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Introduction. In this note we continue the study begun in [4] of incremental finite element methods for finding approximate solutions of quasi-static plasticity problems. We shall here prove convergence of a mixed finite element method for finding approximations to the stresses in a body made up by a hardening elastic-plastic material and acted upon by a time dependent load. Mixed finite element methods are often used in practice, see e.g. [1], [7]. In a mixed method the displacements and stresses are approximated independently using two finite dimensional spaces. This allows for greater flexibility and often makes the construction of the finite element method easier as compared with a more orthodox use of Galerkin's method using approximations of either the displacements or the stresses.

In order to be able to prove convergence of the present mixed method, we have to assume a certain type of hardening behavior of the material. The reason is that under this assumption the exact solution is known to have a certain for the proof required regularity (see [5] and the remark after Theorem 1 in Section 3).

The finite element method will produce approximations to the stresses successively at a finite number of time levels. At each time level one has to solve a finite dimensional saddle point problem. We also discuss an iterative method (Uzawa's method) for solving this problem. At each step of the iteration one has to solve a convex minimization problem and a linear problem, both finite dimensional. With the particular choice of finite element spaces made in this note, the convex minimization problem is easy to solve (see the remark in Section 4).

By C we will denote a positive constant not necessarily the same at each occurrence.

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1. The plasticity problem

Let Ω be a polyhedral domain in $R^3 = \{x = (x_1,x_2,x_3):x_i \in R\}$ and let $I = [0,T]$, $T > 0$, be a time interval. By $\sigma = \{\sigma_{ij}\} \in R^9$, $i,j = 1,2,3$, we shall denote stress vectors depending on $(x,t) \in \Omega \times I$ with components σ_{ij} such that $\sigma_{ij} \equiv \sigma_{ji}$. Stress vectors will also be denoted by τ and χ. Hardening parameters will be denoted by scalars $\xi,\eta,\zeta \in R$ depending on $(x,t) \in \Omega \times I$. Define

$$\bar{R}^6 = \{\tau = \{\tau_{ij}\} \in R^9 : \tau_{ij} = \tau_{ji}\},$$

$$\bar{H} = \{\tau = \{\tau_{ij}\} \in [L^2(\Omega)]^9 : \tau_{ij} = \tau_{ji}\},$$

$$H = \bar{H} \times L^2(\Omega).$$

For $u = (u_1,\cdots,u_n)$, $w = (w_1,\cdots,w_n) \in R^n$, we define

$$(u,w) = \sum_{i=1}^{n} u_i w_i, \quad |u| = (u,u)^{1/2},$$

and for $(u_1,\cdots,u_n)$, $w = (w_1,\cdots,w_n) \in [L^2(\Omega)]^n$, we define

$$(u,w) = \int_\Omega (u(x),w(x))\,dx, \quad ||u|| = (u,u)^{1/2}.$$

In particular, we have, using the convention that repeated indices indicate summation from 1 to 3,

$$(\tau,\chi) = \int_\Omega \tau_{ij}\chi_{ij}\,dx, \quad \tau,\chi \in \bar{H},$$

and $||\tau|| = (\tau,\tau)^{1/2}$, $\tau \in \bar{H}$. Furthermore, let

$$a(\tau,\chi) = \int_\Omega A_{ijkh}\tau_{ij}\chi_{kh}\,dx,$$

where the A_{ijkh} are elasticity constants such that $A_{ijkh} = A_{jikh} = A_{khij}$ and for some positive constant μ

(1.1) $a(\tau,\tau) \geq \mu \, ||\tau||^2, \quad \tau \in \bar{H}.$

We shall assume that the material satisfies the von Mises yield condition. More precisely, we shall assume that the yield condition is given by the function $F: \bar{R}^6 \times R \to R$ defined by

(1.2) $$F(\tau,\eta) = |\bar{\tau}| - (\gamma\eta+1),$$

$$\bar{\tau}_{ij} = \tau_{ij} - \delta_{ij}\tau_m, \quad \tau_m = \frac{1}{3}\tau_{ii},$$

where γ is a positive constant and $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Here $\bar{\tau} = \{\bar{\tau}_{ij}\}$ is the so called <u>stress deviatoric</u> given by τ. The set of admissible pairs (τ,η) is then given by

$$P = \{(\tau,\eta) \in H: F(\tau(x),\eta(x) \leq 0 \text{ a.e. in } \Omega\}.$$

In our formulation of the plasticity problem, the above choice of the function F will correspond to an <u>isotropic</u> <u>strain-hardening</u> material (see [5], where also yield conditions of more general form are considered).

