} P[u(n) < 1] = 0$ because the process A is a finite cadlag process. Moreover, $\sup_{\omega\in\Omega}. [A_{u(n)-}] \leq n$. Let $Z^{u(n)}$ be the cadlag stochastic integral process $\int Y.dX^{u(n)}$ defined, as above, up to indistinguishability. Let Z be the process defined up to indistinguishability by

$$Z.1_{[o,u(n)[} = Z^{u(n)}.1_{[o,u(n)[}$$

and $\quad Z_1 - Z_{1-} = Y_{1-}.(X_1 - X_{1-})$.

Let B be a process which satisfies all the properties of the process A. Let v be a stopping time such that $\sup_{\omega\in\Omega} (B_{v-}) < +\infty$. We can build Z^v with the help of the process B. Then, we can see as above that $Z.1_{[o,v[} = Z^v.1_{[o,v[}$ a.e.

Then the process Z, defined up to indistinguishability, depends only on the processes Y and X ; it does not depend on the process A ; *We shall call it the cadlag stochastic integral process of the process Y with respect to the process X.*

B.7. REMARK

We shall see after that there exists a process A fulfiling the condition B-6-(i) for a very large class of processes X (specially the class of all semi-martingales in the finite-dimensional case).

Now, we can see that the class of processes X for which tehre exists a process A fulfiling the condition B-6-(i) is a vector space and contained all the cadlag processes of finite variation (by the Cauchy-Schwartz inequality applied for each sample function).

B.8 - OPTIONNAL SET AND PROCESS (definitions)

Let $\mathcal{O}$ be the σ- algebra generated by the stochastic intervalls $]o,u[$, for all the stopping times u. This σ-algebra is called the σ-algebra of the optionnal sets. One says that X is an optionnal process if X is measurable with respect to this σ-algebra $\mathcal{O}$.

Of course, the σ-algebra $\mathcal{P}$ of the predictable sets is contained in the σ-algebra $\mathcal{O}$ of the optionnal sets (because $]o,u] = \bigcap_{n>o}]o,u+\frac{1}{n}[$). Conversely, let $\mathcal{O}'$ be a σ-algebra such that $\mathcal{P}$ is contained in $\mathcal{O}'$ and such that, for each stopping time u, $[u]$ belongs to $\mathcal{O}'$, then $\mathcal{O}$ is contained in $\mathcal{O}'$.

B.9 - RIGHT CONTINUOUS AND OPTIONNAL PROCESS (proposition)

Let H be a Banach space ; let X be an H-valued adapted cadlag process ; then X is an optionnal process with respect to the family $(\mathcal{F}_{t+})_{t\in T}$

Proof

1°/ At first, we prove that $Y = X_u.1_{[u,1[}$ is an optionnal process if u is a stopping time ; X being adapted, X_u is an $\mathcal{F}_u$-measurable random variable, thus it is sufficient to consider the case where X_u is an $\mathcal{F}_u$-simple random variable. Thus, we can suppose that :

$$X_u = \sum_{i\in I} a_i.1_{F(i)}$$

with, for each element i of I, $a_i \in H$ and $F(i) \in \mathcal{F}_u$; in this case, $Y = \sum_{i\in I} a_i.1_{F(i)}.1_{[u,1[}$; if we put $u(i) = u$ if $\omega\in F(i)$ and $u(i) = 1$ if $\omega\notin F(i)$, we have $Y = \sum_{i\in I} a_i.1_{[u(i),1[}$ and that proves that Y is an optionnal process (u(i) being a stopping time for each element i of I).

2°/ Now, we consider the general case. For each integer $n>o$, let $(u(n,k))_{k>o}$ be the sequence of stopping times (with respect to the family $(\mathcal{F}_{t+})_{t\in T}$: cf. A.9) defined by $u(n,O) = O$ and :

$$u(n,k+1)=\inf\{t:t\geq u(n,k), \|X_t - X_{u(n,k)}\| > \frac{1}{n}\}$$

Let X^n be the process defined by $X^n_t = X_{u(n,k)}$ for $u(n,k) \leq t < u(n,k+1)$. The process X^n is well defined because $[u(n,k) < 1] \underset{k\to\infty}{\downarrow} \emptyset$ and it is optionnal (cf. the 1°/ above) ; but the sequence $(X^n)_{n>o}$ converges uniformly to the process X ; thus X is an optionnal process.

B.10 - STOCHASTIC INTEGRAL WITH RESPECT TO A CONTINUOUS PROCESS (proposition)

Let X be a Banach space valued continuous process which satisfies the properties given in B.6, the process A, considered in B.6, being continuous.

Let $(u(n))_{n>o}$ be the sequence of stopping times defined by $u(n) = \inf.\{t : A_t > n\}$. For each integer n, let $\hat{a}_n$ be the measure defined on $(\Omega', \mathcal{F}\otimes\mathcal{B}_{[o,1]})$ by :

$$\hat{a}_n(B) = E\{\int_{]o,u(n)]} 1_B(t).dA_t\}$$

By the Fubini theorem, $\hat{a}_n$ is a finite positive measure ; let a_n be the restriction of $\hat{a}_n$ to the σ-algebra $\mathcal{O}$ of the optionnal sets ; for each stopping time u, we have :

$$\hat{a}_n([u]) = a_n([u]) = 0$$

Then, the adherence of $\mathcal{E}(H)$ (cf. B.2) in $L^H_2(\Omega',\mathcal{O},a_n)$ contained all the uniformly bounded optionnal processes (cf. the end of B.8).

Then, if Y is a uniformly bounded optionnal process, it is possible to define the stochastic integral process $Z = \int Y.dX$ exactly as in B.6 ; moreover Z is a continuous process.

C - ITO FORMULA

C.1 - INTRODUCTION

We put $T = [0,1]$.

Let X and f two real functions, X being defined on T and f being defined on the real line. Under the adequate hypothesis, we have

$$d\,f(X) = f'(X)\,dX$$

and this formula is fundamental for all calculations in differential equations. This formula can be also written more precisely

$$f(X_t)-f(X_o) = \int_{]o,t]} f'(X_s).\,dX_s$$

Now, we consider the case where X is a real continuous process, f being a real function defined on the real line ; then, in general, we have not the previous equalities, but we have :

$$df(X) = f'(X)\,d\,X + \frac{1}{2} f''(X)\,d\,\langle X\rangle$$

or, more precisely :

$$f(X_t) - f(X_o) = \int_{]o,t]} f'(X_s)\,dX_s + \frac{1}{2}\int_{]o,t]} f''(X_s)\,d\langle X\rangle_s$$

where $\langle X\rangle$ is an increasing process associated to the quadratic variation of X. This equality is called the ITO FORMULA : it was proved for the first time for the brownian motion in [Ito].

Of course, this formula is fundamental for all calculations in differential stochastic equations.

Before proving this formula, we give the fundamental idea of the proof.

If X is a function, let us recall a proof of the equality given above :
if $(t(k))_{1\leqslant k\leqslant n}$ is an increasing sequence of times such that $t_1=0$ and $t_n=t$, we have :

$$f(X_t)-f(X_o) = \sum_{k=1}^{n-1} \left[f(X_{t(k+1)}).f(X_{t(k)})\right]$$

$$= \sum_{k=1}^{n-1} f'(X_{t(k)}).\left[X_{t(k+1)}-X_{t(k)}\right] + \sum_{k=1}^{n-1} R_k$$

Now, if $\sup_k \left[t(k+1) - t(k)\right]$ goes to zero, for some functions f and X, the first sum converges to $\int_{]o,t]} f'(X_s).dX_s$ and the second sum converges to zero.

Now, if X is a process, in general, the second sum $\sum_{k=1}^{n-1} R_k$ does not go to zero.

Then, we use the Taylor formula and we have :

$$f(X_t) - f(X_o) = \sum_{k=1}^{n-1} f'(X_{t(k)}) \cdot \left[X_{t(k+1)} - X_{t(k)}\right] + \frac{1}{2} \sum_{k=1}^{n-1} f''(X_{t(k)}) \cdot \left[X_{t(k+1)} - X_{t(k)}\right]^2 + \sum_{k=1}^{n-1} R_k^*$$

For some functions f and for some processes X, when $\sup_k \left[t(k+1) - t(k)\right]$ goes to zero, the first sum converges to the stochastic integral $\int_{]o,t]} f'(X_{s-}) \cdot dX_s$, the second sum converges to $\frac{1}{2}\int_{]o,t]} f''(X_{s-}) \cdot d\langle X\rangle_s$ and the third sum converges to zero.

We shall prove the Ito formula for processes with values in a separable Hilbert space H. In our context, to suppose that H is separable is not a restriction ; moreover, it is not more difficult to prove the Ito formula when H is an Hilbert space that when H is a finite-dimensionnal vector space. It is also possible to prove this formula when H is a Banach space (cf. [Gr P]).

In the following, $(h_n)_{n>0}$ will be an orthogonal base of H. Moreover, as in the previous paragraphs, we shall consider a probabilized stochastic basis$(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in T})$ and we shall suppose that this basis is complete and right continuous (cf. A-1). We shall suppose also that $T = [0,1]$.

C.2 - TENSOR PRODUCT AND HILBERT-SCHMIDT NORM :

We shall note $H \otimes H$ the tensor product of H by itself. If x and y are two elements of H, we shall note $x \otimes y$ the tensor product of x and y. If x = y, we shall note $x \otimes x = x^{\otimes 2}$.

Let $(x_i, y_i)_{i\in I}$ be a finite family of pairs of elements of H ; let $z = \sum_{i\in I} x_i \otimes y_i$ be the element of $H \otimes H$ associated to this family.

We consider also similarly $z' = \sum_{j\in J} x'_j \otimes y'_j$

If we put

$$\langle z,z'\rangle = \sum_{i\in I} \sum_{j\in J} \langle x_i, x'_j\rangle . \langle y_i, y'_j\rangle$$

this defines a scalar product on $H \otimes H$. We shall note $H \hat{\otimes} H$ the space $H \otimes H$ completed for the topology associated to this scalar product ; the norm on $H \hat{\otimes} H$ associated to this scalar product is called the Hilbert-Schmidt norm and will be noted $||.||_{H.S.}$. With the canonic extension of the scalar product defined above, $H \hat{\otimes} H$ is a separable Hilbert space : more precisely, $(h_n \otimes h_m)_{n>0,n>0}$ is a base of $H \hat{\otimes} H$. If x and y are two elements of H, we have :

$$||x \otimes y||_{H.S} = ||x||_H . ||y||_H$$

At last, the mapping $(x,y) \rightsquigarrow x \otimes y$ from $(H \times H)$ into $(H \otimes H)$ is a continuous bilinear mapping.

All the previous properties are well-known and easy to prove. Let us recall also that, if H is finite dimensionnal, $H \otimes H = H \hat{\otimes} H$ is isomorphic to the space of all $d \times d$ matrix : more precisely, let $(h_n)_{1\leq n\leq d}$ be an orthonormal base of H, $(x_i, y_i)_{i\in I}$ be a finite family of pairs of elements of H, with $x_i = \sum_{n=1}^{d} x_{i,n} . h_n$ and $y_i = \sum_{n=1}^{d} y_{i,n} . h_n$, and (X^i, Y^i) be the pairs of matrix defined by $X^i = \begin{pmatrix} x_{i,1} \\ \vdots \\ x_{i,d}\end{pmatrix}$ and $Y^i = \begin{pmatrix} y_{i,1} \\ \vdots \\ y_{i,d}\end{pmatrix}$ then the one-to-one mapping which associates the $d \times d$ matrix

$$\Big(\big(\sum_{i\in I} x_{i,j}\, y_{i,k}\big)\Big)_{j,k} = \sum_{i\in I} X^i . (Y^i)^{tr}$$ to the element $\left(\sum_{i\in I} x_i \otimes y_i\right)$ of $H \otimes H$ is an isomorphism from $H \otimes H$ into the vector space of all $d \times d$ matrix.

For the convenience of the reader, we shall explicit the Ito formula when H is finite dimensionnal in C-8 after.

C.3 - QUADRATIC VARIATION :

Let X be an H-valued cadlag process. We shall call the quadratic variation of X the positive increasing right continuous process D defined (up to an indistinguishability) by :

$$D_t = \limsup_{n\to\infty} \sum_{k>0} ||X_{(k+1).2^{-n}\wedge t} - X_{k.2^{-n}\wedge t}||^2 \quad a.e.$$

For each pair (s,t) of elements of T with $s < t$ and $D_t < +\infty$ a.e., we have :

$$D_t - D_s = \limsup_{n\to\infty} \sum_{k>0} ||X_{(k+1)2^{-n}\wedge t\vee s} - X_{k.2^{-n}\wedge t\vee s}||^2 \quad a.e.$$

We shall say that the process X is of finite quadratic variation if $D_1 < +\infty$ a.e.

The set of the processes which are of finite quadratic variation is clearly a vector space. Moreover, if X is a cadlag process of finite variation, then X is also a process of finite quadratic variation.

C.4 - DIFFERENTIAL (CONVENTIONS) :

Let H and K be two Hilbert spaces, f be a K-valued function defined on H and twice differentiable. We shall note f' and f" the first and second differential respectively : the second differential will be considered as a K-valued linear mapping defined on $H \hat{\otimes} H$; if (x,y) is an element of $(H, H \hat{\otimes} H)$, we shall note $[f''(x)](y)$ the value of this second differential considered at the point x and applied to the vector y.

C.5 - ITO FORMULA :

Let X be a cadlag process, adapted to the complete stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in T})$ *and with values in the separable Hilbert space H. We suppose that X is of finite quadratic variation D. Moreover, we suppose that there exists a positive increasing right continuous adapted process A such that (cf. B-6) :*

(i) for each Hilbert space K, for each $\mathcal{A}$*-simple process Y with values in* $\mathcal{L}(H,K)$*, for each stopping time u, we have :*

$$E\{||\int_{]o,u[} Y.dX||^2\} \leqslant E(A_{u-} \{\int_{]o,u[} ||Y_t||^2.dA_t\})$$

Let f be a K-valued twice differentiable function, defined on the Hilbert space H ; we suppose that the second differential f" of f is uniformly continuous on all the bounded subsets of H.

Let S, Q, V and C the processes defined by :

$$S(t) = \sum_{s\leqslant t} (X_s - X_{s-})^{\otimes 2}$$

$$Q(t) = \sum_{s\leqslant t} [f(X_s) - f(X_{s-}) - f'(X_{s-})(X_s - X_{s-})]$$

$$V(t) = X_t^{\otimes 2} - X_o^{\otimes 2} - \int_{]o,t]} (X_{s-} \otimes dX_s + dX_s \otimes X_{s-})$$

$$C(t) = V(t) - S(t)$$

where $\int_{]o,t[} (X_{s-} \otimes dX_s + dX_s \otimes X_{s-})$ *is a stochastic integral and S,V and C are* $(H \hat{\otimes} H)$*-valued.*

Then the processes S, Q, V and C are well defined, adapted, cadlag, of finite variation and C is continuous($(H \hat{\otimes} H)$ being with its Hilbert-Schmidt norm).

Moreover, we have the Ito formula :

$$f(X_t) - f(X_o) = Q(t) + \int_{]o,t]} f'(X_{s-}).dX_s - \frac{1}{2} \sum_{s\leqslant t} f''(X_{s-})(X_s - X_{s-})^{\otimes 2} + \frac{1}{2}\int_{]o,t]} f''(X_{s-}).dV_s$$

$$= Q(t) + \int_{]o,t]} f'(X_{s-}).dX_s + \frac{1}{2}\int_{]o,t]} f''(X_s).dC_s$$

Proof

The proof has two parts ; in the first part (C-6), we study the processes S, Q, V and C ; in the second one (C-7), we prove the Ito formula.

Before, we remark that :

a) we can suppose that the family $(\mathcal{F}_t)_{t\in T}$ is right continuous (cf. A-11).

b) by prelocalization (cf. A-13), we can suppose that the quadratic varation D of X is uniformly bounded by the real number d and the norm of the process X is uniformly bounded by the real number a.

C.6 - THE PROCESSES S, Q, V AND C

Now, we prove the first part of C.5.

1°/ S and Q are well defined.

For almost all the elements ω of Ω, we have

$$\sum_{s\leqslant t} ||X_s(\omega) - X_{s-}(\omega)||^2 \leqslant d$$

thus S(ω) is well defined (for each element ω of Ω) (let us recall that $||x^{\otimes 2}||_{H.S.} \leqslant ||x||_H^2$).

Moreover, the Taylor formula gives :

$$f(X_t) - f(X_{t-}) - f'(X_{t-}) (X_t - X_{t-}) =$$

$$\{ \int_0^1 \frac{1-s}{2} f'[X_{t-} + s(X_t - X_{t-})] ds \} (X_t - X_{t-})^{\otimes 2}$$

The function f" being bounded by the real number C on the domain $||x|| \leqslant a$, the terms of the equality above are less than $\frac{1}{2} C.||X_t - X_{t-}||^2$. Thus, the process Q is well defined as above.

Of course, the processes S and Q are cadlag and of finite variation ; thus, they are defined up to indistinguishability.

2°/ The processes S *and* Q *are adapted*

We shall prove that the process S is adapted ; the proof is about the same for the process Q. Let t be a real positive number. Let b(n) a decreasing sequence of real positive members such that $\lim_{n\to\infty} b(n) = 0$.

For each integer n>o, let $(u(n,k))_{k>0}$ be the increasing sequence of stopping times defined by recurrence by $u(n,1) = 0$ and :

$$u(n,k+1) = \inf \{s : s\leq t,\ s\geq u(n,k),\ ||X_s - X_{u(n,k)}|| > b(n)\}$$

and $u(n,k+1) = t$ if the set above is void.

The process X being cadlag, for each integer n, $[u(n,k) < t] \underset{k\to\infty}{\downarrow} \emptyset$. For each integer n, let W_n be the random variable defined by :

$$W_n = \sum_{k>0} (X_{u(n,k)} - X_{u(n,k)-})^{\otimes 2}$$

(W_n is well defined : cf. the 1°/ above).

The sequence of random variables $(W_n)_{n>0}$ converges a.e. to a random variable W (cf. the 1°/ above), then W is $\mathcal{F}_t$-measurable ; now W = S(t) a.e.; thus S is an adapted process.

3°/ The process V *is the "tensor quadratic variation" of the process* X

Let b(n) be a decreasing sequence of real positive numbers such that $\lim_{n\to\infty} b(n) = 0$. For each integer n, let $(v(n,k))_{k>0}$ be the sequence of stopping times defined by recurrence by $v(n,1) = 0$ and

$$v(n,k+1) = \inf.\{s : s\geq v(n,k),\ ||X_s - X_{v(n,k)}|| > b(n)\}$$

and $v(n,k+1) = 1$ if the set above is void. We have

$$\lim_{k\to\infty} P([v(n,k) < t]) = 0.$$

For each integer n, let V^n be the cadlag process defined, for each element (ω,t) of the set $[v(n,k),\ v(n,k+1)[$, by :

$$V^n_t = \sum_{j=0}^{k-1} (X_{v(n,j+1)} - X_{v(n,j)})^{\otimes 2} \quad \text{and}$$

$$V^n_1 = \sum_{j=0}^{\infty} (X_{v(n,j+1)} - X_{v(n,j)})^{\otimes 2}$$

We have :

$$V^n(1) = \sum_{k=0}^{\infty} (X^{\otimes 2}_{v(n,k+1)} - X^{\otimes 2}_{v(n,k)}) - \sum_{k=0}^{\infty} (X_{v(n,k+1)} - X_{v(n,k)}) \otimes X_{v(n,k)} - \sum_{k=0}^{\infty} X_{v(n,k)} \otimes (X_{v(n,k+1)} - X_{v(n,k)})$$

(the first sum is, of course, equal to $X_1^{\otimes 2} - X_0^{\otimes 2}$)

Let $Z^n(t)$ be the predictable process defined, for each element (t,ω) of the stochastic interval $]v(n,k),\ v(n,k+1)]$, by $Z^n(t) = X_{v(n,k)}$.

We put :

$$\bar{V}^n(t) = X_t^{\otimes 2} - X_0^{\otimes 2} - \int_{]0,t]} [Z^n(u) \otimes dX_u + dX_u \otimes Z^n(u)]$$

We have $V^n(1) = \bar{V}^n(1)$ and, if $(t,\omega) \in [v(n,k), v(n,k+1)[$,

$$V^n(t) = \bar{V}^n(t) - [X_t - X_{v(n,k)}]^{\otimes 2}$$

If the sequence $(b(n))_{n>0}$ decreases sufficiently quickly to zero, the sequence of processes $(\bar{V})_{n>0}$ converges a.e. uniformly (cf. B.6) to the process V ; then, it is the same for the sequence $(V^n)_{n>0}$.

Moreover, that proves that, for each element ω of Ω, the total variation of the process V is less than d.

4°/ The process C *is continuous*

We choise the sequence $(b(n))_{n>0}$ such that, for each integer n, $b(n) \leq (\frac{1}{n})^2$ and we define the sequence of stopping times $(v(n,k))_{k>0}$ as in the 2°/ above.

For each pair of integers (n,k), we put :

$$A(n,k) = \{\omega : ||X_{v(n,k)} - X_{v(n,k)-}|| > \frac{1}{n}\}$$

$$B(n,k) = \Omega \setminus A(n,k)$$

$$E_{n,k} = X_{v(n,k)} \cdot 1_{B(n,k)} + X_{v(n,k)-} \cdot 1_{A(n,k)}$$

For each integer n, let $S^n(t)$ and $W^n(t)$ be the processes defined, for (t,ω) element of $[v(n,k),\ v(n,k+1)[$, by :

$$S^n(t) = \sum_{j=1}^{k} (X_{v(n,j)} - E_{n,j})^{\otimes 2} = \sum_{j=1}^{k} (X_{v(n,j)} - X_{v(n,j)-})^{\otimes 2} \cdot 1_{A(n,j)}$$

$$W^n(t) = \sum_{j=1}^{k} (X_{v(n,j)} - X_{v(n,j-1)})^{\otimes 2} \cdot 1_{A(n,j)}$$

The total variation of the process $(S-S^n)$ converges a.e. to zero. Moreover, on A(n,j), we have :

$$||X_{v(n,j-1)}-X_{v(n,j)}|| \leq (\frac{1}{n})^2 \quad \text{and}$$

$$||X_{v(n,j)} - X_{v(n,j)-}|| \geq \frac{1}{n}$$

we have

$$||W^n(t)-S^n(t)|| \leq \frac{1}{n} \cdot \left\{ \sum_{j=1}^{k} ||X_{v(n,j)}-X_{v(n,j)-}||^2 \cdot 1_{A(n,j)} \right\}$$

and that shows that the sequence of processes $(W^n)_{n>o}$ converges, to the process S, a.e. uniformly.

For each integer n, let C^n be the process defined, for (t,ω) element of $[v(n,k),v(n,k+1)[$, by :

$$C^n(t) = \sum_{j=1}^{k} [X_{v(n,j)} - X_{v(n,j-1)}]^{\otimes 2} \cdot 1_{B(n,j)}$$

$$= V^n(t) - W^n(t)$$

If the sequence $(d(n))_{n>o}$ converges sufficiently quickly to zero, the previous results show that the sequence of processes $(C^n)_{n>o}$ converge a.e. uniformly to $V(t) - S(t) = C(t)$: thus C is continuous, the jumps of C^n being less than 1/n.

5°/ *At last, we remark that :*

$$X_t^{\otimes 2} - X_o^{\otimes 2} = V(t) + \int_{]o,t]} [X_{s-} \otimes dX_s + dX_s \otimes X_{s-}]$$

thus the stochastic integral can be defined with respect to the process $X_t^{\otimes 2}$ (we are in the situation given in B.6).

C.7 - PROOF OF THE ITO FORMULA

Now, we prove the second part of C.5.

1°/ All the integral and processes considered in C.5 are well defined (up to an indistinguishbility) ; moreover, these processes are cadlag ; to prove the Ito formula, it is sufficient to prove that the two members of this formula are equal a.e. for each element t of T (t fixed). It is sufficient to prove that for t = 1.

We consider a decreasing sequence of real positive numbers $(b(n))_{n>o}$ which converges "sufficiently quickly" to zero.

We define the stopping times v(n,k) as in C.6.3°/ and the sets A(n,k) and B(n,k) as in C.6.4°/.

2°/ For each integer n, we have :

$$f(X_1) - f(X_o) = \sum_{k>o} [f(X_{v(n,k+1)})-f(X_{v(n,k)})] [1_{A(n,k+1)}+1_{B(n,k+1)}]$$

Using the Taylor formula, for each n, k and ω, there exists $R_{n,k}(\omega)$ bounded as after (cf. 5°/) and such that :

$$f(X_{v(n,k+1)})-f(X_{v(n,k)})=f'(X_{v(n,k)})[X_{v(n,k+1)}-X_{v(n,k)}]$$
$$+ \frac{1}{2} f''(X_{v(n,k)}) (X_{v(n,k+1)} - X_{v(n,k)})^{\otimes 2}$$
$$+ R_{n,k}(\omega)$$

(Actually, we shall use this identity only on the set B(n,k+1)).

Thus, we have : $f(X_1)-f(X_o) = \sum_{k>o} \sum_{i=1}^{5} a^i_{n,k}$ with

$$a^1_{n,k}=f'(X_{v(n,k)}).(X_{v(n,k+1)} - X_{v(n,k)})$$

$$a^2_{n,k}= -\frac{1}{2} f'' X_{v(n,k)} .(X_{v(n,k+1)}-X_{v(n,k)})^{\otimes 2} 1_{A(n,k+1)}$$

$$a^3_{n,k} = R_{n,k} \cdot 1_{B(n,k+1)}$$

$$a^4_{n,k} = [-f'(X_{v(n,k)})(X_{v(n,k+1)} - X_{v(n,k)})+ f(X_{v(n,k+1)}) -f(X_{v(n,k)})] \cdot 1_{A(n,k+1)}$$

$$a^5_{n,k} = \frac{1}{2} f''(X_{v(n,k)}).(X_{v(n,k+1)}-X_{v(n,k)})^{\otimes 2}$$

Now, we prove that $\sum_{k\geq o} a^i_{n,k}$ converges a.e. for $1\leq i\leq 5$, when n goes to the infinity.

