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MIXING PROPERTIES

OF

SUBSTITUTIONS

by

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Substitution minimal set provide useful examples in topological dynamics (as minimal flows) and in ergodic theory (as strictly ergodic dynamical systems). In general, the topological and measure-theoretic structure of a substitution minimal set is difficult to determine. Throughout this paper, we shall be primarily concerned with the substitution

$$\eta : \begin{array}{l} 0 \rightarrow 001 \\ 1 \rightarrow 11100 \end{array} ,$$

its orbit closure  $X$  under the shift transformation  $T$ , and its unique invariant probability measure  $\mu$  ([12]). For basic definitions and a coherent development of the theory of substitutions, we refer to [4].

The first part of our investigation concerns measure-theoretic mixing properties. After using an idea due to Kakutani [11] to show that the dynamical system  $(X, \mu, T)$  is weakly mixing, we prove that no system arising from a substitution can be strongly mixing. Thus our particular system arising from  $\eta$  provides yet another example (and perhaps the simplest to describe) of weak but not strong mixing ([8], [5], [11], [1]).

The second section of our work is concerned with topological mixing. We show that the minimal flow  $(X, T)$  arising from  $\eta$  is topologically strongly mixing of order two, but not of order three, providing the first example of a minimal flow with these properties. (If minimality is not required, an example can be found in [6].) It is also proved that no minimal flow arising from a substitution can be topologically strongly mixing of all orders.

The last paragraph contains a remark on a substitution closely related to  $\eta$ , whose corresponding dynamical system has arisen in [10] and [15], and we indicate how our method applies to this case.

§1 - Measure-theoretic mixing properties.

Throughout the paper,  $(X, \mu, T)$  will be used to denote the dynamical system arising from the substitution

$$\eta : \begin{array}{l} 0 \rightarrow 001 \\ 1 \rightarrow 11100 \end{array} .$$

We prepare the first result by introducing a change of symbols. Writing  $a = 00$  and  $b = 1$ , we see that

$$\begin{aligned} \eta(a) &= \eta(00) = 001\ 001 = abab \\ \eta(b) &= \eta(1) = 11100 = bbba . \end{aligned}$$

This leads to the consideration of the substitution

$$\theta : \begin{array}{l} a \rightarrow abab \\ b \rightarrow bbba \end{array} .$$

The dynamical system arising from  $\theta$  will be denoted by  $(Y, \nu, S)$ .

It is convenient to prove results for  $\eta$  by using  $\theta$ . This is possible because  $(X, \mu, T)$  can be represented as a tower over  $(Y, \nu, S)$  with respective heights 2 over the cylinder  $[a]$  and 1 over the cylinder  $[b]$  (cf. [3] Th. 4.1 and Ex 4.8).

(Here and in the sequel, a set  $\{y \in Y : y_0 = c_0, \dots, y_n = c_n\}$  will be

denoted by  $[c_0 \dots c_n]$  or by  $[C]$ , where  $C = c_0 c_1 \dots c_n$  is a block (finite sequence) of symbols over the alphabet considered, in this case  $\{a, b\}$ . These sets and their translates by powers of the shift are called cylinders. A cylinder  $[C]$  will be non-empty if and only if the block  $C$  occurs in  $\theta^n a$  for some  $n$ ; such blocks are called admissible.)

Theorem 1 *The dynamical system  $(X, \mu, T)$  is weakly mixing.*

Proof: Let  $f$  be an eigenfunction of  $T$  with eigenvalue  $\lambda$ . Since  $T$  is ergodic, we may assume  $|f| \equiv 1 \equiv |\lambda|$ , and we must show that  $\lambda = 1$ . By the tower representation of  $X$  over  $Y$ , there is a function  $g$  on  $Y$  of constant modulus one (obtained by restricting  $f$  to the base of the tower) such that

$$Sg = \lambda^h g, \quad ,$$

where

$$h(y) = \begin{cases} 2 & \text{for } y \in [a] \\ 1 & \text{for } y \in [b]. \end{cases}$$

For  $n \geq 1$ , set

$$h_n = \sum_{t=0}^{4^n - 1} S^t h.$$

Then

$$S^{4^n} g = \lambda^{h_n} g.$$

It is known [2] that the system  $(Y, \nu, S)$  has discrete spectrum and that its eigenvalue group is the group of all  $4^n$ -th roots of unity. Thus

$$\lim_{n \rightarrow \infty} \left\| 1 - \lambda^{h_n} \right\|_2 = \lim_{n \rightarrow \infty} \left\| S^{4^n} g - g \right\|_2 = 0 .$$

