P. MICHEL

Coincidence Values and Spectra of Substitutions

Publications des séminaires de mathématiques et informatique de Rennes, 1976, fascicule 2
« Séminaire de probabilité I », , exp. nº 10, p. 1-35
<http://www.numdam.org/item?id=PSMIR_1976___2_A10_0>

COINCIDENCE VALUES AND SPECTRA OF SUBSTITUTIONS

by

P. MICHEL

soumis pour publication au

Z. Wahr. verw. Geb.

mars 1977

COINCIDENCE VALUES AND SPECTRA OF SUBSTITUTIONS

In the recent upswing of ergodic theory, symbolic dynamics has come to play an increasingly important role, both in proving general theorems and in providing concrete examples of dynamical systems with desired properties. The earliest examples of this type (Morse [21] , 1921) are constructed by the use of substitutions, and the idea of substitution dynamical systems was formalized by Gottschalk and Hedlund ([5] , 1955). Their topological properties have been studied extensively by Gottschalk ([4] , 1963), Kamae ([9], 1972) and Martin ([15], 1971). On the other hand, measure - theoretic properties of substitution dynamical systems have only recently been investigated (Kakutani [7] , 1967, Keane [10] , 1968, Jacobs-Keane [6] , 1969, Neveu [23] , 1969, Coven-Keane [1] , 1971, Keane [11] , 1972, Klein [14] , 1972).

These results all deal with metric properties of substitution dynamical systems generated by substitutions of constant length.

In general, metric properties of dynamical systems are of more interest for ergodic theory as well as more difficult to establish. In particular, the interesting case of substitutions of non-constant length has (with the exception of the classical special cases considered by Morse and Hedlund [22] , 1940, and Kakutani [8] , 1972) scarcely been touched.

In [17] , [18], it was shown that any substitution minimal set possesses a unique invariant probability measure, thus providing a canonical dynamical system associated with the substitution. In the author ' s thesis [19] ergodic properties of certain classes of substitutions of non-constant length were developed, and this article contains essentially these results.

A substitution θ over a finite alphabet I is a map from I to $\bigcup_{n\geq 2} I^n$. Here we shall principally be interested in the case $I = \{0,1\}$, and θ can be represented as

$$\theta : \begin{array}{l} 0 \longrightarrow a_o^o\, a_1^o \,\ldots\ldots\, a_{\ell_o-1}^o \\ 1 \longrightarrow a_o^1\, a_1^1 \,\ldots\ldots\, a_{\ell_1-1}^1 \end{array}$$

If $a_o^o = 0$ and $a_o^1 = 1$, then two one-sided infinite 0-1 sequences

$$\theta\, 0 = w^o = w_o^o\, w_1^o\, w_2^o \,\ldots\ldots$$

and

$$\theta\, 1 = w^1 = w_o^1\, w_1^1\, w_2^1 \,\ldots\ldots$$

can be generated in an obvious manner by successive replacement of a symbol i by the block $\theta\, i$.

In the first paragraph, we study the coincidence density $d(\theta)$ of such a substitution, defined as the density of the set of integers k for which $w_k^o = w_k^1$. A method is developed for calculating $d(\theta)$, and this method suffices to calculate $d(\theta)$ for the classes of substitutions which are studied in the sequel. It is a rather surprising fact that the coincidence density does not always exist.

In the second paragraph, we study the class of substitutions defined by

$$\theta : \begin{array}{l} 0 \longrightarrow 0^{n+1-p}\, 1\, 0^p \\ 1 \longrightarrow 1\, 0^n \end{array} \qquad , \quad n \geq p \geq 0.$$

It is shown by using a modified continued fraction expansion developed in [12] , [13] that the associated dynamical systems have discrete spectrum and that all eigenfunctions are continuous. The proof is rather complicated, but we have not succeeded in finding a simpler one.

In the last section, an example is given of a substitution dynamical system with partly continuous spectrum. The methods here have been used subsequently by M. Dekking and he has been able to extend this result to a much larger class of substitutions.

Many questions remain to be answered, and in [3] a systematic study of substitution dynamical systems and their topological and metric properties will be published.

The author is grateful to M. Keane for valuable advice in the writing of this paper.

I - COINCIDENCE VALUES. -

A substitution θ over a finite alphabet I is a map θ from I to $\bigcup_{n\geq 2} I^n$.

The substitution θ associates to each letter $i \in I$ a block $\theta i = a_o^i a_1^i \dots a_{\ell_i-1}^i$. We say that θ is of constant lenght if $\ell_i = \ell_j$ for all $i, j \in I$; otherwise θ is of non-constant lenght.

If θ is a substitution, then for any block $b = b_o b_1 \dots b_{n-1} \in I^n$ we define $\theta b = \theta b_o \theta b_1 \dots \theta b_{n-1}$.

In particular, we may define the substitution θ^s for $s \geq 1$ inductively by setting $\theta^{s+1} i = \theta(\theta^s i)$ $(s \geq 1)$

The k^{th} element of $\theta^s i$ will be denoted by $\theta^s i(k)$.

Now let $i \in I$ such that for some $s \geq 1$, the block $\theta^s i$ begins with the letter i. Then the block $\theta^{2s} i$ begins with the block $\theta^s i$, the block $\theta^{3s} i$ begins with $\theta^{2s} i$, etc, and we can define an infinite sequence $\theta^\infty i$ as the " limit " of $\theta^{ns} i$. This sequence will be denoted by

$$w^i = \theta^\infty i = w_o^i w_1^i w_2^i \dots$$

In this paragraph, we shall assume that $I = \{0, 1\}$ and that $\theta 0$ and $\theta 1$ begin with 0 and 1 respectively. Our goal is to study the subset of $\mathbb{N}$ defined by $\{n \in \mathbb{N} : w_n^o = w_n^1\}$

and in particular we shall calculate the relative density of this set in certain cases. We call

$$d(\theta) = \lim_{N\to\infty} \frac{\text{card } \{n \in \mathbb{N} : n < N \text{ and } w_n^o = w_n^1\}}{N}$$

whenever this limit exists, the coincidence density of θ .

I - 1 Balanced blocks and balanced substitutions.

Definition 1.

The blocks $b = b_o b_1 \dots b_{m-1}$ and $c = c_o c_1 \dots c_{n-1}$ are said to be equivalent $(b \sim c)$ iff $m = n$ and $\text{card } \{k : b_k = 0\} = \text{card } \{k : c_k = 0\}$.

Lemma 2

Each pair b, c of equivalent blocks of length n admits a unique decomposition into a sequence of pairs of minimal equivalent blocks, in the following sense :

1) There exist integers $r \geq 1$ and $n_o = 0 < n_1 < n_2 < \dots < n_r = n$ such that for each $0 \leq t < r$, the blocks

$$b_{n_t} \; b_{n_t+1} \dots \; b_{n_{t+1}-1}$$

and

$$c_{n_t} \; c_{n_t+1} \dots \; c_{n_{t+1}-1}$$

are equivalent, and

2) The sequence $n_o, \dots, n_r$ is maximal with respect to the property 1).

Proof : Define n_1 as the minimal number for which $b_o, \dots, b_{n_1-1}$ and $c_o, \dots, c_{n_1-1}$ are equivalent, etc.

Lemma 3

If $b \sim c$, then $\theta b \sim \theta c$

Proof : Let k and n-k be respectively the number of zeros and ones in b (and in c). If u_o and v_o are the number of zeros in $\theta 0$ and $\theta 1$ respectively, then the number of zeros in θb (and in θc) is $u_o k + v_o(n-k)$.

A similar calculation holds for the number of ones.

