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Numerical Analysis of the Semi-linear Heat Equation of Blow-up Type

by

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§1 Introduction.

Let Ω be a bounded open set in $\mathbb{R}^n$ with a smooth boundary $\Gamma = \partial\Omega$. We take the continuous problem of

$$(E) \quad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u), & x \in \Omega \\ u(t,x) = 0, & x \in \Gamma \end{cases} \quad t > 0$$
$$u(0,x) = a(x), \quad x \in \Omega.$$

For simplicity we assume that f is twice continuously differentiable satisfying that

(1) $f(u)$ and $f''(u) \geq 0$ for any $u \in R^1$

and that for some positive γ and C

(2) $f(u) \geq Cu^{1+\gamma}$ as $u \to \infty$.

The initial data $a(x)$ is continuous on $\bar{\Omega}$ vanishing at Γ, the totality of such functions is denoted by $C_0(\bar{\Omega})$. By Kaplan's classical argument [6], the solution $u(t,x)$ tends to infinity at a finite time T for some $a(x)$. This fact is called the blowing-up of solution, and the time T is called the blowing-up time or the finite escape time. Fujita studied extensively this problem in [3], [4] and so forth. There are also some works based on different criteria by other authors, for example, Tsutsumi [9], [10], Ito [5], among others.

The purpose of this paper is to provide a numerical method of (E) by making use of the finite element approximation of lumped mass type, based on Kaplan-Fujita's criterion. One of the authors already investigated an algorithm for the difference approximation to (E) based on Tsutsumi's criterion in [7]. The present work is one of its continuations and based on the recent work of the other author [11]. The finite element version of the algorithm in [7] will be justified in the forthcoming paper.

In Section 2, a reformulation of Kaplan-Fujita's criterion will be presented so that it is appropriate to our purpose. The approximate problem will be stated in Section 3, and an algorithm for controlling time steps will be described in Section 4. The rigorous justification will be done in Sections 5 and 6. Finally in Section 7 some numerical illustrations will be given. Details of numerical results will be reported elsewhere.

We remark that it is straightforward to modify the present result to the case of the condition (1) holding only $u \geqq 0$ under the restriction of the initial data $a(x) \geqq 0$.

We should like to express our sincere thanks to Mr. Y. Yuzurihara for his help in the computation of our model problem.

§2 Kaplan-Fujita's criterion.

Let λ denote the smallest eigenvalue of $-\Delta$ with the Dirichlet boundary condition, and let $\phi(x)$ denote the eigenfunction associated with λ, $\phi(x)$ being normalized as

$$\begin{cases} \phi(x) > 0, & x \in \Omega, \\ \int_\Omega \phi(x)dx = 1. \end{cases}$$

Denote by $J(t)$ the inner product of $u(t,x)$ and $\phi(x)$, i.e.,

$$J(t) = (u(t,x), \phi(x))_{L^2(\Omega)} = \int_\Omega u(t,x)\, \phi(x)dx.$$

Definition 1. The classical solution $u(t,x)$ of (E) J-blows up at $t = T$ if and only if

$$(3) \qquad \begin{cases} u(t,x) \in C([0,T), C_0(\bar{\Omega})) \text{ satisfies (E)}, \\ \lim_{t \uparrow T} J(t) = \infty. \end{cases}$$

Let J^1 be the largest positive root of the equation of

$$-\lambda J + f(J) = 0.$$

If the equation has no positive roots, then let $J^1 = 0$.

Proposition 1. The solution $u(t,x)$ J-blows up at a finite time T if and only if there exists a $t_0 \geqq 0$ such that

$$(4) \qquad \begin{cases} u(t,x) \in C([0,t_0], C_0(\bar{\Omega})) \text{ satisfies (E)}, \\ J(t_0) > J^1. \end{cases}$$

Proof. The necessity of the condition (4) is obvious. The sufficiency criterion due to Kaplan is as follows. Let $[0,T)$ be the maximal existence interval of $u(t,x)$. By Jensen's inequality, we have the following differential inequality for $J(t)$:

$$(5) \qquad \frac{d}{dt}J(t) \geqq -\lambda J(t) + f(J(t)) \quad t \in [0,T),$$

on account of the convexity of f and the normalization condition of ϕ . This inequality implies first that

$$\frac{d}{dt}J(t) > 0 \quad \text{for } t \geq t_0. \tag{6}$$

In fact, unless this were true, there would be $t_1 > t_0$ such that $\frac{d}{dt}J(t_1) = 0$ and $\frac{d}{dt}J(t) > 0$ for $t_0 \leq t < t_1$. The first condition and the inequality (5) shows

$$0 \geq -\lambda J(t_1) + f(J(t_1)).$$

Since the quantity $-\lambda J + f(J)$ is nonnegative for $J \geq J^1$, we have

$$J(t_1) \leq J^1 < J(t_0).$$

But this is impossible since

$$J(t_1) - J(t_0) = \int_{t_0}^{t_1} \frac{dJ(s)}{ds} ds > 0.$$

The condition (6) implies that

$$-\lambda J(t) + f(J(t)) > 0 \quad \text{for any } t \geq t_0,$$

which in turn implies

$$t - t_0 \leq \int_{J(t_0)}^{J(t)} \frac{dJ}{-\lambda J + f(J)} \quad \text{for any } t \geq t_0. \tag{7}$$

Since we assumed that $f(u) \geq Cu^{1+\gamma}$ for $u \to \infty$, the right-hand side of (7) is uniformly bounded. Hence we have that the solution $u(t,x)$ blows up at finite T.