For each  $n \geq 1$ , consider the set  $A_n$  defined by

$$A_n = \bigcup_{t=0}^{4^n-1} S^{-t} [\theta^n(bb)] .$$

We shall show that

$$1) \quad \nu(A_n) \geq \nu([\text{bb}])$$

$$2) \quad h_n = \frac{1}{3} (4^{n+1} - 1) \quad \text{on } A_n$$

for each  $n \geq 1$ . It then follows from  $\nu([\text{bb}]) > 0$  and from

$$\lim_{n \rightarrow \infty} \int_{A_n} |1 - \lambda^{h_n}|^2 d\nu = 0$$

that

$$\lim_{n \rightarrow \infty} \lambda^{\frac{1}{3}(4^{n+1} - 1)} = 1 ,$$

and this implies easily  $\lambda = 1$ .

To prove 1), we use the fact [4] that the measure of  $[\text{bb}]$  is just the limit of the occurrence frequency of  $\text{bb}$  in  $\theta^m a$  as  $m$  tends to infinity, and the corresponding fact for the measure of  $A_n$ . Obviously any occurrence of  $\text{bb}$  in  $\theta^m a$  yields an occurrence of  $\theta^n(\text{bb})$  in  $\theta^{n+m} a$ , and hence  $4^n$  "occurrences" of  $A_n$  in this block. The length of  $\theta^{n+m} a$  being  $4^n$  times the length of  $\theta^m a$ , 1) follows.

To see 2), note that for any  $y \in A_n$

$$h_n(y) = 2N_a(\theta^n b) + N_b(\theta^n b) ,$$

where  $N_i(\cdot)$  denotes the number of occurrences of  $i$  in a block.

If now

$$L = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} N_a(\theta a) & N_b(\theta a) \\ N_a(\theta b) & N_b(\theta b) \end{bmatrix},$$

then

$$L^n = \begin{bmatrix} N_a(\theta^n a) & N_b(\theta^n a) \\ N_a(\theta^n b) & N_b(\theta^n b) \end{bmatrix},$$

and an easy calculation yields

$$L^n = \frac{1}{3} \begin{bmatrix} 4^n + 2 & 2 \cdot 4^n - 2 \\ 4^n - 1 & 2 \cdot 4^n + 1 \end{bmatrix}.$$

Therefore

$$h_n(y) = \frac{2}{3} (4^n - 1) + \frac{1}{3} (2 \cdot 4^n + 1) = \frac{1}{3} (4^{n+1} - 1)$$

for any  $y \in A_n$ .

□

We indicate briefly the range of application of this method to substitutions. Let  $\zeta$  be a substitution on two symbols whose matrix has relatively prime integral eigenvalues and such that the lengths of  $\zeta 0$  and  $\zeta 1$  are also relatively prime. (If these conditions are not fulfilled, it can be shown that a non-constant continuous eigenfunction exists). In this situation, it is always possible ([3] Th. 5.1, Rem. 5.4) to find a substitution  $\theta$  of constant length related to  $\zeta$  as above, but in general  $\theta$  will be defined on an alphabet of more than two symbols. If the dynamical system arising from  $\theta$  has discrete spectrum, then an analogous argument shows that the system corresponding to  $\zeta$  has continuous spectrum. In particular, if we consider substitutions  $\zeta$  having the same matrix as  $\eta$ , we obtain 16 substitutions which are not obviously isomorphic. One of these is periodic, and 13 have continuous spectrum by the above. However, the substitution



$$\zeta : \begin{array}{l} 0 \rightarrow 001 \\ 1 \rightarrow 10110 \end{array}$$

and the coding  $a = 001$ ,  $b = 101$ ,  $c = 10$  yield

$$\theta : \begin{array}{l} a \rightarrow aabc \\ b \rightarrow bcab \\ c \rightarrow cbca \end{array}$$

(noting that  $c$  always follows  $b$  and is always preceded by  $b$ ), and  $\theta$  is known to have partly continuous spectrum [3]. We do not know whether  $\zeta$  has continuous spectrum. One other substitution of this class has an analogous behavior.

We next prove a general result. Let  $\zeta$  be a substitution which, for simplicity in the proof, we shall assume to be defined on  $\{0,1\}$ , and let  $(Z, \pi, U)$  be the dynamical system arising from  $\zeta$ . We suppose that  $\text{card } Z > 1$ .