3°/ We have $\sum_{k\geq o} a^1_{n,k} = \int_{]o,1]} f'[Z^n(t)]. dX_t$

where Z^n is the process defined as in C.6.3°/ above. If the sequence $(b(n))_{n>o}$ converges to zero, $\int_{]o,1]} f'[Z^n(t)].dX_t$ converges a.e. to $\int_{]o,1]} f'(X_{t-}).dX_t$

4°/ $\sum_{k\geq o} a^2_{n,k}$ converges a.e. to $-\frac{1}{2}\int_{]o,1]} f''(X_{t-}).dS_t$ because f" is uniforly continuous and the proof C.6.2°/ (the total variation of the process $(S-S^n)$ converges a.e. to zero).

5°/ The function f" being uniformly continuous, (for $||x|| \leq a$), for each $\varepsilon > o$, we have, for n sufficiently large, if $\omega \in B(n,k+1)$:

6°/ $\sum_{k\geq 0} a^4_{n,k}$ converges a.e. to Q(1) (cf. the proof of C.6.2°/)

7°/ At last, we have :

$$a^5_{n,k} = \frac{1}{2} f''(X_{v(n,k)}) . \Big\{ X^{\otimes 2}_{v(n,k+1)} - X^{\otimes 2}_{v(n,k)} - X_{v(n,k)} \otimes [X_{v(n,k+1)} - X_{v(n,k)}] - [X_{v(n,k+1)} - X_{v(n,k)}] \otimes X_{v(n,k)} \Big\}$$

If we define Z^n as in C.6.3°/ above, we have :

$$2 \sum a^5_{n,k} - \int_{]o,1]} f''(X_{t-}).dV_t = \int_{]o,1]} [f''(Z^n) - f''(X_{t-})].dX^{\otimes 2} - \int_{]o,1]} [f''(Z^n).Z^n - f''(X_{t-}).X_{t-}] \otimes dX_t - \int_{]o,1]} f''(Z^n).dX_t \otimes Z^n - f''(X_{t-}).dX_t . \otimes X_{t-}$$

By the dominated convergence theorem (cf. B.4), when n goes to the infinity, all the previous integrals converge a.e. to zero (cf. C.6. 5°/ above) and that completes the proof.

C.8 - <u>ITO FORMULA : FINITE DIMENSIONNAL CASE</u>

We suppose that H *is a finite dimensionnal vector space and* $(h_j)_{1\leq j\leq n}$ *is a base of* H. *Let* X *be an* H*-valued cadlag process adapted to the complete stochastic base* $(\Omega, \mathfrak{F}, P, (\mathfrak{F}_t)_{t\in T})$ *with* $T = [0,1]$ *we suppose that* X *is of finite quadratic variation and satisfies the condition C.5.(i). We shall write*

$$X = \sum_{j=1}^{n} X^j . h_j$$

Let f be a real function defined on the real line and twice continuously differentiable.

We consider the following real processes with $1\leq i\leq n$ *and* $1\leq j\leq n$:

$$S_{i,j}(t) = \sum_{s\leq t} (X^i_s - X^i_{s-})(X^j_s - X^j_{s-}) = S_{j,i}(t)$$

$$Q(t) = \sum_{s\leq t} \Big[f(X_s) - f(X_{s-}) - \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(X_{s-}).(X^i_s - X^i_{s-})\Big]$$

$$V_{i,j}(t) = X^i_t . X^j_t - X^i_o . X^j_o - \int_{]o,t]} (X^i_{s-}.dX^j_s + X^j_{s-}.dX^i_s)$$

$$C_{i,j}(t) = V_{i,j}(t) - S_{i,j}(t)$$

Then the processes $S_{i,j}$, Q, $V_{i,j}$ *and* $C_{i,j}$ *are real, well defined, cadlag processes of finite variation and* $C_{i,j}$ *is continuous ; moreover, the processes* $S_{i,i}$, $V_{i,i}$ *and* $C_{i,i}$ *are increasing.*

Moreover, we have :

$$f(X_t) - f(X_o) = Q(t) + \int_{]o,t]} \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(X_{s-}).dX^i_s - \frac{1}{2} \sum_{s\leq t} \sum_{i,j} \frac{\partial^2 f(X_{s-})}{\partial x^i \partial x^j} (X^i_s - X^i_{s-})(X^j_s - X^j_{s-}) + \frac{1}{2} \int_{]o,t]} \sum_{i,j} \frac{\partial^2 f(X_{s-})}{\partial x^i \partial x^j} . dV_{i,j}(s)$$

$$= Q(t) + \int_{]o,t]} \sum_{i=1}^{n} \frac{\partial f(X_{s-})}{\partial x^i} . dX^i_s + \frac{1}{2} \int_{]o,t]} \sum_{i,j} \frac{\partial^2 f(X_{s-})}{\partial x^i \partial x^j} . dC_{i,j}(s)$$

<u>Proof</u>

This theorem is a particular case of the theorem C.5 above ; we have only to prove that the processes $V_{i,i}$ and $C_{i,i}$ are increasing ; that proceeds of the proofs C.5.3°/ and C.5.4°/.

C.9 - <u>REMARK</u>

In C.5, we supposed that f" is uniformely continuous on all the bounded subsits of H. Actually, in the proof of C.5, we exactly used the following property :
for each pair (a,ε) of positive numbers, there exists a positive number η such that :
$||x|| \leq a$, and $||y|| \leq \varepsilon$ implies
$||f(x+y) - f'(x).y - \frac{1}{2} f''(x).y^{\otimes 2}||_K \leq \eta \, ||y||^2_H$

C.10 - <u>BROWNIAN MOTION (DEFINITION)</u>

Let X be a real process. One says that X is a real brownian motion, with respect to the stochastic basis $(\Omega, \mathfrak{F}, P, (\mathfrak{F}_t)_{t\in T})$ with $T = [0,1]$, if X satisfies the following properties :
(i) X is a continuous process
(ii) X is a square integrable martingale, id est, $E(X_1^2) < +\infty$ and, for each pair (s,t) of elements of T with s < t, we have : $E(X_t | \mathfrak{F}_s) = X_s$ a.s
(iii) the quadratic variation $[X,X]$ of X is defined by $[X,X]_t = t$ (for each element ω of Ω).

We see in the following paragraph E (cf. E-11) that X satisfies all the hypothesis given in C.5 ; thus, it is possible to apply the ITO Formula to a brownian motion.

C.11 - <u>NOTATION</u> $[X,Z]$

Let X and Z be two real processes which satisfy all the properties given in C.5. In the exercises, we note $[X,Z]$ the "quadratic variation" associated to the processes X and Z. More precisely, $[X,Z]$ is the cadlag process defined by :

$$[X,Z]_t = X_t.Z_t - X_o.Z_o - \int_{]o,t]} X_{s-}.dZ_s - \int_{]o,t]} Z_{s-}.dX_s$$

D - STOCHASTIC DIFFERENTIAL EQUATIONS

D.1. GENERALITIES :

In this paragraph D, we consider :

- $T = [0,1]$, the unit interval of the real line
- H,K two separable Banach spaces and $\mathcal{L}_a(K,H)$ a subspace of $\mathcal{L}(K,H)$, the space of the linear operators from K to H ; on this subspace $\mathcal{L}_a(K,H)$, we consider a norm such that, if u is an element of $\mathcal{L}_a(K,H)$, $||u|| \geq \underset{||k|| \leq 1}{\text{Sup.}} ||u(k)||_H$
- $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in T}) = B^I$ a "stochastic basis" with the usual assumptions, i.e. for each element t of T, $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ and $A \in \mathcal{F}_t$ if $P(A) = 0$. We shall call this basis the "initial basis". We note $\mathcal{A}$ the algebra generated by the sets $F \times]s,t]$ with $F \in \mathcal{F}_s$; the σ-algebra generated by $\mathcal{A}$ is the σ-algebra of predictable sets.

D.2. CANONICAL BASIS (DEFINITION) :

We shall use the french notations "cadlag" "caglad", and so on ; more precisely, let f be a real function defined on T ; we say that f is cadlag if, for each element of T, f is right continuous and has left limit. (in french : continu à droite et a une limite à gauche). We say that a process X is cadlag if, for each element ω of Ω , the sample function $t \rightsquigarrow X_t(\omega)$ is cadlag.

Let D^H be the space of all H-valued cadlag functions defined on T. For each element t of T, let $\mathcal{D}^H_t$ the σ-algebra generated by the sets $\{\omega : X_s(\omega) \in H_o\}$ with $s \leq t$ and H_o borelian set of H ; we define $\mathcal{D}^H = \mathcal{D}^H_1$. The family $(D^H \times \Omega, \mathcal{D}^H \otimes \mathcal{F}, (\mathcal{D}^H_t \otimes \mathcal{F}_t)_{t \in T})$ is a "stochastic basis" that we shall note B^H and we shall call the canonical basis (for the H-valued processes defined with respect to the basis B^I).

D.3 REMARKS AND CONVENTIONS :

a) The σ-algebra of predictable sets of the canonic basis B^H is generated by the sets $G \times F \times]s,t]$ where G is an element of $\mathcal{D}^H_s$, and F is an element of $\mathcal{F}_s$; actually, it is sufficient to consider the case where G is the set of the cadlad functions x such that $x_u = x(u)$ is an element of H_o with $u < s$ and H_o borelian set of H.

b) Let $a(x, \omega, t)$ be an $\mathcal{L}_a(H,H)$-valued process defined with respect to the canonic basis B^H. Let X be an H-valued process defined with respect to the initial basis B^I. In the following, we consider processes Z such that $Z_t(\omega) = a[X(\omega),\omega,t]$; in this situation, for the commodity of notations, we shall not write the symbol ω; then, we shall write $Z_t = a(X,t)$.

c) we shall consider stopping times such that :

$w = \inf \{t : t \geq u, t \leq v, ||X_t|| > \varepsilon\}$

In this situation, if the set above is empty, we define $w(\omega) = v(\omega)$.

d) Let u and v be two stopping times. We define the stochastic integral $\int_{]u,v]} Y.dX$ as usual and we define :

$$\int_{]u,v[} Y.dX = \int_{]u,v]} Y.dX - Y_{v-}(X_v - X_{v-})$$

(when these terms are well defined).

If v is a predictable stopping time, the set $]u,v[$ is a predictable set, then we have :

$$\int_{]u,v[} Y.dX = \int 1_{]u,v[} Y.dX$$

e) Let u be a stopping time and let X and Y be two processes, the process X being cadlag ; then, we shall note $\underset{t<u}{\text{Sup.}} ||\int_{]0,t]} Y_s.dX_s||^2$ the random variable U defined more precisely by :

$$U(\omega) = \underset{t<u(\omega)}{\text{Sup}} . ||Z_t(\omega)||^2$$

where Z is the cadlag process, unique up to indistinguishability, stochastic integral of Y with respect to X, i.e. defined by

$$Z_t = \int_{]0,t]} Y_s.dX_s.$$

D.4. PROPOSITION :

Let K be a Banach space. Let X be a cadlag H-valued process, defined and adapted with respect to the initial basis B^I. Let $a(x,\omega,t)$ be a K-valued process, defined and predictable with respect to the canonical basis B^d. Let Y be the process defined by : $Y_t(\omega) = a(X(\omega),\omega,t)$. Then, Y is a K-valued process, predictable with respect to the initial basis B^I. Moreover, $Y_t(\omega)$ is depending only on the values $X_s(\omega)$ for $s < t$ (then, it is possible to define $Y_t(\omega)$, when X_s is known only for $s < t$).

Proof :

1°) First, we consider the case where there exists k element of K, $u < v < w$ elements of T, H_o borelian set of H, F element of $\mathcal{F}_v$ such that, if

$J = \{x : x_u \in H_o\}$,

then, $a(x,\omega,t)=k.1_J(x).1_F(\omega).1_{]v,w]}(t)$

Let F' be the set defined by $F'= \{\omega:X_u(\omega) \in H_o\}$. The process X being adapted F' belongs to $\mathcal{F}_u$; we have also :

$$Y_t(\omega)=a[X.(\omega),\omega,t]=k.1_J[X.(\omega)].1_F(\omega).1_{]v,w]}(t)$$
$$= k.1_F(\omega).1_{F'}(\omega).1_{]v,w]}(t)$$

then Y is a predictable process and $Y_t(\omega)$ is only depending on $X_s(\omega)$ for $s < t$.

2°) Then, we consider an H-valued process X, adapted with respect to the initial basis B^I. Let $\mathcal{C}_X$ be the family of all the K-valued processes a, defined with respect to the canonic basis B^H and such that, if $Y = a(X,t)$, Y is a predictable process with Y_t only depending on X_s for $s < t$. The space $\mathcal{C}_X$ is a vector space and a monotone class ; moreover, $\mathcal{C}_X$ contains all the processes $a=k.1_J.1_F.1_{]v,w]}$ as defined in the 1°) above. Then, $\mathcal{C}_X$ contains all the predictable processes (cf. the remark D.3-a).

D.5 : THEOREM :

Let H and K be two separable Hilbert spaces. Let $B^I = (\Omega,\mathcal{F}, P, (\mathcal{F}_t)_{t\in T})$ be a stochastic basis with the usual assumptions, (cf. D.1 above), that we shall call the initial basis. Let Z be a K-valued cadlag process, defined and adapted with respect to the initial basis B^I. We suppose that there exists a real positive increasing process Q, defined and adapted with respect to the initial basis B^I, such that, for each (strongly) predictable $\mathcal{L}_a(K,H)$-valued uniformly bounded process Y, and for each stopping time u, we have (cf. D.3.e. above) :

$$(i) \quad E\left\{\sup_{t<u}\left\|\int_{]o,t]} Y_s.dZ_s\right\|^2\right\} \leq E\left(Q_{u-}.\left\{\int_{]o,u[}\|Y_t\|^2.dQ_t\right\}\right)$$

Let $a(x,\omega,t)$ be an $\mathcal{L}_a(K,H)$-valued process, defined and predictable with respect to the canonical basis B^H. We suppose that a is locally lipschitzian in the following sense :

(ii) For each real positive number d, there exists a right continuous increasing adapted process L_d such that, if (ω,t) is an element of $(\Omega\times T)$, if (x,x') is a pair of elements of D^H with $\sup_{s\leq t}\|x_s\| \leq d$ and $\sup_{s\leq t}\|x'_s\| \leq d$, then we have :

$$\|a(x,\omega,t)-a(x',\omega,t)\|^2 \leq L_d(\omega).\sup_{s<t}\|x_s-x'_s\|^2(\omega)$$

Let u be a stopping time and X^u be an H-valued cadlag adapted process stopped at u. Then, there exists a predictable (cf.A.14) stopping time v and an H-valued cadlag process X, defined and adapted with respect to be initial basis B^I, unique up to indistinguability, with the following properties :

(iv) if ω belongs to the set $\{v < 1\}$,

$$\lim.\sup_{t\uparrow v(\omega)} \|X_t(\omega)\| = +\infty$$

$$(v) \quad X_t = X^u + \int_u^t a(X,s).dZ_s$$

On the stochastic interval $]u,v[$, this integral being an usual stochastic integral.

Then, we say that X is a strong solution of the stochastic differential equation $dX_t= a(X,t).dZ_t$ on the stochastic interval $]u,v[$, with the initial value X^u.

Proof :

By localization, it is sufficient to consider the case where Q is uniformly bounded ; then, in the right term of the inequality D.5(i), we can write :

$E\left\{\int_{]o,u[}\|Y_t\|^2.dQ_t\right\}$ instead of $E\left(Q_{u-}.\left\{\int_{]o,u[}\|Y_t\|^2.dQ_t\right\}\right)$

that we shall do henceforth. In the following we shall omit the symbol ω if there is no possible confusion. The following proof is a natural generalization of the classical study of ordinary differential equations based on the fixed point theorem. This proof has three steps :

1°/ Unicity : Lemma D.6

2°/ Extension principle for solutions (D.7)

3°/ Maximal solution (D.8)

D.6 - UNICITY

We consider the hypothesis and notations given in the theorem D.5 above. Let X and X' be two adapted cadlag processes which are solutions of the equation D.5.(v) on the stochastic intervals $]u,v]$ *and* $]u,v']$ *respectively and which are equal on the stochastic interval* $1_{]0,u]}$

Then, $X.1_{[u,v\wedge v']}$ *and* $X'.1_{[u,v\wedge v']}$ *are two indistinguishable processes.*

Proof :

Let X and X' two solutions on $]u,v]$ and $]u,v']$ respectively. We define :

$u' = \inf\{t : ||X_t - X'_t|| > 0,\ t \geq u,\ t \leq (v\wedge v')\}$.

If $P[u' < (v\wedge v')] = 0$, the lemma is proved. Then, we suppose that $P[u' < (v\wedge v')] > 0$. The processes X and X' being cadlag, there are a real number d and a stopping time w' such that :

$$\sup_{u'\leq s<w'} .(||X_s|| + ||X'_s||) \leq d$$

$$P([w' > u']) > 0 \quad \text{and} \quad w' \leq (v\wedge v').$$

Let L_d be the "Lipschitz process" associated to d which appears in the condition D.5.(ii). Let w be the stopping time defined by :

$$w = \inf.\{t : t \geq u',\ t \leq w',\ Q_t - Q_{u'} > \frac{1}{2L_d}\}$$

The processes Q and L_d being right continuous we have $P([w > u']) > 0$. Then we define :

$$h = E\{\sup_{u\leq s<w} ||X'_s - X_s||^2\}$$

Then, we have (cf. D.3-e) :

$$h = E\{\sup_{u\leq s<w} ||\int_{]u,s]} [a(X,r)-a(X',r)].dZ_r||^2\}$$

$$\leq E\{\int_{]u,w[} ||a(X,r)-a(X',r)||^2.dQ_r\} \quad \text{(cf D.5-(i))}$$

$$\leq E\{\int_{]u,w[} L_d \cdot \sup_{u\leq r<w} .||X_r - X'_r||^2.dQ_r\} \quad \text{(cf.D.5-(ii))}$$

$h \leq \frac{1}{2} h$ (cf. the building of w); then h = 0 and that proves the unicity.

D.7 - EXTENSION PRINCIPLE FOR SOLUTIONS

We consider the hypothesis and notations given in the theorem D.5. Then for each $\varepsilon > 0$ *there exist a stopping time v and an H-valued cadlag adapted process X, defined on the stochastic interval* $[u,v]$ *, which satisfy the following two properties :*

(i) $P([v > u]) > P([u < 1]) - \varepsilon$

(ii) $X_t = X^u + \int_u^t a(X,s)dZ_s$ *on the stochastic interval* $\{(t,\omega) : u(\omega) \leq t \leq v(\omega)\}$

Proof

1°/ Let X^0 be the process defined by :

$$X^0_t(\omega) = X^u(\omega)$$

Let d be a real number such that

$$P\left[\sup_{t\leq u} ||X_t|| > d\right] < \varepsilon.$$

Let $L = L_{2d}$ be the "Lipschitz process" associated to 2d which appears in D.5.(ii).

Let v" the stopping time defined by :

$$v'' = \inf.\{t : t \geq u,\ Q_t - Q_u > \frac{1}{8L}\}$$

The process Q being right continuous, we have $P([v'' > u]) = P([u < 1])$

Let w be the T-valued random variable defined on $(D^H, \mathcal{D}^H)$ by $w(x) = \inf.\{t : ||x_t|| > 2d\}$.

We put $a'(x,\omega,t) = a(\hat{x},\omega,t)$ where $\hat{x}_t = x_{t\wedge w}$

It is easily seen that a' is predictable with respect to the canonical basis B^H (actually, if we put $\hat{w}(x,\omega) = w(x)$, $\hat{w}$ is a stopping time with respect to B^H) ; moreover :

$$a(x,\omega,t).1_{[0,w]}(x) = a'(x,\omega,t).1_{[0,w]}(x) \quad \text{and}$$

$$||a'(x,\omega,t)-a'(x',\omega,t)||^2 \leq L_{2d}(\omega) . \sup_{s<t} ||x_s - x'_s||^2$$

for each pair (x,x') of elements of D^H.

2°/ Now, we can define a process X on the stochastic interval $[u,v']$ such that

$$X_t = X^u + \int_{]u,t]} a'(X,s)\ dZ_s$$

on this same stochastic interval by the classical procedure ; we recall this procedure for the convenience of the reader. We define the sequence $(X^n)_{n>0}$ by the following way :

$$X^{n+1}_t = X^u + \int_{]u,t]} a'(X^n,s).dZ_s$$

If we put $h_n = E\left\{\sup_{u\leq s<v'} ||X^{n+1}_s - X^n_s||^2\right\}$, we have :

$$h_n = E\left\{\sup_{u\leqslant s<v'} \left|\left|\int_{]u,s]} [a'(X^n,r)-a'(X^{n-1},r)].dZ_r\right|\right|^2\right\}$$
(cf. D.5.(i))
$$\leqslant E\left\{\int_{]u,v'[} ||a'(X^n,r) - a'(X^{n-1},r)||^2.dQ_r\right\}$$
(cf. D.5.(ii))
$$\leqslant E\left\{\int_{]u,v'[} \sup_{u\leqslant r<v'} ||X^n - X^{n-1}||^2.L_{2d}.dQ_r\right\}$$
$$\leqslant \frac{1}{2} h_{n-1} \qquad \text{(cf. the definition of v')}$$

Thus we have $h_n \leqslant 2^{-n}.h_o$. That implies that the sequence of processes $(X^n.1_{]u,v[})_{n>o}$ converges almost everywhere, uniformly for each sample function, to a cadlag process X on the stochastic intervall $]u,v'[$; we have the same property on the stochastic intervall $]u,v']$ because

$$X^n_{v'} - X^n_{v'-} = a'(X^{n-1},v').(Z_{v'} - Z_{v'-}) \quad ;$$

then, on the stochastic intervall $]u,v']$, we have :

$$X_t = X^u + \int_{]u,t]} a'(X_s,s).dZ_s$$

At last, we put

$$v = \inf.\{t : t \geqslant u,\ u \leqslant v', ||X_t|| > 2d\}.$$

The process X being right continuous, we have

$$P([v > u]) \geqslant 1 - \varepsilon \ ;$$

but, $a.1_{]o,v]} = a'.1_{]o,v]}$ and that proves the lemma D.7.

D.8 - MAXIMAL SOLUTION

Now, we prove the theorem D.5. Then, we consider the hypothesis given in D.5.

Proof

We consider the family $\mathcal{J}$ of the pairs (v,X) where v is a stopping time and X is a solution of D.5.(v) on $]u,v]$. The set $\mathcal{J}$ is not empty according to lemma D.7. We denote by w the essential supremum of these stopping times v and by $(w(n),X^n)_{n>o}$ a sequence of elements of $\mathcal{J}$ such that $(w(n))_{n>o}$ is a sequence increasing (a.s.) to w ; such a sequence exists because of the following property : if (v',X') and (v",X") are two elements of $\mathcal{J}$,

$$(v'\vee v'',\ X'.1_{]o,v']} + X''.1_{]v',v'']})$$

is also an element of $\mathcal{J}$ (see D.6).

According to the lemma D.6, it is possible to define the process X on $[o,w[$ by $X.1_{]o,w(n)]} = X^n.1_{]o,w(n)]}$

For each integer k, let r(k) be the stopping time defined by $r(k) = \inf.\{t : t \leqslant w \text{ and } ||X_t|| > k\}$. If, for each integer k, $P([r(k) = w \text{ and } w < 1]) = 0$, the theorem D.5 is proved.

Now, we suppose that there exists an integer k such that $P([r(k) = w \text{ and } w < 1]) = 2\varepsilon > 0$. According to the lemma D.7, we can extend the solution $(r(k),X.1_{]o,r(k)]})$ on a stochastic intervall $]o,r(k)\vee r']$ where $P([r'>r(k)]) > P([r(k)<1]) - \varepsilon$

But that implies that $P([r' > w]) > \varepsilon$ and this is impossible by the definition of w.

D.9 - REMARK

Let c be a measurable mapping from $(H \times T)$ into a Banach space K which is continuous with respect to the first variable. For each element (x,ω,t) of $(D^H\times \Omega \times T)$, we put $a(x,\omega,t) = \lim_{s\uparrow t} .c(x_s,t)$ (actually, a does not depend on ω) ; it is easily seen that a is well defined and is a K-valued predictable process with respect to the canonic basis B^H. Thus, this situation is a particular case of the situation studied before.

D.10 - LEMMA

Let $\mathcal{W}$ *be a family of elements of* $\mathcal{L}_\infty(\Omega,\mathcal{F},P)$ *such that* $w\in\mathcal{W}$ *and* $w = w'$ *P.a.e. implies* $w'\in\mathcal{W}$; *for each element* w *of* $\mathcal{W}$, *we suppose that* $0 \leqslant w \leqslant 1$. *Than there exists an increasing family* $(w_n)_{n>o}$ *of elements of* $\mathcal{W}$ *such that, if we have*

$w\in\mathcal{W}$ *and* $w \geqslant \sup_n .w_n$ *P.a.e.*

then $w = \sup_n .w_n$ *P.a.e.*

Proof

Let f be the canonical mapping from $\mathcal{L}_\infty(\Omega,\mathcal{F},P)$ into $L_\infty(\Omega,\mathcal{F},P)$ and $\mathcal{W}'$ be the subset of L_∞ defined by $\mathcal{W}' = f(\mathcal{W})$. On $\mathcal{W}'$ we consider the usual partial order $\leqslant$; according to Zorn Lemma there exists a "maximal" ordered family $(w'_i)_{i\in I}$ of elements of $\mathcal{W}'$; if $b = \sup_{i\in I} E(w'_i)$, there exists an increasing sequence $(w'_n)_{n>o}$ extracted from the previous family such that $b = \sup_{n>o} E(w'_n)$ (there exists a cofinal sequence)

and that proves the lemma (consider a sequence $(w_n)_{n>o}$ such that $w'_n = f(w_n)$ for each integer n).