Definition 4

Let θ be a substitution over $I = \{0,1\}$ such that $\theta 0$ and $\theta 1$ begin with 0 and 1 respectively. We set $n_o = 0$ and define n_t inductively for $t \geq 1$ by

$$n_t = \inf \{ n : n \geq n_{t-1} + 1, \; w^o_{n_{t-1}} \; w^o_{n_{t-1}+1} \dots w^o_{n_t-1} \sim w^1_{n_{t-1}} \; w^1_{n_{t-1}+1} \dots w^1_{n_t-1} \}$$

We distinguish two cases :

Case 1. $n_1 = \infty$.

In this case, we say that Θ does not possess balanced blocks.

Case 2. $n_1 < \infty$.

In this case, lemmas 2 and 3 imply that $n_t < \infty$ for all t, and we say that Θ possesses balanced blocks.

In case 2, we set

$$\hat{I} = \left\{ \begin{pmatrix} w^o_{n_{t-1}} & w^o_{n_{t-1}} + 1 & \cdots & w^o_{n_t -1} \\ w^1_{n_{t-1}} & w^1_{n_{t-1}} + 1 & \cdots & w^1_{n_t -1} \end{pmatrix} \; : \; t \geq 1 \right\}$$

and we call the elements of $\hat{I}$ <u>balanced blocks</u> for Θ .
It can happen that $\hat{I}$ is finite or infinite. In case 1, we set simply $\hat{I} = \emptyset$.

<u>Lemma 5</u> .

If $\begin{pmatrix} b \\ c \end{pmatrix} \in \hat{I}$, then $\Theta b \sim \Theta c$.

Any pair of minimal equivalent blocks b', c' of Θ b and Θ c given by lemma 2 are such that $\begin{pmatrix} b' \\ c' \end{pmatrix} \in \hat{I}$.

<u>Proof</u> :

If $\begin{pmatrix} b \\ c \end{pmatrix} \in \hat{I}$, then by definition $b \sim c$ and hence $\Theta b \sim \Theta c$ by lemma 3. Now let b" and c" be 0-1-blocks such that w^o begins with b"b and w^1 begins with c"c. Since $\Theta w^o = w^o$ and $\Theta w^1 = w^1$, w^o and w^1 begin respectively with Θ (b"b) and Θ (c"c).
Moreover, by the definition of $\hat{I}$, we may choose b" and c" such that b" $\sim$ c". Then b"b $\sim$ c"c and Θ (b"b) = Θ(b") Θ(b)

$\sim \Theta$(c") Θ (c) = Θ(c"c).

If $n_o = 0 < n_1 \ldots . < n_r$ is the minimal equivalent decomposition of Θ (b"b) and Θ (c"c), then since Θ (b") $\sim$ Θ(c"), the construction of lemma 2 shows that for some t, n_t is the beginning index of Θ (b) in Θ (b"b), and hence $n_t < \ldots < n_r$ yields the minimal decomposition for Θ(b) and Θ (c).
It follows from definition 4 that if b', c' is a pair of minimal equivalent blocks of Θ(b) and Θ (c) (given by lemma 2), then $\begin{pmatrix} b' \\ c' \end{pmatrix} \in \hat{I}$.

Corollary 6. If $\hat{I} \neq \emptyset$, then Θ induces a map

$$\hat{\Theta} : \hat{I} \longrightarrow \bigcup_{n \geq 1} \hat{I}^n$$

where $\hat{\Theta}\begin{pmatrix} b \\ c \end{pmatrix}$ is the minimal decomposition of $\begin{pmatrix} \Theta b \\ \Theta c \end{pmatrix}$.

If $\hat{I}$ is finite, $\hat{\Theta}$ will be called the balanced substitution associated with Θ .

Remark : $\hat{\Theta}$ does not satisfy strictly our definition of " substitution ", since $\hat{\Theta}\,\hat{\imath}$ can be of length one for some $\hat{\imath} \in \hat{I}$.

I-2 Coincidence density.

In this paragraph, we shall assume that Θ is a substitution over $I = \{0,1\}$ such that $\Theta 0$ and $\Theta 1$ begin with 0 and 1 respectively, and also that $\hat{I}$ is non-empty and finite. If $\hat{\imath}_o \in \hat{I}$ is the first balanced block of the minimal decomposition of w^o and w^1 (as described in definition 4), then it is obvious that $\hat{\Theta}\,\hat{\imath}_o$ begins with $\hat{\imath}_o$.

Thus $\hat{\Theta}^{\infty}\,\hat{\imath}_o = \hat{w} = \hat{w}_o\,\hat{w}_1 \;\ldots$ exists, where $\hat{w}_n \in \hat{I}$ for each $n \in \mathbb{N}$.

For any $\hat{\imath} \in \hat{I}$, we set

$$d(\hat{\imath}) = \lim_{N\to\infty} \frac{\text{card}\,\{\, n \in \mathbb{N} : n < N \text{ and } \hat{w}_n = \hat{\imath} \,\}}{N}$$

A simple application of the Perron- Frobenius theorem (see e.g. [18]) shows that this limit exists for each $\hat{\imath} \in \hat{I}$ and that the convergence rate is exponential. Obviously $d(\hat{\imath}) \geq 0$ and $\sum_{\hat{\imath}\in\hat{I}} d(\hat{\imath}) = 1$.

In order to formulate our next theorem, we shall need the following notation. Let $\hat{\imath} \in \hat{I}$ with $\hat{\imath} = \begin{pmatrix} b \\ c \end{pmatrix}$ and $b = b_o \ldots b_{\ell-1}$, $c = c_o \ldots c_{\ell-1}$.

We then set $\ell(\hat{\imath}) = \ell$ and

$$c(\hat{\imath}) = \text{card}\,\{n : 0 \leq n < \ell \quad \text{and} \quad b_n = c_n\} \quad .$$

Theorem 7

Let Θ be a substitution over $I = \{\,0,1\,\}$ such that $\Theta 0$ begins with 0 and $\Theta 1$ with 1. If $\hat{I}$ is non-empty and finite, then the coincidence density $d(\Theta)$ exists and is given by the formula :

$$d(\sigma) = \frac{\sum_{\hat{\imath} \in \hat{I}} c(\hat{\imath})\ d(\hat{\imath})}{\sum_{\hat{\imath} \in \hat{I}} \ell(\hat{\imath})\ d(\hat{\imath})}$$

Proof :

Let (n_t) be the sequence of definition 4. Since $\hat{I}$ is finite, $n_{t+1} - n_t$ is bounded, and thus

$$d(\sigma) = \lim_{t \to \infty} \frac{\text{card}\ \{ n \in \mathbb{N} : n < n_t \text{ and } w_n^o = w_n^1 \}}{n_t}$$

if the right - hand limit exists. Now

$$\text{card}\ \{n \in \mathbb{N} : n < n_t \text{ and } w_n^o = w_n^1\} = \sum_{\hat{\imath} \in \hat{I}} c(\hat{\imath})\ \text{card}\ \{ m \in \mathbb{N} : m < t \text{ and } \hat{w}_m = \hat{\imath} \}$$

and $n_t = \sum_{\hat{\imath} \in \hat{I}} \ell(\hat{\imath})\ \text{card}\ \{ m \in \mathbb{N} : m < t \text{ and } \hat{w}_m = \hat{\imath} \}$

Thus

$$\text{card}\ \{ n \in \mathbb{N} : n < n_t \text{ and } w_n^o = w_n^1 \}$$

$$= \frac{\sum_{\hat{\imath} \in \hat{I}} c(\hat{\imath})\ \dfrac{\text{card}\ \{m \in \mathbb{N} : m < t \text{ and } \hat{w}_m = \hat{\imath} \}}{t}}{\sum_{\hat{\imath} \in \hat{I}} \ell(\hat{\imath})\ \dfrac{\text{card}\ \{m \in \mathbb{N} : m < t \text{ and } \hat{w}_m = \hat{\imath} \}}{t}}$$

and the latter expression tends to

$$\frac{\sum_{\hat{\imath} \in \hat{I}} c(\hat{\imath})\ d(\hat{\imath})}{\sum_{\hat{\imath} \in \hat{I}} \ell(\hat{\imath})\ d(\hat{\imath})}$$

as t tends to infinity.