Theorem 2 *The dynamical system  $(Z, \pi, U)$  is not strongly mixing.*

Proof : We may obviously restrict our attention to  $Z$  infinite and, interchanging 0 and 1 if necessary, suppose that  $\pi([00]) > 0$ . Let us denote by  $s_n$  the length of  $\zeta^n 0$ .

Let  $B$  be an admissible block for  $\zeta$ . For each  $n \geq 1$ , we set

$$D_n = [B] \cap U^{-s_n} [B].$$

If  $(Z, \pi, U)$  were strongly mixing, we would have

$$\lim_{n \rightarrow \infty} \pi(D_n) = \pi([B])^2.$$

We shall show, on the contrary, that there exists a constant  $\gamma > 0$  independent of the choice of  $B$  such that

$$(*) \quad \liminf_{n \rightarrow \infty} \pi(D_n) \geq \gamma \pi([B]) .$$

Since the measure of  $[B]$  can be made arbitrarily small by choosing  $B$  long enough, the proof will be finished if we succeed in establishing  $(*)$ .

Now the measure of  $D_n$  is just the limit of the occurrence frequency of two  $B$ 's at distance  $s_n$  in the block  $\zeta^m 0$ , as  $m$  tends to infinity. Combining this with the fact that such an occurrence inside of a block of the form  $\zeta^n 0 \zeta^n 0$  happens at least as often as  $B$  occurs in  $\zeta^n 0$ , we obtain

$$\pi(D_n) \geq \lim_{m \rightarrow \infty} \frac{N_B(\zeta^n 0) N_{00}(\zeta^{m-n} 0)}{s_m} ,$$

where  $N_E(F)$  denotes the number of occurrences of  $E$  in  $F$ .

Let  $\lambda$  denote the largest eigenvalue of the matrix of  $\zeta$ . Then  $s_n$ , which is the first row sum of the  $n$ -th power of this matrix, is asymptotically equal to  $\alpha \lambda^n$  for some positive constant  $\alpha$ . Therefore

$$\lim_{m \rightarrow \infty} \frac{N_{00}(\zeta^{m-n} 0)}{s_m} = \pi([00]) \lim_{m \rightarrow \infty} \frac{s_{m-n}}{s_m} = \pi([00]) \lambda^{-n}$$

and

$$\begin{aligned} \liminf_{n \rightarrow \infty} \pi(D_n) &\geq \pi([00]) \lim_{n \rightarrow \infty} N_B(\zeta^n 0) \lambda^{-n} = \\ &= \pi([00]) \pi([B]) \alpha . \end{aligned}$$

Setting  $\gamma = \alpha \pi([00])$ , we obtain the desired result. □

§2. Topological mixing properties.

Recall that the substitution

$$\begin{aligned} & 0 \rightarrow 001 \\ \eta : & \\ & 1 \rightarrow 11100 \end{aligned}$$

gives rise to a minimal flow denoted by  $(X, T)$ . Our first goal in this paragraph is to prove the following theorem. The proof will consist of a series of lemmas.

Theorem 3 *The minimal flow  $(X, T)$  is topologically strongly mixing.*

Let  $\theta$  and  $(Y, \nu, S)$  be as in §1. To prove the theorem, it suffices to show that for any fixed  $n$  and for all  $t$  sufficiently large, a block of the form

$$\eta^n(1) C \eta^n(1)$$

and of length  $t$  is admissible for  $\eta$  (i.e., appears in some  $\eta^m 0$ ). In terms of  $\theta$ , this means that for any fixed  $n$  and for all sufficiently large  $t$ , there is a  $\theta$ -admissible block of the form

$$B_t = \theta^n(b) C_t \theta^n(b)$$

such that

$$2N_a(B_t) + N_b(B_t) = t.$$

This is the content of lemma 8.

In the following, the length of a block  $B$  will be denoted by  $N(B) = N_a(B) + N_b(B)$ . We also define the excess  $e(B)$  of  $B$  by

$$e(B) = 2N_a(B) - N_b(B) \quad .$$

Its properties are :

- Lemma 4
- (i)  $e(AB) = e(A) + e(B)$
  - (ii)  $e(\theta B) = e(B)$
  - (iii)  $\int_Y e(y_0 y_1 \dots y_{k-1}) d\nu(y) = 0$

for all blocks  $A$  and  $B$  and all  $k \geq 1$  .