Corollary 2. The blowing-up time T is bounded from above as

$$T \leq t_0 + \int_{J(t_0)}^{\infty} \frac{dJ}{-\lambda J + f(J)}.$$

Proof. This follows from the estimate (7).

§3 Setting of the approximating problem.

Let $\{\Omega_h;\ h > 0\}$ be the family of polyhedral open domains contained in Ω satisfying

$$\begin{cases} \Omega_h > \Omega_{h'}, & \text{if } h \leq h', \\ \max\limits_{x \in \Gamma_h} \operatorname{dist}(x, \Gamma) \to 0 & \text{as } h \to 0. \end{cases} \tag{8}$$

Here Γ_h is the boundary of Ω_h.

Definition 2. The set $\mathfrak{S}_h = \{S^{(k)}\}$ is a triangulation of the polyhedral open domain Ω_h if and only if

$$\begin{cases} \text{(i)} & S^{(k)},\ k=1,2,\cdots, \text{ are nondegenerate closed } n\text{-simplices, the number of which is finite,} \\ \text{(ii)} & \overline{\Omega}_h = \bigcup\limits_{S^{(k)} \in \mathfrak{S}_h} S^{(k)}, \\ \text{(iii)} & \text{the face of } S^{(k)} \text{ is either a face of another } n\text{-simplex of } \mathfrak{S}_h \text{ or else is a portion of the boundary of } \Omega_h. \end{cases} \tag{9}$$

In the following, we shall omit the superscript (k) of $S^{(k)}$ for simplicity of notation. Let $b_0, \cdots, b_n$ denote $n+1$ vertices (or nodal points) of an n-simplex S. In terms of the barycentric coordinates $(\lambda_0(x), \cdots, \lambda_n(x))$, $x \in S$, define the barycentric subdivision $B_{b_0(S)}$ of b_0 in S such that

$$B_{b_0(S)} = \{x;\ 1 \geq \lambda_0(x)/(\lambda_0(x) + \lambda_i(x)) > 1/2,\ 1 \leq i \leq n\}.$$

The "lumped mass region" B_b associated with node b is given as

$$B_b = \bigcup_S B_{b(S)}$$

in which $\bigcup_S$ denotes the union with respect to all n-simplices S

having b as their vertex. The characteristic function of B_b is denoted by $\overline{w}_b(x)$.

By appropriate renumbering of all nodal points of all simplices in $\mathfrak{S}_h$, let $b_1, \cdots, b_N$ denote the interior nodal points of Ω_h, and let $b_{N+1}, \cdots, b_{N+M}$ denote the boundary nodal points of Ω_h. Define the two functions $\hat{w}_j(x)$ and $\overline{w}_j(x)$, $j=1,\cdots,N$, such that

$$\begin{cases} \hat{w}_j(x) \text{ is a linear function on each S satisfying} \\ \qquad \hat{w}_j(b_k) = \delta_{jk} \quad \text{for } k=1, \cdots, N+M, \\ \overline{w}_j(x) = \overline{w}_{b_j}(x). \end{cases}$$

Let $\hat{V}_h$ and $\overline{V}_h$ be the sets of linear combinations of $\hat{w}_j(x)$ and $\overline{w}_j(x)$ for $j = 1, \cdots, N$, respectively, i.e.,

$$\begin{cases} \hat{V}_h = \{\hat{u}_h;\ \hat{u}_h = \sum_{j=1}^{N} \alpha_j \hat{w}_j\}, \\ \overline{V}_h = \{\overline{u}_h;\ \overline{u}_h = \sum_{j=1}^{N} \alpha_j' \overline{w}_j\}. \end{cases}$$

The two functions $\hat{u}_h \in \hat{V}_h$ and $\overline{u}_h \in \overline{V}_h$ specified by the same coefficients $\alpha_j = \alpha_j'$, $j = 1, 2, \cdots, N$, are said to be associate each other. Introduce the mappings K_h and K_h^{-1} such that

$$\begin{cases} K_h: \quad \overline{u}_h \to \hat{u}_h, \\ K_h^{-1}: \quad \hat{u}_h \to \overline{u}_h, \end{cases}$$

where $\hat{u}_h \in \hat{V}_h$ and $\overline{u}_h \in \overline{V}_h$ are associate each other. Introduce the space X which is the space $C_0(\overline{\Omega})$ normed with the maximum norm. Similarly, introduce the finite dimensional spaces $\hat{X}_h$ and X_h which are $\hat{V}_h$ and $\overline{V}_h$, respectively, normed with the maximum norm. Define the operator $\tilde{P}_h$ by

$$(\tilde{P}_h u)(x) \equiv \sum_{j=1}^{N} u(b_j)\hat{w}_j(x) \quad \text{for } u \in X.$$

Clearly the mapping $P_h = K_h^{-1}\tilde{P}_h$,

$$P_h: \ X \to X_h,$$

is the projection of X onto X_h.