Theorem 8

Let σ be a substitution over $I = \{ 0,1 \}$ such that $\sigma\ 0$ begins with 0 and $\sigma\ 1$ begins with 1 . If $\hat{I}$ is non empty, finite, and if $\hat{0} = \binom{0}{0} \in \hat{I}$, then $d(\sigma) = 1$.

<u>Proof</u> : Suppose $\hat{0}$ appears at place s_o in $\hat{w}$. The block in $\hat{w}$, beginning at place $s_o \ell_o^{(k)}$ and of length $\ell_o^{(k)}$ is the block $\sigma^k(0)$, and thus :

$$\frac{\operatorname{card}\{n \in \mathbb{N} : s_o \ell_o^{(k)} \leq n < (s_o+1)\ell_o^{(k)} \text{ and } w_n^o = w_n^1\}}{\ell_o^{(k)}} = 1 \quad (k \in \mathbb{N})$$

But :

$$\frac{\operatorname{card}\{n \in \mathbb{N} : s_o \ell_o^{(k)} \leq n < (s_o + 1)\ell_o^{(k)} \text{ and } w_n^o = w_n^1\}}{\ell_o^{(k)}}$$

$$= (s_o + 1)\,\frac{\operatorname{card}\{n \in \mathbb{N} : n < (s_o + 1)\ell_o^{(k)} \text{ and } w_n^o = w_n^1\}}{(s_o + 1)\ell_o^{(k)}} - s_o\,\frac{\operatorname{card}\{n \in \mathbb{N} : n < s_o \ell_o^{(k)} \text{ and } w_n^o = w_n^1\}}{s_o \ell_o^{(k)}}$$

and the latter expression tends to

$$(s_o + 1)\, d(\Theta) - s_o\, d(\Theta) = d(\Theta)$$

as k tends to infinity and thus $d(\Theta) = 1$.

Remark : We have $\hat{0} \in \hat{I}$ iff $\hat{1} \in \hat{I}$.

I - 3 Examples and counterexamples

In this section we shall determine $\hat{I}$ and $d(\theta)$ for some cases of substitutions on $I = \{0,1\}$. In general, this seems to be a difficult problem, and it would be interesting in view of our applications in II and III to have a method for determining $d(\theta)$ for any substitution θ .

The first case to be considered is when θ is of constant length ℓ. According to [1] , we separate substitutions into two classes, discrete and continuous.

If

$$\theta : \begin{array}{l} 0 \longrightarrow a_o a_1 \ \dots \ a_{\ell-1} \\ 1 \longrightarrow b_o b_1 \ \dots \ b_{\ell-1} \end{array}$$

then θ is continuous iff $a_k \neq b_k$ for all $0 \leq k < \ell$, and discrete if for some k , $a_k = b_k$. We recall that only the case $a_o = 0$ and $b_o = 1$ is being considered. (This is not really a restriction, since using the normal form of [1] we may always find another θ satisfying this condition with the same orbit closure.)

Proposition 9

If θ is of constant length $\ell \geq 2$ and if $\theta\,0(1) = 0$ and $\theta\,1(1) = 1$, then

i) $d(\theta) = 0$ if θ is continuous and

ii) $d(\theta) = 1$ if θ is discrete.

Proof : If θ is continuous, then $w_n^o \neq w_n^1$ for all n , so $d(\theta) = 0$. If θ is discrete, then

$$\text{card } \{n < \ell^k : w_n^o \neq w_n^1 \} \leq (\ell - 1)^k$$

and this implies $d(\theta) = 1$. (See also [1])

In the case of constant length, the relation between $\hat{\theta}$ on $\hat{I}$ and $d(\theta)$ is not as essential as in the case of non-constant length. It is not hard to see that if θ is of constant length ℓ with $\theta\,0(1) = 0$ and $\theta\,1(1) = 1$, then $\hat{I}$ is finite and non-empty iff the number of ones

in $\theta 0$ is the same as the number of ones in $\theta 1$.
We now investigate the more interesting and difficult case of non-constant length. Let θ be a substitution over the finite alphabet I.
If $i,j \in I$, we set

$$\ell_{i,j} := \text{card } \{ k : 0 \leq k \leq \ell_i \quad \text{and } \theta\, i(k) = j \}$$

The matrix $M = M(\theta) = (\ell_{ij})_{i,j \in I}$ is called the <u>θ - matrix.</u>
For any $s \geq 1$, if

$$M^s = [\ell_{ij}^{(s)}]$$

then $\ell_{ij}^{(s)} = \text{card } \{ k : 0 \leq k \leq \ell_i^{(s)} \text{ and } \theta^s i(k) = j \}$
where $\ell_i^{(s)} = \sum_{j \in I} \ell_{ij}^{(s)}$ denotes the length of the block $\theta^s i$.

If we take $I = \{0,1\}$, then the matrix $M(\theta)$ has positive integral entries. Therefore its eigenvalues λ_1 and λ_2 are real and distinct, and the larger eigenvalue λ_1 is larger than 1.
By replacing θ by θ^2 if necessary, we may assume that $\lambda_2 \geq 0$.
(This changes neither $\hat{I}$ nor $d(\theta)$, since w^0 and w^1 remain the same.)

We distinguish four cases :

1. $\lambda_1 > 1 > \lambda_2 = 0$. This means $\det(M) = 0$ and $\text{tr}(M) = \lambda_1$.

In this case, we can see that $d(\theta)$ always exists, is rational, and there is a method for calculating $d(\theta)$. Since we shall only need to calculate $d(\theta)$ for the θ of section III, we adopt a simpler technique which may not work for the general case. Suppose that

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then there are three possibilities for θ :

$\theta_1 : \begin{array}{l} 0 \longrightarrow 01 \\ 1 \longrightarrow 1001 \end{array}$ $\qquad \theta_2 : \begin{array}{l} 0 \longrightarrow 01 \\ 1 \longrightarrow 1010 \end{array}$ $\qquad \theta_3 : \begin{array}{l} 0 \longrightarrow 01 \\ 1 \longrightarrow 1100 \end{array}$

In the case of θ_1, we have

$$\hat{I} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Thus by theorem 8, $d(\theta_1) = 1$ since $\hat{0} = \binom{0}{0} \in \hat{I}$.

In the case of θ_2, $\hat{I}$ remain the same and $d(\theta_2) = 1$.

For θ_3, the situation is different. We have

$$\hat{I} = \left\{ \binom{011}{110}, \binom{10}{01}, \binom{01}{10}, \binom{100}{001}, \binom{110}{011}, \binom{001}{100}, \binom{1100}{0101}, \binom{0101}{1100} \right\}$$

$$= \{\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}, \hat{f}, \hat{g}, \hat{h}\} \quad \text{with} \quad \hat{i}_0 = \hat{a}.$$

The calculation of $d(\theta_3)$ can be simplified (also in the other cases) by identifying the pairs $(\hat{a}, \hat{e})$, $(\hat{b}, \hat{c})$, $(\hat{d}, \hat{f})$ and $(\hat{g}, \hat{h})$. This yields

$$\theta_3 : \quad \begin{array}{l} \hat{a} \longrightarrow \hat{a}\,\hat{b}\,\hat{b}\,\hat{d} \\ \hat{b} \longrightarrow \hat{a}\,\hat{d} \\ \hat{d} \longrightarrow \hat{g}\,\hat{g} \\ \hat{g} \longrightarrow \hat{a}\,\hat{b}\,\hat{d}\,\hat{g} \end{array}$$

(where we have modified θ_3 and $\hat{I}$ according to our identification).