Proof : (i) is immediate from the definition ; (iii) follows from  $\nu([a]) = 1 - \nu([b]) = \frac{1}{3}$  and from (i) . To show (ii) it suffices to remark that  $(2, -1)$  is a right eigenvector of the  $\theta$ -matrix

$$L = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{with eigenvalue one.} \quad \square$$

Lemma 5 For each  $k \geq 2$  we can choose an admissible block  $B_k$  of length  $k$  , beginning in  $b$  and ending in  $a$  , such that

$$\lim_{k \rightarrow \infty} e(B_k) = +\infty .$$

Proof : Suppose that  $B_k$  has been chosen for some fixed  $k$  . Then

$$\theta B_k = bbba D abab ,$$

and we can define

$$B_{4k-1} = bbba D aba$$

$$B_{4k-2} = bba D aba$$

$$B_{4k-3} = ba D aba$$

$$B_{4k-4} = bba D a .$$

If  $e(B_k) = e$  , then by lemma 4 (i) , (ii) we have

$$e(B_{4k-1}) = e + 1$$

$$e(B_{4k-2}) = e + 2$$

$$e(B_{4k-3}) = e + 3$$

$$e(B_{4k-4}) = e + 1 .$$

Now set  $B_2 = ba$  and  $B_3 = bba$  , and proceed by induction. □

Our next task is to investigate the possible excess values for all admissible blocks of a given length  $k$  . If  $B$  and  $B'$  are admissible blocks of length  $k$  , then they both appear in some  $\theta^m a$  . Therefore we can proceed from  $B$  to  $B'$  by a sequence of admissible blocks, each block in the sequence being obtained from the preceding one by deleting a symbol from one end and adding a symbol to the other end. This addition-deletion process either leaves the excess unchanged or changes it by  $\pm 3$  . Thus for given  $k$  ,  $B$  , and  $B'$  , necessarily  $e(B) = e(B') \pmod{3}$  , and all values  $e$  between  $e(B)$  and  $e(B')$  with  $e = e(B) \pmod{3}$  occur as excess values of admissible blocks of length  $k$  . Examining blocks of length one, it is now obvious that  $e(B) = -k \pmod{3}$  if  $B$  has length  $k$  .

Lemma 6 *Let  $e_0 \geq 0$  . Then for all  $k$  sufficiently large and each  $0 \leq e \leq e_0$  with  $e = -k \pmod{3}$  , there exists an admissible block of length  $k$  and excess  $e$  .*

Proof : By lemma 4 (iii), there exists for each  $k$  an admissible block  $B$  of length  $k$  such that  $e(B) \leq 0$  . Our conclusion then follows from lemma 5 and the above remarks. □

Unfortunately, lemma 6 is not powerful enough to conclude the proof of theorem 3, since we do not know whether such blocks can be chosen also to begin and end with a fixed symbol.

Lemma 7 *Let  $e_0 \geq 5$ . Then for all sufficiently large  $k$  and each  $5 \leq e \leq e_0$  with  $e \equiv -k \pmod{3}$ , there exists an admissible block of length  $k$  and excess  $e$  which begins and ends in the symbol  $b$ .*

Proof : The idea of the proof is to take a block as in lemma 6 and apply  $\theta^3$ , obtaining an admissible block with the same excess but 64 times as long. Then by deleting some symbols from the end and adding some symbols to the beginning of this block, we can obtain blocks beginning and ending in  $b$  with the desired properties.

We begin by examining  $\theta^3 a$  and  $\theta^3 b$ . For each  $m$  with  $1 \leq m \leq 63$  define blocks  $L(a,m)$ ,  $R(a,m)$ ,  $L(b,m)$  and  $R(b,m)$  by requiring that

$$N(L(a,m)) = N(L(b,m)) = m$$

and

$$\theta^3 a = L(a,m) R(a,m)$$

$$\theta^3 b = L(b,m) R(b,m)$$

Now set

$$S = \{4, 5, 6, 16, 20, 21, 22, 24, 25, 26, 29, 52, 53\}.$$

It is easily verified that :

- 1)  $R(a,s)$  and  $R(b,s)$  begin in  $b$  for each  $s \in S$ ,
- 2)  $e_s := e(R(a,s)) = e(R(b,s))$  for each  $s \in S$ ,
- 3)  $\max_{s,t \in S} |e_s - e_t - 1| = 5$ , and

- 4) any integer from  $-31$  to  $+32$  is a difference of two elements of  $S$ .