We shall assume that the displacements of the body are zero on the boundary of Ω. It is therefore natural to introduce the space $V = [H_o^1(\Omega)]^3$, where $H_o^1(\Omega)$ is the closure of $C_o^\infty(\Omega)$ in the norm

$$||\cdot||_V = (\sum_{|\alpha|\leq 1} ||D^\alpha \cdot||^2)^{1/2}.$$

The <u>displacement</u> <u>rate</u> will be given by a function $v: I \to V$. Given $w = (w_1,w_2,w_3) \in V$, we define $\varepsilon(w)$, the <u>strain</u> <u>rate</u> associated with w, to be the vector function $\varepsilon(w)$ in $\bar{H}$ with components

$$\varepsilon_{ij}(w) = \frac{1}{2}(\frac{\partial w_i}{\partial x_j} + \frac{\partial w_j}{\partial x_i}).$$

We shall assume that the body is acted upon by a time dependent (volume) force f of the form $f(x,t) = g(t)G(x)$, where $g:I \to R$ is a smooth nonnegative function with $g(0) = 0$, and $G:\Omega \to R^3$ is smooth. For technical reasons we shall also assume that G has a smooth extension to a domain $\tilde{\Omega}$ with smooth boundary such that $\Omega \subset \tilde{\Omega}$.

Let us also introduce the following notation: For $\hat{\tau} \equiv (\tau,\eta)$, $\hat{\chi} \equiv (\chi,\zeta) \in H$, we define

$$[\hat{\tau},\hat{\chi}] = a(\tau,\chi) + \gamma(\eta,\zeta), \qquad |||\hat{\tau}||| = [\hat{\tau},\hat{\tau}]^{1/2},$$

where γ is the constant in (1.2). For X a normed space, let $L^p(X) \equiv L^p(I;X)$, $1 \leq p \leq \infty$, be the set of L^p - integrable functions from I to X. Let $C(X)$ denote the set of continuous functions from I to X. Finally, write u' instead of $\frac{du}{dt}$.

We can now formulate the <u>plasticity problem</u>: Find $(\hat{\sigma},v):I \to P \times V$, $\hat{\sigma} = (\sigma,\xi)$, such that a.e. on I,

$$\text{(1.3a)} \qquad [\hat{\sigma}',\hat{\tau} - \hat{\sigma}] - (\varepsilon(v),\tau - \sigma) \geq 0, \quad \hat{\tau} \in P,$$

$$\text{(1.3b)} \qquad (\varepsilon(w),\sigma) = (w,f), \qquad w \in V,$$

$$\text{(1.3c)} \qquad \hat{\sigma}(0) = 0.$$

Existence of a unique solution of this problem satisfying $\hat{\sigma} \in C(H)$, $\hat{\sigma}' \in L^2(H)$ and $v \in L^2(V)$ follows from Theorems 1 and 2 in [5] and the following lemma. Here

$$E(t) = \{\hat{\tau} = (\tau,\eta) \in H: (\varepsilon(w),\tau) = (w,f(t)), \ w \in V\},$$

$$K(t) = E(t) \cap P,$$

and for a vector function $u:\Lambda \to R^n$,

$$||u||_{\infty,\Lambda} = \sup_{y \in \Lambda} |u(y)|.$$

Lemma 1. There exists $\hat{\chi}(t) \in K(t)$, $t \in I$, such that

$$||\frac{\partial^j \hat{\chi}}{\partial t^j}||_{\infty,\Omega\times I} \leq C, \qquad j = 0,1,2,$$

$$(1+\delta)\hat{\chi}(t) \in P, \quad t \in I,$$

for some positive constants C and δ.