The matrix of θ_3 is

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The corresponding frequencies, lengths and coincidences are :

$d(\hat{a}) = d(\hat{b}) = d(\hat{d}) = d(\hat{g}) = \frac{1}{4}$

$c(\hat{a}) = c(\hat{d}) = 1$ $\qquad \ell(\hat{a}) = \ell(\hat{d}) = 3$

$c(\hat{b}) = 0$ $\qquad \ell(\hat{b}) = 2$

$c(\hat{g}) = 2$ $\qquad \ell(\hat{g}) = 4$.

This yields according to theorem 7, $d(\theta_3) = \frac{1}{3}$.

We note that these examples show that $d(\theta)$ does not depend only on the θ - matrix, but also on the distribution of zeros and ones in $\theta 0$ and $\theta 1$.

2. $\lambda_1 > 1 > \lambda_2 > 0$. This means $0 < \det(M) < \operatorname{tr}(M) - 1$.

In this case, we do not know whether $\hat{I}$ is finite, infinite or empty, or whether $d(\theta)$ exists for a general substitution θ. We shall restrict our attention to the case where

$$M(\theta) = \begin{bmatrix} n+1 & 1 \\ n & 1 \end{bmatrix} \quad , \quad n \geq 1 .$$

Then the possibilities for θ are

$$\theta : \begin{array}{l} 0 \longrightarrow 0^{n+1-p} \; 1 \; 0^p \\ 1 \longrightarrow 1 \; 0^n \end{array} \qquad 0 \leq p \leq n.$$

where $0^k = 0 \ldots 0$ k times.

If $p = n$, $\hat{I} = \{ \hat{a},\hat{0},\hat{1} \}$ with $\hat{a} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{1}_o$,

and

$$\hat{\theta} : \begin{array}{l} \hat{a} \longrightarrow \hat{a}\; \hat{0}^n \; \hat{1} \; \hat{0}^n \\ \hat{0} \longrightarrow \hat{0} \; \hat{1} \; \hat{0}^n \\ \hat{1} \longrightarrow \hat{1} \; \hat{0}^n \end{array}$$

Thus $\hat{0} \in \hat{I}$ and $d(\theta) = 1$.

If $p = n-1$, $\hat{I} = \{\hat{a}, \hat{b}, \hat{0}, \hat{1}\}$ with $\hat{a} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \hat{1}_o$, $\hat{b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

and

$$\hat{\theta} : \begin{array}{l} \hat{a} \longrightarrow \hat{a} \; \hat{0}^n \; \hat{b} \; \hat{0}^{n-1} \; \hat{b} \; \hat{0}^{n-1} \\ \hat{b} \longrightarrow \hat{a} \; \hat{0}^{n-1} \; \hat{b} \; \hat{0}^{n-1} \\ \hat{0} \longrightarrow \hat{0}^2 \; \hat{1} \; \hat{0}^{n-1} \\ \hat{1} \longrightarrow \hat{1} \; \hat{0}^n \end{array}$$

This yields again $\hat{0} \in \hat{I}$ and $d(\theta) = 1$.

Finally, if $0 \leq p \leq n-2$, we have

$\hat{I} = \{\hat{a},\hat{b},\hat{c},\hat{0},\hat{1}\}$ with $\hat{a} = \begin{pmatrix} 0^{n+1-p} & 1 \\ 1 & 0^{n+1-p} \end{pmatrix} = \hat{1}_o$,

$\hat{b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\hat{c} = \begin{pmatrix} 0^{n-p} & 1 \\ 1 & 0^{n-p} \end{pmatrix}$,

$$\hat{\theta} : \begin{array}{lll} \hat{a} & \longrightarrow & \hat{a} \; (\hat{0}^n \; \hat{b})^{n-p} \; \hat{0}^p \; \hat{c} \; \hat{0}^p \\ \hat{b} & \longrightarrow & \hat{a} \; \hat{0}^p \; \hat{c} \; \hat{0}^p \\ \hat{c} & \longrightarrow & \hat{a} \; (\hat{0}^n \; \hat{b})^{n-p-1} \; \hat{0}^p \; \hat{c} \; \hat{0}^p \\ \hat{0} & \longrightarrow & \hat{0}^{n+1-p} \; \hat{1} \; \hat{0}^p \\ \hat{1} & \longrightarrow & \hat{1} \; \hat{0}^n \end{array}$$

Here again, $\hat{0} \in \hat{I}$ and $d(\theta) = 1$. Thus we conclude for any θ with matrix M that $d(\theta) = 1$.

If M is any matrix with positive integral entries and $\det(M) = 1$, then we can find at least one substitution σ such that $M(\sigma) = M$ and $d(\sigma) = 1$. We omit the proof since we shall not use this fact.

3 - $\lambda_1 > \lambda_2 = 1$. This means $\det(M) = tr(M) - 1$.
In this case, we conjecture that $\hat{I}$ is empty or infinite. A calculation due to M. Dekking shows that if

$$\sigma_1 : \begin{array}{l} 0 \longrightarrow 0\ 0\ 1 \\ 1 \longrightarrow 1\ 1\ 0\ 0\ 1 \end{array}$$

and

$$\sigma_2 : \begin{array}{l} 0 \longrightarrow 0\ 1\ 0 \\ 1 \longrightarrow 1\ 1\ 0\ 1\ 0, \end{array}$$

then $d(\sigma_1) = \frac{1}{2}$ and $d(\sigma_2)$ does not exist . Note that

$M(\sigma_1) = M(\sigma_2)$. We also conjecture in this case that $d(\sigma)$ cannot be equal to 1.

4 - $\lambda_1 > \lambda_2 > 1$. This means $\det(M) > tr(M) - 1$.
We conjecture here that $d(\sigma)$ does not exist. The only thing we can prove is that if $\ell_{oo} \geq \ell_{o1} + 2$

and $\ell_{11} \geq \ell_{1o} + 2$,

then $\hat{I}$ is empty or infinite. We have not succeeded in calculating $d(\sigma)$ for any σ satisfying this condition.

II - A CLASS OF SUBSTITUTIONS WITH DISCRETE SPECTRUM.

In this paragraph, we consider the substitutions

$$\theta_p : \begin{array}{l} 0 \longrightarrow 0^{n+1-p} \; 1 \; 0^p \\ 1 \longrightarrow 1 \; 0^n \end{array}$$

for $0 \leq p \leq n$. As we have seen in I.3, $d(\theta_p) = 1$. This will enable us to prove that θ_p has discrete spectrum.

In general, if θ is a substitution over $I = \{0,1\}$ and if $\theta 0$ and $\theta 1$ both contain 0 and 1, then the subset

$X(\theta) = \{x \in I^{\mathbb{Z}}$: for all $p \leq q$, $x_p x_{p+1} \ldots x_{p+q}$ appears in some $\theta^s 0\}$

of $I^{\mathbb{Z}}$ is compact, invariant under the <u>shift</u> T (defined by $(Tx)_k = x_{k+1}$), and for each $x \in X(\theta)$, the orbit

$\mathrm{Orb}(x) = \{T^s x : s \in \mathbb{Z}\}$ is dense in $X(\theta)$. (see e.g. [5]).

By [17] [18] , there is a unique probability measure μ_θ such that $\mu_\theta(X(\theta)) = 1$ and $T\mu_\theta = \mu_\theta$. The <u>spectrum</u> of θ is the spectrum of the unitary operator (which we shall also denote by T) induced by T on the space $\mathbb{L}^2(X(\theta), \mu_\theta)$. The substitution θ has discrete spectrum iff $\mathbb{L}^2(X(\theta),\mu_\theta)$ is spanned by the eigenfunctions of T. We shall prove the following result :

<u>Theorem 10</u>.