In terms of the above defined concepts and notations, define the operator A_h in X_h by

$$(A_h\overline{u}_h, \overline{v}_h)_{L^2(\Omega_h)} = -(\nabla\hat{u}_h, \nabla\hat{v}_h)_{L^2(\Omega_h)}$$

$$\text{for } \overline{u}_h \in X_h,\ \overline{v}_h \in X_h,$$

$$\hat{u}_h = K_h\overline{u}_h,\ \hat{v}_h = K_h\overline{v}_h,$$

and the nonlinear mapping f in X_h by

$$f(\overline{u}_h) = \sum_{j=1}^{N} f(\alpha_j)\overline{w}_j \quad \text{for } \overline{u}_h = \sum_{j=1}^{N} \alpha_j\overline{w}_j.$$

Let τ denote the ordered set $(\tau_0, \tau_1, \tau_2, \cdots)$ with elements $\tau_n > 0$, $n = 0, 1, 2, \cdots$. The set τ will be called the time mesh vector. Now we state our approximating scheme.

$$(E_h^\tau) \quad \begin{cases} t_{n+1} = t_n + \tau_n,\ t_0 = 0, \quad \tau_n > 0, \\ u_h(t) = u_h(t_n),\ t_n \leq t < t_{n+1}, \\ \dfrac{u_h(t_{n+1}) - u_h(t_n)}{\tau_n} = A_h u_h(t_n) + f(u_h(t_n)),\ n=0, 1, 2, \cdots, \\ u_h(0) = a_h, \end{cases}$$

where $a_h = P_h a$.

<u>Proposition 3.</u> If $(\nabla\hat{w}_i, \nabla\hat{w}_j)_{L^2(\Omega_h)} \leqq 0$ for $i \neq j$, $1 \leqq i \leqq N$, $1 \leqq j \leqq N+M$, then it holds that the smallest eigenvalue λ_h of $-A_h$ is simple, and that there is the associated

eigenfunction $\phi_h(x)$ normalized as $\phi_h(x) \geqq 0$ $(x \in \Omega_h)$ and $\int_{\Omega_h} \phi_h(x)dx = 1$.

Proof. As is well known, the operator A_h is invertible. By the criterion due to Ciarlet-Raviart [1], the operator $(-A_h)^{-1}$ is a nonnegative element of $L(X_h)$. Namely we have that $(-A_h)^{-1}u_h \geqq 0$ for any $u_h \in X_h$ with $u_h \geqq 0$. Consider the matrix expression of the eigenvalue problem $-A_h\phi_h = \lambda_h\phi_h$. The problem is reduced to $-\mathbb{A}\phi = \lambda\mathbb{M}\phi$ where $\mathbb{A}$ and $\mathbb{M}$ are the stiffness matrix and the mass matrix, respectively, and ϕ is an N-vector. Applying Frobenius' Theorem to the nonnegative matrix $(-\mathbb{A})^{-1}\mathbb{M}$, we have the conclusion.

§4 An algorithm for controlling time steps.

Define $J_h(t)$, the discrete analogue to $J(t)$, by

$$J_h(t) = (u_h(t,x),\ \phi_h(x))_{L^2(\Omega_h)}.$$

Let $J_h^{\,1}$ denote the largest positive root of the equation of

$$-\lambda_h J + f(J) = 0.$$

If the equation has no positive roots, then let $J_h^{\,1} = 0$.

Define τ_h by the formula,

(10) $$\tau_h = \min_{1 \leqq i \leqq N} \|\bar{w}_i\|^2 / \|\nabla\hat{w}_i\|^2.$$

Choose a fixed value of τ which is not greater than τ_h. Then our algorithm for controlling the time step τ_n is given by

(11) $$\begin{cases} \tau_0 = \tau, \text{ and} \\ \tau_n = \begin{cases} \tau & \text{if } J_h(t_{n-1}) < J_h^{\,1}, \\ \min\left\{\tau,\ \dfrac{J_h(t_n) - J_h(t_{n-1})}{-\lambda_h J_h(t_n) + f(J_h(t_n))}\right\} & \text{otherwise,} \end{cases} \end{cases}$$

for $n = 1, 2, 3, \cdots$.

Fig. 1 shows the general flow chart to calculate $u(t_n)$ by $(E_h^{\,\tau})$ with (11).

Definition 4. The solution $u_h(t,x)$ of $(E_h^{\,\tau})$ where τ is the time mesh vector obtained by the algorithm described above, J_h-blows up at $t = T_h$ if and only if

$$T_h = \sum_{n=0}^{\infty} \tau_n < \infty .$$

Proposition 4. The solution $u_h(t,x)$ J_h-blows up at a finite time T_h if and only if there is a $t_n \geqq 0$ such that

(12) $$J_h(t_n) > J_h^{\,1}.$$

Proof. The necessity of the condition (12) is obvious.

The sufficiency criterion, in parallel to the proof of Proposition 1, is as follows. By Jensen's inequality we have

$$\frac{J_h(t_{k+1}) - J_h(t_k)}{\tau_k} \geq -\lambda_h J_h(t_k) + f(J_h(t_k)) \tag{13}$$

for $k = 0, 1, \cdots$.