D.11 - CONDITION FOR NON EXPLOSION

We consider the hypothesis and notations given in the theorem D.5. Moreover, we suppose that the three following conditions are fulfiled :

(i)' for each (strongly) predictable $\mathcal{L}_a(K,H)$*-valued uniformly bounded process* Y, *and for each stoppint time* u, *we have :*

$$E\left\{\sup_{t\leq u}\left\|\int_{]0,t]} Y_s.dZ_s\right\|^2\right\} \leq E\left\{Q_u \int_{]0,u]} \|Y_t\|^2.dQ_t\right\}$$

(i)" Q *is locally integrable, id est there exists an increasing sequence* $(u(n))_{n>0}$ *of stopping times such that* $\lim_{n\to\infty} .P[u(n) < 1] = 0$ *and, for each integer n,* $E(Q_{u(n)}) < +\infty$

(iii) there exists a positive number C *such that, for each element* (x,ω,t) *of* $(D^H \times \Omega \times T)$ *, we have :*

$\|a(x,\omega,t)\|^2 \leq C(1 + \sup_{s<t} \|x_s\|^2)$

Then $P([v = 1]) = 1$ *if* v *is the stopping time considered in the theorem D.5. Moreover, we have the following inequality :*

$E(\|X_1\|^2) \leq E\left\{\sup_{t\leq u} \|X_t\|^2\right\}. 3C\, q.\exp(12Cq)$

where X *is the unique solution as considered in the theorem D.5 and where* $q = E(Q_1 - Q_u)$

Proof

For each stopping time w, we put $X^*_w = \sup_{t\leq w} \|X_t\|^2$. For each $\varepsilon > 0$, there exists a positive number r such that

$$P(F) \geq 1-\varepsilon \quad \text{if } F = \{\omega: X^*_u \leq r\} ;$$

Thus, by considering the process X only on the set $F \times T$, we can suppose that $g_0 < +\infty$ if we put $g_0 = E(X^*_u)$.

Now we consider the set $\mathcal{W}$ of all the stopping times w such that if w' is a stopping time with $w' \leq w$, we have :

$$E(X^*_{w'}.1_{[w'<1]}) \leq E(X^*_u).3Cq_{w'} \exp(12Cq_{w'})$$

where $q_{w'} = E(Q_{w'} - Q_u)$.

According to the lemma D.10, there exists a "maximal" increasing sequence $(w(n))_{n>0}$ of elements of $\mathcal{W}$; if we put $w = \sup_{n>0} w(n)$, we see that $w \in \mathcal{W}$ (Lebesgue theorem).

Now, we suppose that $P([w < 1]) > 0$; then, there exists a positive number d such that $P(F) > 0$ if $F = \{\omega: w(\omega) < 1 \text{ and } d \leq X^*_w \leq 2d\}$ and such that $d \leq E(X^*.1_{[w<1]})$.

Let w' be the stopping time defined by :

if $\omega \notin F$, $w'(\omega) = w(\omega)$

if $\omega \in F$, $w'(\omega) = \inf.\{t : t \geq w(\omega), X^*_t > 4d, t \leq v(\omega)\}$.

On the set $[w' < 1]$, $X^*_{w'} \geq 4d$ (because X is right continuous) then we have :

$$X^*_{w'} \leq X^*_w + 3. \sup_{w\leq t\leq w'} \|X_t - X_w\|^2$$

(because of the inequality $(a+b)^2 \leq a^2 + 3b^2$ if $|b| \geq |a|$).

We put :

$$y = E(X^*_{w'}.1_{[w' < 1]}) \text{ and } x = E(X^*_w.1_{[w < 1]})$$

Then, we have :

$$\begin{aligned} y - x &\leq 3\, E\left\{\sup_{w\leq t\leq w'} \left\|\int_{]w,t]} a(X,s).dZ_s\right\|^2\right\} \\ &\leq 3\, E\left\{\int_{]w,w']} C(1 + \sup_{w\leq t<w'} \|X_t\|^2).dQ_s\right\} \\ &\leq 3\, E\left\{\int_{]w,w']} C(1 + 4d).dQ_s\right\} \\ &\leq 3\, C(1 + 4d)\, E(Q_{w'} - Q_w) \end{aligned}$$

But $4d \leq 4E(X^*.1_{[w<1]}) = 4x$;

thus, we obtain :

$$y \leq x + 3\, C(1 + 4x)\, E(Q_{w'} - Q_w)$$

we have also :

$$x \leq 3C.E(Q_w - Q_u) \exp.\{12C\, E(Q_w - Q_u)\}$$

and that gives :

$$y \leq 3C.E(Q_{w'} - Q_u) \exp.\{12C\, E(Q_{w'} - Q_u)\}$$

Then w' belongs to $\mathcal{W}$: this is impossible because w was an element "maximal" in $\mathcal{W}$: thus $P([w < 1]) = 0$ and that proves the theorem.

E - MARTINGALE AND DOLEANS MEASURE

In all this paragraph E, we consider a probalilized stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in T})$; we shall note $T^\infty = \text{Sup.}\{t : t \in T\}$ and we suppose that T^∞ is an element of T.

E.1. DOLEANS FUNCTION (lemma and definition)

Let X be a process, or a process defined up to a modification, with values in the Banach space H and such that, for each element t of T, X_t is an element of $L_1^H(\Omega, \mathcal{F}, P)$. For each element $A = F \times]s,t]$ of $\mathcal{R}$, we put $x(A) = E\left[1_F.(X_t - X_s)\right]$.

It is easily seen that x can be entended, on a unique way, in a function defined and additive on $\mathcal{A}$.

We shall note d(X) this function and we shall call it the Doléans function of the process X.

Actually, we are chiefly interested in the case where d(X) is σ-additive : in this case, one calls it *the Doléans measure of the process X.*

The following lemma is fundamental to possibly prove that it is so.

E.2. LEMMA (sufficient condition to have an outer "Doléans measure")

Let v be a positive function defined on $\mathcal{A}$ which satisfies the following three properties :

(i) for each pair (A,B) of elements of $\mathcal{A}$,
$v(A) \leq v(A \cup B) \leq v(A) + v(B)$

(ii) for each element s of T, $\lim_{t \downarrow s} v(\Omega \times]s,t]) = 0$

(iii) for each increasing sequence $(u(n))_{n>o}$ of simple stopping times such that $\lim_{n\to\infty} P\left[u(n) < T^\infty\right] = 0$, *we have :*
$\lim_{n\to\infty} v(]u(n), T^\infty]) = 0$

Then, the following property is fulfiled :

(iv) for each sequence $(A_n)_{n>o}$ of elements of $\mathcal{A}$ such that $A_n \downarrow_{n\to\infty} \emptyset$, we have $\lim_{n\to\infty} v(A_n) = 0$.

Proof

1°) Let $(A_n)_{n>o}$ be a sequence of elements of $\mathcal{A}$ such that $A_n \downarrow_{n\to\infty} \emptyset$; we put $a = \frac{1}{4} \lim_{n\to\infty} v(A_n)$; we suppose that a > o and we shall prove that there is an impossibility.

2°) For each integer n, let $(B(n,k))_{1\leq k\leq b(n)}$ be a finite partition of A_n such that, for each integer k, B(n,k) is an element of $\mathcal{R}$ (cf. A-6).

For each pair (n,k) of integers, we have $B(n,k) = F(n,k) \times]s(n,k), t(n,k)]$.

Let s'(n,k) be an element of T such that $s(n,k) < s'(n,k) < t(n,k)$ and

$$v(]s(n,k), s'(n,k)]) \leq a.2^{-n}.\frac{1}{b(n)}$$

For each integer n, we put :

$$C(n) = \bigcup_{k=1}^{b(n)} (F(n,k) \times]s'(n,k), t(n,k)])$$

$$\bar{C}(n) = \bigcup_{k=1}^{b(n)} (F(n,k) \times [s'(n,k), t(n,k)])$$

$$D(n) = \bigcap_{k=1}^{n} C(k) \quad , \quad \bar{D}(n) = \bigcap_{k=1}^{n} \bar{C}(k)$$

$$S(n) = \bigcup_{k=1}^{b(n)} (\Omega \times]s(n,k), s'(n,k)])$$

We have :

$A(n) \subset \{S(n) \cup C(n)\}$.

If we remember that $A(n) \downarrow \emptyset$, we have

$$A(n) \subset \left\{D(n) \cup \left\{\bigcup_{i=1}^{n} S(i)\right\}\right\}$$

and that implies $v\left[A(n)\right] \leq v\left[D(n)\right] + a$.

3°) Moreover, for each integer n, $\bar{C}(n)$ being contained in A(n), we have $\bar{D}(n) \downarrow_{n\to\infty} \emptyset$.

For each integer n, let u(n) be the simple stopping time which is the "beginning" of the set $\bar{D}(n)$, i.e.

$$\left[u(n)\right](\omega) = \inf.\{t : t \in T, (\omega,t) \in \bar{D}(n)\}$$

Let ω be an element (fixed) of Ω and let $\bar{D}(n,\omega)$ the compact subset of Tdefined by :

$$\bar{D}(n,\omega) = \{t : t \in T, (\omega,t) \in \bar{D}(n)\}$$

We have (ω being fixed), $\bar{D}(n,\omega) \downarrow \emptyset$; then there exists an integer k such that $\bar{D}(k,\omega) = \emptyset$ (property of the compact sets) ; that means that $\left[u(k)\right](\omega) = T^\infty$;

thus, we have proved that

$$[u(n) < T\infty] \underset{n\to\infty}{\downarrow} \emptyset$$

Then, there exists an integer j such that

$$v\,(]u(j),\ T\infty]) \leq a \qquad \text{(cf. (iii))}$$

we have :

$$\begin{aligned} v\,[A(j)] &\leq v\,[D(j)] + a \qquad \text{(cf. 2°) above)} \\ &\leq v\,(]u(j),T\infty]) + a \\ &\leq 2a \end{aligned}$$

and this is impossible by the definition of a.

E.3. REMARKS

1°) In this paragraph E, we shall use the lemma above for an additive function v : in this case, if the conditions E.2-(i), (ii) and (iii) are satisfied, v is a Doléans measure.

2°) The proof of this lemma E.2, is a natural generalization of the associated basic lemma when Ω has only one element (deterministic case).

E.4. EXISTENCE OF A CADLAG MODIFICATION (theorem)

Let X be an adapted process defined up to modification, with values in a finite dimensional vector space H and right continuous in probability (i.e., for each element s of T and for each $\varepsilon > 0$, $\lim_{t\downarrow s} .P\,[\,||X_t - X_s|| > \varepsilon\,] = 0$). *We suppose that X satisfies one of the following two properties :*

(i) for each element t of T, X_t *is an element of* $L_1^H(\Omega,\mathfrak{F},P)$ *and the set* $\{z : z = [d(X)](A),\ A \in \mathcal{R}\}$ *is bounded in H, i.e. there exists a real number a such that, for each element A of* $\mathcal{R}$, $||\,[d(X)]\,(A)\,|| \leq a$.

(ii) the set $\{z : z = \int 1_A.dX, A \in \mathcal{R}\}$ *(this integral being defined as in B-2) is bounded (in the Bourbaki sense) in* $L_0^H(\Omega,\mathfrak{F},P)$.

Then there exists a process Y, defined up to indistinguishability, which is a modification of X.

Proof

1°) It is sufficient to consider the case where $T = [0,1]$. It is also sufficient to consider the case where X is a real process (look at the projections on a base of H).

The condition (ii) is the same as the following one :

(ii)' there exists a positive decreasing function f defined on $\mathbb{R}^+$ such that $\lim_{x\to\infty} f(x) = 0$ and, for each element A of $\mathcal{R}$ and for each real strictly positive number d, we have :

$$P(\ |\int 1_A.dX| > d\ ) \leq f(d)$$

Let Q' be the set of the rational numbers belonging to T. For each element t of Q', we put $Z_t = X_t$.

At first, we shall prove that the process $(Z_t)_{t\in Q'}$ is ladlag.

Let (a,b) a pair of rational numbers with a < b.

2°) Let S be a finite part of Q' ; let $\{t(k)\}_{1\leq k\leq n}$ be the increasing sequence of the elements of S. Let $\{u(k)\}_{1\leq k\leq 2n}$ be the associated family of simple stopping times defined by recurrence by u(1) = 0 and :

$$u(2k+1) = \inf.\{\,t : t\in S,\ t \geq u(2k), Z_t \geq b\,\}$$
$$u(2k) = \inf.\{\,t : t\in S,\ t \geq u(2k-1), Z_t \leq a\,\}$$

and u(j) = 1 if the sets above are void.

Let A(j,S) be the domain where the process $(Z_t)_{t\in S}$ has more than (j-1) upcrossings of the interval $[a,b]$; if $\omega\in\Omega$, we have

- either $\omega \notin A(j,S)$ and this implies

$$\sum_{k=1}^{j} [Z_{u(2k+1)} - Z_{u(2k)}] \geq j.(b-a)$$

- or $\omega\in A(j,S)$ and this implies

$$\sum_{k=1}^{j} [Z_{u(2k+1)} - Z_{u(2k)}] < -(Z_1 - a)^-$$

(we have $Z_{u(2k+1)} - Z_{u(2k)} < 0$ only if $Z_{u(2k)} < a$ and $Z_{u(2k+1)} = X_1$).

Then, we have :

$$\sum_{k=1}^{j} [Z_{u(2k+1)} - Z_{u(2k)}] \geq j(b-a).1_{A(j,S)} - (Z_1 - a)^-.1_{\Omega\setminus A(j,S)}$$

If the condition (i) is fulfiled, we put :

$$C_j = \frac{1}{j(b-a)}\,[a + E\,(|z_1 - a|)]$$

If the condition (ii) (i.e. (ii)') is fulfiled, we put :

$$C_j = f\,[j(b-a)]$$

In all the cases, we have
$P\,[A(j,S)] \leq C_j$ and $\lim_{j\to\infty} . C_j = 0$.

3°) Now, we consider an increasing sequence $(S(n))_{n>o}$ of finite parts of Q' such that $Q' = \bigcup_{n>o} S(n)$. Let $A(j,Q')$ the domain where the process $(Z_t)_{t\in Q'}$ has more than (j-1) upcrossings of the interval $[a,b]$; we have $A(j,S(n)) \uparrow A(j,Q')$, and this implies $P\,[A(j,Q')] \leq C_j$. Thus, if $A(Q')$ is the domain where the process $(Z_t)_{t\in Q'}$ has an infinity of upcrossigns of the interval $[a,b]$, we have $P\,[A(Q')] = 0$.

4°) If we consider the family of all the pairs of rational numbers a and b, we see that, with the probability one, there do not exist two different real number c and d such that X upcrosses infinitely the interval $[c,d]$.

But, a classical result says that this is equivalent to say that the process X is a.e. ladlag.

5°) Then, for each element (ω,t) of $([\Omega \setminus A(Q')]\times T)$, we can put

$Y_t(\omega) = Z_{t+}(\omega) = \lim_{s\downarrow t} . Z_s(\omega)$

Let t be an element of T and $\{t(k)\}_{k>o}$ be a sequence of elements of Q' decreasing to t ; the sequence of random variables $(Y_{t(k)})_{k>o}$ converges a.e. to Y_t (by the definition of Y_t) and in probability to X_t ; then $Y_t = X_t$ a.e. and Y is a modification of X.

E.5. MARTINGALE (definition and lemma)

Let X be a process, or a process defined up to modification, with values in the Banach space H and such that, for each element t of T, X_t belongs to $L_1^H(\Omega,\mathfrak{F}_t,P)$. One says that X is a martingale if the associated Doléans function is identically null.

It is easily seen that this condition is equivalent to say that, for each pair (s,t) of elements of T, with s < t, we have :

$$E(X_t \mid \mathfrak{F}_s) = X_s \qquad \text{a.e.}$$

More generally, if u and v are two simple stopping times, we have $X_u = E(X_v \mid \mathfrak{F}_u)$ if $u \leq v$ (cf. A-2 for the definition of $\mathfrak{F}_u$ and A-7).

If X is a real process, one says that X is a supermartingale (resp. a submartingale) if the Doléans function is negative (resp. positive).

E.6. EXAMPLE OF SUBMARTINGALE (proposition)

Let M be a martingale, defined up to modification, with values in the Banach space H. Let f be a convex real positive function defined on the real line. Let X be the real process defined up to modification by $X_t = f(\|M_t\|)$.

Then, X is a submartingale .

Proof

Applying the definition above, we see that this proposition is a corollary of the following inequality :

Jensen inequality :

Let Y be an element of $L_1(\Omega,\mathfrak{F},P)$ and $\mathfrak{G}$ a sub-σ-algebra of $\mathfrak{F}$; let f be a convex real positive function defined on the real line ; we have :

$$f\,[E(Y \mid \mathfrak{G})] \leq E\,[f(Y) \mid \mathfrak{G}]$$

This inequality is obvious if $f(x) = ax+b$; thus, we have the same inequality in the general case because a convex function is the supremum of a family $(f_n)_{n>o}$ of functions such that

$f_n(x) = a_n\,x + b_n$

E.7. EQUI-INTEGRABILITY

Let H be a Banach space.

Let $(A_n)_{n>o}$ be a family of elements of $L_1^H(\Omega,\mathfrak{F},P)$. One says that this family is equi-integrable if, for each $\varepsilon > o$, there exists $\eta > o$ such that $P(F) \leq \eta$ implies (for each integer n), $E(1_F.\|A_n\|) \leq \varepsilon$.

It is well known and easy to verify that an equi-integrable sequence $(A_n)_{n>o}$ of random variables which converges a.e. to a random variable A converges also to A in $L_1^H(\Omega,\mathfrak{F},P)$.

Moreover, let A be an element of $L_1^H(\Omega,\mathfrak{F},P)$ and $(\mathfrak{G}_n)_{n>o}$ be a family of sub-σ-algebras of $\mathfrak{G}$; if we put $A_n = E(A \mid \mathfrak{G}_n)$, the sequence $(A_n)_{n>o}$ is equi-integrable.

E.8. THEOREM

Let X be a cadlag process, with values in the Banach space H, such that, for each element t of T, X_t belongs to $L_1^H(\Omega,\mathfrak{F},P)$. We suppose that the Doléans function d(X) of X is σ-additive. We suppose also that, for each decreasing sequence $(u(n))_{n>o}$ of simple stopping times, the associated sequence $(X_{u(n)})_{n>o}$ of random variables is equi-integrable.

Let u be a stopping time. Let X' be the process X stopped at u (i.e. $X'_t = X_{t\wedge u}$). Then, for each element B of $\mathcal{R}$, we have :

$$[d(X')](B) = [d(X)]\ (B \cap\ ]o,u])$$

Specially, if X is a cadlag martingale and if u is a stopping time, the process X stopped at u is also a martingale : in this case, we have :

$$E(X_u \mid \mathfrak{F}_t).1_{]o,u]} = E(X_{T\infty} \mid \mathfrak{F}_t).1_{]o,u]}$$

Proof

1°) The proof of the first part of the theorem is easy when u is a simple stopping time.

2°) We consider any stopping time u and a process X fulfiling the properties given in the beginning of the theorem. Let $(u(n))_{n>o}$ be a sequence of simple stopping times which decreases to u (cf. the end of the proof of A-9). Let $B = F \times\ ]s,t]$ an element of $\mathfrak{R}$; we have :

$$[d(X')](B) = E\ [1_F.(X_{t\wedge u}-X_{s\wedge u})]$$

$$= \lim_{n\to\infty} E\ [1_F.(X_{t\wedge u(n)}-X_{s\wedge u(n)})]$$

because the sequence $(X_{t\wedge u(n)})_{n>o}$ is equi-integrable and converges a.e. to $X_{t\wedge u}$

$$= \lim_{n\to\infty} [d(X)]\ (B \cap\ ]o,u(n)])$$

$$= [d(X)]\ (B \cap\ ]o,u])$$

3°) Now, we suppose that X is a cadlag martingale. For each simple stopping time u(n), we have (cf. E.5)

$$X_{u(n)} = E(X_{T\infty} \mid \mathfrak{F}_{u(n)})$$

then, the family $(X_{u(n)})_{n>o}$ is equi-integrable ; thus, we can use the first part of the theorem and we have d(X') = 0 ; that means that X' is a martingale.

E.9. DOLEANS MEASURE FOR A SUB-MARTINGALE

Let X be a real positive process defined up to a modification and fulfiling the following three conditions :

(i) for each element t of T, X_t belongs to $L_1(\Omega,\mathfrak{F},P)$

(ii) $(X_t)_{t\in T}$ is a sub-martingale

(iii) X is right continuous in mean, i.e., for each element s of T, we have :
$$\lim_{t\downarrow s} E(|X_t-X_s|) = 0$$

Then the Doléans function x of X is σ-additive.

Proof

We can suppose that T = [0,1]. The Doléans function x of X is positive (cf. (ii)) and additive ; then, it is sufficient to prove the condition E.2-(iii).

Let $(u(n))_{n>o}$ be a sequence of simple stopping times such that $[u(n) < 1] \downarrow_{n\to\infty} \emptyset$. We have :

$$x(]u(n),1]) = E\ [X_1-X_{u(n)}]$$

$$\leq E\ [(X_1-X_{u(n)})^+] \leq E\ [X_1.1_{A(n)}]$$ and

that implies $\lim_{n\to\infty} x(]u(n),1]) = 0$.

E.10. LEMMA

Let s and t be two elements of T with s < t. Let H be an Hilbert space. Let X_s and X_t be an element of $L_2^H(\Omega,\mathfrak{F}_s,P)$ and of $L_2^H(\Omega,\mathfrak{F}_t,P)$ respectively. We suppose that $E(X_t \mid \mathfrak{F}_s) = X_s$ a.e.

Let Z be an element of $L_2^H(\Omega,\mathfrak{F}_s,P)$. Then we have :

1°) The random variables Z and (X_t-X_s) are orthogonal in $L_2^H(\Omega,\mathfrak{F},P)$

2°) $E(||X_t-X_s||_H^2) = E(||X_t||_H^2 - ||X_s||_H^2)$

Proof

We note <.,.> the scalar product in H.

1°) $E(\langle Z,X_t-X_s\rangle) = E(\langle Z,E[(X_t-X_s) \mid \mathfrak{F}_s]\rangle)$

because Z is $\mathfrak{F}_s$-measurable

$= 0$.

2°) $b = E(||X_t - X_s||_H^2) = E(<X_t - X_s,\ X_t - X_s>)$

$= E(<X_t, X_t>) - 2.E(<X_s, X_t>) + E(<X_s, X_s>)$

Now, $E(<X_s, X_t>) = E(<X_s,\ E(X_t \mid \mathcal{F}_s)>)$

because X_s is $\mathcal{F}_s$-measurable ; then

$E(<X_s, X_t>) = E(<X_s, X_s>)$ and

$b = E(<X_t, X_t>) - E(<X_s, X_s>)$

$= E(||X_t||_H^2) - E(||X_s||_H^2)$

E.11 - SQUARE INTEGRABLE MARTINGALE :

Let H be an Hilbert space. Let M be an H-valued martingale defined up to modification We suppose that M is a square integrable martingale, i.e. for each element t of T, $E(||M_t||_H^2) < +\infty$. We suppose also that M is right continuous in quadratic mean, i.e., for each element s of T, $\lim_{t \downarrow s} E(||M_t - M_s||^2) = 0$.

Let v the Doléans function of the process N defined up to a modification by $N_t = ||M_t||^2$. Then, we have :

1°) v is positive and σ-additive.

2°) let J and K be two Hilbert spaces ; we consider a bilinear continuous mapping from (J × H) into K which, to (y,x) element of (J × H) associates y.x element of K. We suppose that $||y.x||_K \leqslant ||y||_J.||x||_H$. Let Y be an $\mathcal{A}$-simple J-valued process. Then, we have :

$E(||\int Y.dM||_K^2) \leqslant \int ||Y||_J^2.dv$ (cf. B-3).

3°) The mapping which, to $Y \in L_2^{\mathbb{R}}(\Omega', \mathcal{P}, v)$, associates the random variable stochastic integral $\int Y.dM$ is an isometry from $L_2^H(\Omega', \mathcal{P}, v)$ into $L_2^H(\Omega, \mathcal{F}, P)$.

4°)M is a process of finite quadratic variation.

Proof :

1°) N is a sub-martingale (cf. E-6). Moreover, if s belongs to T and if t is decreasing to s, $\lim_{t \downarrow s} E(N_t - N_s) = 0$ (cf. E.10-2°)).

Then, we can use E.9 and this proves the 1°).

2°) For the notations, see the paragraph B.

Let Y be an $\mathcal{A}$-simple J-valued process. We have :

$Y = \sum_{i \in I} a_i.1_{A(i)}$ where, for each element i of the finite set I, a_i belongs to J and $A(i) = F(i) \times]s(i), t(i)]$ belongs to $\mathcal{R}$; we can also suppose that the sets $(A(i))_{i \in I}$ are disjoint. If $i \neq j$, we have $F(i) \cap F(j) = \emptyset$ or $t(i) \leqslant s(j)$ or $t(j) \leqslant s(i)$; then the random variables $(\int a_i.1_{A(i)}.dM)_{i \in I}$ are orthogonal in $L_2^K(\Omega, \mathcal{F}, P)$.