Proof. By assumption G has a smooth extension $\tilde{G}$ to a domain $\tilde{\Omega}$ with smooth boundary such that $\Omega \subset \tilde{\Omega}$. Let $(\tilde{\chi},\tilde{v})$ be the solution of the linear elastic problem

$$\tilde{a}(\tilde{\chi},\tau) = (\varepsilon(\tilde{v}),\tau)_{\sim}, \quad \tau \in \underset{\sim}{\bar{H}},$$

$$(\varepsilon(w),\tilde{\chi})_{\sim} = (w,\tilde{G})_{\sim}, \quad w \in \underset{\sim}{V},$$

where $\tilde{a}(\cdot,\cdot),(\cdot,\cdot)_{\sim}$, $\underset{\sim}{\bar{H}}$ and $\underset{\sim}{V}$ are defined as above with $\tilde{\Omega}$ replacing Ω. Then $\tilde{\chi}$ is smooth and in particular $||\tilde{\chi}||_{\infty,\Omega} \leq C$. Also, extending $w \in V$ by zero outside Ω, it follows that

$$(\varepsilon(w),\tilde{\chi}) = (w,G), \quad w \in V.$$

Therefore, defining

$$\chi(t) = g(t)\tilde{\chi}, \quad \zeta(t) = C_1 g(t), \quad \hat{\chi} = (\chi,\zeta),$$

we have $\hat{\chi}(t) \in E(t)$, $t \in I$,

$$||\frac{\partial^j \hat{\chi}}{\partial t^j}||_{\infty,\Omega\times I} \leq C, \quad j = 0,1.2,$$

and, chosing the constant C_1 sufficiently large,

$$|\bar{\chi}| - \gamma\zeta \leq 0 \quad \text{on} \quad \Omega \times I.$$

This shows that $\hat{\chi}$ satisfies the requirements of the lemma (with δ an arbitrary positive number).

2. A mixed finite element method

Let h be a small positive parameter and let $\mathcal{T}_h = \{T\}$ be a triangulation of Ω_h,

$$\Omega_h = \bigcup_{T \in \mathcal{T}_h} T,$$

such that the diameters of the tetrahedrons T are less than h. Let us also assume that the triangulation is regular, i.e., there is a positive constant ρ independent of h such that the ratio between the inscribed and circumscribed sphere for any $T \in \mathcal{T}_h$ is bounded below by ρ.

Let us now introduce the finite element spaces we shall use. For $k = 0$ $(k=1)$ let $P_k(T)$ be the set of constant (linear) functions defined on T and define

$$H_h = \{\hat{\tau} \in H : \hat{\tau}|_T \in [P_0(T)]^9 \times P_0(T),\ T \in \mathcal{T}_h\},$$

$$V_h = \{w \in V : w|_T \in [P_1(T)]^3,\ T \in \mathcal{T}_h\},$$

$$P_h = H_h \cap P.$$

Let us also introduce a time discretization: For N a natural number, let $k = T/N$, $t_n = nk$ for $n = 0,1,\cdots,N$, let $I_k = \{t_0, t_1, \cdots, t_N\}$ and write $u^n = u(t_n)$. Define the difference quotient $\partial u^n = (u^n - u^{n-1})/k$.

We can now formulate the finite element method: Find $(\hat{\sigma}_h, v_h): I_k \to P_h \times V_h$, $\hat{\sigma}_h = (\sigma_h, \xi_h)$, such that for $n = 1,\cdots,N$,

$$\text{(2.1a)} \qquad [\partial\hat{\sigma}_h^n, \hat{\tau} - \hat{\sigma}_h^n] - (\varepsilon(v_h^n), \tau - \sigma_h^n) \geq 0, \quad \hat{\tau} \in P_h,$$

$$\text{(2.1b)} \qquad (\varepsilon(w), \sigma_h^n) = (w, f^n), \qquad w \in V_h,$$