The convexity of f and the condition (12) imply that

$$J_h^1 < J_h(t_n) < J_h(t_{n+1}) < J_h(t_{n+2}) < \cdots .$$

Let k be an integer greater than or equal to n+1. Then our algorithm implies

$$\tau_k \leq \frac{J_h(t_k) - J_h(t_{k-1})}{-\lambda_h J_h(t_k) + f(J_h(t_k))}$$

$$= \int_{J_h(t_{k-1})}^{J_h(t_k)} \frac{dJ}{-\lambda_h J_h(t_k) + f(J_h(t_k))}$$

$$\leq \int_{J_h(t_{k-1})}^{J_h(t_k)} \frac{dJ}{-\lambda_h J + f(J)} ,$$

where the last inequality follows from the convexity of f.

Therefore we have

$$T - t_n - \tau_n = \sum_{k=n+1}^{\infty} \tau_k$$

$$\leq \int_{J_h(t_n)}^{\infty} \frac{dJ}{-\lambda_h J + f(J)} < \infty .$$

Corollary 5. The blowing-up time T_h is bounded from above as

$$T_h \leq t_n + \tau_n + \int_{J_h(t_n)}^{\infty} \frac{dJ}{-\lambda_h J + f(J)} .$$

§5 Convergence of the blowing up time.

Theorem 1. Assume the following two conditions:

(i) $\lambda_h \to \lambda$ and $\phi_h \to \phi$ in $L^2(\Omega)$ as $h \to 0$.

(ii) Let the solution u of (E) J-blow up at a finite time T. For any $T' < T$ and for any sufficiently small h, there is a solution $u_h(t)$ of $(E_h^{\tau_h})$ for $0 \leq t \leq T'$ satisfying $\max_{0\leq t\leq T'} \| u_h(t) - u(t) \|_{L^2(\Omega)} \to 0$ as $h \to 0$. Here τ_h is the time mesh vector obtained by (11).

Then it holds that

$$T_h \to T \text{ as } h \to 0$$

provided that $\| \tau_h \|_\infty \to 0$ as $h \to 0$.

Proof. Fix $T' < T$ arbitrarily. Then we have from the conditions of Theorem 1

$$\lim_{h\to 0} J_h(t) = J(t) \tag{14}$$

uniformly in $t \in [0,T']$. This implies that $T' \leq \liminf_{h\to 0} T_h$. Since T' is arbitrarily close to T, we have

$$T \leq \liminf_{h\to 0} T_h .$$

Suppose next that $T'' = \limsup_{h\to 0} T_h > T$ would hold. By Condition (i) we can find numbers J^2 and h_0 in such a way that $J^2 > J_h^1$ and $\int_{J^2}^{\infty} \frac{dJ}{-\lambda_h J + f(J)} \leq \frac{T'' - T}{2}$ hold for any $h \leq h_0$. By (14) there is a number $t' < T$ such that $J_h(t') > J^2$ for any $h \leq h_0$. We may assume that $t_n^h \leq t' < t_{n+1}^h < T$ for $h \leq h_0$. Then $J_h(t') = J_h(t_n^h)$. By Corollary 5, it holds that

$$T_h - t_n^h \leq \tau_n^h + \int_{J_h(t_n^h)}^{\infty} \frac{dJ}{-\lambda_h J + f(J)} .$$

Hence we have

$$T_h \leqq t_n^h + \tau_n^h + \frac{T'' - T}{2}$$

$$\leqq T + \frac{T'' - T}{2} = T'' - \frac{T'' - T}{2} .$$

Therefore we have $\limsup_{h\to 0} T_h < T'' = \limsup_{h\to 0} T_h$, which is a contradiction.

§6 Convergence of the approximate solution.

Let $\{\overline{\tau}_h : h > 0\}$ be a sequence tending to zero as h tends to zero, and satisfying $\overline{\tau}_h \leqq \tau_h$ where τ_h is defined by the formula (10). Let T' be a positive number specified in Theorem 1. Consider a family $\lambda = \{\boldsymbol{\tau}_h : h > 0\}$ of time mesh vectors $\boldsymbol{\tau}_h$ satisfying that

$$\| \boldsymbol{\tau}_h \|_\infty = \sup \{\tau : \tau \in \boldsymbol{\tau}_h\} \leqq \overline{\tau}_h ,$$

and that

$$\| \boldsymbol{\tau}_h \|_1 = \sum_{\tau \in \boldsymbol{\tau}_h} \tau > T' .$$

We can construct a family of solutions $\{u_h^\lambda(t,x) : 0 \leqq t \leqq T'\}$ of $(E_h^{\boldsymbol{\tau}})$ choosing $\boldsymbol{\tau}_h$ as the time mesh vector for each h. Let Λ be the totality of the above index λ. Let h(S) denote the diameter of an n-simplex S, and let $\rho(S)$ denote the maximum of diameters of the inscribed spheres of S.

Theorem 2. Assume (8) and the following three conditions

(i) $(\nabla \hat{w}_i, \nabla \hat{w}_j) \leqq 0$ for $i \neq j$, $1 \leqq i \leqq N$, $1 \leqq j \leqq N+M$

(ii) $\max_{S \in \mathfrak{S}_h} h(S) \leqq h$

(iii) $\inf_h \min_{S \in \mathfrak{S}_h} \rho(S)/h(S) = \mu > 0.$

If the unique classical solution u(t,x) of (E) exists in $t \in [0,T']$ then

$$\lim_{h\to 0} \max_{0 \leqq t \leqq T'} \max_{x \in \Omega_h} |u_h^\lambda(t,x) - u(t,x)| = 0$$

uniformly in $\lambda \in \Lambda$.