For any $0 \leq p \leq n$, θ_p has discrete spectrum.

The proof is rather long and we shall separate it into several parts.

II. 1 - <u>The sturmian case</u>

If $p = n$, that is, $\theta_n : \begin{array}{l} 0 \longrightarrow 0 \; 1 \; 0^n \\ 1 \longrightarrow 1 \; 0^n \end{array}$, it is not hard to see that the substitution

$$\eta : \begin{array}{l} 0 \longrightarrow 0^{n+1} \; 1 \\ 1 \longrightarrow 0^n \; 1 \end{array}$$

satisfies $X(\theta_n) = X(\eta)$ and $\mu_{\theta_n} = \mu_\eta$. (In general, if the two blocks

θ 0 and θ 1 of a substitution end with the same symbol, we may " transfer " this symbol to the beginning of the blocks without changing $X(\theta)$ or μ_θ .)

Let $$M = M(\eta) = \begin{bmatrix} n+1 & 1 \\ n & 1 \end{bmatrix}$$

Then the characteristic polynomial $\lambda^2 - (n+2)\lambda + 1$ of M admits roots $\lambda_1 > 1 > \lambda_2 > 0$, with λ_1, λ_2 irrational and

$$(*) \qquad \lambda_k = \frac{1-(n+1)\lambda_k}{1-\lambda_k} \qquad (k = 1, 2).$$

We consider now the compact space $Y = \mathbb{R}/_{\mathbb{Z}}$, provided with normalized Haar measure ν . The transformation

$$S : Y \longrightarrow Y$$

defined by

$$Sy = y + \lambda_2 \pmod 1,$$

satisfies $S\nu = \nu$, and the spectrum of the dynamical system (Y, ν, S) is discrete with eigenvalues $\exp(2\pi i k\lambda_2)$, $k \in \mathbb{Z}$ ([8]).

Proposition 11.

There is a continuous map $\varphi : X(\eta) \longrightarrow Y$ such that $\varphi \circ T = S \circ \varphi$ and such that $\varphi\mu_\eta = \nu$. Moreover, $\{ y \in Y : \text{card } \varphi^{-1}(y) > 1\}$ is countable, and card $\varphi^{-1}(y) \leq 2$ for each $y \in Y$.

Proof : Let $w \in X(\eta)$ be the point for which $w_0 = 0$, $w_{-1} = 1$, and $w = \eta w$. A simple calculation using $(*)$ for $k = 2$ shows that $w_t = 0$ if $S^t(0) \in [0, (1-\lambda_2)[$ and $w_t = 1$ if $S^t(0) \in [1-\lambda_2, 1[$. Since the orbit of w is dense in $X(\eta)$, for any $x \in X(\eta)$, we may find a sequence of integers t_k such that $x = \lim_{k\to\infty} T^{t_k} w$.

Then $\lim_{k\to\infty} t_k \lambda_2 \pmod 1$ exists, and if we set $\varphi(x) = \lim_{k\to\infty} t_k\lambda_2 \pmod 1$, then φ has the desired properties.

Corollary 12

If $p = n$, then θ_p has discrete spectrum.

This result is essentially contained in the results of Hedlund and Morse on Sturmian sequences (see [22]).

See also [7] , [8] .

II - 2 Martin's result

In the case $0 \leq p < n$, we shall need a result given in [16]. Let $Y = \mathbb{R}/\mathbb{Z}$, let $\lambda = \lambda_2$ be the smaller root of $\lambda^2 - (n+2)\lambda + 1 = 0$, and denote by S the rotation

$$S\, y = y + \lambda \quad \text{mod } 1 \qquad (y \in Y).$$

Theorem 13 (Martin [16])

There exists a continuous map h from $X(\theta_p)$ to Y such that

$$h(T x) = S\, h(x) \qquad (x \in X(\theta_p)).$$

Our procedure in the following will be to show that h is one-to-one on a set of measure one, so that h actually represents an (almost-continuous) isomorphism between $(X(\theta_p), T)$ and (Y, S). It then follows immediately that θ_p has discrete spectrum.

We define $w = w^{oo} \in X(\theta_p)$ by setting

$$w = (\dots, w_{-1}, w_o, w_1, \dots)$$

with $(w_o, w_1, w_2, \dots) = \lim_{s\to\infty} \theta_p^s(0) = 0^{n+1-p}\ 1\ 0^p \dots$

and $(\dots, w_{-2}, w_{-1}) = \text{" } \lim_{s\to\infty} \text{ "}\ \theta_p^s(0) = \dots 0^{n+1-p}\ 1\ 0^p.$

By composition of h with a rotation of $\mathbb{R}/\mathbb{Z}$, we may obviously assume that the h of theorem 13 satisfies

$$h(w) = 0 \in \mathbb{R}/\mathbb{Z}.$$

We remark that the result of Martin applies to a much more general situation, but that his methods yield little information concerning non-continuous eigenvalues.

II - 3 Continued fraction expansion

In this section we define a symbolic system (Ω, τ) and relate this system to the rotation (Y, S), where $S\, y = y + \lambda$ and λ is the smaller root of $\lambda^2 - (n+2)\lambda + 1 = 0$.

Details of the proofs can be found in [13] .

Since $\lambda = \dfrac{1}{n+2-\lambda}$,

we have the following continued fraction expansion for :

$$\lambda = \cfrac{1}{n+2-\cfrac{1}{n+2-\cfrac{1}{n+2-\ddots}}}$$

Consider a sequence

$(\omega_1, \omega_2, \dots, \omega_k) \in \{0, 1, \dots, n+1\}^k$.

Such a sequence will be called <u>admissible</u> if it contains no block of the form $n+1 \;\; n^j \;\; n+1 = n+1 , \underbrace{n, n, \dots, n,}_{j \text{ times}} n+1$, $j \geq 0$

We set

$$\Omega = \{ \omega = (\omega_1, \omega_2, \dots) : 0 \leq \omega_k \leq n+1 , \\ (\omega_1, \dots, \omega_k) \text{ admissible for all } k \geq 1 \}$$

Obviously, Ω is a compact subset of $\{0,1,\dots,n+1\}^{\mathbb{N}}$.

A map $\tau : \Omega \longrightarrow \Omega$ is defined by setting

$$\tau(\omega) = (\omega_1+1, \omega_2, \omega_3, \dots)$$

if $(\omega_1+1, \omega_2, \omega_3, \dots)$ is admissible, and by setting $\tau(\omega)$ be the first admissible element of Ω following ω in the lexicographical ordering otherwise. This defines the pair (Ω,τ). We remark that τ is injective, $\tau(\Omega) = \Omega \setminus (0,0,0,\dots)$, and τ is continuous except at $\bar{\omega} = (n,n,n,\dots)$ (see [13]).

Now define $\pi : \Omega \longrightarrow \mathbb{R} / \mathbb{Z}$ by

$$\pi(\omega) = \sum_{k=1}^{\infty} \omega_k \lambda^k$$

<u>Theorem 14</u> [13]

a) π is continuous, onto, and $\pi \circ \tau = S \circ \pi$

b) π is one-to-one on $\pi^{-1}(\mathbb{R} / \mathbb{Z} - \mathbb{Z}\lambda)$

c) If $[\alpha_1, \dots, \alpha_k] = \{ \omega \in \Omega : \omega_i = \alpha_i$ for $1 \leq i \leq k \}$, then $\pi([\alpha_1, \dots, \alpha_k])$ is an interval in $\mathbb{R}/\mathbb{Z}$ and $\nu(\pi([\alpha_1, \dots, \alpha_k]))$ takes one of the two values λ^k or $\lambda^k(1-\lambda)$.