$$\text{(2.1c)} \qquad \hat{\sigma}_h^0 = 0.$$

Let us now show that this problem has a solution $(\hat{\sigma}_h, v_h)$ with $\hat{\sigma}_h$ uniquely determined. At the same time we also establish an a priori estimate which will be used below in proving convergence of the finite element method. We shall use the following notation:

$$E_h(t) = \{\hat{\tau} \in H_h: (\varepsilon(w),\tau) = (w,f(t)),\ w \in V_h\} ,$$

$$K_h(t) = E_h(t) \cap P,$$

$$||\hat{\tau}||_{\ell^2(H)} = (\sum_{n=1}^{N} ||\hat{\tau}^n||^2 k)^{1/2}.$$

We shall also refer to the following lemma.

Lemma 2. Let π be the orthogonal projection in H onto H_h, let $t \in I$, and suppose that $\hat{\tau} \equiv (\tau,\eta) \in K(t)$. Then

(i) $||\pi\hat{\tau}||_{\infty,\Omega} \leq ||\hat{\tau}||_{\infty,\Omega}$,

(ii) $\pi\hat{\tau} \equiv (\tau_h,\eta_h) \in K_h(t)$.

Proof. Note that $\tau_h|_T$ is the projection of $\tau|_T$ onto $[P_0(T)]^9$ in $[L^2(T)]^9$ and that $\eta_h|_T$ is the projection of $\eta|_T$ onto $P_0(T)$ in $L^2(T)$. It follows easily that

$$(2.2)\qquad \begin{aligned} |\tau_h|_T| &\leq \frac{1}{\text{area}(T)} \int_T |\tau| dx, \\ \eta_h|_T &= \frac{1}{\text{area}(T)} \int_T \eta dx, \end{aligned}$$

which clearly proves (i). To prove (ii), observe that since $\varepsilon(w)$ is piecewise constant if $w \in V_h$ and $\hat{\tau} \in E(t)$, we have,

$$(\varepsilon(w),\tau_h) = (\varepsilon(w),\tau) = (w,f(t)),\ w \in V_h,$$

i.e., $\hat{\tau}_h \in E_h(t)$. Further, since (2.2) will be true also if we replace τ and τ_h by the stress deviatorics $\bar{\tau}$ and $\bar{\tau}_h$, respectively, and by assumption $|\bar{\tau}(x)| \leq 1 + \gamma\eta(x)$ for a.e. $x \in \Omega$, we obtain that

$$|\bar{\tau}_h|_T| \leq 1 + \gamma\eta_h|_T,$$

which proves that $\pi\hat{\tau} \in P$. Thus, $\pi\hat{\tau} \in K_h(t)$ and the proof is complete.

<u>PROPOSITION 1</u>. There is a solution $(\hat{\sigma}_h, v_h)$ of (2.1) and $\hat{\sigma}_h$ is uniquely determined. Moreover, there is a constant C, independent of h and k such that,

$$||\partial\hat{\sigma}_h||_{\ell^2(H)} \leq C.$$

<u>Proof</u>. The result will follow, using the technique of proof of Theorem 1 in [5], if we can show that there exists $\hat{\chi}_h(t) \in K_h(t)$, $t \in I$, such that for some constants C and δ independent of h,

$$||\frac{\partial^j \hat{\chi}_h}{\partial t^j}||_{\infty,\Omega\times I} \leq C, \quad j = 0,1,2,$$

$$(1+\delta)\hat{\chi}_h(t) \in P, \quad t \in I.$$

But if we take $\hat{\chi}_h = \pi\hat{\chi}$, where $\hat{\chi}$ is given in Lemma 1, it follows by Lemmas 1 and 2 that $\hat{\chi}_h$ satisfies the desired requirements, and the result follows.

3. Convergence of the finite element method

We shall prove the following result:

Theorem 1. Let $(\hat{\sigma}, v)$ and $(\hat{\sigma}_h, v_h)$ be solutions of (1.3) and (2.1), respectively. Then there is a constant C, independent of h and k, such that

$$\max_n |||\hat{\sigma}^n - \hat{\sigma}_h^n||| \le C(\alpha(h) + \sqrt{k}),$$

where

$$\alpha(h) = \inf\{||v - w||_{L^2(V)} : w \in L^2(V_h)\}.$$

Remark. By a density argument it follows that $\alpha(h) \to 0$ as $h \to 0$.