Proof. This is a slight variant of Theorem 1.2 of [11], in which the uniform dependence of λ was not discussed. If one checks its proof, the present result can be obtained. As in [11], we have the expression

$$u_h^\lambda(t) = U_h^\lambda(t,0)a_h + \int_0 \hat{U}_h^\lambda(t,s)f(u_h^\lambda(s))ds$$

regarding $u_h^\lambda(t,x)$ as an X_h-valued step function where $U_h^\lambda(t,s)$ and $\hat{U}_h^\lambda(t,s)$ are suitably defined approximating operators of $e^{(t-s)A_h}$. It is to be noted that the proof of Theorem 1.2 in [11] implies that the families of operator valued sequences $\{U_h^\lambda(t,s): h > 0\}_{0\leq s\leq t\leq T',\ \lambda\varepsilon\Lambda}$ and $\{\hat{U}_h^\lambda(t,s): h > 0\}_{0\leq s\leq t\leq T',\ \lambda\varepsilon\Lambda}$ K-converge to $e^{(t-s)A}$ as h tends to 0 uniformly with respect to the parameters s, t, and λ. Here, the operator A is the generator of the semi-group $e^{(t-s)A}$ in $X = C_0(\bar{\Omega})$ corresponding to the heat equation in Ω with the Dirichlet boundary condition. This fact assures the validity of the present Theorem.

<u>Remark.</u> Although the twice differentiability of f was assumed in [11], the local Lipshitz continuity is sufficient for the present conclusion apart from the assurance of the unique existence of the smooth solution of (E). See [8].

Now we establish the condition (ii) of Theorem 1 to our solution $u_h(t)$ obtained by the algorithm described in Section 4. Because of Theorem 2, it suffices to show that for any fixed $T' < T$ one can choose h_0 in such a way that $u_h(t)$ never blows-up within the interval $[0,T']$ if $h \leq h_0$. This fact is also implied by Theorem 2. In fact, let h_0 be such that

$$\max_{0\leq t\leq T'} \max_{x\varepsilon\Omega} | u_h^\lambda(t,x) - u(t,x) | \leq 1$$

for any $\lambda \varepsilon \Lambda$ and $h \leq h_0$ in the situation of Theorem 2. This

implies that there is a finite number M satisfying

$$(15) \qquad \sup_{0\leq t\leq T',\ \lambda\varepsilon\Lambda, h\leq h_0} \| u_h^\lambda(t)\|_{L^2(\Omega)} = M < \infty.$$

Assume that there is a solution $u_h(t)$ J_h-blowing up at $t = T_h < T'$. Then there is a mesh point t_n such that $\| u_h(t_n)\| > M$ since $\|\phi_h\| \, \|u_h(t)\| \geq J_h(\phi_n) \uparrow \infty$. This contradicts the condition (15), since there is a λ containing the time mesh vector in the form

$$\tau_h = (\tau_0, \tau_1, \cdots, \tau_{n-1}, \tau_{n-1}, \cdots)$$

where τ_j, $0 \leq j \leq n-1$, are the mesh lengths determined by our algorithm.

It is seemingly well known that the condition (i) of Theorem 1 holds under the same conditions of Theorem 2. We skip its proof though we have not known literatures containing its proof.

Finally we remark the following two Propositions in the literature which concern the conditions of Theorem 2.

<u>Proposition 6.</u> (Ciarlet-Raviart [1].) Define σ_S by

$$\sigma_S = \max_{i\neq j} \{(\nabla\lambda_i, \nabla\lambda_j)_{R^n} / |\nabla\lambda_i|_{R^n} \cdot |\nabla\lambda_j|_{R^n}\}$$

in which $\lambda_i = \lambda_i(x)$ is the barycentric coordinate of point $x \varepsilon S$ with respect to vertex P_i of S, and $(\ ,\ )_{R^n}$ and $|\ \ |_{R^n}$ respectively denote the Euclidean scalar product and the Euclidean norm in R^n. If $\sigma_S \leq 0$ for any $S \varepsilon S_h$, then Condition (i) of Theorem 2 holds.

<u>Proposition 7.</u> (Fujii [2].) Define κ_h by

$$\kappa_h = \min_{S\varepsilon S_h} \kappa_S$$

in which κ_S is given as

$$\kappa_S = \min_{b_i \varepsilon S} \text{dist } (b_i, \text{ the face of } S \text{ not containing } b_i)$$

then τ_h is estimated as

$$\tau_h \geqq \kappa_h^2/(n+1).$$

§7 Numerical Examples.

As an numerical illustration, we take a spatially 1 dimensional problem with $f(u) = u^2$. Let $\Omega = (0,1)$. And $S_h = \{[jh, (j+1)h]: 0 \leq j < 2^N\}$ for $h = 2^{-N}$ with positive integer N. Then our approximate equation (E_h^τ) is nothing but the usual explicit difference scheme in which $\frac{d^2}{dx^2}u(x)$ is approximated by the central difference: $\frac{u(x+h)-2u(x)+(x-h)}{h^2}$.