Next we consider the orbit of the point $\tilde{\omega} = (0,0,0,\dots)$. By the definition of τ, $\tau^k \tilde{\omega}$ $(k \geq 0)$ are points of the form $(\omega_1, \omega_2, \dots, \omega_j, 0, 0, 0, \dots)$, and all points of this nature belong to the forward orbit of $\tilde{\omega}$. In particular, the point

$$\tilde{\omega}(j) = (\underbrace{0, 0, 0, \dots, 0}_{j-1 \text{ times}}, 1, 0, 0, \dots)$$

corresponds to an integer which we shall call C_j. That is, $\tau^{C_j} \tilde{\omega} = \tilde{\omega}(j)$. It is easy to see that

$$C_1 = 1$$

$$C_2 = n+2$$

and

$$C_{j+1} = (n+2)C_j - C_{j-1} \quad (j \geq 1).$$

Lemma 15.

The sequence $C_j \lambda^j$ is bounded.

Proof : Obviously, $C_j = a\lambda^j + b\dfrac{1}{\lambda^j}$ for some constants a and b.

Thus $C_j \lambda^j = a\lambda^{2j} + b \leq |a| + |b|$.

It follows also from the construction in [13] that $C_j\lambda = \lambda^j$ mod 1. Hence $C_j\lambda \longrightarrow 0$ mod 1 and

$$\pi(\omega) = \sum_{j=1}^{\infty} \omega_j \lambda^j = \sum_{j=1}^{\infty} \omega_j C_j \lambda$$

For ω of the form $(\omega_1, \dots, \omega_j, 0, 0, 0, \dots)$ this just means

$$\pi(\omega) = \sum_{j=1}^{k} \omega_j C_j \lambda = S^{\sum_{j=1}^{k} {}_j C_j}(0),$$

which also follows from the definition of the C'_{js}. We shall need also the following notation. Let β $\mathbb{R}/\mathbb{Z} - \mathbb{Z}\lambda$. By theorem 146), there exists a unique $\omega = (\omega_1, \omega_2, \dots) \in \Omega$ such that $\pi(\omega) = \beta$. We set

$$\tilde{\beta}_m = \sum_{j=1}^{m-1} \omega_j \; C_j \qquad (m \geq 1).$$

Then $\tilde{\beta}_m$ is a non-negative integer and

$$\lim_{m\to\infty} S^{\tilde{\beta}_m}(0) = \beta .$$

II - 4 A null set in Ω.

In this section we shall prove a technical lemma necessary for the proof of theorem 10.

Let $N = \{ \omega \in \Omega : \{k : \omega_k = \omega_{k+1} = 0 \}$ is finite $\}$.

Lemma 16

$$\nu(\pi(N)) = 0$$

Proof : If we set

$$N_o = \{ \omega \in \Omega : 00 \text{ does not appear in } \omega \},$$

then by the definition of τ, $N \subseteq \bigcup_{s\geq 0} \tau^s (N_o)$.

Thus it suffices to show that $\nu(\pi(N_o)) = 0$.

Consider now a cylinder set

$$[\alpha_1, \dots, \alpha_k] = \{ \omega \in \Omega : \omega_i = \alpha_i \text{ for } 1 \leq i \leq k \} .$$

Let N_k denote the union of all such cylinders of length k with the property that no two successive zeros occur in $\alpha_1, \dots, \alpha_k$ and $\alpha_1, \dots, \alpha_k$ is admissible. Then for each k,

$$N_o \subseteq N_k ,$$

and hence

$$\nu(\pi(N_o)) \leq \inf_k \nu(\pi(N_k)).$$

We shall calculate the number of cylinders in N_k. Let p_k, q_k, r_k, s_k denote respectively the number of cylinders $[\alpha_1, \dots, \alpha_k]$ in N_k such that α_k is $0, i, n, n+1$ (where i denotes any symbol with $1 \leq i \leq n-1$).

Then $p_1 = 1$, $q_1 = n-1$, $r_1 = 1$, $s_1 = 1$

and $p_2 = n+1$, $q_2 = (n+2)(n-1)$, $r_2 = n+2$, $s_2 = n+1$.

Moreover, for $k \geq 1$,

$$\begin{bmatrix} p_{k+1} \\ q_{k+1} \\ r_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ n-1 & n-1 & n-1 & n-1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_k \\ q_k \\ r_k \\ s_k \end{bmatrix}$$

The first three lines of this matrix are obvious ; the fourth line is obtained by noting that

$$s_{k+1} = p_k + q_k + r_k - \sum_{j=1}^{k-1} s_j ,$$

and by the corresponding matrix for $k - 1$, one gets

$$r_k = p_{k-1} + q_{k-1} + r_{k-1} + s_{k-1}$$

$$s_k = p_{k-1} + q_{k-1} + s_{k-1}$$

and $r_k - s_k = r_{k-1}$

Hence

$$r_k - \sum_{j=1}^{k-1} s_j = s_k \text{ by induction.}$$

If we now calculate the characteristic polynomial of the above matrix, we get

$$P(\xi) = \xi\,(\xi^3 - (n+1)\xi^2 - n\,\xi + 1).$$

If $\frac{1}{\lambda}$ is the larger root of $\lambda^2 - (n+2)\,\lambda + 1 = 0$, then one finds that $P\,(\frac{1}{\lambda}) = \frac{1}{\lambda} > 0$; since $P\,(0) > 0$ and $P(1) < 0$, this implies that the largest root ξ_o of $P\,(\xi) = 0$ (positive by the Perron - Frobenius theorem) satisfies $\xi_o < \frac{1}{\lambda}$.

By theorem 14c, each cylinder $[\alpha_1, \ldots, \alpha_k]$ of N_k satisfies $\nu\,(\pi\,([\alpha_1, \ldots, \alpha_k]))\leq \lambda^k$.

Since there are asymptotically $K\,.\,\xi_o^k$ such cylinders, we obtain

$$\nu\,(\pi(N_k)) \leq K\,.\,\xi_o^k\,.\,\lambda^k \xrightarrow[k\to\infty]{} 0.$$

II - 5 Subsets of $\mathbb{N}$ related to θ_p, η and Ω.

We recall that η is the substitution defined by

$$\eta : \begin{array}{l} 0 \longrightarrow 0^{n+1}\ 1 \\ 1 \longrightarrow 0^n\ 1. \end{array}$$

If we consider $\eta^\infty(0) = \eta^s(\eta^\infty(0)) = v_o\ v_1\ v_2\ \dots$, then we see that $\eta^\infty(0)$ is made up of a sequence of blocks of the form $\eta^s(0)$ and $\eta^s(1)$, for any fixed integer $s \geq 1$. Suppose that $0 = k_o < k_1 < k_2 < k_3 < \dots$ is the sequence of integers such that for each j,

$$v_{k_j}\ v_{k_j+1} \cdots v_{k_{j+1}-1}$$

is either $\eta^s(0)$ or $\eta^s(1)$. We then set

$$\mathcal{R}_s^* = \{ k_o , k_1, k_2 , \dots \}$$

In the same fashion we define $\mathcal{R}_s^p$ for $s \geq 1$ and for the substitution θ_p $(0 \leq p \leq n)$, by decomposing $\theta_p^\infty(0)$ into its blocks $\theta_p^s(0)$ and $\theta_p^s(1)$.

Our purpose in this paragraph is to relate the sets $\mathcal{R}_s^p$ and $\mathcal{R}_s^*$ with the sets Υ_s defined by

$$\Upsilon_s = \{k \in \mathbb{N} : k = \sum_{j=s+1}^{\infty} \omega_j C_j \ , \omega = (\omega_j) \in \Omega \ , \{j : \omega_j \neq 0\} \text{ finite } \}$$

$$(s \geq 0)$$

Lemma 17.