Remark. In the proof of this result we shall use the fact that $v \in L^2(V)$. In the case of an elastic-perfectly plastic material it is only known that $v \in L^2(W)$, where $W = [L^{3/2}(\Omega))^3$, see [3], [5].

Proof of Theorem 1. Let us first extend $\hat{\sigma}_h$ linearly to I. Since $\hat{\sigma}_h^n \in P$, $n = 0, \cdots, N$, and P is convex, we then clearly have $\hat{\sigma}_h(t) \in P$, $t \in I$. Taking $\hat{\tau} = \hat{\sigma}_h(t)$ in (1.3), integrating over $I_n \equiv [t_{n-1}, t_n]$, and dividing by k, we find that

$$[\partial\hat{\sigma}^n, \hat{\sigma}_h^n - \hat{\sigma}^n] - (\widetilde{\varepsilon(v^n)}, \sigma_h^n - \sigma^n)$$

$$\text{(3.1)} \quad \ge \frac{1}{k} \int_{I_n} [\hat{\sigma}', \hat{\sigma}_h^n - \hat{\sigma}_h(t) + \hat{\sigma}(t) - \hat{\sigma}^n]dt$$

$$- \frac{1}{k} \int_{I_n} (\varepsilon(v(t)), \sigma_h^n - \sigma_h(t) + \sigma(t) - \sigma^n)dt$$

where

$$\widetilde{\varepsilon(v^n)} = \frac{1}{k}\int_{I_n} \varepsilon(v(t))\,dt.$$

Next, by Lemma 2 we have $\hat{\tau}_h^n \equiv \pi\hat{\sigma}^n \in K_h(t_n) \subset P_h$ so that we may take $\hat{\tau} = \hat{\tau}_h^n \equiv (\tau_h^n,\eta_h^n)$ in (2.1). Adding the so-obtained inequality to (3.1), we get, writing $e = \hat{\sigma} - \hat{\sigma}_h$ and denoting by r_n the right hand side at (3.1),

$$\begin{aligned}
r_n &\le [\partial\hat{\sigma}^n,\hat{\sigma}_h^n - \hat{\sigma}^n] - (\widetilde{\varepsilon(v^n)},\sigma_h^n - \sigma^n) \\
&\quad + [\partial\hat{\sigma}_h^n,\hat{\tau}_h^n - \hat{\sigma}_h^n] - (\varepsilon(v_h^n),\tau_h^n - \sigma_h^n) \\
&= - [\partial e^n,e^n] - [\partial\hat{\sigma}_h^n,\hat{\sigma}^n - \hat{\tau}_h^n] \\
&\quad - (\widetilde{\varepsilon(v_n)},\sigma_h^n - \sigma^n) - (\varepsilon(v_h^n),\tau_h^n - \sigma_h^n) .
\end{aligned} \tag{3.2}$$

Since $\hat{\tau}_h^n$ is the projection in H of $\hat{\sigma}^n$ onto H_h, we have

$$[\hat{\tau},\hat{\sigma}^n - \hat{\tau}_h^n] = 0, \qquad \hat{\tau} \in H_h, \tag{3.3}$$

so that in particular $[\partial\hat{\sigma}_h^n,\hat{\sigma}^n - \hat{\tau}_h^n] = 0$. Also, since $\hat{\tau}_h^n \in E_h(t_n)$, we have

$$(\varepsilon(w),\tau_h^n - \sigma_h^n) = 0, \qquad w \in V_h . \tag{3.4}$$

Therefore, it follows from (3.2) that

$$[\partial e^n,e^n] \le |r_n| - (\widetilde{\varepsilon(v^n)},\sigma_h^n - \sigma^n) .$$