As an initial value we take the function

$$u(x) = A \sin \pi x.$$

In Figure 2, a comparison between the controlling time mesh algorithm and the fixed time mesh algorithm is shown in the case of $h = 2^{-3}$, $\tau_0 = 2^{-7}$ and $A = 12$. We denote by $\cdot$, and $\times$, the values of $u(t,x)$ at $x = 1/2$, calculated by the controlling mesh algorithm and the fixed mesh algorithm, respectively. We obtained $t_n = 0.307$ as the approximate value of T_h when the controlling mesh algorithm stopped by the condition of $\tau_n < 2^{-20}$, whereas the fixed mesh algorithm worked until $t_n = 0.367$ when the calculation became unable because of machine overflow $(u_{max} > 10^{65})$.

In Figure 3, the convergence of $u_h(t,1/2)$ as $h \to 0$ is shown for $A = 12$, with $\tau_0 = 2^{-1}h^2$. The stopping criterion is that $\tau_n < 2^{-20}$. Seemingly the round off error might effect the calculation in the case $h = 2^{-6}$.

Our calculation was performed by the HITAC 8250, a medium-sized machine, in single precision arithmetic with a 23 bit mantissa.

Finally in Figure 4, we present the result of a numerical search for the threshold of blowing up by changing the coefficient A of the initial data $a(x)$, in the case of $h = 2^{-4}$ and $\tau_0 = 2^{-9}$. Numerically the solution decays exponentially if $A \leq 11.47417$, and blows up at finite time if $A \geq 11.47418$. It is worth recalling that if $J(0) > J^1$ the solution of (E) blows up at finite time. In our sample problem, $J(0) > J^1$ is equivalent to the condition of $A > 4\pi = 12.56637$.

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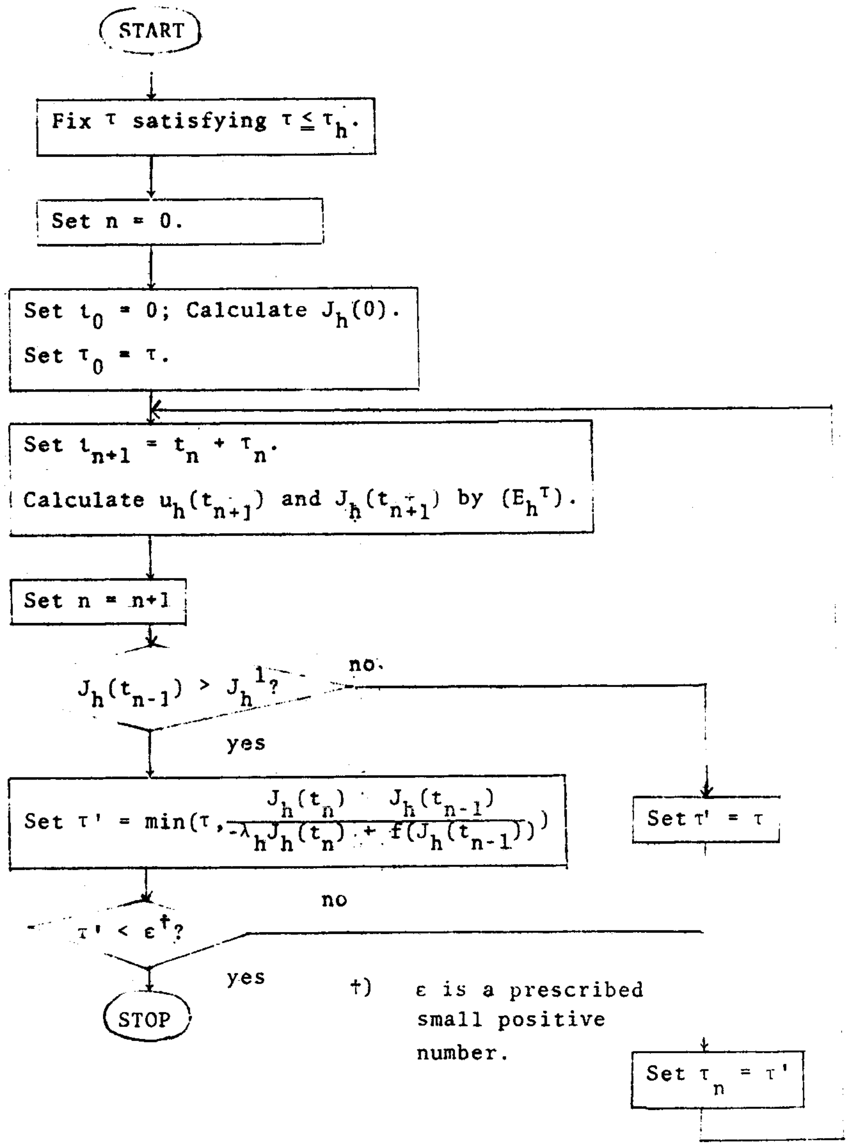

Fig. 1 Flow chart for computing $u_h(t_n)$ by controlled time steps τ_n.