For any $s \geq 2$, we have $\Upsilon_s \subseteq \mathcal{R}^p_{s-1}$

Proof : 1. $\Upsilon_s \subseteq \mathcal{R}_s^*$

We recall that $\ell_o^{(s)}$ denotes the length of the block $\eta^s(0)$, and $\ell_1^{(s)}$ the length of $\eta^s(1)$. Since

$$\eta^s(0) = \eta^{s-1}(0)\ \eta^{s-1}(0^n 1) = \eta^{s-1}(0)\ \eta^s(1)$$

and $\eta^{s+1}(0) = [\eta^{s}(0)]^{n+1} \eta^{s}(1)$,

we have

$$\ell_o^{(s)} = \ell_o^{(s+1)} + \ell_1^{(s)}$$

$$\ell_o^{(s+1)} = (n+1)\,\ell_o^{(s)} + \ell_1^{(s)} .$$

This yields the recurrence relation

$$\ell_o^{(s+1)} = (n+2)\,\ell_o^{(s)} - \ell_o^{(s-1)},$$

with

$$\ell_o^{(o)} = 1 ,\ \ell_o^{(1)} = n+2.$$

Thus we see that for each $s \geq 0$, $\ell_o^{(s)} = C_{s+1}$, where C_{s+1} is the number defined in II.3 (by the same recurrence relation).

Any element in $\mathcal{S}_s$ is of the form

$$k = \sum_{j=s+1}^{t} \omega_j C_j .$$

If $t = s+1$, that is if $k = \omega_{s+1} C_{s+1}$, then since $C_{s+1} = \ell_o^{(s)}$ and $\omega_{s+1} \leq n+1$, and since $\eta^{\infty}(0)$ begins with $n+1$ blocks $\eta^{s}(0)$ followed by one block $\eta^{s}(1)$ we see that $k \in \mathcal{R}_s^*$.

Moreover, $k \in \mathcal{R}_s^*$ corresponds to the beginning of a block $\eta^{s}(1)$ if and only if $\omega_{s+1} = n+1$. Now suppose that $t = s+2$, that is, $k = \omega_{s+1} C_{s+1} + \omega_{s+2} C_{s+2} = \omega_{s+1} C_{s+1} + k'$. Then $k' \in \mathcal{R}_{s+1}^*$ and is the beginning of a block $\eta^{s+1}(0)$ or $\eta^{s+1}(1)$ in $\eta^{\infty}(0)$, the latter occurring only if $\omega_{s+2} = n+1$.

Now $$\eta^{s+1}(0) = \underbrace{\eta^{s}(0)\ \eta^{s}(0)\ \ldots\ \eta^{s}(0)}_{n+1 \text{ times}}\ \eta^{s}(1),$$

so that if k' is the beginning of a block $\eta^{s+1}(0)$, then k is the beginning of a block $\eta^{s}(0)$ or $\eta^{s}(1)$ (the latter only if $\omega_{s+1} = n+1$) and $k \in \mathcal{R}_s^*$. If on the other hand k' is the beginning of a block $\eta^{s+1}(1)$ then $\omega_{s+1} \leq n$ (since ω must be admissible) and

$$\eta^{s+1}(1) = \underbrace{\eta^{s}(0)\ \ldots\ \ldots\ \eta^{s}(0)}_{n \text{ times}}\ \eta^{s}(1).$$

Therefore $k = k' + \omega_{s+1} C_{s+1}$ is the beginning of a block $\eta^s(0)$ or $\eta^s(1)$ (the latter only if $\omega_{s+1} = n+1$), and $k \in \mathcal{R}^*_s$.

Continuing in this fashion, we have $\mathcal{J}_s \subseteq \mathcal{R}^*_s$.

2. $\mathcal{R}^*_{s+1} \subseteq \mathcal{R}^p_s$

We use here a method similar to that of I) to compare the sequences $\eta^\infty(0)$ and $\theta^\infty_p(0)$. As in I), we defined a balanced substitution η^* associated with η and θ_p by setting

$$a = \begin{pmatrix} \eta(0) \\ \theta_p(0) \end{pmatrix} = \begin{pmatrix} 0^{n+1} & 1 & \\ 0^{n+1-p} & 1 & 0^p \end{pmatrix}$$

$$b = \begin{pmatrix} \eta(0) & \\ \theta_p(1) & 0 \end{pmatrix} = \begin{pmatrix} 0^{n+1} & 1 \\ 1 & 0^{n+1} \end{pmatrix}$$

$$c = \begin{pmatrix} \eta(0) & \\ \bar{\theta}_p(0) & 0 \end{pmatrix} = \begin{pmatrix} 0^{n+1} & 1 & \\ 0^{n-p} & 1 & 0^{p+1} \end{pmatrix}$$

$$d = \begin{pmatrix} \eta(1) \\ \bar{\theta}_p(0) \end{pmatrix} = \begin{pmatrix} 0^n & 1 & \\ 0^{n-p} & 1 & 0^p \end{pmatrix}$$

(Note that a, b, c and d consist of pairs of equivalent blocks of the same length, but for this purpose we have not decomposed

$\begin{pmatrix} \eta^\infty(0) \\ \theta^\infty_p(0) \end{pmatrix}$ into <u>minimal</u> balanced blocks.)

Now set $I^* = \{ a, b, c, d \}$

and define η^* on I^* by

$$\eta^* : \begin{array}{l} a \longrightarrow a^{n+1-p}\ b\ c^{p-1}\ d \\ b \longrightarrow b\ c^n\ d \\ c \longrightarrow a^{n-p}\ b\ c^p\ d \\ d \longrightarrow a^{n-p}\ b\ c^{p-1}\ d\ . \end{array}$$

Then for each $i^* = \begin{pmatrix} A \\ B \end{pmatrix} \in I^*$, we have

$$\eta^* (i^*) = \begin{pmatrix} \eta (A) \\ \theta_p (B) \end{pmatrix} .$$

Therefore

$$\eta^{*\infty} (a) = \begin{pmatrix} \eta^{\infty} (0) \\ \theta_p^{\infty}(0) \end{pmatrix} = (x_o \; x_1 \; x_2 \; \dots) = x$$

with

$$x_i = \begin{pmatrix} y_i \\ z_i \end{pmatrix} , \quad x_i \in I^* .$$

Since for any $s \geq 1$, we have

$$\eta^{*s} (x) = x = \begin{pmatrix} \eta^s(y_o) \\ \theta_p^s(z_o) \end{pmatrix} \begin{pmatrix} \eta^s(y_1) \\ \theta_p^s(z_1) \end{pmatrix} \cdots$$

and since y_i is either $\eta (0)$ or $\eta (1)$, we see that this decomposition of $\eta^{*s} (x)$ corresponds to the set $\mathfrak{R}^*_{s+1}$. On the other hand, the elements z_i are finite sequences of zeros and ones, so that $\theta_p^s (z_i)$ is a sequence of blocks of the form $\theta_p^s (0)$ or $\theta_p^s (1)$. Thus the same decomposition corresponds also to a subset of the set $\mathfrak{R}_s^p$, and hence $\mathfrak{R}^*_{s+1} \subseteq \mathfrak{R}^p_s$.

II - 6 A null set in Y.

The last preparation for the proof of theorem 10 is a technical result. We recall that w^o and w^1 were defined in I) for σ_p as

$$w^o = \sigma_p^{\infty}(0)$$
$$w^1 = \sigma_p^{\infty}(1).$$

It was proved there that the coincidence density $d(\sigma_p) = 1$ for each of the substitutions σ_p, $0 \leq p \leq n$, and that the convergence to $d(\sigma_p)$ is exponentially fast.