Now, using (3.3) and (3.4) again, we see that for $w \in L^2(V_h)$,

$$\begin{aligned}
(\widetilde{\varepsilon(v^n)},\sigma_h^n - \sigma^n) &= \frac{1}{k}\int_{I_n} (\varepsilon(v),\sigma_h^n - \sigma^n)\,dt \\
&= \frac{1}{k}\int_{I_n} (\varepsilon(v) - \varepsilon(w),\tau_h^n - \sigma^n)\,dt \\
&\quad - \frac{1}{k}\int_{I_n} (\varepsilon(v) - \varepsilon(w),\tau_h^n - \sigma_h^n)\,dt \\
&= \frac{1}{k}\int_{I_n} (\varepsilon(v) - \varepsilon(w),\sigma_h^n - \sigma^n)\,dt .
\end{aligned}$$

Thus, for any $w \in L^2(V_h)$, we have for $n = 1,2,\cdots,N$,

(3.5) $$[\partial e^n, e^n] \leq r_n + \frac{1}{k}\int_{I_n} ||\varepsilon(v) - \varepsilon(w)||\,||\sigma_h^n - \sigma^n||dt \ .$$

Observing that for $M = 1,\cdots,N$,

$$2\sum_{n=1}^{n} [e^n - e^{n-1}, e^n] = |||e^M|||^2 + \sum_{n=1}^{M} |||e^n - e^{n-1}|||^2,$$

we find multiplying (3.5) by k, summing over n and using Cauchy's inequality that for $w \in L^2(V_h)$,

$$\frac{1}{2}\max_n |||e^n|||^2 \leq \sum_{n=1}^{N} |r_n|k + C\max_n |||e^n|||\ ||\varepsilon(v) - \varepsilon(w)||_{L^2(\bar{H})}$$

$$\leq \sum_{n=1}^{N} |r_n|k + \frac{1}{4}\max_n |||e^n|||^2 + C||\varepsilon(v) - \varepsilon(w)||^2_{L^2(\bar{H})},$$

where we also used the fact that by (1.1), $||\sigma_h^n - \sigma^n|| \leq C|||e^n|||$. Finally, using the facts that $||\partial\hat{\sigma}_h||_{L^2(H)} \leq C$ and $\sigma' \in L^2(H)$, it is easy to see that (c.f. [4]),

$$\sum_{n=1}^{N} |r_n|k \leq Ck,$$

which completes the proof of the theorem.

4. An iterative method.

To find the finite element solution $(\hat{\sigma}_h, v_h)$ we have to solve a finite dimensional saddle point problem at each time level t_n; given $\hat{\sigma}_h^{n-1}$ we have to find $(\hat{\sigma}_h^n, v_h^n) \in P_h \times V_h$ satisfying (2.1), which is equivalent to finding a saddle point $(\hat{\sigma}_h^n, v_h^n)$ of the convex-concave functional $L: P_h \times V_h \to R$, defined by

$$L(\hat{\tau}, w) = \frac{1}{k}[\hat{\tau}, \hat{\tau}] - \frac{1}{k}[\hat{\tau}, \hat{\sigma}_h^{n-1}] - (\varepsilon(w), \tau) + (w, f^n).$$

We shall consider the following <u>iterative method</u> (Uzawa's method, see e.g. [2]) for solving this problem (here we drop the subscript h and write $(\hat{\sigma}_j^n, v_j^n)$ instead of $(\hat{\sigma}_{h,j}^n, v_{h,j}^n)$): Find $(\hat{\sigma}_j^n, v_j^n)$, $j = 1,2,\cdots,$ such that

(4.1a) $\frac{1}{k}[\hat{\sigma}_j^n - \hat{\sigma}_h^{n-1}, \hat{\tau} - \hat{\sigma}_j^n] - (\varepsilon(v_{j-1}^n), \tau - \sigma_j^n) \geq 0. \quad \tau \in P_h,$

(4.1b) $(\varepsilon(v_j^n), \varepsilon(w)) = (\varepsilon(v_{j-1}^n), \varepsilon(w)) + \rho((f^n, w) - (\varepsilon(w), \sigma_j^n)), \; w \in V_h,$

where ρ is a positive constant and v_0^n is an initial guess, e.g., $v_0^n = 0$. Note that $\hat{\sigma}_j^n$ is determined by (4.1a) as the function which minimizes the convex functional $L(\cdot, v_{j-1}^n)$ over the convex set P_h. Furthermore, to find v_j^n amounts to solving the linear elliptic problem (4.1b). This proves that (4.1) can be solved for $j = 1,2,\cdots,$.