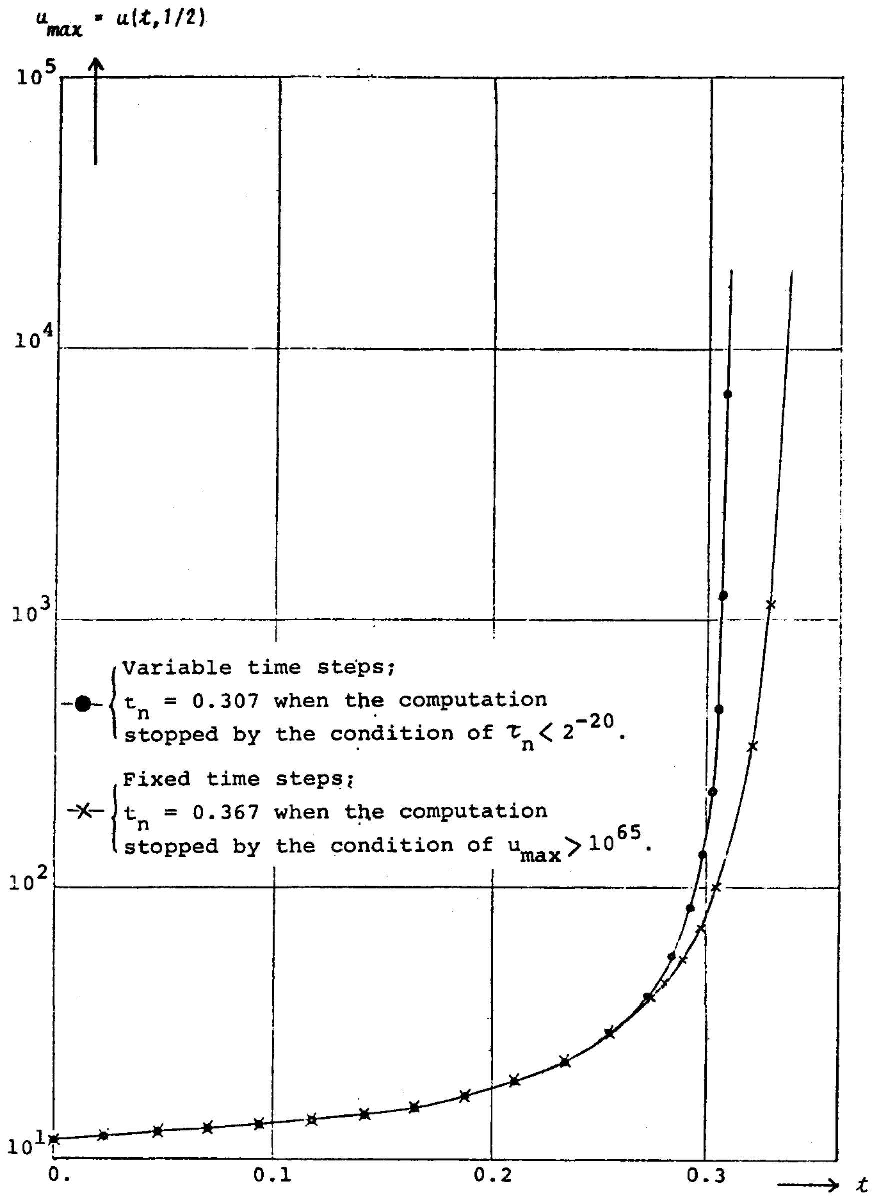

Fig.2 The computed value of u by the variable time-step algorithm and the fixed-time step algorithm in the case of $a(x) = 12\sin\pi x$. Parameters: $h = 2^{-3}$ and $\tau_0 = 2^{-7}$.