Now we can describe the announced deleting and adding. Let  $k'$  be a positive integer,  $B'$  a block of length  $k'$  ending in the symbol  $j$  ( $j = a$  or  $j = b$ ), and  $i$  a symbol ( $i = a$  or  $i = b$ ) such that the block

$$iB' = iA'j$$

is admissible. Then if  $A = \theta^3 A'$ , the block

$$\theta^3 i A \theta^3 j$$

is also admissible, and hence for any  $s, t \in S$ , the block

$$C(s,t) := R(i,s) A L(j,t) b$$

is admissible (since  $R(j,t)$  begins in  $b$ ), and begins and ends in  $b$ . Moreover,

$$N(C(s,t)) = 64k' + t - s + 1$$

and

$$e(C(s,t)) = e(B') + e_s - e_t - 1.$$

It is important to note that by 2), the excess of  $C(s,t)$  does not depend on the values of  $i$  and  $j$ .

Next we choose  $k_0$  such that the conclusion of lemma 6 is valid for  $e_0 + 5$  and all  $k' \geq k_0$ ,  $e_0$  being the number of our hypothesis. Let  $5 \leq e \leq e_0$  and choose any  $k \geq 64k_0$  with  $e = -k \pmod{3}$ . Then by 4) there exist  $k' \geq k_0$  and  $s, t \in S$  such that

$$k = 64k' + t - s + 1.$$

Now set

$$e' = e - e_s + e_t + 1.$$

By 3),  $0 \leq e' \leq e_0 + 5$ , so that we can apply lemma 6 for  $e'$  and  $k'$  to obtain an admissible block  $B' = A'j$  of length  $k'$

and excess  $e'$  ( $e' = -k' \pmod{3}$  because  $e = -k \pmod{3}$ .) The block  $C(s,t)$  corresponding to  $B'$ ,  $s$  and  $t$  as above, where  $i$  is chosen so that  $iB'$  is admissible, obviously has length  $k$ , excess  $e$ , and begins and ends in  $b$ . □

Lemma 8 *Let  $n \geq 1$ . Then for sufficiently large  $t$  there exists an admissible block of the form*

$$B_t = \theta^n(b) C_t \theta^n(b)$$

*such that*

$$2N_a(B_t) + N_b(B_t) = t.$$

*Proof :* For any block  $B$ , the definition of  $e(B)$  yields

$$2N_a(B) + N_b(B) = \frac{1}{3} (4N(B) + e(B)).$$

Let  $e_0 = 4^{n+1} + 4$  and choose  $k_0$  such that the conclusion of lemma 7 holds for  $k \geq k_0$ . Then for any  $t \geq \frac{4^{n+1}k_0}{3} + 2$  we can find  $k \geq k_0$  and  $5 \leq e \leq 4^{n+1} + 4$  such that

$$3t = k \cdot 4^{n+1} + e$$

Since  $e = -k \pmod{3}$ , lemma 7 gives us an admissible block  $B$  of length  $k$  and excess  $e$  beginning and ending in  $b$ .

Thus  $B_t := \theta^n B$  is of the desired form, and

$$2N_a(B_t) + N_b(B_t) = \frac{4 \cdot 4^n k + e}{3} = t. \quad \square$$

This completes the proof of Theorem 3. It is not surprising that the proof was somewhat delicate, in view of the following result.



Theorem 9    *The minimal flow  $(X,T)$  is not topologically strongly mixing of order three.*

Proof :    It is easily verified (by a calculation using the matrix of  $\eta$  that for each  $n \geq 1$ ,

$$N(\eta^n(00)) = N(\eta^n(1)) + 1 .$$

Our proof leans heavily on this fact. Now set

$$Z_0 = [0], Z_1 = [11], Z_2 = [010], \text{ and } t_n = N(\eta^n 1) \text{ for each } n \geq 1 .$$

It suffices to show that

$$Z_0 \cap T^{-t_n} Z_1 \cap T^{-2t_n} Z_2 = \emptyset$$

for all  $n \geq 1$ . If  $x$  belongs to this set, then the following five inequalities hold :

- 1)  $x_0 \neq x_{t_n}$
- 2)  $x_{t_n} \neq x_{2t_n}$
- 3)  $x_0 \neq x_{2t_n+1}$
- 4)  $x_{t_n+1} \neq x_{2t_n+1}$
- 5)  $x_0 \neq x_{t_n+1}$  .