Then, we have :

$$E(||\int Y.dM||_K^2) = \sum_{i \in I} E(||\int a_i.1_{A(i)}.dM||_K^2)$$

$$\leqslant \sum_{i \in I} ||a_i||^2.E(1_{F(i)}.||M_{t(i)} - M_{s(i)}||^2)$$

$$\leqslant \sum_{i \in I} ||a_i||^2.E\{1_{F(i)}.(||M_{t(i)}||^2 - ||M_{s(i)}||^2)\}$$

(cf. E.10-2°)

$$\leqslant \int ||Y||_J^2 . dv$$

3°) If $||x.y||_K = ||x||_H.||y||_J$, the inequality above becomes an equality and this proves the isometry.

4°) At first, we consider the case where M is a real martingale. By the 2°) above, we can use C.6- 3°) and the quadratic variation V of M is the cadlag process V defined by $V_t = M_t^2 - M_o^2 - 2\int_{]o,t]} M_{s-}\, dM_s$

and $E(V_t) = E(M_t^2) - E(M_o^2)$.

Now, if M is an H-valued martingale, then M takes its values in separable subset H_o of H ; let $(h_n)_{n>o}$ be an orthonormal basis of H_o ; for each integer n, let M^n be the real martingale such that $M = \sum_{n>o} M^n h_n$; let V^n be the quadratic variation of M^n ; then, we have $V = \sum_{n>o} V^n$ and $E(V_t) = E(||M_t - M_o||^2)$.

E.12 - A DOOB INEQUALITY (PROPOSITION)

Let p and q be two real positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, $(1<p<+\infty)$. Let $(X_t)_{t \in [o,1]}$ be a real positive right continuous (for each sample function) sub-martingale such that $E(X_1^p) < +\infty$. We put : $Y(\omega) = \operatorname{Sup}_{t \in T} X_t(\omega)$. Then we have : $E(Y^p) \leqslant q^p.E(X_1^p)$.

Proof

Let Q' be the set of the rationnal numbers belonging to $[o,1]$; then, we have $Y = \operatorname{Sup.}_{t\in Q'} X_t$ and that proves that Y is $\mathcal{F}$-measurable.

Now, let d be a real number with $d > 1$.

For each n, let v(n) be the stopping time defined by $v(n) = \inf.\{ t : X_t > d^n \}$ (and, of course, $v(n)=1$ if this set is void). We note $A(n) = [Y \geqslant d^n]$ and $a_n = P[A(n)]$. We have :

$$E(X_1) \geqslant E(X_{v(n)}) \geqslant d^n.a_n + \int_{\Omega\setminus A(n)} X_1.dP$$

and $X_{v(n)}.1_{\Omega\setminus A(n)} = X_1.1_{\Omega\setminus A(n)}$ then

$$a_n \leqslant d^{-n}. \int_{A(n)} X_1.dP$$

Moreover :

$$E(Y^p) \leqslant d^p. \sum_{n\in\mathbb{Z}} (a_n - a_{n+1}).d^{np}$$

$$\leqslant d^p \sum_{n\in\mathbb{Z}} a_n. [d^{np}-d^{(n-1)p}]$$

Now, if we use the inequality obtained above, we have :

$$E(Y^p) \leqslant d^p. \sum_{n\in\mathbb{Z}} d^{-n}.(\int_{A(n)} X_1.dP).(d^{np}-d^{(n-1)p})$$

If we put $B(n) = A(n) \setminus A(n+1)$, we have :

$$E(Y^p) \leqslant d^p. \sum_{n\in\mathbb{Z}} . \sum_{k\geqslant n} (\int_{B(k)} X_1.dP).d^{-n}.(d^{np}-d^{(n-1)p})$$

$$\leqslant d^p \sum_{k\in\mathbb{Z}} \sum_{n\leqslant k} (\int_{B(k)} X_1.dP).d^{-n}.(d^{np}-d^{(n-1)p})$$

But, we have also :

$$\sum_{n\leqslant k} d^{-n}.(d^{np}-d^{(n-1)p}) \leqslant \int_o^{d^{kp}} x^{-1/p}\,dx = \frac{p}{p-1} . d^{k(p-1)}$$

then :

$$E(Y^p) \leqslant \frac{p}{p-1} . d^p. \sum_{k\in\mathbb{Z}} .(\int_{B(k)} X_1.d^{k(p-1)}.dP)$$

but, on B(k), $d^k \leqslant Y$, then we have :

$$E(Y^p) \leqslant \frac{p}{p-1} . d^p.\int_\Omega X_1. Y^{p-1}. dP$$

But, this is true for each real number such that $d > 1$; then we have the same inequality for $d = 1$, i.e. :

$$E(Y^p) \leqslant \frac{p}{p-1} .\int_\Omega X_1 . Y^{p-1} . dP$$

Now, if we use the Holder inequality, we obtain :

$$E(Y^p) \leqslant \frac{p}{p-1} [E(X_1^p)]^{1/p}. [E(Y^{q(p-1)})]^{1/q}$$

but $q(p-1) = p$, then, we have :

$$E(Y^p) \leqslant \frac{p}{p-1} [E(X_1^p)]^{1/q}. [E(Y^p)]^{1/q}$$

At first, we suppose that $E(Y^p) < +\infty$; in this case, we obtain :

$$[E(Y^p)]^{1-1/q} \leqslant \frac{p}{p-1} [E(X_1^p)]^{1/p}$$

but $1-1/q = 1/q$, then we have :

$$E(Y^p) \leqslant (\frac{p}{p-1})^p.E(X_1^p)$$

Now, if we have $E(Y^p) = +\infty$, we consider the cadlag martingale X(n) defined by :

$X(n)= E[(X_1 \wedge n) | \mathcal{F}_t]$ and we put $Y(n) = \operatorname{Sup.}_{t\in T} X_t(n)$;

by the previous proof, we have :

$$E(Y^p(n) \leqslant q^p . E [(X_1 \wedge n)^p]$$

but, $Y \leqslant \operatorname{Sup.}_{n>o} Y(n)$, thus (Lebesgue theorem) :

$$E(Y^p) \leqslant q^p . E(X_1^p)$$

Exceptionnally, in the two following propositions, we suppose that the set T is open on the right.

E.13 - CONVERGENCE OF A SUB-MARTINGALE (PROPOSITION)

Let $(X_t)_{t\in[o,1[}$ be a sub-martingale defined up to modification and such that $\operatorname{Sup.}_{t\in[o,1[} E(|X_t|)=K<+\infty$. Then, there exists a cadlag process $(Y_t)_{t\in[o,1[}$ which is a modification of X and there exists a random variable Z such that $Z = \lim._{t\uparrow 1} Y_t$.

Moreover, if the values of X are negative, for each element t of T, we have $E(Z | \mathcal{F}_t) \geqslant X_t$ p.s.

Proof

We put $X_1 = O$; let x be the Doleans function of the process $(X_t)_{t\in[o,1]}$; let A be an element of $\mathcal{A}$; there exists a partition (B,C) of A and an element t of T with $t < 1$, such that $B \subset]o,t]$ and $C = H\times]t,1]$ with $H\in\mathcal{F}_t$ (consider the last time t where the value of 1_A changes).

We have $o \leq x(B) \leq x(]o,t]) \leq K$;
Moreover, $x(C) = E\left[1_H.(X_1 - X_t)\right] = -E\left[1_H.X_t\right]$,
then $|x(C)| \leq K$; then, the Doleans function of X is bounded and it is possible to apply the theorem E.4 to the process $(X_t)_{t\in[o,1]}$ and that proves the first part of the theorem.

Now, we suppose that X is a negative sub-martingale. Let t be an element of T ; we put $Y_n = Y_{1-1/n}$. We have :

$E(Z|\mathfrak{F}_t) \geq E(Y_n|\mathfrak{F}_t) \geq X_t$ for $1-1/n \geq t$.

But, $\{E(Y_n|\mathfrak{F}_t)\}_{n>o}$ is an increasing family of random variables, then (for $1-\frac{1}{n} \geq t$), we have (Fatou lemma) : $E(Z|\mathfrak{F}_t) \geq E(Y_n|\mathfrak{F}_t) \geq X_t$.

E.14 - <u>CONVERGENCE OF A MARTINGALE</u> (PROPOSITION)

Let H be a finite dimensionnal vector space. Let $(X_t)_{t\in[o,1[}$ an H-valued martingale defined up to modification. We suppose that the family $(X_t)_{t\in[o,1[}$ is equi-integrable. Then, there exists a cadlag process Y, which is a modification of X, and a random variable Z such that :

(i) $Z = \lim_{t\uparrow 1} Y_t$

(ii) for each element t of T, $X_t = E(Z|\mathfrak{F}_t)$ a.s.

<u>Proof</u>

It is sufficient to consider the case where X is a real process. The equi-integrability implies

$$\sup_{t\in[o,1[} E(|X_t|) = K < +\infty.$$

Thus, we can use the previous proposition E.13, and that permits us to define Y and Z. Now, the equi-integrability implies that the sequence $(Y_{1-1/n})_{n>o}$ converges in mean to Z when n goes to the infinity.

F - MEYER PROCESS AND DECOMPOSITION THEOREMS

F.1 - GENERALITIES

In this paragraph F, we suppose that $T = [0,1]$ and we consider a complete probabilized stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in T})$: if we say that X is an adapted process (or a martingale, or a predictable process, and so on ...), that means that X is adapted with respect to this stochastic basis.

Moreover, we suppose that the family $(\mathcal{F}_t)_{t\in T}$ is right continuous. If F is an element of $\mathcal{F}$, we note $E(1_F|\mathcal{F}_{t-})$ the left continuous martingale M such that $M_t = E(1_F|\mathcal{F}_{t-})$ for each element t of T.

F.2 - PREDICTABLE STOPPING TIME AND $\mathcal{F}_{u-}$ (Definitions)

Let u be a stopping time. One says that u is predictable if there exists a sequence $(u(n))_{n>0}$ of stopping times increasing to u and such that, for each (n,ω), $[u(n)](\omega) < u(\omega)$. In this case, one notes $\mathcal{F}_{u-}$ the σ-algebra generated by the σ-algebras $(\mathcal{F}_{u(n)})_{n>0}$ and one says that the sequence $(u(n))_{n>0}$ is "announcing" u.

If u is a predictable stopping time, the stochastic intervall $]0,u[$ is predictable because $]0,u[= \bigcup_{n>0}]0,u(n)]$. In the same way, $[u]$ is a predictable set. Moreover, let Z be an $\mathcal{F}_{u-}$-measurable random variable : then $Z.1_{[u,1]}$ is a predictable process ; indeed, if the sequence of random variables $(Z^n)_{n>0}$ converges to Z and, if, for each integer n, Z^n is $\mathcal{F}_{u(n)}$-measurable, then the sequence of predictable processes $(Z^n.1_{]u(n),1]})_{n>0}$ converges to the process $Z.1_{[u,1]}$.

F.3 - DOLEANS MEASURE (Definition)

We say that a is a Doléans measure if a is a real measure defined (thus finite) on the σ-algebra of the predictable sets and such that $a(B) = 0$ for all the evanescent sets B (i.e. the sets B such that 1_B is indistinguishable of 0).

In E.2, we gave some conditions such that a is a Doléans measure ; notably, if A is a right continuous increasing process, with $E(A_1-A_0) < +\infty$, the Doléans function $a = d(A)$ associated to A (i.e. the function a defined by $a(F\times]s,t]) = E[1_F.(A_t-A_s)]$ for each element $F\times]s,t]$ of $\mathcal{R}$) can be extended in a Doléans measure : this is a special case of E.9 and can be directly proved with the Fubini theorem. *In this case, we say that a is the Doléans measure associated to A.*

In this paragraph, if a is a Doléans measure, we construct a predictable increasing process A such that $a = d(A)$ (cf. F.12). A fundamental step of this study is the "projection lemma" F.8.

F.4 - TOTALLY INACESSIBLE STOPPING TIME (lemma and definitions)

Let u be a stopping time. The two following properties are equivalent :

(i) $P[w=u$ and $u < 1] = 0$ for each predictable stopping time w.

(ii) For each sequence $(v(n))_{n>0}$ of stopping times increasing to a stopping time v, the sequence of the sets $([v(n)\geq u$ and $u<1])_{n>0}$ is, P-a.s., increasing to the set $[v\geq u$ and $u<1]$

If these properties are satisfied, one says that u is a totally inaccessible stopping time.

Attention : With this definition, the stopping time u=1 is predictable and totally inacessible.

Proof

At firt , we suppose that the condition (ii) is satisfied ; let w be a predictable stopping time and $(v(n))_{n>0}$ be a sequence of stopping times which is announcing w;we have :

$$[v(n)\geq u \text{ and } u<1] \uparrow_{n\to\infty} [v\geq u \text{ and } u<1] \quad P\text{-a.s.}$$

and

$$[v(n)\geq u \text{ and } u<1] \uparrow_{n\to\infty} [w>u \text{ and } u<1]. \quad P\text{-a.s.}$$

thus

$P([w=u$ and $u<1]) = 0.$

Now, let $(v(n))_{n>0}$ a sequence of stopping times which is increasing to v.

For each integer n, we put :

$v'(n) = v(n)$ if $v(n) < v$

and

$v'(n) = 1$ if $v(n) = v$

It is easily seen that $v'(n)$ is a stopping time. The sequence $(v'(n))_{n>o}$ of stopping times is increasing to the predictable stopping time v' and $[v(n) \geq u \text{ and } u < 1] \underset{n\to\infty}{\uparrow} ([v \geq u] \setminus [v' = u])$.

If the condition (i) is satisfied, we have $P([v' = u]) = 0$: thus the condition (ii) is satisfied.

F.5 - DECOMPOSITION OF A STOPPING TIME (lemma)

Let u be a stopping time. Then there exists a sequence $(v_n)_{n>o}$ of predictable stopping times and a totally inacessible stopping time w such that :

$[u] \subset \{[w] \cup (\bigcup_{n>o} [v_n])\}$. *Moreover, it is possible to suppose that $P[v_j = v_k < 1] = 0$ for each pair (j,k) of integers. We have also $P[v_j = w < 1] = 0$ for each integer j.*

Proof

Let a be the supremum of the positive numbers b such that there exists a sequence $(u_n)_{n>o}$ of predictable stopping times with

$$b = P\{\omega : \exists n,\ u_n(\omega) = u(\omega)\}$$

This supremum a is reached for a sequence $(v_n)_{n>o}$ of stopping times. Let w be the random variable defined by :

$$w(\omega) = \begin{cases} u(\omega) & \text{if, } \forall n,\ u(\omega) \neq v_n(\omega) \\ 1 & \text{if } \exists n \text{ with } u(\omega) = v_n(\omega) \end{cases}$$

It is easily seen that w is a stopping time and w is totally inaccessible.

To have the last property, it is sufficient to consider the sequence $(v'_n)_{n>o}$ of stopping times defined by :

$$v'_n(\omega) = \begin{cases} v_n(\omega) & \text{if, } \forall k < n,\ v_n(\omega) \neq v_k(\omega) \\ 1 & \text{if } \exists k < n,\ v_n(\omega) = v_k(\omega) \end{cases}$$

F.6 - MEYER PROCESS AND NOTATION $\mathcal{C}$ (Definitions)

1°) Let A be an increasing (real) process ; we say that A is a Meyer process if $A_o = o$, $E(A_1) < +\infty$ and A is a predictable right continuous process.

2°) We note $\mathcal{C}$ the set of the processes A which satisfy the following properties :

(i) A is real increasing and right continuous

(ii) $A_o = o$ and $E(A_1) < +\infty$

(iii) For each element F of $\mathfrak{F}$ and for each stopping time u, we have :

$$E(1_F . A_u) = \int_{]o,u]} E(1_F | \mathfrak{F}_{t-})\, da$$

where a is the Doléans measure associated to the process A.

If A is an element of $\mathcal{C}$, if u is a stopping time, the process A^u, i.e. the process A stopped at the stoppingtime u, is also an element of $\mathcal{C}$ and the Doléans measure a^u associated to A^u is defined by $a^u(B) = a(]o,u] \cap B)$ for each predictable set B. Of course, we have the same property if A is a Meyer process.

We shall see in F.12 that the conditions given in the 1°) and in the 2°) are equivalent.

F.7 - CONSTRUCTION OF A (Proposition)

Let a be a Doléans measure. Then there exists an element A of $\mathcal{C}$ such that a is the Doléans measure of A ; This process A is unique up to indistinguishability.

Proof

1°) A is unique up to modification by the condition F.6 (iii) ; then A is unique up to indistinguishability because A is right continuous.

2°) Let t be an element of T ; for each element H of $\mathfrak{F}$, we put :

$$v_t(H) = \int_{]o,t]} E(1_H | \mathfrak{F}_{s-})\, .\, da$$

The function $v_t(.)$ is σ-additive (Lebesgue theorem) and dominated by P ; then we can put $V_t = \frac{dv_t}{dP}$ and this defines, up to modification, an increasing process V right continuous in mean. Let A be a right continuous process (in the strict sense) which is a modification of V.

3°)Let Y be an element of $L_\infty(\Omega, \mathfrak{F}, P)$; we have :

$$\int_{]o,t]} E(Y | \mathfrak{F}_{s-}) . da = \int_\Omega Y . v_t(d\omega)$$

Indeed, we have this equality if $Y = 1_F$ with $F \in \mathfrak{F}$

(by the definition of v) ; then, we have this equality in the general case by linearity and density.

4°) A is an adapted process, indeed :

$$v_t(H)=\int_{]o,t]} E(1_H|\mathcal{F}_{s-}).da=\int_{]o,t]} E\left[E(1_H|\mathcal{F}_t)|\mathcal{F}_{s-}\right].da$$

$$=\int_{\Omega} E(1_H|\mathcal{F}_t).v_t(d\omega) \qquad \text{(cf. the 3°) above)}$$

5°) For each element (F,t) of $\mathcal{F} \times T$, we have :

$$E(1_F.A_t) = v_t(F) = \int_{]o,t]} E(1_F|\mathcal{F}_{s-}).da$$

Then the property F.6 (iii) is satisfied if u is a simple stopping time ; thus, we have this same property if u is a general stopping time because such a stopping time is the limit of a decreasing sequence of simple stopping times (cf. the 2°/ of the proof of A.12).

F.8 - "PROJECTION" LEMMA

Let u be a totally inaccessible stopping time. Let B be the process defined by $B = 1_{[u,1]}.1_{[u<1]}$. For each integer n, let B^n(resp. C^n) the right (resp. left) continuous process defined by :

$$B^n_t = E(B_{(k+1).2^{-n}}|\mathcal{F}_{t+}) \text{ if } k.2^{-n} \leq t < (k+1).2^{-n}$$

$$C^n_t = E(B_{(k+1).2^{-n}}|\mathcal{F}_{t-}) \text{ if } k.2^{-n} < t \leq (k+1).2^{-n}$$

When n goes to the infinity, the sequence $(B^n)_{n>o}$ (resp. $(C^n)_{n>o}$) converges P- a.s. uniformly to the process B(resp. $C=1_{]u,1]}$).

Proof

At first, we can remark that this lemma is a corollary of the properties of the "predictable projections" as studied in [DEL] ; this lemma is sufficient for the following and allows us to do not use the "section and projection theorems" and the "capacitability theorem" as done in [DEL].

1°) The processes B^n and C^n are defined up to indistinguishability, the sequences $(B^n)_{n>o}$ and $(C^n)_{n>o}$ are decreasing and, for each integer n, $B^n \geq B$.

Let ε be a positive number. For each integer n, let v(n) be the stopping time defined by :

$$v(n) = \inf. \{ t : B^n_t - B_t > \varepsilon \}$$

The sequence $(v(n))_{n>o}$ is increasing to the stopping time v.

For the convenience of notations, we note $[v = u]$ the set $\{\omega : v(\omega) = u(\omega)\}$, and so on ...

2°) For the convenience of notations, we put

$\bar{u} = 2^n.u$, $\bar{v} = 2^n.v$, $\overline{v(n)} = 2^n.v(n)$, $\mathcal{G} = \mathcal{F}_{(k+1).2^{-n}}$

$D = \{\omega : k \leq \overline{v(n)}(\omega)\}$ and

$$a(n,k) = E\{B^n_{v(n)}.1_{k\leq\overline{v(n)}<k+1}|\mathcal{G}\}$$

$$\leq E(B^n_{v(n)}.1_D|\mathcal{G})$$

For $k.2^{-n} \leq t$, $1_D.B^n$ is a sub-martingale (because $D \in \mathcal{F}_{k.2^{-n}}$) then (cf. the "stopping theorem" D.6), we have :

$$a(n,k) \leq E(B_{(k+1).2^{-n}}.1_D|\mathcal{G})$$

$$\leq E(1_{[\bar{u}\leq k+1]}.1_D|\mathcal{G})$$

This implies :

$$E[a(n,k)] = E\left[a(n,k).1_{[\overline{v(n)}<k+1]}\right]$$

$$\leq E\left[1_{\bar{u}\leq k+1}.1_D.1_{[\overline{v(n)}<k+1]}\right]$$

$$\leq E\left[1_{\bar{u}\leq\overline{v(n)}+1}.1_{[k\leq\overline{v(n)}<k+1]}\right]$$

Thus, we have :

$$E\left[B^n_{v(n)}\right] = \sum_{k=o}^{2^n-1} E\left[a(n,k)\right]$$

$$\leq E\left[1_{[\bar{u}\leq\overline{v(n)}+1]}\right] \leq P([\bar{u}\leq\bar{v}+1])$$

Then :

$$\lim_{n\to\infty}. E\left[B^n_{v(n)}\right] \leq P([u\leq v \text{ and } u<1])$$

3°) The definition of v(n) implies :

$$E\left[B^n_{v(n)} - B_{v(n)}\right] \geq \varepsilon . P([v(n)<1]) \geq \varepsilon . P([v<1])$$

If we consider the limit of this inequality when n goes to the infinity, we obtain (cf. the 2°) above) :

$P([u \leq v \text{ and } u<1]) - \lim_{n\to\infty}.P([u \leq v(n) \text{ and } u<1])$

$\geq \varepsilon.P([v<1])$

But $[u\leq v(n) \text{ and } u<1] \uparrow [u\leq v \text{ and } u<1]$ (cf. F.4) thus $\varepsilon.(P[v<1]) = 0$.

4°) That proves that the sequence $(B^n)_{n>o}$ converges P.a.s. uniformly for each sample function to the process B. But, for each integer n, C^n is the left continuous process associated to the right continuous process B^n ; thus the sequence $(C^n)_{n>o}$ converges P.a.s. uniformly for each sample function to the process C defined by $C_t = B_{t-}$, id est $C = 1_{]u,1]}$.

F.9 - <u>WHEN A IS CONTINUOUS</u> (proposition)

Let A be an element of $\mathcal{C}$ and a be its Doléans measure. We suppose that, for each predictable stopping time u, $a([u]) = 0$. Then A is continuous (up to indistinguishability) , thus A is a Meyer process.

<u>Proof</u>

1°) Let u be a predictable stopping time and $(u(n))_{n>o}$ be a sequence announcing u. We have :

$$0 = a([u]) = \lim_{n\to\infty} . a(]u(n),u]) = \lim_n .E[A_u - A_{u(n)}]$$

$$= E(A_u - A_{u-})$$

2°) Let u be a totally inaccessible stopping time. We define the sequence $(C^n)_{n>o}$ of processes as in F-8. For each pair (n,k) of integers, we put :

$$D(n,k) = [k.2^{-n} < u \leq (k+1).2^{-n}]$$

and $$w(n) = \sum_k (k+1).2^{-n} . 1_{D(n,k)} .$$

We have :

$$E(A_u - A_{u-}) = \lim_{n\to\infty} \left\{ \sum_k [1_{D(n,k)} . (A_{(k+1).2^{-n}} - A_{k.2^{-n}})] \right\}$$

this and the property F.6 (iii) implies :

$$E(A_u - A_{u-}) = \lim_{n\to\infty} . \int [C^n - 1_{]w(n),1]}] . da$$

and this limit is equal to zero (cf. F.8)

3°) Then, for each stopping time u, we have $E(A_u - A_{u-}) = 0$ (cf. F.5 and the 1°) and 2°) above) Thus, A is continuous (for each $\varepsilon > o$, consider the stopping time u defined by $u = \inf.\{t : (A_t - A_{t-}) > \varepsilon\}$

F.10 - <u>WHEN $a([u]) = a(\Omega')$</u> (proposition)

Let A be an element of $\mathcal{C}$. Let u be a predictable stopping time. We suppose that $a([u]) = a(\Omega')$. Then the process A is predictable (i.e. A is a Meyer process).

<u>Proof</u>

We have :

$$E(A_1 - A_o) = a(\Omega') = a([u]) = E(A_u - A_{u-})$$

(cf. F.9.1°)).

That means A has a jump on the stopping time u and A is constant elsewhere. Let $(u(n))_{n>o}$ be a sequence announcing u. Let F be an element of $\mathcal{F}$. We have (F.6.(iii)) :

$$E(1_F.A_u) = \int E(1_F|\mathcal{F}_{t-}) . 1_{]o,u]} . da$$

But the martingale $E(1_F|\mathcal{F}_{t-})$ stopped at u(n) is indistinguishable of the martingale $E[E(1_F|\mathcal{F}_{u-})|\mathcal{F}_{t-}]$ stopped at u(n) ; thus, we have the same property when we stop these martingales at u ; then, we obtain :

$$E(1_F.A_u) = \int E[E(1_F|\mathcal{F}_{u-})|\mathcal{F}_{t-}] . 1_{]o,u]} . da$$

$$= E[E(1_F|\mathcal{F}_{u-}) . A_u]$$

That proves that the random variable A_u is $\mathcal{F}_{u-}$- measurable and $A_u.1_{[u,1]}$ is predictable (cf. the end of F.2).