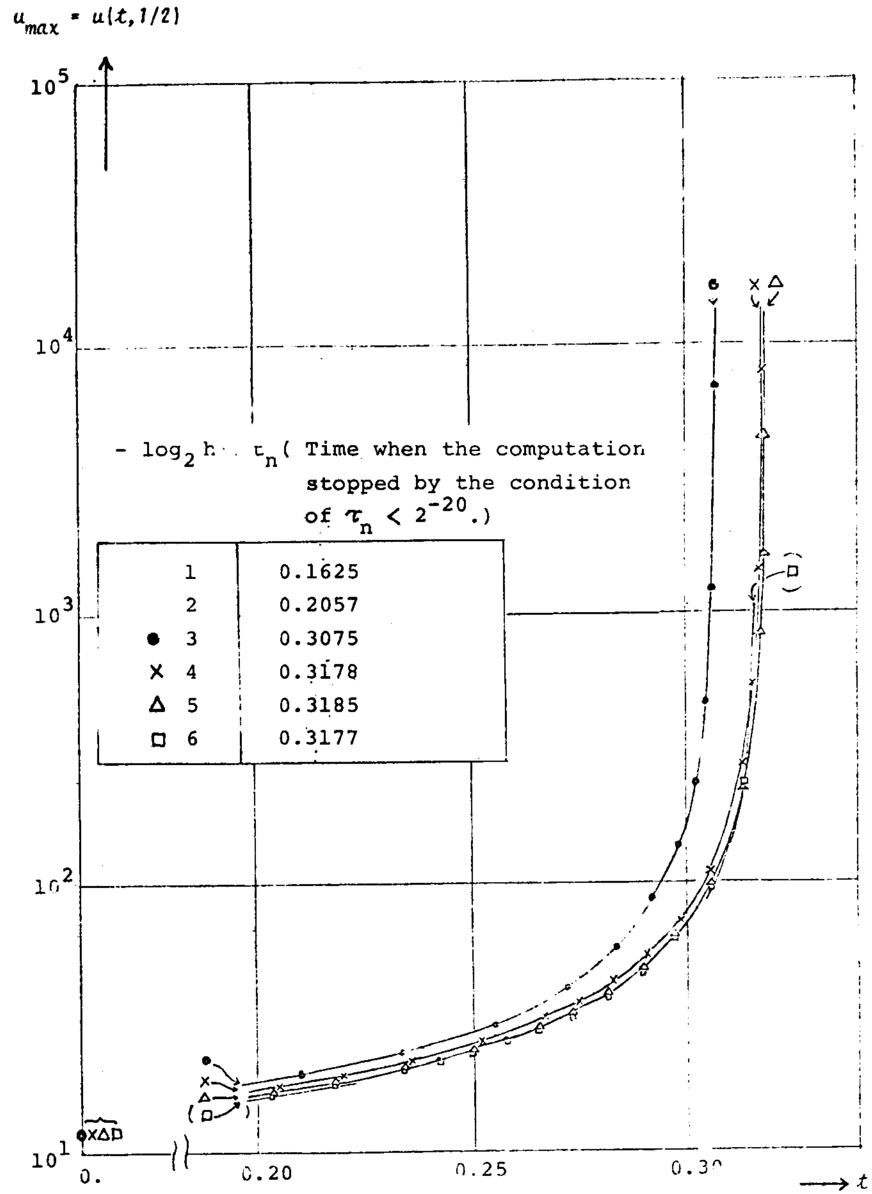

Fig.3 Convergence of profile of u as $h \to 0$ in the case of $a(x) = 12\sin\pi x$.

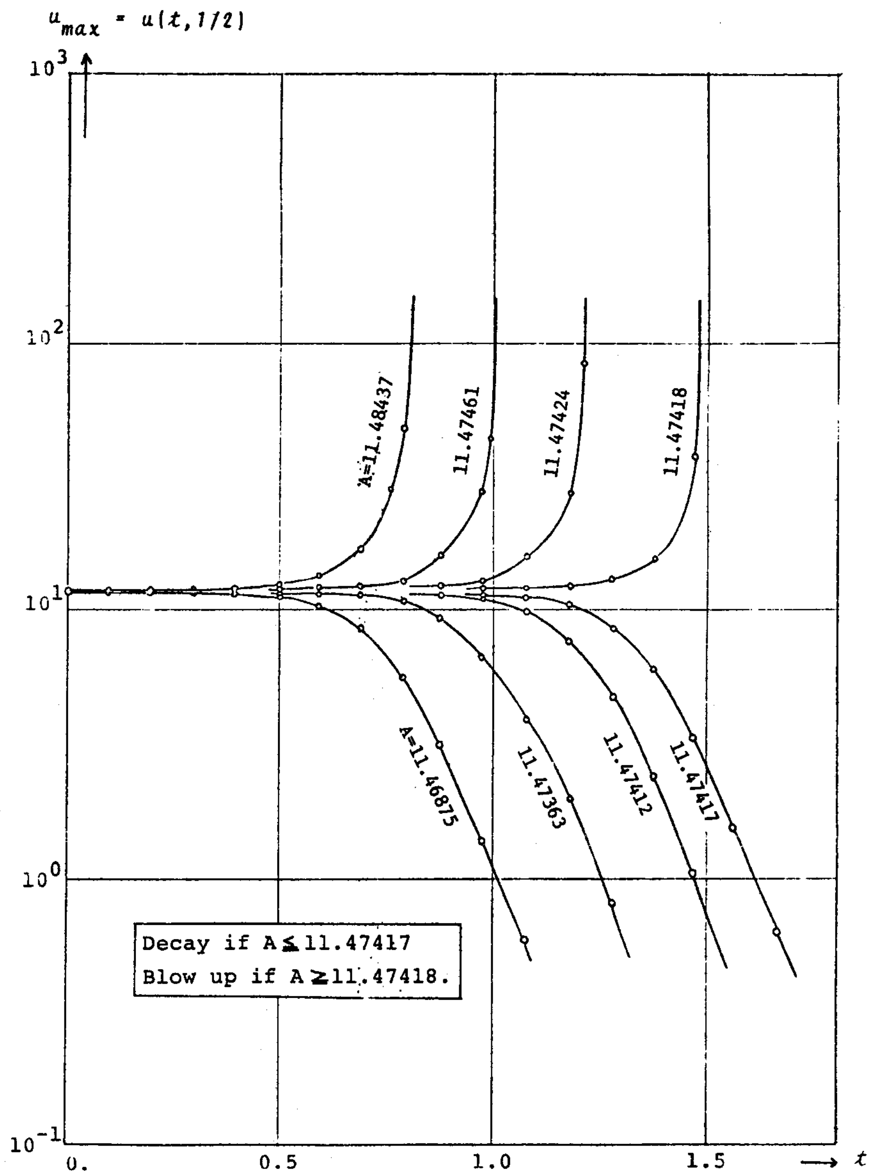

Fig. 4 Numerical search for the threshold of blowing-up in the case of $a(x) = A \sin \pi x$.
Parameters: $h = 2^{-4}$ and $\tau_0 = 2^{-9}$.