Now decompose  $x$ , for fixed  $n$ , into a sequence of blocks of the form  $\eta^n(00)$  or  $\eta^n(1)$ . The coordinate  $x_0$  occupies a certain place, say  $k$ , in one of the blocks of this decomposition. In each of the following six cases, we arrive at the contradiction indicated :

$x_0$ occupies place $k$ in the first block of	contradicts inequality
$\eta^n(1)\eta^n(1)$	1)
$\eta^n(1)\eta^n(00)\eta^n(1)$	3)
$\eta^n(1)\eta^n(00)\eta^n(00)$	4)
$\eta^n(00)\eta^n(1)\eta^n(1)$	2) if $k > 0$
$\eta^n(00)\eta^n(1)\eta^n(00)$	3)
$\eta^n(00)\eta^n(00)$	5)

This clearly exhausts all possibilities except  $k = 0$  in the fourth case. In this case our hypothesis implies however that  $\eta^n(1)$  begins with  $10$ , which is not true.  $\square$

The idea of the proof of Theorem 9 suffices to obtain a more general statement, at the cost of precision. Let  $\zeta$  be a substitution on two symbols which gives rise to a minimal flow  $(Z, U)$ . We suppose that  $\text{card } Z > 1$ .

Theorem 10 *There exists an integer  $m$  such that the minimal flow  $(Z, U)$  is not topologically strongly mixing of order  $m$ .*

Proof : If  $Z$  is finite, the result is obvious. Hence we may suppose with no loss of generality that the block  $00$  occurs with bounded gap in all points of  $Z$ . Since  $\lim_{n \rightarrow \infty} N(\zeta^n 0)/N(\zeta^n 1)$  is finite and non-zero, there exists an integer  $m \geq 3$  such that for all  $n \geq 1$ , the block  $\zeta^n(00)$  occurs in all points of  $Z$  with a gap bounded by  $(m-3)N(\zeta^n 0)$ . Now let  $Z_0 = [0]$  and  $Z_1 = [1]$ , and con-

sider for each  $n$  the set

$$Z_0 \cap U^{-s_n} Z_1 \cap U^{-2s_n} Z_0 \cap \dots \cap U^{-(m-1)s_n} Z_i ,$$

where  $i$  is 0 if  $m$  is odd and 1 if  $m$  is even. By our choice of  $m$ , this set is empty for all  $n \geq 1$ , so that  $(Z,U)$  cannot be strongly mixing of order  $m$ . □

The same proof (modulo some technical difficulties) goes through for substitutions on a finite number of symbols. We do not know whether there exists substitutions on two symbols which are topologically strongly mixing of order three.

### §3. A remark about the Toeplitz sequence.

At the end of the proof of Theorem 1, it was noted that the argument applies to other substitutions. In this vein, it is interesting to look at

$$\eta' : \begin{array}{l} 0 \rightarrow 001 \\ 1 \rightarrow 11001 \end{array} ,$$

whose matrix is the same as the matrix of  $\eta$ . Setting  $a = 00$  and  $b = 1$  as in §1, we obtain

$$\theta' : \begin{array}{l} a \rightarrow abab \\ b \rightarrow bbab \end{array}$$

which we recognize as the Toeplitz substitution ([7], [9], [10], [13], [15]). It was first announced in [10] that the system given by  $\widetilde{\text{dōu}}$

bling the symbol  $a$  in  $\theta'$  is measure-theoretically weakly but not strongly mixing, and in [15] it was proved that this system is not topologically strongly mixing. We remark that the system obtained by doubling  $a$ 's in  $\theta'$  is just the system corresponding to  $\eta'$ . (This seems to have gone unnoticed up to now.) Since  $\theta'$  has discrete spectrum ([13]), the result announced in [11] follows from our work in the first paragraph. It is also quite simple to derive the result of [15] using the technique of our second paragraph. In fact, a calculation of the excess values for  $\theta'$ -admissible blocks of length  $4^n$  yields exactly two possible values, 2 and -1, in contrast to Lemma 5 for  $\theta$ . From this follows (by using that blocks  $\theta'$ -b necessarily occur at intervals divisible by 4) that if

$$2N_a(B) + N_b(B) = \frac{4^n + 5}{3},$$

then the block  $\theta'(B)B\theta'(b)$  cannot occur. Thus  $\eta'$  is not topologically strongly mixing.

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## Summary

Measure-theoretical and topological mixing properties of dynamical systems arising from substitutions are considered. It is shown that such systems are neither measure-theoretically strongly mixing, nor topologically strongly mixing of all orders. In the case of the substitution

$$\eta : \begin{array}{l} 0 \rightarrow 001 \\ 1 \rightarrow 11100 \end{array}$$

it is shown that the corresponding dynamical system is measure-theoretically weakly but not strongly mixing, and topologically strongly mixing of order two but not of order three.

Key words : Substitution, mixing, topological mixing.

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