F.11 - <u>INTEGRATION OF A MARTINGALE WITH RESPECT TO AN INCREASING PROCESS</u> (proposition)

Let M be an uniformly bounded right continuous martingale and A be an adapted integrable increasing right continuous process. For each element t of T, we have :

$$E\left[\int_{]o,t]} M_s . dA_s\right] = E\left[M_t.(A_t - A_o)\right]$$

<u>Proof</u>

We note $T^* = (T \cap [o,t])$. Let $(T_n)_{n>o}$ be an increasing sequence of finite subsets of T^* such that $\bigcup_{n>o} T_n$ is dense in T^* and $t \in T_1$ and $0 \in T_1$.
For each integer n, let $(t(k))_{1\leq k\leq q}$ be the increasing family of the elements of T_n and let M^n be the process defined by :

$$M^n = \sum_{k=1}^{q-1} M_{t(k+1)} . 1_{]t(k),t(k+1)]}$$

The sequence of processes $(M^n)_{n>o}$ converges to the process M ; by the dominated convergence theorem, it is sufficient to prove the equality for each process M^n. But we have :

$$E\Big[\int_{]o,t]} M^n_s . dA_s\Big] = \sum_{k=1}^{q} E\Big\{M_{t(k+1)} . \big[A_{t(k+1)} - A_{t(k)}\big]\Big\}$$

$$= \sum_{k=1}^{q} E\Big\{M_t . \big[A_{t(k+1)} - A_{t(k)}\big]\Big\}$$

$$= E\big[M_t . (A_t - A_o)\big] \ .$$

F.12 - MEYER PROCESS:EXISTENCE AND UNICITY (theorem)

a) If A and B are two increasing Meyer processes which have the same Doléans measure (i.e. (A-B) is a martingale), then A and B are indistinguishable.

b) If a is a positive Doléans measure, there exists a Meyer process A such that a is the Doléans measure of the process A. Moreover this process A is continuous (up to indistinguishability) if and only if $a([u]) = 0$ for all the predictable stopping times u.

c) At last, an increasing process A is a Meyer process if and only if A is an element of $\mathcal{C}$.

Proof

1°) Let A and B be two Meyer processes such that (A-B) is a martingale. At first, we suppose that A and B are uniformly bounded. We have (cf. F.11 above) :

$$E(M_t . A_t) = E\Big(\int_{]o,t]} M_s . dA_s \Big)$$

if a is the Doléans measure of A, we have (because M is a predictable process) :

$$E\Big(\int_{]o,t]} M_s . dA_s\Big) = \int_{]o,t]} M_s . da$$

$$= \int_{]o,t]} M_s . db = E(M_t . B_t) \qquad \text{(as above)}$$

At last, we obtain $0 = E\big[M_t . (A_t - B_t)\big] = E(M_t^2)$

thus M is indistinguishable of 0.

2°) Now, we suppose that A and B are two Meyer processes such that (A-B) is a martingale, but we do not suppose that A and B are uniformly bounded. For each integer n, we consider the predictable set C(n) where $(A_t + B_t) \geq n$ and $D(n) = \Omega' \setminus C(n)$; we put

$$A^n = n.1_{C(n)} + A.1_{D(n)}$$

and $\quad B^n = n.1_{C(n)} + B.1_{D(n)}$

The processes A^n and B^n are two Meyer processes which have the same Doléans measures, thus (cf. the 1°)) A^n and B^n are indistinguishable ; that proves that A and B are also indistinguishable.

3°) Let a be a positive Doléans measure. To prove that there exists a Meyer process A such that a is its Doléans measure, we begin to prove that $a = d + \sum_{n>o} b_n$ where d is as in F.9 and b_n is as in F.10.
For that, we consider the supremum $\bar{c}$ of the positive numbers $\bar{b}$ such that :

(i) there exists a sequence $(u(n))_{n>o}$ of predictable stopping times and a sequence $(b_n)_{n>o}$ of Doléans measures such that $\bar{b} = \sum_{n>o} b_n(\Omega')$,

$\sum_{n>o} b_n(.) \leq a(.)$ and, for each integer n,

$b_n(\Omega') = b_n([u(n)])$.

It is easily seen that this supremum $\bar{c}$ is reached for a pair of sequences $(u(n),b_n)_{n>o}$ satisfiing to the condition (i), with $\bar{b} = \bar{c}$. We put $d = a - \sum_{n>o} b_n$. Let D and B_n be the processes belonging to $\mathcal{C}$ associated to d and b_n respectively as built in F.7. These processes are also Meyer processes (cf. F.9 and F.10). Moreover, $\sum_{n>o} E(B_1^n) = \sum_{n>o} b_n(\Omega') < +\infty$, then the sequence $(A^n)_{n>o}$ of processes defined by $A^n = D + \sum_{k=1}^{n} B_k$ converges P.a.s. uniformly, for each sample function, (Borel-Cantelli lemma) to a process A wich is a Meyer process and wich belongs to $\mathcal{C}$. Moreover, the Doléans measure of A is equal to $d + \sum_{n>o} b_n = a$.

4°) If A is continuous, for each predictable stopping time u, $a([u]) = E(A_u - A_{u-}) = 0$ (cf. F.9.1°))
Conversely, if $a([u]) = 0$ for each predictable stopping time, we saw in F.9 that A is continuous.

5°) Now, we have only to prove the c). Let A' be an increasing right continuous process with $A'_o = 0$ and $E(A'_1) < +\infty$. Let a be its Doléans measure. Let A be the associated process as built in the 3°) above ; A is a Meyer process and belongs to $\mathcal{C}$: then A is indistinguishable of A' if A' is a Meyer process or if A' belongs to $\mathcal{C}$ and that proves the c).

F.13 - BOUND FOR A PREDICTABLE JUMP (lemma)

Let H be a Hilbert space and a a Doléans measure associated to the H-valued process Z. Let A be the Meyer process associated to a. Let u be a predictable stopping time. We suppose that Z is uniformly bounded by d (i.e. $\|Z_t(\omega)\| \leq d$ for each element (ω,t) of $(\Omega \times T)$). Then we have :

$$E\big[\|A_u - A_{u-}\|^2\big] \leq 2d.E(|A_u - A_{u-}|)$$

Proof

Of course, the Meyer process A is a right continuous predictable process of bounded variation associated to a ; its existence is easily seen. We have :

$$x = E\left[(A_u - A_{u-})^2\right] = E \int 1_{[u]} <(A_s - A_{s-}), dA_s>$$

But $(1_{[u]}(A_s - A_{s-}))_{s\in T}$ is a predictable process and A and Z have the same Doléans measures, thus we have :

$$x = E \int 1_{[u]} <(A_s - A_{s-}), dZ_s>$$

$$\leqslant 2d.E(|A_u - A_{u-}|)$$

F.14 - DECOMPOSITION OF A MARTINGALE (proposition)

Let H *be a Hilbert space and* M *be an* H*-valued martingale. Then there exist an* H*-valued cadlag locally square integrable martingale* W *and an* H *-valued cadlag process of bounded variation* Q *such that* M = W + Q

Proof

By localization, we have only to consider the case where there exists a stopping time u and a positive number d such that $|M_t(\omega)| < d$ if $t < u(\omega)$ and $M_t(\omega) = M_{u(\omega)}(\omega)$ if $t \geqslant u(\omega)$ (consider the sequence $(u(n))_{n>o}$ of stopping times defined by $u(n) = \inf.\{ t : |M_t| \geqslant n \}$ and stop M at u(n)). In this case, we put :

$$Z = - M.1_{[o,u[} \quad \text{and} \quad B = M.1_{[u,1]}$$

Let a be the Doléans measure of B (and of Z) and A be the Meyer process associated to a. For each integer n, we put :

$v(n) = \inf.\{ t : |A_t| \geqslant n \}$; since $\lim_{n\to\infty}.P[v(n) < 1] = 0$ it is sufficient to prove the decomposition for the process M stopped at v(n).

Because the Meyer process associated to the process B stopped at u(n) is also the Meyer process stopped at u and associated to B, we can suppose that v(n) = u (for the convenience of notations). In this case, we have :

$E\left[(A_{u-})^2\right] \leqslant n^2$ (cf. the definition of v(n))

$E\left[(A_u - A_{u-})^2\right] \leqslant 2d.E(\|A_u - A_{u-}\|)$ (cf F.13)

thus $E(A_1^2) = E(A_u^2) < +\infty$

But M = (A-Z) + B - A where (B-A) is a process of bounded variation and (A-Z) is a square integrable martingale ($E(Z_1^2) \leqslant d^2$ and $E(A_1^2) < +\infty$).

F.15 - IF $\mathcal{F}_t \neq \mathcal{F}_{t+}$ (remark)

For the convenience of notations, we supposed that the family $(\mathcal{F}_t)_{t\in T}$ is right continuous. In the general case, it is always possible to consider the family $(\mathcal{F}_{t+})_{t\in T}$ and to use the results of this paragraph F ; in this case, a Meyer process with respect to the family $(\mathcal{F}_{t+})_{t\in T}$ is also a Meyer process with respect to the family $(\mathcal{F}_t)_{t\in T}$ (cf. A.11).

G - AN INEQUALITY FOR SEMI-MARTINGALES

G.1. GENERALITIES :

The main result of this paragraph is the theorem G.6 after. It is fundamental to note that the inequality G.6-(i) is only concerned by the values of processes Z and A "strictly before" the stopping time u. The theorem G.6 gives an example where we have the condition (i) of the theorem D.5 above.

In this paragraph, we consider a stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in T})$ and we use the french notation cadlag and the conventions given in D.3-d) and D.3-e). Moreover, if M is an H-valued square integrable cadlag martingale, we note $[M]$ the increasing positive cadlag adapted process which is the quadratic variation of M, i.e.

$$[M]_t = \lim_{n\to\infty} \left\{ \sum_{k=0}^{\infty} ||M_{(k+1).2^{-n}\wedge t} - M_{k.2^{-n}\wedge t}||^2 \right\}$$

and we shall note $\langle M\rangle$ the "Meyer" process associated to $[M]$ (i.e. the predictable increasing right continuous process such that $[M]_0 = 0$ and $[M]_t - \langle M\rangle_t$ is a real martingale).

G.2. A LEMMA ON THE CONDITIONNAL EXPECTATIONS :

We consider $(\Omega, \mathcal{F}, P)$ *a probability space,* $\mathcal{G}$ *a sub-σ-algebra of* $\mathcal{F}$, A *an element of* $\mathcal{F}$ *and* $\mathcal{G}^*$ *the σ-algebra generated by* $\mathcal{G}$ *and* A. *Let* Z *be an element of* $L_1^H(\Omega, \mathcal{G}^*, P)$ *such that* $E(Z|\mathcal{G}) = 0$.
If we note $B = \Omega \setminus A$, *we have :*

(i) $E(1_A|\mathcal{G}).E(||Z||^2.1_A|\mathcal{G}) = E(1_B|\mathcal{G}).E(||Z||^2.1_B|\mathcal{G})$ a.e.

(ii) $E(1_B||Z||^2) = E\{1_A.E(||Z||^2|\mathcal{G})\}$

Proof

1°/ The elementary following proof was suggested by J.Jacod.
We can write $Z = X.1_A + Y.1_B$ where X and Y belong to $L_1^H(\Omega, \mathcal{G}, P)$. The property $E(Z|\mathcal{G}) = 0$ implies

$$X.E(1_A|\mathcal{G}) = E(X.1_A|\mathcal{G}) = -E(Y.1_B|\mathcal{G}) = -Y.E(1_B|\mathcal{G}).$$

Then, we have also :

$$E(1_A|\mathcal{G}).E(||Z||^2.1_A|\mathcal{G}) = E(1_A|\mathcal{G}).E(||X||^2.1_A|\mathcal{G})$$
$$= [E(1_A|\mathcal{G})]^2\ ||X||^2 = ||Y||^2.[E(1_B|\mathcal{G})]^2$$
$$= E(1_B|\mathcal{G}).E(||Z||^2.1_B|\mathcal{G})$$

2°/

$$E\{1_A.E(||Z||^2.1_A|\mathcal{G})\} = E\{E(1_A|\mathcal{G}).E(||Z||^2.1_A|\mathcal{G})\}$$
$$= E\{E(1_B|\mathcal{G}).E(||Z||^2.1_B|\mathcal{G})\} = E\{1_B.E(||Z||^2.1_B|\mathcal{G})\}$$

Then, we have also :

$$E\{1_A.E(||Z||^2|\mathcal{G})\}$$
$$= E\{1_A.E(||Z||^2.1_A|\mathcal{G})\} + E\{1_A.E(||Z||^2.1_B|\mathcal{G})\}$$
$$= E\{1_B.E(||Z||^2.1_B|\mathcal{G})\} + E\{1_A.E(||Z||^2.1_B|\mathcal{G})\}$$
$$= E(||Z||^2.1_B)$$

G.3. LEMMA (if only one jump)

Let H *be a Banach space. Let* u *and* v *be two stopping times,* v *being predictable. Let* S *be an* H*-valued square integrable random variable which is* $\mathcal{F}_v$ *- measurable. We put* $C = S - E(S|\mathcal{F}_{v-})$.

Let M *be the cadlag martingale defined by* $M_t = E(C|\mathcal{F}_t)$. *Then* M *has a jump on the stopping time* v *and is "fixed" elsewhere. Moreover there exists an* H*-valued square integrable cadlag martingale* W *with the following properties :*

(i) $W.1_{[0,u[} = M.1_{[0,u[}$
(this implies $[W].1_{[0,u[} = [M].1_{[0,u[}$*)*

(ii) the random measures $d\langle W\rangle$ *and* $d\langle M\rangle$ *are such that* $d\langle W\rangle \leq d\langle M\rangle$

(iii) for each predictable real positive process Y :

$$E\left\{\int_{]0,u]} Y.d[W]\right\} = E\left\{\int_{]0,u]} Y.d\langle W\rangle\right\} \leq E\left\{\int_{]0,u[} Y.\ (d[W] + d\ \langle W\rangle)\right\}$$

Proof :

1°/ Let $(v(k))_{k>o}$ be a sequence of stopping times "announcing" v (i.e. $v(k)\uparrow v$ and, $\forall k, P([v(k)<v])=1$)

For each integer k, $E(C|\mathcal{F}_{v(k)}) = 0$, then

$$M.1_{[o,v(k)]} = 0 \quad ;$$

this implies $M.1_{[o,u[} = 0$. Moreover, C being $\mathcal{F}_v$ - measurable, $M_1 = M_v$. This proves that M has a jump on the stopping time v and is "fixed" elsewhere.

2°/ For the building of W, we can suppose that M is stopped at u (i.e. $M_1 = M_u$). Then we consider the sets $B = \{\omega : v(\omega) = u(\omega)\}$ and $A = \Omega \setminus B$, the σ-algebra $\mathcal{G} = \mathcal{F}_{v-}$, the σ-algebra $\mathcal{G}^*$ generated by $\mathcal{G}$ and by the set B, the randon variable

$$D_1 = C.1_B - E(C.1_B|\mathcal{G}^*) = 1_B.\{C-E(C|\mathcal{G}^*)\}$$

and the cadlag martingale D defined by $D_t = E(D_1|\mathcal{F}_t)$. We note that D has a jump on the stopping time v and is "fixed" elsewhere (cf. 1°/ above).

Now we put $W = M - D$ and we shall prove the properties (ii) and (iii) (the property (i) is proved above). We note that $W_v = W_1 = 1_A.C + 1_B.\ E(C|\mathcal{G}^*)$.

3°/ The stopping time v being predictable, we have :

$$\langle W\rangle_v = \langle W\rangle_v - \langle W\rangle_{v-}$$

$$= E([W]_v \ |\mathcal{G})$$
$$= E(\{1_A.||C||^2 + 1_B.\ ||E(C|\mathcal{G}^*)||^2\}|\mathcal{G})$$
$$\leqslant E(\{1_A.||C||^2 + 1_B.\ ||C||^2\}|\mathcal{G})$$
$$\leqslant E([M]_v|\mathcal{G}) \quad \leqslant \quad \langle M\rangle_v$$

Actually, we have $d\langle W\rangle \ \leqslant \ d\langle M\rangle$, i.e. the property (ii).

4°/ Let Y be a predictable real positive process. Then, the random variable Y_v is $\mathcal{F}_{v-}$-measurable. We have :

$$E(\int_{]o,u]} Y.d[W])$$
$$= E(Y_v.1_A.||C||^2) + E(Y_v.1_B.||E(C|\mathcal{G}^*)||^2)$$

The first term is bounded by $E(\int_{]o,u[} Y.d[W])$.

By the lemma G.2, if we put $Z = (Y_v)^{1/2}.E(C|\mathcal{G}^*)$, the second term is equal to

$$E(Y_v.1_A.E\{||E(C|\mathcal{G}^*)||^2|\mathcal{G}\})$$

which is bounded by $E(\int_{]o,u[} Y.d\langle W\rangle)$ (see above).

This proves the property (iii).

G.4. PROPOSITION

Let H be an Hilbert space. Let M be an H-valued cadlag square integrable martingale. Let u be a stopping time. Then, there exists an H-valued cadlag square integrable martingale W with the following properties :

(i) $W.1_{[o,u[} = M.\ 1_{[o,u[}$ *(this implies :* $[W].1_{[o,u[} = [M].1_{[o,u[}$*)*

(ii) the "random measures" $d\langle W\rangle$ *and* $d\langle M\rangle$ *are such that* $d\langle W\rangle \leqslant d\langle M\rangle$.

(iii) for each predictable real positive process :

$$E\left\{\int_{]o,u]} Y.d[W]\right\} = E\left\{\int_{]o,u]} Y.d\langle W\rangle\right\}$$
$$\leqslant E\left\{\int_{]o,u[} Y.(d[W] + d\langle W\rangle)\right\}$$
$$\leqslant E\left\{\int_{]o,u[} Y.(d[M] + d\langle M\rangle)\right\}$$

Proof :

We can assume that $M_1 = M_u$. Let w be a "totally inacessible" stopping time and let $(v(n))_{n>o}$ be a sequence of predictable stopping times such that $[u] \subset [w] \cup \{\bigcup_{n>o} [v(n)]\}$; we can assume that the sets $[v(n)]_{n>o}$ are disjoint.

For each integer n we define the random variable $C_n = M_{v(n)} - M_{v(n)-}$ and $M^n_t = E(C_n|\mathcal{F}_t)$.

We can define $\overline{M} = M - \sum_{n>o} M^n$ (convergent serie in the space of square integrable martingales) and we have $[M] = [\overline{M}] + \sum_{n>o} [M^n]$ (the sets $[v(n)]_{n>o}$ being disjoint). Moreover, $\langle\overline{M}\rangle$ being a predictable process, $\langle\overline{M}\rangle_w = \langle\overline{M}\rangle_{w-}$; this implies $\langle\overline{M}\rangle_u = \langle\overline{M}\rangle_{u-}$. Then, for each predictable real positive process Y, we have :

$$E(\int_{]o,u]} Y.d\langle\overline{M}\rangle) = E(\int_{]o,u]} Y.d\langle\overline{M}\rangle)$$
$$= E(\int_{]o,u[} Y.d\langle\overline{M}\rangle)$$

(if we define $\overline{W} = \overline{M}$, the properties (i), (ii) and (iii) are satisfied for the pair $(\overline{W},\overline{M})$).

Then, for each integer n, we build a martingale W^n associated to M^n as in the lemma B-3. By additivity, the proposition is proved if we put

$$W = \overline{W} + \sum_{n>o} W^n = \overline{M} + \sum_{n>o} W^n$$

G.5. COROLLARY

Let M be an Hilbert space valued square integrable cadlag martingale. Then, for each stopping u and for each real bounded predictable process Y, we have :

$$E(\sup_{o\leq t<u} . \|\int_{]o,t]} Y_s.dM_s\|^2)$$
$$\leq 4. \left(E\left\{\int_{]o,u[} Y_s^2.(d\langle M\rangle_s + d[M]_s)\right\}\right)$$

(cf. D.3.e) for the notation above).

Proof :

Let W be a martingale associated to M and u as in the proposition G.4 above. The stochastic integral $\int_{]o,u]} Y_s.dW_s$ is well defined (see the property (ii)) and we have :

$$\left(\int Y.dM\right).1_{]o,u[} = \left(\int Y.dW\right)1_{]o,u[}$$

(this is obvious if Y is an $\mathcal{H}$- simple process, and it is true in the general case by linearity and density).

Then, we have :

$$E\left(\sup_{o\leq t<u} \|\int_{]o,t]} Y_s.dM_s\|^2\right)$$
$$= E\left(\sup_{o\leq t<u} \|\int_{]o,t]} Y_s.dW_s\|^2\right)$$
$$\leq E\left(\sup_{o\leq t\leq u} \|\int_{]o,t]} Y_s.dW_s\|^2\right)$$
$$\leq 4.E\left(\|\int_{]o,u]} Y_s.dW_s\|^2\right) \qquad \text{(Doob inequality)}$$
$$\leq 4.E\left(\int_{]o,u]} Y_s^2 . d[W]\right)$$
$$\leq 4.E\left(\int_{]o,u[} Y_s^2 .\{d[M] + d\langle M\rangle\}\right) \text{ (cf. G.4).}$$

G.6. THEOREM

We consider a Banach space H, two Hilbert spaces J and K, and a bilinear mapping of H × J into K which, to (x,y) element of (H × J), associates y.x element of K. We suppose that we have, for each element (y.x) of (H × J), $\|y.x\| \leq \|y\|.\|x\|$.

Let M be a J-valued cadlag locally square integrable martingale. Let V be a J-valued cadlag adapted process of finite variation (i.e., for each element ω of Ω, the function $s \rightsquigarrow V_s(\omega)$ has a bounded variation). We put Z = M + V. Then, there exists an increasing cadlag adapted process A such that,

(i) for each stopping time u and for each H-valued predictable process Y :

$$E\{\sup_{t<u}\|\int_{]o,t]} Y.dZ\|_K^2 \leq E(A_u - \{\int_{]o,u[}\|Y_t\|^2.dA_t\})$$

(see D-3-e) for the notation above).

Actually, this property is fulfiled for a process Z which is the sum of a local martingale M and a process of bounded variation V (i.e. a general semi-martingale).

Proof :

1°) The set of processes Z for which there exists a process A with the property (i) is clearly a vector space. Moreover, if Z = V, is a process of finite variation $B_t = \int_{]o,t]} d\|V_s\|$ and if we put $A_t = B_t$, the condition (i) is satisfied by the Cauchy-Schwartz inequality (applied for each "sample function").Then, it is sufficient to prove the theorem when Z = M is a locally square integrable martingale. In this case, the condition

(i) is satisfied if we put $A_t = (\langle M\rangle_t + [M]_t) + 1$
Indeed, ⟨M⟩ can be defined by localization and the corollary G.5 is available for a locally square integrable martingale (Fatou lemma).

2°) It is sufficient to apply the proposition F.14.

G.7 - SUMMABLE PROCESS (définitions)

One says (cf [Kus]) that **X** *is a* p-*summable (with* p≥o) *process if the mapping* $A \rightsquigarrow \int 1_A.dX$, *defined on the algebra* $\mathcal{A}$, *can be extended in a measure σ-additive for the strong topology of* $L_p(\Omega,\mathcal{F},P)$. *We say that* X *is a prelocally (cf. A.13)* **p**-*summable process if there exists a sequence* $(u(n))_{n>o}$ *of stopping times such that*

$$\lim_{n\to\infty} .P([u(n) < T^\infty]) = 0$$

and, for each integer n, $X.1_{]o,u(n)[}$ *is a* p-*summable process (in [Kus], such a process is called locally* p-*summable).*

G.8 - CHARACTERIZATIONS OF A SEMI-MARTINGALE

(proposition)

Let H be a Hilbert space and Z be an H-valued cadlag adapted process. The following properties are equivalent :

(i) Z is a semi-martingale, in other words Z = M+V where M is a local martingale and V is a process of bounded variation

(ii) Z is prelocally 2-summable (cf. G.7 above)

(iii) Z is prelocally 1-summable

(iv) Z satisfies the condition G.6.(i) for each Banach space J and each Hilbert space K.

(v) there exists a real positive finite increasing adapted cadlag process A such that for each real $\mathcal{A}$-simple process Y and for each stopping u ,

$$E\{\|\int_{]0,u[} Y.dZ\|_H^2\} \leq E(A_{u-}.\{\int_{]0,u[} \|Y\|^2.dA_s\})$$

Proof

At first, we can remark that this proposition generalizes the theorem 12.3 of [Kus].

We saw in G.6 that (i) implies (iv) ; we saw in B.6 that (v) implies (ii) ; it is obvious that (ii) implies (iii) and that (iv) implies (v). Let Z be a 1-summable process ; the Doléans function d(X) of X is σ-additive ; thus, there exists an H-valued Meyer process A associated to this Doléans measure ; Z-A is a martingale, then Z = A+(Z-A) is a semi-martingale. Now, we suppose that Z is a prelocally 1-summable process ; let $(u(n))_{n>0}$ be an increasing sequence of stopping times such that $\lim_{n\to\infty} P([u(n) < 1]) = 0$ and, for each integer n, $X.1_{]0,u(n)[}$ is a 1-summable process ; for each integer n, we have

$$X.1_{]0,u(n)[} = X.1_{]0,u(n)[} + X.1_{[u(n)]}$$

thus $X.1_{]0,u(n)]}$ is a semi-martingale ; then, X is also a semi-martingale and that completes the proof.

H - BURKHOLDER INEQUALITIES

H.1 - THEOREM

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in T})$ be a stochastic basis and X be a martingale with respect to this basis.

For the convenience of notations, we suppose that $T = [o,\infty]$.