<u>Remark</u>. Since the functions in P_h are defined independently on each triangle $T \in \mathcal{T}_h$, the minimization problem (4.1a) can be solved by solving a simple minimization problem in $\bar{R}^6 \times R$ for each $T \in \mathcal{T}_h$. Let us also note that (4.1) is similar to an iterative method described in [6].

It is well-known that Uzawa's method will converge in a situation such as the present one if ρ is sufficiently small (see e.g. [2]). For completeness we include a proof of convergence of the proposed method.

<u>Theorem 2</u>. Let $\hat{\sigma}_j^n, j = 1,2,\cdots,$ be the sequence given by (4.1). Then if $\rho < 2\mu/k$ (with μ given in (1.1)), one has

$$||\hat{\sigma}_h^n - \hat{\sigma}_j^n|| \to 0 \quad \text{as} \quad j \to \infty.$$

<u>Proof</u>. First, taking $\hat{\tau} = \hat{\sigma}_j^n$ in (2.1a) and $\hat{\tau} = \hat{\sigma}_h^n$ in (4.1a) and adding, we obtain,

$$(4.2) \qquad [\hat{\sigma}_h^n - \hat{\sigma}_j^n, \hat{\sigma}_h^n - \hat{\sigma}_j^n] - k(\varepsilon(v_h^n) - \varepsilon(v_{j-1}^n), \sigma_h^n - \sigma_j^n) \leq 0.$$

Next, multiplying (2.1b) by ρ, subtracting from (4.1b) and choosing $w = v_j^n - v_h^n$, we get

$$(\varepsilon(v_j^n - v_h^n), \varepsilon(v_j^n - v_h^n)) = (\varepsilon(v_{j-1}^n - v_h^n), \varepsilon(v_j^n - v_h^n))$$
$$+ \rho(\sigma_h^n - \sigma_j^n, \varepsilon(v_j^n - v_h^n)).$$

By Cauchy's inequality, it follows that

$$||\varepsilon(v_j^n - v_h^n)||^2 \leq ||\varepsilon(v_j^n - v_h^n) + \rho(\sigma_h^n - \sigma_j^n)||^2$$
$$= ||\varepsilon(v_{j-1}^n - v_h^n)||^2 + 2\rho(\varepsilon(v_{j-1}^n - v_h^n), \sigma_h^n - \sigma_j^n) + \rho^2||\sigma_h^n - \sigma_j^n||^2.$$

Now, multiplying (4.2) by $2\rho/k$ and adding to the above inequality, we find that

$$\frac{2\rho}{k}|||\hat{\sigma}_k^n - \hat{\sigma}_j^n|||^2 + ||\varepsilon(v_h^n - v_j^n)||^2$$
$$\leq ||\varepsilon(v_h^n - v_{j-1}^n)||^2 + \rho^2||\sigma_h^n - \sigma_j^n||^2,$$

But $||\tau||^2 \leq \frac{1}{\mu}|||\hat{\tau}|||^2$ and thus we obtain by summation,

$$\sum_{j=1}^{M}(\frac{2\rho}{k} - \frac{\rho^2}{\mu}\ |||\hat{\sigma}_k^n - \hat{\sigma}_j^n|||^2 + ||\varepsilon(v_h^n - v_M^n)||^2$$
$$\leq ||\varepsilon(v_h^n - v_0^n)||^2, \qquad M = 1,2,\cdots,.$$

This shows that if $\rho < 2\mu/k$, then

$$|||\hat{\sigma}_h^n - \hat{\sigma}_j^n||| \to 0 \quad \text{as} \quad j \to \infty$$

which proves the lemma.

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