We consider

$$N_1(X) = E\,\{\mathrm{Sup.}_{t\in T} |X_t|\}$$

$$N_2^+(X) = \lim.\sup._{n\to\infty} E\left[\left\{\sum_{k\geq o} (X_{(k+1).2^{-n}} - X_{k.2^{-n}})^2\right\}\right]^{1/2}$$

$$N_2^-(X) = \lim.\inf._{n\to\infty} E\left[\left\{\sum_{k\geq o} (X_{(k+1)2^{-n}} - X_{k\,2^{-n}})^2\right\}\right]^{1/2}$$

$$N_3(X) = \mathrm{Sup.}\ E\left[\left|\int_{|o,\infty|} Y.dX\right|\right]$$

this supremum being considered for all the processes Y such that

$$Y = \sum_{k=1}^{n-1} 1_{]u(2k),u(2k+1)]} \quad \text{where } (u(j))_{1\leq j\leq 2n}$$

is an increasing family of stopping times.

$$N_4(X) = \mathrm{Sup.}\ E\left[\left|\int_{]o,\infty]} f(t).dX_t\right|\right]$$

this supremum being considered for all the functions f such that

$$f = \sum_{k=1}^{n-1} 1_{]t(2k),t(2k+1)]} \text{ where } (t(j))_{1\leq j\leq 2n}$$

is an increasing family of elements of T.

Then, these five semi-norms are uniformly equivalent in the space of all the martingales such that $X_o = o$. *More precisely :*

$N_1(X) \leq 4.N_3(X)$; $N_3(X) \leq 8\,N_2^-(X)$

$N_2^-(X) \leq N_2^+(X)$; $N_2^+(X) \leq 45.N_1(X)$

$N_4(X) \leq N_3(X)$; $N_2^+(X) \leq 180.N_4(X)$

Proof

The inequalities $N_2^-(X) \leq N_2^+(X)$ and $N_4(X) \leq N_3(X)$ are trivial. The inequality $N_1 \leq 4.N_3$ is proved in H.2. The inequality $N_3 \leq 8\,N_2^-$ is proved in H.4. The inequality $N_2^+ \leq 45.N_1$ is proved in H.6. The inequality $N_2^+ \leq 180.N_4$ is proved in H.9 (actually, it is possible to do better by using the Khintchine lemma).

Moreover, let $[X,X]$ be the quadratic variation process associated to X as defined and studied in C.6 3°) : the hypothesis given in C.5 are satisfied (cf. G.6) ; thus, we have $N_2^+(X) = E\,(\sqrt{[X,X]_\infty}) = N_2^-(X)$; actually, in this paragraph H, we do not use this property : we use only E.10.

H.2 - PROOF OF $N_1(X) \leq 4.N_3(X)$

If Z is a random variable, we put $Z^+ = \mathrm{Sup.}(Z,o)$ and $Z^- = \sup.(-Z,o)$.

1°) Let $(\Omega,\mathcal{F},P,(\mathcal{F}_k)_{1\leq k\leq n})$ be a probabilized stochastic basis. We note $\mathcal{S}_n$ the space of all the martingales X (with respect to this stochastic basis) such that $X_1 = o$ and such that, for each element ω of Ω, the sample function $k \to X_k(\omega)$ is "fixed" after its first "going down", id est :

- X_n is an element of $L_1(\Omega,\mathcal{F},P)$ and $E(X_n|\mathcal{F}_1) = o$
- for each integer k, $X_k = E(X_n|\mathcal{F}_k)$
- $X_{k+1}(\omega) < X_k(\omega)$ implies, $\forall\, j \geq 1$, $X_{k+j}(\omega) = X_n(\omega)$

If X is an element of $\mathcal{S}_n$, we put $X^* = \mathrm{Sup.}_{1\leq k\leq n} X_k$, we say that a martingale Y is a "transform" of X (cf. [Bur]) if Y is a martingale (with respect to $(\Omega,\mathcal{F},P,(\mathcal{F}_k)_{1\leq k\leq n})$ and if there exists a subset J of $K = \{k : 1\leq k\leq n-1\}$ such that $Y_n = X_1 + \sum_{k\in J}(X_{k+1} - X_k)$.

If X is an element of $\mathcal{S}_n$ and if X' is a transform of X, X' is also an element of $\mathcal{S}_n$; moreover, if X" is a transform of X', X" is also a transform of X.

2°) Now, we prove that, if X is an element of $\mathcal{S}_n$, we have :

$$E(X^*) \leq 2\ \mathrm{Sup.}\ \{E(|Y|)\}$$

this supremum being considered for all the transforms Y of X.

Then, we consider an element X of $\mathcal{S}_n$. Let d be a positive number. For each integer k, we put

$A(k) = \{\omega : X_{k+1}(\omega) < X_k(\omega)\}$; the sets $(A(k))_{1\leq k\leq n}$ are disjoint (because $X \in \mathcal{S}_n$).

At first, we suppose that there exists an integer j such that :

$E(\bar{X}_{j+1} . 1_{A(j)}) \leqslant d.E(X_j . 1_{A(j)})$.

Let k be the first such integer j and Y be the transform of X defined by $Y_n = X_n - (X_{k+1} - X_k)$. We put $Y^* = \sup_{1 \leqslant k \leqslant n} Y_k$ and $B(k) = \{\omega : X_{k+1}(\omega) > X_k(\omega)\}$ if $\omega \notin B(k)$, $X^*(\omega) = Y^*(\omega)$ and, if $\omega \in B(k)$ $X^*(\omega) = Y^*(\omega) + X_{k+1}(\omega) - X_k(\omega)$. Then, we have :

$$E(X^*) = E(Y^*) + E\left[1_{B(k)} . (X_{k+1} - X_k)\right]$$

$$= E(Y^*) + E\left[1_{A(k)} . (X_{k+1} - X_k)\right]$$

(X being a martingale)

$$\leqslant E(Y^*) + E(1_{A(k)} . X_k) + E(X^-_{k+1} . 1_{A(k)})$$

(because $X_k(\omega) \geqslant 0$ if $\omega \in A(k)$)

$$\leqslant E(Y^*) + (1 + d) . E(1_{A(k)} . X_k)$$

$$\leqslant E(Y^*) + (1 + d) . E(1_{A(k)} . Y_n)$$

Now, we use the same argumentation for the martingale Y : if there exists an integer such that $E\left[Y^-_{j+1} . 1_{A(j)}\right] \leqslant d.E\left[Y_j . 1_{A(j)}\right]$ (the sets $(A(k))_{1 \leqslant k \leqslant n}$ being always the same); we consider k, the first such integer j and we consider the martingale Z transform of Y defined by $Z_n = Y_n - (Y_{k+1} - Y_k)$; at last, we obtain a subset J of $K = \{k : 1 \leqslant k \leqslant n\}$ and a martingale M transform of X such that :

(i) $M_n = X_n - \sum_{j \in J} (X_{j+1} - X_j)$

(ii) $E(X^*) \leqslant E(M^*) + (1 + d) \sum_{j \in J} E\left[1_{A(j)} . M_n\right]$

(ii) if $j \notin J$, $E\left[M^-_{j+1} . 1_{A(j)}\right] > d.E\left[M_j . 1_{A(j)}\right]$

If we put $D = \Omega \setminus \bigcup_{j \in J} A(j)$, we obtain :

$E(M^*) = E(M_n . 1_D) + \frac{1}{d} \sum_{j \in J} E(1_{A(j)} . M_j)$ and

$E(X^*) \leqslant E(M^*) + (1 + d) \sum_{j \in J} E(1_{A(j)} . M_j)$ and

$$\sum_{j \notin J} E(1_{A(j)} . M_j) < \frac{1}{d} \sum_{j=1}^{n-1} E(1_{A(j)} . M^-_{j+1}) < \frac{1}{d} . E(M^-_n)$$

Thus, we have :

$$E(X^*) \leqslant \frac{1}{d} E(M^-_n) + (2 + d) E(M^+_n)$$

But $E(M^-_n) = E(M^+_n) = \frac{1}{2} E(|M_n|)$; then, if we choose $d = 1$, we obtain :

$$E(X^*) \leqslant 2 . E(|M_n|)$$

3°) Now, we consider a real cadlag martingale X with respect to the stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in T})$ with $T = [0,1]$. Let a be a real number such that $a > 1$; let $(u(n))_{n > 0}$ be the sequence of stopping times defined by :

$u(n) = \inf. \{t : X_t > a^n\}$ (and $u(n) = 1$ if the set $X_t > d^n$ is void) ; the process $(X_{u(k)})_{k \leqslant n}$ is a martingale with respect to the stochastic basis $(\Omega, \mathcal{F}, P, \mathcal{F}_{u(k)})_{1 \leqslant k \leqslant n}$ and satisfies the hypothesis given in the 1°) above (with u(k) instead of k) ; then, we have :

$$E\{\sup_{1 \leqslant k \leqslant n} X_{u(k)}\} \leqslant \sup. E\{|\int Y.dX|\}$$

where this supremum is considered for all the processes Y such :

that $Y = \sum_{j \in J} 1_{]u(j), u(j+1)]}$ with $J \subset K = \{k : 1 \leqslant k \leqslant n-1\}$

Moreover, $\sup_{t \in T} X_t \leqslant a . \sup_{n > 0} \sup_{k \leqslant n} X_{u(k)}$

At last, we obtain :

$$E\{\sup_{t \in T} X_t\} \leqslant 2a . \sup. E\{|Y.dX|\}$$

this supremum being considered for all the processes Y such that

$$Y = \sum_{k=1}^{n} 1_{]u(2k), u(2k+1)]}$$

where $(u,(k))_{0 \leqslant k \leqslant 2n+1}$ is an increasing family of stopping times. This inequality being satisfied for all the real numbers a with $a > 1$, it is also satisfied for $a = 1$.

Of course, we have also :

$$E(\sup_{t \in T} |X_t|) \leqslant E(\sup_{t \in T} X_t) + E(\sup_{t \in T} - X_t)$$

and that gives the inequality $N_1 \leqslant 4 N_3$.

H.3 - LEMMA

Let $(\Omega, \mathcal{F}, P)$ be a probabilized space and $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Let V, X and Z be three elements of $L_1(\Omega, \mathcal{F}, P)$. We suppose that $E(Z|\mathcal{G}) = 0$, V and X are $\mathcal{G}$-measurable and $|X| . 1_A \geqslant |V| . 1_A$ if $A = \{\omega : Z(\omega) \neq 0\}$. Then, we have :

$$E(|X+Z| - |X|) \leqslant 6 E(\sqrt{V^2 + Z^2} - |V|)$$

Proof

We can suppose that $0 \leqslant V \leqslant X$ (in the general case, one considers the sets where V and X are positive or negative). In this case, we put $B = \{\omega : (X+Z)(\omega) < 0\}$. We have :

$E(|X + Z| - |X|) = 2 E(|X+Z| . 1_B)$ (because $E(Z|\mathcal{G}) = 0$).

But, if $\omega \in B$, $|Z(\omega)| \geq |X(\omega)| \geq |V(\omega)|$ thus $3(\sqrt{V^2+Z^2} - |V|)(\omega) \geq |Z(\omega)|$

Indeed, if v and z are two real numbers with $o \leq v \leq z$, we have $3\sqrt{v^2+z^2} \geq 3v + z$. Then, we obtain :

$$6\,E(\sqrt{V^2+Z^2} - |V|) \geq 2\,E(|Z|.1_B) \geq 2\,E(|X+Z|.1_B)$$
$$\geq E(|X+Z| - |X|)$$

H.4 - PROOF OF $N_3(X) \leq 8.N_2(X)$

Let Y be a predictable process such that $Y = \sum_{k=1}^{n-1} 1_{]u(2k),u(2k+1)]}$ where $(u(k))_{1\leq k\leq 2n}$ is an increasing family of stopping times ; if we put $Z = \int Y.dX$, it is easily seen that $N_2^-(Z) \leq N_2^-(X)$ (cf. E.11 - 4°) ; then, it is sufficient to prove that, if X is a martingale and if $(t(k))_{1\leq k\leq q}$ is an increasing family of dyadic numbers belonging to T, we have :

$$E\left\{\left|\sum_{k=1}^{q-1}(X_{t(k+1)}-X_{(k)})^2\right|^{1/2}\right\} \geq 8\,E(|X_q - X_1|)$$

Now, we suppose that $(X_k)_{1\leq k\leq q}$ is a real martingale with $X_1 = o$ (and $E(|X_q|) < +\infty$).

Let V be the quadratic variation of X, id est $V_1 = o$ and

$$V_{k+1} = \left\{\sum_{j=1}^{k-1}(X_{j+1}-X_j)^2\right\}^{1/2}$$

For each integer k, we put $A(k) = \{\omega : V_k \leq |X_k|\}$. Let S be the random variable defined by

$$S = \sum_{k=1}^{q-1}[1_{A(k)}.(|X_{k+1}| - |X_k|)]$$

We have (cf. H.3) :

$$E(S) \leq 6\sum_{K=1}^{q-1} E\left[1_{A(k)}.\sqrt{V_k^2 + (X_{K+1}-X_k)^2} - V_k\right]$$
$$\leq 6\sum_{k=1}^{q-1} E\left[\sqrt{V_k^2 + (X_{k+1}-X_k)^2} - V_k\right]$$
$$\leq 6\;E(V_q)$$

Now, let ω be an element "fixed" of Ω : in the following we do not write the symbol ω for the convenience of notation ; let k be the first integer, $1\leq k\leq q$, such that $\omega \in A_{k+j}$ for all the integers $j>o$; we have :

$$|X_q| \leq \sum_{k=1}^{q-j-1}(|X_{k+j+1}|-|X_{k+j}|)+(|X_{k+1}|-|X_k|)+|X_k|$$

But :

$$\sum_{k=1}^{q-j-1}(|X_{k+j+1}| - |X_{k+j}|) \leq S$$
$$|X_{k+1}| - |X_k| \leq |X_{k+1}-X_k| \leq V_q$$
$$|X_k| \leq V_k \quad \text{(see the definition of k)}$$

That implies :

$|X_k| \leq S + 2\,V_q$

Thus : $E(|X_q|) \leq 8.\,E(V_q)$

H.5 - LEMMA

Let $(\Omega,\mathcal{F},P,(\mathcal{G}_k)_{1<k<q})$ *be a probabilized stochastic basis. Let* $(Y_k)_{1\leq k\leq q}$ *be a square integrable martingale (with respect to this basis). Let* W_1 *be an element of* $L_2(\Omega,\mathcal{G}_1,P)$ *such that* $W_1 \geq o$ *and* $W_1 \geq Y_1$.

We put :

$$W_q = \left[W_1^2 + \sum_{k=1}^{q-1}(Y_{k+1} - Y_k)^2\right]^{1/2}$$

We suppose that, if $|Y_k| > 2.W_1$, *then, for each integer* $j \geq 1$, $Y_{k+j} = Y_k = Y_q$. *We put :* $W'_q = W_q \vee |Y_q|$. *Then, we have :*

$$E(W'_q - W_1) \leq 9.E(|Y_q - Y_1|)$$

Proof

It is sufficient to prove that :

$$E(W_q - W_1) \leq 8\,E(|Y_q - Y_1|)$$

For each integer, we put :

$A(k) = \{\omega : |Y_k|(\omega) \geq 6W_1 \text{ and } |Y_{k-1}(\omega)| < 2W_1\}$

The sets $(A(k))_{2\leq k\leq q}$ are disjoint ; moreover

$|Y_k - Y_{k-1}|.1_{A(k)} \geq 4W_1.1_{A(k)}$ and

$Y_k.1_{A(k)} = Y_q.1_{A(k)}$. We put $B(k) = \Omega\setminus A(k)$.

We have :

$$W_q = \left\{W_1^2 + \sum_{k=2}^{q}(Y_k-Y_{k-1})^2.1_{A(k)} + \sum_{k=2}^{q}(Y_k-Y_{k-1})^2.B(k)\right\}^{1/2}$$

The inequality $\sqrt{a^2+x^2} \leq a + \frac{1}{2}\frac{x^2}{a}$ (for $a > o$) implies :

$$W_q \leq C + D \qquad \text{with}$$

$$C = \left\{W_1^2 + \sum_{k=2}^{q}(Y_k - Y_{k-1})^2.1_{A(k)}\right\}^{1/2}$$

$$D = \frac{1}{2W_1} \sum_{k=2}^{q} (Y_k - Y_{k-1})^2 . 1_{B(k)}$$

But if w and y are two positive numbers, we have :

$$\sqrt{w^2 + y^2} \leqslant w + y \quad \text{then}$$

$$C \leqslant W_1 + \sum_{k=2}^{q} |Y_k - Y_{k-1}| . 1_{A(k)}$$

Moreover :

$$D = \frac{1}{2W_1} \sum_{k=2}^{q} (Y_k - Y_{k-1})^2 - \frac{1}{2W_1} \sum_{k=2}^{q} (Y_k - Y_{k-1})^2 . 1_{A(k)}$$

$(Y_k)_{1 \leqslant k \leqslant q}$ being a martingale and W_1 being $\mathcal{G}_1$-measurable, the expectation of the first sum above is equal to

$$E\left[\frac{1}{2W_1} . (Y_q - Y_1)^2\right] ,$$

Then, we have $E(D) = E(D')$ with :

$$D' = \frac{1}{2W_1} (Y_q - Y_1)^2 - \frac{1}{2W_1} \sum_{k=2}^{q} (Y_k - Y_{k-1})^2 . 1_{A(k)}$$

$$= \frac{1}{2W_1} \sum_{k=2}^{q} \left[(Y_q - Y_1)^2 - (Y_k - Y_{k-1})^2\right] . 1_{A(k)}$$

$$+ \frac{1}{2W_1} (Y_q - Y_1)^2 . 1_B$$

where

$$B = \bigcap_{k=2}^{q} B(k) = \Omega \setminus \left[\bigcup_{k=2}^{q} A(k)\right]$$

But, if $\omega \in A(k)$

$$|Y_k - Y_{k-1}| = |Y_q - Y_{k-1}| \leqslant 2 |Y_q - Y_1|$$

(because $|Y_q| \geqslant 6W_1$, $|Y_{k-1}| \leqslant 2W_1$ and $|Y_1| \leqslant 2W_1$)

At last, we obtain :

$C + D' \leqslant W_1 + 8 |Y_q - Y_1|$, id est :

$E(W_q) \leqslant E(C + D') \leqslant E(W_1) + 8 E(|Y_q - Y_1|)$.

H.6 - PROOF OF $N_2^+ (X) \leqslant 45. N_1 (X)$

1°) It is sufficient to prove this inequality for a martingale X with respect to a stochastic basis $(\Omega, \mathcal{F}, P, (\mathcal{F}_k)_{1 \leqslant k \leqslant q})$ (id est card (T) < +∞).

For each integer n, we put $X_q^n = X.1_{\{|X| \leqslant n\}}$ and we consider the martingale X^n defined by $X_k^n = E(X_q^n | \mathcal{F}_k)$. By the Lebesgue theorem, we have $N_2^+(X) = \lim_{n\to\infty} N_2(X^n)$ and $N_1(X) = \lim_{n\to\infty} N_1(X^n)$. Thus it is sufficient to prove the inequality for each martingale X_n ; thus, it is sufficient to prove the inequality if X is a square integrable martingale. Let V be the quadratic variation of X, id est :

$$V_k = \left[\sum_{j=1}^{k-1} (X_{j+1} - X_j)^2\right]^{1/2}$$

We put $X^* = \sup_{1 \leqslant k \leqslant q} |X_k|$

2°) We define the sequence $(u(n), W_n, W'_n)_{n>o}$ by the following way :

$u(1) = 1$ and $W_1 = \varepsilon + |X_1|$ with $\varepsilon > o$

$u(n+1) = \inf.\{ t : t \geqslant u(n), |X_t| > 2.W'n\}$

$$W_{n+1} = \left\{ W'^2_n + \sum_{k=1}^{q-1} (Y^n_{k+1} - Y^n_k)^2 \right\}^{1/2}$$

with $Y^n_k = X_{[k \vee u(n) \wedge u(n+1)]}$

$W'_{n+1} = \sup.\{ W_{n+1}, |X_{u(n+1)}|\}$

We put $\mathcal{G}^n_k = \mathcal{F}_{[k \wedge u(n) \vee u(n+1)]}$

For each integer n, we can apply the lemma H.5 to the martingale $(Y^n_k)_{1 \leqslant k \leqslant q}$ which is a martingale with respect to the stochastic basis $(\Omega, \mathcal{F}, P, \mathcal{G}^n_k)_{1 \leqslant k \leqslant q})$; thus, we have :

$$\sum_{n=1}^{\infty} E(W'_{n+1} - W'_n) \leqslant \varepsilon + 9. \sum_{n=1}^{\infty} E\left|[X_{u(n+1)} - X_{u(n)}|\right|$$

But, for each integer n, we have $V_{u(n)} \leqslant W'_n$, thus

$$E(V_q) \leqslant \varepsilon + 9 \sum_{n=1}^{\infty} E(|X_{u(n+1)} - X_{u(n)}|)$$

For each integer, we put $A(n) = \{ u(n) < u(n+1)\}$,

$B(n) = \{A(n) \cap u(n+1) = q\}$ and $G(n) = A(n) \setminus B(n)$.

By the definition of u(n), if $\omega \in G(n)$ we have : $|X_{u(n+1)}(\omega)| \geqslant 2 W'_n(\omega) \geqslant 2 |X_{u(n)}(\omega)|$; then, in this case, we have also :

$$|X_{u(n+1)} - X_{u(n)}| (\omega) \leqslant 3.\left[| X_{u(n+1)}(\omega)| - |X_{u(n)}(\omega)|\right]$$

This implies :

$$\sum_{n=1}^{q} |X_{u(n+1)} - X_{u(n)}| \leqslant 3 \sum_{n=1}^{\infty} \left[|X_{u(n+1)}| - |X_{u(n)}|\right] . 1_{G(n)}$$

$$+ \sum_{n=1}^{\infty} |X_{u(n+1)} - X_{u(n)}| . 1_{B(n)}$$

The sets $(B(n))_{n>o}$ being disjoint, the second sum is bounded by $2 X^*$; the sequence of sets $(G(n))_{n>o}$ being decreasing, the first sum is bounded by $3 \sup_{k \leqslant q} (|X_k| - |X_1|) \leqslant 3 X^*$

At last, we have : $E(V_q) \leqslant \varepsilon + 45. E(X^*)$

and that proves the expected inequality when ε goes to zero.

H.7 - LEMMA

Let $(\Omega,\mathcal{F},P)$ *be a probabilized space ; let* A *and* B *be two elements of* $\mathcal{F}$ *and* f *be a one-to-one measurable mapping from* A *into* B *which preserves the measure* P *, i.e.* $P(G) = P[f(G)]$ *and* $P(D) = P[f^{-1}(D)]$ *if* G *and* D *are two elements of* $\mathcal{F}$ *with* $G \subset A$ *and* $D \subset B$ *; let* X *be an element of* $L_1(\Omega,\mathcal{F},P)$ *such that* $X.1_A = X$ *; we put* $Y(\omega) = X(\omega) - X[f(\omega)]$*; let* a *be a real number. Then we have :*

$$E(|Y + a.1_A - a.1_B|) \geqslant E(|Y|)$$

Proof

We have :

$$E(|Y+a.1_A-a.1_B|)=E(|X+a|1_A)+E(|X-a|.1_A)$$

and $E(|Y|) = 2\,E(|X|) = 2\,E(|X|.1_A)$

Then, we put :

$$q = E[(X-a)^+] \ , \quad s = E[(X+a)^-] \ ,$$

$$p = E(X^+) - q \ , \quad r = E(X^-) - s$$

$$u = P(\{X > o\}), \quad v = P(\{X < o\})$$

Then, we have :

$$E(|X|)=p+q+r+s \ ; \ E(|X+a|.1_A) \geqslant p+q+au+s$$

$$E(|X-a|.1_A) \geqslant q+r+s+av \ ; \ p \leqslant au \ ; \ r \leqslant av$$

and that implies :

$$E(|X+a|.1_A) + E(|X-a|.1_A) \geqslant 2\,E(|X|.1_A)$$

H.8 - LEMMA

Let $(Z_k)_{1\leqslant k\leqslant q}$ *be a martingale with respect to the stochastic basis* $(\Omega,\mathcal{F},P,(\mathcal{F}_k)_{1\leqslant k\leqslant n}$ *; we suppose that* $(Z_k-Z_{k-1})_{1\leqslant k\leqslant q}$ *is a family of Rademacher functions. Let* $(d_k)_{1\leqslant k\leqslant q-1}$ *be a family of real numbers. We put*

$$S_k = \sum_{j=1}^{k-1} d_j.(Z_{j+1} - Z_j) = E(S_q|\mathcal{F}_k)$$

Then, we have :

$$E(|S|) = N_4(S) = N_3(S) \geqslant \frac{1}{180} N_2^+(S) = \frac{1}{180} \sum_{j=1}^{q-1} (d_j)^2$$

Proof

Let us recall that (Z_k-Z_{k-1}) is a Rademacher function if Z_k-Z_{k-1} = +1 or -1 with the probability 1/2 respectively.

The inequality $N_3(S) \geqslant \frac{1}{180} N_2^+(S)$ is a corollary of the inequalities given in H.2 and H.6.

Actually, it is possible to say better (Khintchine lemma).

Now, we have to prove that $N_3(S) \leqslant E(|S|)$ (of course, we have $N_3(S) \geqslant E(|S|)$).

For each integer k, with $1\leqslant k\leqslant q$, let V_k be an $\mathcal{F}_k$-measurable random variable such that $o\leqslant V_k\leqslant 1$.

Now, se suppose that $V_k = 1$ if $k > p$ and we prove at first, that

$$E\left\{\left|\sum_{k=o}^{q-1} V_k.d_k(Z_{k+1}-Z_k)\right|\right\} \leqslant E\left\{\left|\sum_{k=o}^{q-1} \bar{V}_k.d_k(Z_{k+1}-Z_k)\right|\right\}$$

if $\bar{V}_k = V_k$ for $k \neq p$ and $\bar{V}_p = 1$.

By convexity, we can suppose that $V_k(\omega) = o$ or $V_k(\omega) = 1$ (for each element ω of Ω).

The inequality above is a corollary of the lemma H.7 above if we put $A = \{(1-V_k)(Z_{k+1}-Z_k) > o\}$ and $B = \{(1-V_k)(Z_{k+1}-Z_k) < o\}$ and $a = d_k$

Then, by reasoning by recurence on p, we obtain :

$$E\left(\left|\sum_{k=o}^{q-1} V_k.d_k.(Z_{k+1}-Z_k)\right|\right) \leqslant E\left(\left|\sum_{k=o}^{q-1} d_k(Z_{k+1}-Z_k)\right|\right)$$

H.9 - PROOF OF $N_2^+(X) \leqslant 180.N_4(X)$

It is sufficient to prove this inequality if $(X_k)_{1\leqslant k\leqslant q}$ is a martingale with respect to the stochastic basis $(\Omega,\mathcal{F},P,(\mathcal{F}_k)_{1\leqslant k\leqslant q})$. In this case, let $(Y_k)_{1\leqslant k\leqslant q-1}$ be a family of Rademacher functions defined on another probability space $(\Omega',\mathcal{F}',P')$. We put $D_k = X_{k+1}-X_k$.

For each element ω' of Ω', we have :

$$\int_\Omega \left|\sum_{k=1}^{q-1} Y_k(\omega').D_k(\omega)\right|.\,P(d\omega) \leqslant N_4(X)$$

This implies :

$$\int_\Omega \int_{\Omega'} \left|\sum_{k=1}^{q-1} Y_k(\omega').D_k(\omega)\right|.P(d\omega).P'(d\omega') \leqslant N_4(X)$$

By the Fubini theorem, we have also :

$$\int_\Omega \int_{\Omega'} \left|\sum_{k=1}^{q-1} Y_k(\omega').D_k(\omega)\right|.P'(d\omega').P(d\omega) \leqslant N_4(X)$$

But, the lemma H.8 implies :

$$\int_{\Omega'} \left|\sum_{k=1}^{q-1} Y_k(\omega').D_k(\omega)\right| P(d\omega') \geqslant \frac{1}{180}\left\{\sum_{k=1}^{q-1} [D_k(\omega)]^2\right\}^{1/2}$$

At last, we obtain :

$$N_4(X) \geqslant \int_\Omega \frac{1}{180}\left\{\sum_{k=1}^{q-1} [D_k(\omega)]^2\right\}^{1/2}.P(d\omega)$$

$$\geqslant \frac{1}{180} N_2^+(X)$$

H.10 - THE SPACE H_1 (remarks)

Let $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\in T})$ be a stochastic basis. One notes H_1 the space of all the real cadlag martingales X such that $N_2^+(X) < +\infty$. If Y is an uniformly bounded predictable real process, it is possible to define the stochastic integral $\int YdX$ as in B.6 (cf. F.14). Actually, it is not necessary to use B.6 and F.14 if we know the inequality $N_3(X) \leq 8.N_2^+(X)$; in this case, we can define the stochastic integral $\int YdX$ as follows :

For each $\mathcal{A}$-simple real process Y (cf. B.2), we put

$$N(Y) = \lim_{n\to\infty}\sup.\ E\left\{\left[\sum_{k\leq o} Y^2_{k.2^{-n}}.(X_{(k+1)2^{-n}}-X_{k.2^{-n}})^2\right]^{1/2}\right\}$$

this defines a semi-norm on $\mathcal{E}$ (cf. B.2) such that, for each element B of $\mathcal{A}$, we have :

$$\left\|\int_B Y\cdot dX\right\|_{L_1(\Omega,\mathcal{F},P)} \leq 8.\ N(Y)$$

Thus, the mapping $Y\to\int YdX$ is a linear mapping from $\mathcal{E}$ into $L_1(\Omega,\mathcal{F},P)$ and this mapping is continuous if we consider on H_1 the topology associated to the semi-norm N and, on $L_1(\Omega,\mathcal{F},P)$, the usual topology. Then, this mapping can be extended by density and this defines the stochastic integral $\int YdX$ notably for each uniformly bounded predictable real process ; moreover, it is possible to define the stochastic integral process $Z = \int YdX$ as in B.5 and we see that Z is also an element of H_1. We can also prove, exactly as in B.5, that H_1 is a complete space.

Actually, if we consider the additive mapping $B \to \int 1_B.dX$, defined on $\mathcal{A}$ and with values in $L_1(\Omega,\mathcal{F},P)$, we can see that the semi-norm $N_3(X)$ is exactly the semi-norm of the semi-variation as considered in [Bar] ; thus, this mapping $B\to\int 1_B.dX$ can be extended in vector measure ; that proves notably that the family of random variables $(\int YdX)$, for all the bounded real predictable processes Y, is uniformly equi-integrable (classical property of the vector measures : see [BDS]).

Moreover, let A be the increasing process-defined by :

$$A_t = \lim_{n\to\infty}\sup.\left\{\sum_{k=o}^{\infty}\left[A_{(k+1)2^{-n}\wedge t} - A_{k.2^{-n}\wedge t}\right]^2\right\}$$

if t is a dyadic number and $A_t = \lim_{s\downarrow t} A_s$ in the general case. It is easily seen that A is a cadlag increasing process ; A is the quadratic variation process of X as studied in C.6. All the previous remarks if Y is a bounded predictable process are also available if Y is predictable process such that $E\{\int Y^2 dA\} < +\infty$.

J - STOCHASTIC INTEGRAL CONSIDERED AS A GROUP-VALUED INTEGRAL

J.1 - INTRODUCTION

Let B be a Banach space and X be a B-valued process. We want to build and study the stochastic integral $\int Y dX$ if Y is a real predictable process. In B.2, we saw that this integral is defined in a natural way when Y belongs to the vector space $\mathcal{E}$. The mapping $Y \to \int Y dX$ can be considered as a linear mapping from $\mathcal{E}$ into $L_o^B(\Omega,\mathcal{F},P)$. The problem is to extend this mapping to a family of processes larger than $\mathcal{E}$.

From this point of view, we are in a typical situation where we have to extend a classical integral, with values in the group $L_o^B(\Omega,\mathcal{F},P)$ and defined on the set $\mathcal{E}$ of all the real $\mathcal{A}$-simple functions ($\mathcal{A}$ being an algebra). Then, in such a context, it is possible to use the classical results on vector or group-valued integral. Actually, this method permits to obtain some specific results.

In this paragraph J, we consider only the case where Y is a real process ; the methods given here are still available when Y is a Banach space valued process ; they are also available if the "time" T is not an interval of the real line (cf. [MeP-2]).

J.2 - HYPOTHESIS AND NOTATIONS

In this paragraph, we consider :

- a Banach space B
- a stochastic basis $(\Omega,\mathcal{F},P(\mathcal{F}_t)_{t\in T})$ with $T = [o,1]$
- a B-valued process X defined up to modification

We put :

$L_p^B = L_p^B(\Omega,\mathcal{F},P)$ with $p \geq o$

$\Omega' = \Omega \times (T \setminus \{o\})$

The sets $\mathcal{R}$, $\mathcal{A}$, $\mathcal{P}$ and $\mathcal{E}$ are defined as in A.5 and B.2. We consider also :

$\mathcal{E}_o = \{Y : Y \in \mathcal{E}, \sup_{t,\omega} |Y_t(\omega)| \leq 1\}$

For each element Y of $\mathcal{E}$, we define the stochastic integral $\int Y.dX$ as in B.2. Moreover, for each element A of $\mathcal{A}$, we put $x(A) = \int 1_A.dX$. Then x is an additive function defined on $\mathcal{A}$ and with values in L_o^B ; moreover the stochastic integral $\int Y dX$ is a classical integral of Y, considered as a function defined on Ω', with respect to x.

In this paragraph, we give necessary and sufficient conditions (cf. theorem J.5) such that this integral can be extended to the family of all the uniformly bounded predictable processes We will use the lemma E.2 and the two following lemmas that we recall for the convenience of the reader.

J.3 - A BOUNDED ADDITIVE FUNCTION (LEMMA)

Let $||.||$ be an F-norm (cf. [MaO]) on the vector space $\mathcal{U}$ and x be an $\mathcal{U}$-valued additive function defined on an algebra $\mathcal{A}$. We suppose that $\lim_{n\to\infty} x(A_n) = o$ for each sequence $(A_n)_{n>o}$ of disjoint elements of $\mathcal{A}$. Let v be the function defined on $\mathcal{A}$ by $v(A) = \sup_{B\in\mathcal{A}, B\subset A} ||v(B)||$

Then, for each element A of $\mathcal{A}$, $v(A) < +\infty$

For the proof, see the corollary 4.11 in [Dre].

J.4 - DANIELL THEOREM

We consider the hypothesis and notations given in J.2. Moreover, let p be a non negative real number. We suppose that the following properties are fulfiled :

(i) for each element Y of $\mathcal{E}$, $\int Y dX$ belongs to L_p^B

(ii) for each sequence $(Y_n)_{n>o}$ of elements of $\mathcal{E}$ such that $Y_n \downarrow o$, the sequence $(\int Y_n dX)_{n>o}$ converges to zero in L_p^B

(iii) if $(Y_n)_{n\geq o}$ is a sequence of elements of $\mathcal{E}$ such that $\sum_{n\geq 1} Y_n \leq Y_o$, the sequence $(\int Y_n dX)_{n>o}$ converges to zero in L_p^B.

Then, the mapping $Y \to \int Y dX$ can be extended in a linear mapping, defined on the set of all the uniformly bounded predictable processes, with values in L_p^B and which satisfies the Lebesgue dominated convergence theorem.

This theorem is proved in [Pel-1].

J.5 - EXTENSION THEOREM

We consider the hypothesis and notations given in J.2. *Let* p *be a non negative real number and* X *be* B*-valued process, defined up to modification, which satisfies the following properties :*

(i) *for each element* s *of* T, $\lim_{n\to\infty} (X_t - X_s) = o$ *for the usual topology of* L_p^B.

(ii) $\{Z : Z = \int 1_A.dX,\ A \in \mathcal{A}\}$ is a *bounded (with the Bourbaki meaning) subset of* L_p^B.

(iii) *for each sequence* $(A(n))_{n>o}$ *of disjoint elements of* $\mathcal{A}$*, the sequence* $(\int 1_{A(n)}.dX)_{n>o}$ *converges to zero for the usual topology of* L_p^B.

Then, the mapping $Y \to \int YdX$ *can be extended in a (unique) linear mapping, defined on the set of all the uniformly bounded predictable processes, with values in* L_p^B *and which satisfies the following dominated convergence property :*

(iv) *if* $(Y_n)_{n\geqslant o}$ *is a sequence of uniformly bounded predictable processes such that* $|Y_n| \leqslant |Y_o|$ *for each integer* n *and such that* $Y = \lim_{n\to\infty} Y_n$ *then, we have :* $\int YdX = \lim_{n\to\infty} \int Y_n dX$ *(for the usual topology of* L_p^B *).*

Moreover, if B *is a finite dimensionnal vector space, the condition* (iii) *is necessarly fulfiled.*

Proof

1°) For each random variable which belongs to L_p^H, we put :

$$||f||_p = \left[\int |f(\omega)|^p.P(d\omega) \right]^{1/2} \quad \text{if } p \geqslant 1 \text{ (usual norm)}$$

$$||f||_p = \int |f(\omega)^p|.P(d\omega) \quad \text{if } o < p \leqslant 1$$

$$||f||_o = \int \left[|f(\omega)| \wedge 1 \right].P(d\omega) \quad \text{if } p = o$$

Then $||.||$ is an F-norm (cf. [MaO]) associated to the usual topology of L_p^B.

In the following, we note $||.||$ instead of $||.||_p$ and we put $x(A) = \int 1_A.dx$ if A is an element of $\mathcal{A}$.

2°) At first, we consider the case where $B = \mathbb{R}$ in this case, the spaces $L_p^B = L_p^R$ (pour $p \geqslant o$) satisfy the hypothesis of the theorem 3 of [MaO]; let $(A(n))_{n>o}$ be a sequence of disjoint elements of $\mathcal{A}$: x being an additive function and $x(\mathcal{A})$ being a bounded subset of L_p^R, the serie $(x(A(n)))_{n>o}$ is "perfectly bounded" (cf. [MaO]) ; thus, it is convergent. That proves the end of the theorem when $B = \mathbb{R}$; thus, we have the same property when B is a finite dimensionnal vector space.

3°) Now, we suppose that B is a general Banach space. Let v be the non negative function defined on the subsets of $(\Omega\times T)$ by :

$$v(A) = \sup_{C\in\mathcal{A}, C\subset A} ||x(C)||$$

At first, we prove that the restriction of v to $\mathcal{A}$ satisfies the properties (i), (ii) and (iii) of the lemma E.2. The condition (i) is obviously satisfied and the condition E.2 (ii) is the condition J.5 (i). If the condition E.2 (iii) is not fulfiled, there exist $\varepsilon > o$, an increasing sequence of stopping time $(u(n))_{n>o}$ such that $F(n) = [u(n) < 1] \neq \emptyset$ and a sequence $(A_n)_{n>o}$ of elements of $\mathcal{A}$ such that, for each integer n, $A_n \subset (F(n)\times T)$ and $||x(A_n)|| \geqslant 8\varepsilon$

We have $v(F(1)\times T) = a < +\infty$ (cf. J.3). Let D be a set such that $D \subset (F(1)\times T)$, $D \in \mathcal{A}$ and $||x(D)|| \geqslant a-\varepsilon$; let k be an integer such that $k>1$ and $||x(D).1_{F(k)}|| \leqslant \varepsilon$; let E be a set which belongs to $\mathcal{A}$ and such that $E \subset (F(k)\times T)$ and $||x(E)|| \geqslant 8\,\varepsilon$; We have $||x(E\setminus D)|| \geqslant 4\,\varepsilon$ or $||x(E\cap D)|| \geqslant 4\,\varepsilon$; in the first case, we have :

$$||x(E\cup D)|| \geqslant a+\varepsilon - \varepsilon + (4\varepsilon - \varepsilon) \geqslant a+3\varepsilon$$

In the second case, we have :

$$||x(D\setminus E)|| \geqslant a+2\,\varepsilon$$

In the two cases, this implies $v(F(1)\times T) \geqslant a+2\,\varepsilon$ but this is impossible ; then the condition E.2 (iii) is fulfiled and we can apply the lemma E.2.

4°) Now, we prove the properties J.4 (ii) and (iii). At first, the set $\{Z = Z = \int YdX,\ Y \in \mathcal{E}_o\}$ is a bounded subset of L_p^B (according to [Tur], J.5 - (ii) and the properties of the F-norm considered). That means that, for each $\varepsilon > o$, there exists $\eta(\varepsilon)$ such that $\sup_{t,\omega} |Y_t(\omega)| \leqslant \eta(\varepsilon)$ implies $||\int YdX|| \leqslant \varepsilon$. We fix ε and $\eta(\varepsilon)$.

Let $(Y_n)_{n>o}$ be a sequence of non negative $\mathcal{A}$-simple processes such that $Y_n \downarrow o$; for each integer n, we put $A(n) = [Y_n > \eta(\varepsilon)]$ and $B(n) = \Omega \setminus A(n)$; we have $\| \int Y_n . 1_{B(n)} . dX \| \leqslant \varepsilon$ (because $Y_n . 1_{B(n)} \leqslant \eta(\varepsilon)$) ; moreover, $A(n) \downarrow \emptyset$; thus $v(A(n)) \downarrow o$ (cf. 3°) above) ; then, $\lim_{n\to\infty} \int Y_n . 1_{A(n)} . dX = o$ and that proves the condition J.4 (ii).

5°) Now, we prove that the condition J.4 (iii) is fulfiled. Let $(Y_n)_{n>o}$ be a sequence of non negative $\mathcal{A}$-simple processes such that $\sum_{n>o} Y_n \leqslant 1$. Let ε be a positive real number ; we consider $\eta(\varepsilon)$ as in the 4°) above ; we can suppose that $\eta(\varepsilon) = 1/k$ (k depends on ε). For each integer n, we put $A(n) = [Y_n > \eta(\varepsilon)]$; as in the 4°) above, we have to prove that

$$\lim_{n\to\infty} \int Y_n . 1_{A(n)} . dX = o$$

For that, we consider the following sequences of sets :

$B(o,o) = \Omega$, $B(n,-1) = \emptyset$ if $n \geqslant o$

$B(o,j) = \emptyset$ if $j \geqslant 1$ and, for $n \geqslant 1$ and $j \geqslant 1$:

$B(n,j) = [B(n-1,j-1) \cap A(n)] \cup [B(n-1,j) \setminus A(n)]$

$C(n,j) = B(n-1,j-1) \cap A(n) = B(n,j) \cap A(n)$

If $j>k$, we have $B(n,j) = \emptyset$; moreover, for each integer n, $\{C(n,j)\}_{1\leqslant j\leqslant k}$ is a partition of A(n). At last, for each integer j, the sets $\{C(n,j)\}_{n>o}$ are disjoint. According to [Tur] and E.2 (iv), we have :

$$\lim_{n\to\infty} \int Y_n \, 1_{C(n,j)} \, dX = 0$$

for each integer j with $1\leqslant j\leqslant k$.
But this implies : $\lim_{n\to\infty} \int Y_n . 1_{A(n)} . dX = o$ (because $\{C(n,j)\}_{1\leqslant j\leqslant k}$ is a partition of A(n)) and that proves J.4 (iii).

Then, we can apply the Daniell theorem J.4 and that completes the proof.

J.6 - STOCHASTIC INTEGRAL PROCESS

We consider the hypothesis and notations given in the theorem J.5. Moreover, we suppose that X *is an adapted cadlag process (X is a process in the " strict " sense). Let* $\hat{Z}$ *be the process defined, up to modification, by* $\hat{Z}_t = \int 1_{]o,t]} Y.dX$ *where* Y *is a uniformly bounded predictable real process. Then, there exists a cadlag adapted process* Z *which is a modification of* $\hat{Z}$*. Moreover, we have the following dominated convergence theorem.*

If $(Y_n)_{n>o}$ *is a sequence of predictable real processes such that* $\sup_{n,t,\omega} |Y_n(t,\omega)| \leqslant 1$ *and such that* $(Y_n)_{n>o}$ *converges to* Y*, there exists a subsequence* $(Y_{n(k)})_{k>o}$ *such that the sequence* $(Z_{n(k)})_{k>o}$ *associated as above converges P-a.e. uniformly for each sample function to the process* Z *associated to* Y *as above.*

Proof

This theorem can be proved exactly as in B.5 by using the Borel Cantelli lemma and the "outer" measure v.

J.7 - REMARK

1°) We consider the hypothesis and notations given in the theorem J.5. If $p \geqslant 1$, L_p^B is a Banach space ; thus x is a Banach space valued additive function : then, it is possible to use and apply all the classical results on vector measures ; of course, the hypothesis considered in J.5 are more general.

2°) The stochastic basis $(\Omega,\mathcal{F},P, (\mathcal{F}_t)_{t\in T})$ being fixed, let $\mathcal{S}$ be the space of all the B-valued cadlag adapted processes ; if $p\geqslant o$, for each element Z of $\mathcal{S}$, we put :

$|||Z|||_p = \sup_{Y\in \mathcal{E}_o} \| \int Y.dZ \|_p$ (where $\|.\|_p$ is defined as in the proof of J.5 above and $\mathcal{E}_o$ as in J.2).

We put $\mathcal{S}_p = \{Z : Z \in \mathcal{B}, |||Z|||_p < +\infty \}$

It is easily seen that $\mathcal{S}_p$ is complete for the topology associated to the F-norm $|||.|||_p$ (as in B.5).

Now, let X be an element of $\mathcal{S}_p$; for each element A of $\mathcal{A}$, the stochastic integral process $\hat{x}(A) = \int 1_A dX$ can be considered as an element of $\mathcal{S}_p$; then $\hat{x}$ can be considered as an additive function defined on $\mathcal{A}$ and with values in $\mathcal{S}_p$; moreover, if Y is an element of $\mathcal{E}$, the stochastic integral process $\int YdX$ can be considered as the usual integral of Y, considered as a real function defined on $\Omega' = \Omega \times (T\setminus\{o\})$, with respect to $\hat{x}$; it is possible to write the theorem J.5 in this new context.

EXERCISES

EXERCISE A.1

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in T})$ be the probabilized stochastic basis defined by : $\Omega = \{a,b\}$ (set including two elements a and b), $\mathcal{F} = \mathcal{P}(\Omega)$ (set including all the parts of Ω), $P(\{a\}) = P(\{b\}) = \frac{1}{2}$ $T = [0,1]$ (unit interval of the real line), $\mathcal{F}_t = (\emptyset, \Omega)$ (trivial σ- algebra) if $t \leqslant 1/2$ and $\mathcal{F}_t = \mathcal{F}$ if $t > 1/2$.

1°/ We put $u(a) = 1$ and $u(b) = 1/2$
$v(a) = 0$ and $v(b) = 1/2$

Are u and v stopping times ?

2°/ Have you $\mathcal{F}_t = \mathcal{F}_{t+}$ for each element t of T ?

3°/ Are u and v stopping times for the family $(\mathcal{F}_{t+})_{t \in T}$?

4°/ Is $[0,u[$ a predictable set ?

5°/ Is $X = 1_{[0,u[}$ a predictable process ? an adapted process ?

6°/ Let $(w_n)_{n>0}$ be the sequence of random variables defined by, for each integer n, $w_n(a) = 1$ and $w_n(b) = 1/2 + 1/n$. We put $w = \inf_{n>0} . w_n$. Is w_n a stopping time ?
Is w a stopping time for the family $(\mathcal{F}_t)_{t \in T}$? for the family $(\mathcal{F}_{t+})_{t \in T}$?

EXERCISE A.2

We define $(\Omega, \mathcal{F}, P)$ as in the exercise 1. Let Y be the process defined by $Y_t(b) = 0$ for each element t of T and $Y_t(a) = 0$ for $t \leqslant 1/2$ and $Y(a) = t - 1/2$ if $t \geqslant 1/2$. For each element t of T, let $\mathcal{G}_t$ be the σ-algebra generated by the random variables $(Y_s)_{s \leqslant t}$ (i.e. the smallest σ-algebra for which these random variables are measurable). Compare these σ-algebras $(\mathcal{G}_t)_{t \in T}$ and the σ-algebras $(\mathcal{F}_t)_{t \in T}$ of the exercice A.1. Is the process Y adapted, or predictable, with respect to the stochastic basis $(\Omega, \mathcal{G}, P, (\mathcal{G}_t)_{t \in T})$? In this situation, $(\Omega, \mathcal{G}, P, (\mathcal{G}_t)_{t \in T})$ is often called the canonical stochastic basis of the process Y.

EXERCISE A.3

Let $(u(n))_{n>0}$ be an increasing sequence of stopping times. We put $u = \operatorname{Sup}_{n>0} . u(n)$. Is u a stopping time ?

EXERCISE A.4

You can do the exercices A.4 and A.5 together. Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in T})$ be a stochastic basis with $T = [0,1]$. Say if the following assertions are true or false : (to show that one of the following assertion is false, you can use the exercise A.1).

1°/ Let u be a T-valued random variable defined on $(\Omega, \mathcal{F}, P)$; then u is a stopping time if and only if, for each element t of T, the set $A_t = \{\omega : u(\omega) < t\}$ belongs to $\mathcal{F}_t$.

2°/ Let u be a T-valued random variable defined on $(\Omega, \mathcal{F}, P)$; then u is a stopping time if and only if, for each element t of T, the random variable $(u \wedge t)$ is $\mathcal{F}_t$-measurable.

3°/ Let u and v be two stopping times with $u \geqslant v$. Let w the random variable defined by :

$$w(\omega) = 1 \quad \text{if} \quad u(\omega) > v(\omega)$$
$$w(\omega) = u(\omega) \quad \text{if} \quad u(\omega) = v(\omega)$$

Then w is a stopping time.

EXERCISE A.5

Do the exercise A.4 if the family $(\mathcal{F}_t)_{t \in T}$ is assumed to be right continuous.

EXERCISE B.1

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in T})$ be a stochastic basis. We note K the vector space of all the real cadlag processes adapted to this stochastic basis. We suppose that there exists a positive mapping N defined on K such that :

(i) $N(X + Y) \leqslant N(X) + N(Y)$

(ii) $N(aX) = |a| . N(X)$ for each real number a

(iii) $N(\sum_{n>0} X_n) \leqslant \sum_{n>0} N(X_n)$

We note H the vector space of the elements X of K such that $N(X) < + \infty$. We suppose that, for each element X of H and for each real $\mathcal{A}$-simple process (cf. B.2), the process Z defined by $Z_t = \int_{]0,t]} Y.dX$ is such that : $N(Z) \leqslant N(X) . \operatorname{Sup}_{t,\omega} . |Y_t(\omega)|$.

Prove, by reasoning as in B.5, that H is a complete space.

EXERCISE B.2

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in T})$ be a stochastic basis. Let X be a real cadlag process adapted to this stochastic basis. We suppose that there exists a positive measure a such that, for each real $\mathcal{A}$-simple process (cf. B.2), we have :

$$E\left(\left| \int Y.dX \right| \right) \leq \left\{ \int |Y|^2.da \right\}^{1/2}$$

Is it possible to build, as in B.3, the stochastic integral $\int Y.dX$ for all the processes Y which belong to $L_2(\Omega \times T, \mathcal{P}, a)$?

EXERCISE C.1

Let X and V be two real continuous processes ; we suppose that V is with bounded variation and that X satisfies the assumptions given in C.5 (thus, notably, we can use the Ito formula).

1°/ For each integer n, let $(u(n,k))_{k\geq 0}$ be the sequence defined by recurrence by $u(n,0) = 0$ and
$u(n,k+1) = \inf\{ t : t \geq u(n,k),$
$|M_t - M_{u(n,k)}| + |V_t - V_{u(n,k)}| > 1/n \}$

Calculate $[M,V]_1$,id est :

$$\lim_{n\to\infty} \sum_{k>0} [M_{u(n,k+1)} - M_{u(n,k)}].[V_{u(n,k+1)} - V_{u(n,k)}]$$

2°/ Prove that the quadratic variations $[M,M]$ and $[M+V,M+V]$ of M and (M+V) are equal.

3°/ Prove the following equality (integration by parts) :

$$M_tV_t - M_oV_o = \int_o^t V_s.dM_s + \int_o^t M_s.dV_s$$

Indication : apply the Ito formula to the process (M_t,V_t), considered as an $\mathbb{R}^2$-valued process, and to the function $(x,y) \rightsquigarrow x.y = F(x,y)$

4°/ Let W be a brownian motion ; by admiting that W = X satisfies the assumptions given in C.5 (see the paragraph E), prove that W is not a process with bounded variation (see the 1°/ above)

5°/ Study the case where X and V are two real cadlag processes but X and V are not continuous processes.

EXERCISE C.2

Let X be a cadlag real martingale and Y be a real predictable uniformly bounded process. You will admit that X satisfies all the properties given in C.5. Let Z be the cadlag process defined by :

$$Z_t = \int_{]o,t]} Y.dX.$$

1°/ Prove that Z is a martingale

2°/ Let H be a real predictable uniformly bounded process. Prove that we have :

$$\int_{]o,t]} H_s.dZ_s = \int_{]o,t]} H_s.Y_s.dX_s$$

(id est, with the symbolic differential notation, if $dZ = YdX$, then $HdZ = H.Y.dX$).

3°/ Prove that we have :

$$[Z,Z]_t - [Z,Z]_o = \int_{]o,t]} Y_{s-}^2.d[X,X]_s$$

where $[.,.]$ is the quadratic variation.
(id est, with the symbolic differential notation, $d[Z,Z]_t = Y_{t-}^2.d[X,X]_t$).

Indication : For all these questions, you can begin to consider the case where Y is an $\mathcal{A}$-simple process (cf. B.2).

EXERCISE C.3 (some properties of the brownian motion)

Let $(W_t)_{t\in[0,1]}$ be a real brownian motion. You will admit that this process satisfies all the properties given in C.5 and you can use the exercise C.2 1°/ and 2°/.

1°/ Let F be an element of $\mathcal{F}_s$. We put :

$$f(u) = E[1_F.e^{ia(W_{s+u} - W_s)}]$$

Show that the function f, considered as a function of u, (F, s and a being fixed), satisfies an elementary differential equation.

2°/ Calculate f(u)

3°/ Calculate $E[e^{ia(W_{s+t} - W_s)} \mid \mathcal{F}_s]$

4°/ What does that mean ?

EXERCISE C.4

Let $(W_t)_{t\in T}$ be a real brownian motion with $T = [0,1]$. For each integer n, let Y^n be the real process defined by :

$$Y^n = \sum_{k=0}^{2^{-n}-1} \frac{W_{(k+1).2^{-n}} - W_{k.2^{-n}}}{|W_{(k+1).2^{-n}} - W_{k.2^{-n}}|}.1_{]k.2^{-n},(k+1).2^{-n}]}$$

Is Y an adapted process ? Is Y a predictable process ? Is Y uniformly bounded ? Prove that the sequence of random variables $Z^n = \int_{]o,1]} Y^n.dX$ goes to the infinity, almost surely, when n goes to the infinity (you can use the exercise C.1.4°/).

This exercice shows that it is not possible to build the stochastic integral for processes Y which are only measurable with respect to the σ-algebra $\mathcal{F} \otimes \mathcal{C}$ (where $\mathcal{C}$ is the σ-algebra of the borelian sets of T).

EXERCISE D.1 (The Ornstein-Uhlenbeck process)

Let W be a real brownian motion. Let f be a real continuous function defined on the real line. Let x_o , a , and b three real numbers. We put :

$$Z_t = e^{ta}.x_o + \int_o^t e^{(t-s)a}.f(s)ds + \int_o^t e^{(t-s)a}\, b\, dW_s$$

1°/ Compare the process Z and the process X which is a solution of the following differential equation :

$$X_t = x_o + \int_o^t b\, dW_s + \int_o^t [aX_s + f(s)].ds$$

Indication : you admit (see § E after) that you can apply the theorem D.5 and the Ito formula to $F(t,Y_t) = e^{ta}\{ x_o + \int_o^t e^{-sa}.f(s).ds + Y_t \}$ where $Y_t = \int_o^t e^{-sa}.b.dW_s$

2°/ Study $E(X_t)$.

3°/ Prove that X is a gaussian process, id est : for each finite family $\{t(k)\}_{1\leq k\leq n}$ of elements of T, the random variable $\{X_{t(k)}\}_{1\leq k\leq n}$ is a gaussian random variable

Indication : you can begin to prove that, for some "good" functions f, the process Z, defined by $Z_t = \int_{]o,t]} f(s).dW_s$, is a gaussian process by using the exercise C.3.

EXERCISE D.2

Let $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\in T})$ be a stochastic basis with $T = [0,1]$. Let $(Z_t)_{t\in T}$ and $(V_t)_{t\in T}$ be two real continuous processes such that we can apply C.5 and D.5. We put : $W_t = -V_t + 1/2[V,V]_t$ and

$$C_t = \exp(-W_t).\int_o^t [\exp.(W_s)].(dZ_s - d[Z,V]_s)$$

Compare the process C and the process Y which is a solution of the following differential equation : $Y_t = Z_t - Z_o + \int_o^t Y_s.dV_s$

Indication : You can apply the Ito formula to the function $F(a,b) = b.e^{-a}$ and to the processes (A,B) where $A_t = W_t$ and $B_t = \int_o^t [\exp(W_s)].(dZ_s - d[Z,V]_s)$ Moreover use the exercises C.1 and C.2.

EXERCISE D.3 (see [WoZ] and [All])

Let $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\in T})$ be a stochastic basis with $T = [0,1]$. Let $(B_t)_{t\in T}$ be a real brownian motion. Let n be an integer. Let $(B^n_t)_{t\in T}$ be the process defined by $B^n_o = 0$, and, for each element t of $[k.2^{-n},(k+1).2^{-n}]$.

$$B^n_t = B_{k.2^{-n}} + 2^n.(t-k.2^{-n})[B_{(k+1).2^{-n}} - B_{k.2^{-n}}]$$

Let σ be a real function, defined on the real line, such that its thirst derivative is uniformly bounded. Let X^n the continuous process defined by

$$X^n_t = \int_o^t \sigma(B^n_s).dB^n_s$$

(this integral being defined, as usual, for each element ω of Ω).

The problem is to see if the sequence of processes $(X^n)_{n>o}$ converges, when n goes to the infinity, to the continuous process X defined by $X_t = \int_o^t \sigma(B_s).dB_s$ (this integral being an usual stochastic integral).

1°/ For each integer n, is the process B^n adapted ? is this process continuous ?

2°/ When n goes to the infinity, does the sequence of processes $(B^n)_{n>o}$ converge, uniformly for each sample function, to the process B ?

3°/ For each integer n, we define the process A^n by :

$$A^n_t = \sum_{k=0}^{2^{-n}-1} B_{k.2^{-n}} . 1_{]k.2^{-n},(k+1).2^{-n}]}$$

Is the process A^n adapted ? Is the process A^n predictable ? Is the process A^n continuous ? Does the sequence of processes $(A^n)_{n>o}$ converge, uniformly for each sample function, to the process B when n goes to the infinity ?

4°/ We consider the following processes :

$$C^n_t = \int_o^t \sigma(A^n_s).dB^n_s$$

$$D^n_t = \int_o^t \sigma'(A^n_s).(B^n_s - A^n_s).dB^n_s$$

$$R^n_t = \int_o^t \sigma(B^n_s).dB^n_s - C^n_t - D^n_t$$

Study the convergence of the sequences of processes $(C^n)_{n>o}$, $(D^n)_{n>o}$ and $(R^n)_{n>o}$ when n goes to the infinity ;

Indication : calculate $D^n_{(k+1).2^{-n}} - D^n_{k.2^{-n}}$ and find an adequate bound for $R^n_{(k+1).2^{-n}} - R^n_{k.2^{-n}}$

What does that mean ?

EXERCISE D.4

Let f and g be two real functions defined on the real line ; we suppose that the derivative f' and g' of f and g are continuous ; moreover, we suppose also that, for each real number x, we have $g'(x) = f[g(x)]$. We put $T = [0,1]$

1°/ Let $(X_t)_{t \in T}$ be a continuous adapted process which satisfies the properties given in the theorem C.5 (i.e. such that we can apply the Ito formula) ; let $A = [X,X]$ be the quadratic variation of X. We consider the following differential equation :

$$Y_t = g(o) + \int_o^t f(Y_s).[dM_s + dA_s]$$

Show that the solution of this differential equation can be written $Y_t = g(Z_t)$ where

$$Z_t = X_t + A_t - 1/2 \int_o^t [f'\ g(Z_s)]\ .dU_s$$

and where U is a continuous adapted process with bounded variation (calculate U).

Indication : you can apply the Ito formula to $g(Z_t)$ and show that $g(Z_t)$ is a solution of a differential stochastic equation.

2°/ Verify that you have $g'(x) = f[g(x)]$ in the following two cases :

a) $f(x) = x$ and $g = C\,e^x$ where C is a real number

b) $f(x) = \sin x$ and $g(x) = \arcsin\left[\frac{1}{ch(C-x)}\right]$ where C is a real number and when $x < C$.

EXERCISE E.1

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in T})$ be a stochastic basis, the family $(\mathcal{F}_t)_{t \in T}$ being right continuous. Let $\{u(n)\}_{n>o}$ be a sequence of stopping times which is strictly increasing to a stopping time u. Is the set $]o,u[$ a predictable set ? Let X be a real cadlag process such that its Doléans function d(X) is well defined and σ-additive. Let Z be the process X stopped just before the stopping time u, id est :

$$Z_t(\omega) = X_t(\omega) \quad \text{if} \quad t < u(\omega)$$

$$Z_t(\omega) = X_{u(\omega)-}(\omega) \quad \text{if} \quad t \geq u(\omega)$$

For each predictable set A, compare $d(Z)(A)$ and $d(X)(A \cap]o,u[)$

EXERCISE E.2

Let M be a continuous Hilbert space valued martingale. Prove that M is a locally (cf.A.10) square integrable martingale.

EXERCISE E.3

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in T})$ be a stochastic basis with $T = [0,1]$.

1°/ Let $(N_t)_{t \in T}$ be a real continuous process which is a martingale and a process with bounded variation. Prove that $N_t = N_o$ a.s.

Indication : you can begin to suppose that N is a square integrable martingale ; thus, you can prove that the process $[N,N]$ is equal to zero (cf. the exercise C.1) and study $E[(N_t - N_o)^2]$; at last, you can use the exercise E.2 above.

2°/ Let $(W_t)_{t \in T}$ be a real continuous square integrable martingale and $(V_t)_{t \in T}$ be a real continuous increasing process ; we suppose that $V_o = W_o = 0$. Prove that the two following properties are equivalent :

a) $(W_t^2 - V_t)_{t \in T}$ is a martingale

b) $V = [W,W]$

Indication : to prove that a) implies b) you can study the process $V - [W,W]$ and apply the 1°/ above.

3°/ Let $(M_t)_{t \in T}$ and $(A_t)_{t \in T}$ be two real continuous adapted uniformly bounded processes. We suppose that $A_o = M_o = 0$ and that $(A_t)_{t \in T}$ is an increasing process. Prove that the two following properties are equivalent :

a) for each real number λ , the process Z^λ is a martingale where Z^λ is the continuous process defined by $Z_t^\lambda = \exp.(\lambda M_t - \frac{1}{2} \lambda^2.A_t)$

b) the process M is a martingale and $A = [M,M]$

Indication : to prove that b) implies a), you can use the Ito formula ; to prove that a) implies b), you can derivative twice with respect to λ , consider the case where $\lambda = 0$ and use the 2°/ above.

EXERCISE E.4 (Girsanov theorem)

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in T})$ be a stochastic basis, that we note $B(P)$, with $T = [0,1]$. In the following, we will consider a probability Q defined on $(\Omega, \mathcal{F})$ and such that, if $Z_1 = \frac{dQ}{dP}$ is the Radon-Nikodym derivative of Q with respect to P, then there exists two real numbers α and β such that $0 < \alpha < \beta$ and, for each element ω of Ω, $\alpha \leq Z_1(\omega) \leq \beta$.

In this case, $(\Omega, \mathcal{F}, Q, (\mathcal{F}_t)_{t \in T})$ is also a stochastic basis that we note $B(Q)$.

1°/ Let X be a real process adapted with respect to B(P) ; is X adapted with respect to B(Q) ?

2°/ Let M be a real martingale with respect to B(P); is M a martingale with respect to B(Q) ?

In the following, if M is a martingale with respect to B(P), we say that M is a P-martingale (and the same for Q).

Let $(M_t)_{t\in T}$ be a real continuous P-martingale such that $M_o = 0$. We put $A = [M,M]$ and we suppose that M and A are uniformly bounded.

3°/ Let Y be a real predictable uniformly bounded process. Compare the stochastic integral $\int Y dM$ calculated in B(P) and in B(Q).

4°/ Let $(R_t)_{t\in T}$ be a real predictable uniformly bounded process. We put :

$$X_t = M_t - \int_o^t R_s . dA_s$$

$$Z_t = \exp.\left\{\int_o^t R_s . dM_s - \frac{1}{2}\int_o^t R_s^2 . dA_s\right\}$$

Q measure defined on $(\Omega,\mathcal{F})$ by $\frac{dQ}{dP} = Z_1$

and, for each real number :

$$H_t^\lambda = \exp.\left\{-\lambda\int_o^t R_s\, dA_s + M_t - \frac{1}{2}\lambda^2 A_t\right\}$$

$$K_t^\lambda = \exp.\left\{\int_o^t (\lambda+R_s).dM_s - \frac{1}{2}\int_o^t (\lambda+R_s)^2 . dA_s\right\}$$

We suppose that X is uniformly bounded.

a) Is Z a P-martingale ;
<u>Indication</u> : you can apply the Ito formula to the function $f(x) = e^x$ and to the process $J_t = \int_o^t R_s . dM_s - \frac{1}{2}\int_o^t R_s^2 . dA_s$

b) Does Q satisfy the properties given at the beginning of the exercise ?

c) Is K_t^λ a P-martingale ?
<u>Indication</u> : you can use the 3°/ of the exercice E.3 with $\lambda = 1$.

d) Using the results above and the equality $K_t^\lambda = Z_t . H_t$, prove that H_t^λ is a Q-martingale.

e) Prove that X is a Q-martingale.

<u>EXERCISE E.5</u> (See E.13 and E.14)

We put $\Omega = [0,1]$, $\mathcal{F}$ = σ-algebra of the borelian sets of Ω, P = Lebesgue measure, $T = \mathbb{N}$ and $T^* = \mathbb{N} \cup \{\infty\}$. For each integer n, let X_n be the random variable defined by $X_n = 2^n . 1_{[0,2^{-n}]}$ and $\mathcal{F}_n$ be the σ-algebra generated by the random variables $(X_k)_{0\leq k\leq n}$. We put $\mathcal{F}_\infty = \mathcal{F}$.

1°/ Is $(X_n)_{n\in T}$ a martingale with respect to the stochastic basis $(\Omega,\mathcal{F},P,(\mathcal{F}_n)_{n\in T})$? Is the family $(X_n)_{n\in T}$ uniformly integrable ?

2°/ Calculate $\sup_{n>o} E(|X_n|)$ and $E[\sup_{n>o} |X_n|]$

3°/ We put $X_\infty = \lim_{n\to\infty} X_n$; is $(X_n)_{n\in T^*}$ a martingale with respect to the stochastic basis $(\Omega,\mathcal{F},P,(\mathcal{F}_n)_{n\in T^*})$? is it a supermartingale ?

<u>EXERCISE F.1</u> (cf [DEL])

We put : $\Omega = [0,1]$, $\mathcal{F}$ = σ-algebra of the borelian sets of Ω, P probability on $(\Omega,\mathcal{F})$, $T=[0,1]$, for each element t of T, $\mathcal{G}_t$ = σ-algebra of all the borelian sets contained in $[0,t]$, $\mathcal{F}_t$ = σ-algebra on Ω generated by $\mathcal{G}_t$: more precisely a subset A of Ω belongs to $\mathcal{F}_t$ if and only if A belongs to $\mathcal{G}_t$ or if $A = B \cup]t,1]$ where B belongs to $\mathcal{G}_t$. Then $(\Omega,\mathcal{F},P,(\mathcal{F}_t)_{t\in T})$ is a stochastic basis.

1°/ Is the family $(\mathcal{F}_t)_{t\in T}$ right continuous ?

2°/ Let v be a stopping time ; prove that $v(s) < s$ and $v(t) < t$ implies $v(s) = v(t)$. (almost surely).

3°/ Let u be the T-valued random variable defined on Ω by $u(s) = s$; is u a stopping time ? is u a predictable stopping time ? is u a totally inaccessible stopping time ?

4°/ We suppose that there exists $t \in]0,1[$ such that $P(\{t\}) > 0$; let w the T-valued random variable defined on Ω by :

$w(s) = 1$ if $s \neq t$ and $w(t) = t$

Is w a stopping time ? Is w a predictable stopping time ? Is w a totally inacessible stopping time ?

5°/ For this question, we suppose that P is the Lebesgue measure.
Let M_1 be the random variable defined by $M_1(s) = s$.
Let M be the cadlag martingale defined by $M_t = E(M_1 | \mathcal{F}_t)$. Calculate M_t. Is M a continuous martingale ? What is the quadratic variation of M ? What is the Doléans function of M^2 ? What is the Meyer process associated to this Doléans function ? Is this process continuous ?

6°/ Same questions as in the 5°/ above when $P = \frac{1}{2}(P_1+P_2)$ where P_1 is the Lebesgue measure and P_2 is defined by $P_2(\{\frac{1}{2}\}) = 1$ and $P_2(\Omega\setminus\{\frac{1}{2}\}) = 0$.

EXERCISE F.2

We consider $T = [0,1]$, $\Omega = [0,1]$, $\mathcal{F}$ = σ-algebra of the borelian sets of Ω, $\mathcal{F}_t = \{\emptyset,\Omega\}$ if $t < 1/2$ and $\mathcal{F}_t = \mathcal{F}$ if $t \geq 1/2$, M_1 an element of $L_1(\Omega,\mathcal{F},P)$ such that $E(M_1) = 0$, $M_t = M_1$ if $t < 1/2$ and $M_t = 0$ if $t < 1/2$. We suppose that $E(|M_1|^2) = +\infty$. Is $(M_t)_{t\in T}$ a locally square integrable martingale ? Is $[M,M]$ a locally integrable process ? Is it possible to define a Meyer process associated to $[M,M]$?

EXERCISE G.1

We define : $T = [0,1]$, $\Omega = \{1,2\}$, $\mathcal{F}_t = \{\emptyset,\Omega\}$ if $t < 1/3$ and $\mathcal{F}_t = \mathcal{P}(\Omega)$ if $t \geq 1/3$, $u = 1/3 + \frac{1}{3} \cdot 1_{\{2\}}$, $P(\{1\}) = p > 0$, $P(\{2\}) = q > 0$ with $p + q = 1$, $M_1 = q.1_{\{2\}} - p.1_{\{1\}}$, $M_t = E(M_1|\mathcal{F}_t)$. Is u predictable ? Is u totally inacessible ?

Calculate $E\{\sup_{s<u} |M_s|^2\}$, $<M>$ and $E(<M>_{u-})$.

If we put $A = a+b<M>$, is the condition (i) of the theorem G.6 satisfied for suitable positive numbers a and b (with H = J = K = R in this theorem G.6) ?

EXERCISE G.2

We consider $\Omega = T = \mathbb{N}$ (the set of all the non negative integers) ; for each integer k, let $\mathcal{F}_k$ be the σ-algebra generated by the atoms $\{j\}_{1\leq j\leq k}$ and P be the probability defined by

$$P(\{k\}) = (\tfrac{1}{2})^{k+1} \quad ;$$

We put $u(\omega) = \omega$, $M_0 = 0$ and

$$M_{k+1} = M_k - \frac{1}{2} \cdot 1_{\{k\}} + \frac{1}{2} \cdot 1_{\{j\ :\ j>k\}}$$

Is u predictable ? Is u totally inacessible ? Calculate $E(|M|_{u-})$ and $E(\sup_{0\leq t<u} |M_t|^2)$. If we put $A = a+b[M]$, is the condition (i) of the theorem G.6 satisfied for suitable positive numbers a and b ?

EXERCISE H.1

We consider $T = \Omega = \mathbb{N}$ (the set of all the non negative integers) ; for each integer k, let $\mathcal{F}_k$ be the σ-algebra generated by the atoms $\{j\}_{0\leq j\leq k}$ and P be the probability defined by $P(\{k\}) = 2^{-(k+1)}$; let M_∞ be the random variable defined by

$$M_\infty = \sum_{k>0} \frac{1}{k(k+1)} \cdot 2^k.1_{\{k\}}$$

and M be the martingale defined by $M_k = E(M_\infty|\mathcal{F}_k)$. Is the martingale M equi-integrable ? Calculate $E(M^*)$ if $M^* = \sup_{k\geq 0} M_k$.

Calculate $\lim_{n\to\infty} E\left(\left|\left\{\sum_{k=1}^{n} (M_{2k+1} - M_{2k})\right\}\right|\right)$

Let m be the additive function, with values in $L_1(\Omega,\mathcal{F},P)$, defined on the finite subsets of T by $m(\{k\}) = M_k - M_{k-1}$: can m be extended in a vector measure defined on $\mathcal{P}(T)$ and σ-additive for the usual topology of $L_1(\Omega,\mathcal{F},P)$?

EXERCISE H.2 (cf. [Pel-3])

We consider $\Omega =]0,1]$, $T = \mathbb{N}$ (the set of all the non negative integers), $\mathcal{F}$ = σ-algebra of all the borelian sets of $]0,1]$, P = Lebesgue measure on $]0,1]$. For each pair of integers (n,k) with $0\leq k<2^n$, we put $A(n,k) =]k.2^{-n}, (k+1).2^{-n}]$ and

$$X_n = \sum_{k=1}^{2^{n-1}} 1_{A(n,2k-1)} - 1_{A(n,2k)}$$

(Rademacher functions)

We put $Y_\infty = \sum_{k\geq 0} \frac{1}{k}.X_k$ (which is an element of $L_2(\Omega,\mathcal{F},P)$) and $Y_n = E(Y_\infty|\mathcal{F}_n)$ if $\mathcal{F}_n$ is the σ-algebra generated by the sets $\{A(n,k)\}_{0\leq k<2^n}$.

Does there exist a measure z, σ-additive for the usual topology of $L_2(\Omega,\mathcal{F},P)$, defined on $\mathcal{B}_T$, the σ-algebra of the borelian sets of T, by

$$z(]k.2^{-n},(k+1)2^{-n}]) = Y_{(k+1)2^{-n}} - Y_{k.2^{-n}} \quad ?$$

Let y be the function defined on $\mathcal{A}$ by $y(A) = \int 1_A.dY$; can y be extended in a measure σ-additive for the usual topology of $L_2(\Omega,\mathcal{F},P)$? Does there exist a positive number K such that, for each martingale M, $N_3'(M) \leq K\, N_4'(M)$ if N_3' and N_4' are defined as N_3 and N_4 (cf. H.1) by considering the norm in L_2 instead of the norm in L_1 ?

BIBLIOGRAPHICAL NOTES

A. The notions and properties studied in the paragraph A are now very classical ; the fundamental role of the σ-algebra of predictable sets was disclosed by the Strasbourg school (cf., notably, [Del]) (see also [Bur]) ; the systematic use of the algebra $\mathcal{A}$ is due to the authors (cf. [Pel-2]) ; this idea has been also exploited by Follmer (cf. [Fol]).

B. There are many books and studies on the stochastic integral ([Sko], [ItK], [GiS], [Kus], etc...) ; in the non continuous case, this integral was notably studied in [KuW] and [DoM-1] ; the construction given here is due to the authors.

C. The Ito formula is a fundamental point of this theory ; the first study is, of course, due to K.Ito (cf. [Ito]) ; the general Ito formula in the finite dimensionnal non continuous case was obtained in [DoM-1] ; the proof given here, available for Hilbert space valued processes, is very different from the proof used in [DoM-1] and is due to the authors.

D. The use of the fixed point theorem to obtain strong solutions for stochastic differential equations is very classical ; the theorems given here, available in a very general context, are due to the authors.

The theorem D.5. generalizes [DoM-2], [Dol-2] and [Pro].

E. The theorem E.4 is due to Doob ; the formulation given here is due to the authors (cf. also, [Ore]) ; the Doléans measure was introduced in [Dol-1] ; its systematic use and study, in particular the lemma E.2, are due to the authors (cf. [Pel-2]) ; the stochastic integral with respect to square integrable martingales was considered in [Cou] and [KuW] ; the inequality E.12 is due to Doob.

F. The construction and the properties of Meyer process were obtained by the Strasbourg school (cf. [Del]) ; the presentation given here, specially F.7 and F.8, is due to the authors (cf. [Pel-3]); it does not require the prerequisit on predictable projections and predictable sections as developped by Dellacherie ; cf. also [Rao] for an attempt in this direction.

H. The most important inequalities of the paragraph H were obtained by Burkholder (cf. [Bur]) ; several authors gave simplications with respect to the initial proofs ; ([Dov], [Fef], [Gar], [Kus], etc ...) ; the inequalities H.2 and H.9 are due to the authors : the proof of H.9 uses an idea given in [Mey-2].

The paragraphs G and J are due to the authors (see [MeP-3] and [Pel-4]).

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