Let $\beta \in Y = \mathbb{R}/\mathbb{Z}$ and consider the sequence of positive integers $\tilde{\beta}_m$ corresponding to β as defined in II.3. For each m, let k_m be chosen with $|k_m|$ minimal such that

$$w^o_{\tilde{\beta}_m+k_m} \neq w^1_{\tilde{\beta}_m+k_m} .$$

Now set

$$Y_o = \{ \beta \in Y : \liminf_{m\to\infty} |k_m| < \infty \}$$

<u>Lemma 18.</u> $\nu(Y_o) = 0.$

<u>Proof</u> : By II.3 and [13], the points $0, \lambda, 2\lambda, \ldots, (C_m-1)\lambda$ divide Y into C_m intervals I_j of measures λ^m or $\lambda^m(1-\lambda)$.

Let r be a positive integer. Then

$$Y_o = \bigcup_{r=0}^{\infty} \{ \beta \in Y : \liminf_{m\to\infty} |k_m| \leq r \}$$

If we consider now for fixed r and m the set

$$\{ \beta \in Y : |k_m| \leq r \} ,$$

by the definition of k_m and because $\tilde{\beta}_m$ is constant on each interval I_j, we obtain

$$\nu(\{\beta \in Y : |k_m| \leq r) \leq \lambda^m . \frac{\text{card}\left\{ \begin{array}{l} s : 0 \leq s < C_m ,\ w^o_{s+t} \neq w^1_{s+t} \\ \text{for some } 0 \leq |t| \leq s \end{array} \right\} + r}{C_m}$$

$$\leq \lambda^m . r . \frac{\text{card} \{ s : 0 \leq s < C_m , w_s^o \neq w_s^1 \} + r}{C_m}$$

Since $C_m \longrightarrow \infty$ exponentially and since

$$\frac{\text{card} \{ s : 0 \leq s < C_m, w_s^o \neq w_s^1 \}}{C_m} \longrightarrow 1 - d(\theta_p) = 0$$

exponentially (by I.), we conclude that $\nu(Y_o) = 0$.

II. 7 . Proof of theorem 10.

We first show that if $\beta \in Y$ is such that $\beta \notin Y_o$ (see II.6), $\beta \notin \pi(N)$ (see II.4), $\beta \notin Z\lambda$ (mod 1), and if $\lim_{k\to\infty} n_k \lambda = \beta$, then $\lim_{k\to\infty} T^{n_k} w$ exists, where $w = w^{oo}$ as in II.2. Let r and s be fixed integers ≥ 2. Then $\lim_{k\to\infty} n_k \lambda = \beta$ implies that for sufficiently large k, $n_k - \tilde{\beta}_s \in \mathcal{S}_s$. By lemma 17, we have $n_k - \tilde{\beta}_s \in \mathfrak{R}^p_{s-1}$.

If we write $n_k = (n_k - \tilde{\beta}_s) + \tilde{\beta}_s$, we see that the symbol w_{n_k} occupies a place in the sequence w which is $\tilde{\beta}_s$ to the right of the place $n_k - \tilde{\beta}_s$, and, in w, the place $n_k - \tilde{\beta}_s$ is the beginning of a block of the form $\theta_p^{s-1}(0)$ or $\theta_p^{s-1}(1)$ (since $n_k - \tilde{\beta}_s \in \mathfrak{R}^p_{s-1}$).

Since $\beta \notin \mathbb{Z}\lambda$ mod 1, we have $\tilde{\beta}_s \longrightarrow \infty$ as $s \longrightarrow \infty$.
Moreover, if s is such that $\omega_{s-2}\ \omega_{s-1} = 00$, then
$\tilde{\beta}_s = \sum_0^{s-1} \omega_i C_i = \sum_0^{s-3} \omega_i C_i < C_{s-2}$, and it follows easily that
$\ell_1^{(s-1)} - \tilde{\beta}_s \longrightarrow \infty$ and $\ell_o^{(s-1)} - \tilde{\beta}_s \longrightarrow \infty$ as $s \longrightarrow \infty$.
Now choose a subsequence $s_u \longrightarrow \infty$ such that $\omega_{s_u-2}\ \omega_{s_u-1} = 00$.
This is possible because $\beta = \sum_{s=0}^{\infty} \omega_s \lambda^s \notin \pi(N)$. Then choose u such the number k_{s_u} defined in II.6 corresponding to β satisfies $|k_{s_u}| \geq r$.

This is possible because $\beta \notin Y_o$. Finally, choose $K = K(u,s)$ such that if $k \geq K$, then $n_k - \tilde{\beta}_{s_u} \in \mathcal{S}_{s_u}$. It follows now for u and $k \geq K$ that the blocks

$$w_{n_k-r}\ ,\ w_{n_k-r+1}\ ,\ \ldots\ ,\ w_{n_k+r-1}\ ,\ w_{n_k+r}$$

do not depend on k . Therefore any two accumulation points of the sequence $T^{n_k} w$ agree in the coordinates $-r, \ldots, +r$. Since r was arbitrary, $\lim_{k\to\infty} T^{n_k} w$ exists.

Now let $h : X(\theta_p) \longrightarrow Y$ be the homomorphism of II.2, with $h(w) = 0$, and set

$$\overline{Y} = Y \setminus (Y_o \cup \pi(N) \cup \mathbb{Z}\lambda\ ).$$

Then by lemmas 16 and 18, $\nu(\overline{Y}) = 1$. If $\beta \in Y$ and if $x \in h^{-1}(\beta)$, then there exists a sequence n_k such that $T^{n_k} w \longrightarrow x$ (because $X(\theta_p)$ is minimal) and the corresponding sequence $n_k\lambda \in Y$ tends to β (because h is a homomorphism). If now $\beta \in \overline{Y}$, then $h^{-1}(\beta)$ must consist of a single point, since any sequence n_k such that $n_k\lambda \longrightarrow \beta$ will make $T^{n_k}w$ converge. Therefore h is one-to-one on a set $h^{-1}(\overline{Y})$ of measure 1, and hence θ_p has discrete spectrum.

III - A SUBSTITUTION OF NON-CONSTANT LENGTH WITH PARTLY CONTINUOUS SPECTRUM.

We have seen in the preceding section that a class θ_p of substitutions of non-constant length have discrete spectrum. It is natural to ask whether any substitution of non-constant length has discrete spectrum. In this paragraph, we give an example of a substitution with partly continuous spectrum.

We consider the matrix

$$M = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

and its corresponding substitutions. Among them, only three do not yield periodic orbits. They are :

$$\theta_1 : \begin{array}{l} 0 \longrightarrow 0\ 1 \\ 1 \longrightarrow 1\ 0\ 0\ 1 \end{array}$$

$$\theta_2 : \begin{array}{l} 0 \longrightarrow 0\ 1 \\ 1 \longrightarrow 1\ 0\ 1\ 0 \end{array}$$

$$\theta_3 : \begin{array}{l} 0 \longrightarrow 0\ 1 \\ 1 \longrightarrow 1\ 1\ 0\ 0. \end{array}$$

The coincidence densities of these substitutions have been obtained in I.3 ; they are respectively $d(\theta_1) = d(\theta_2) = 1$ and $d(\theta_3) = \frac{1}{3}$.

Again a result in [16] yields the continuous eigenfunctions for θ_1, θ_2 and θ_3 , and gives the equicontinuous factor $\mathbb{Z}_2 \times \mathbb{Z}(3)$, $\mathbb{Z}_2$ being the cyclic group of order two and $\mathbb{Z}(3)$ the 3-adic integers. It is easy to see why this is so : the $\mathbb{Z}_2$ - part is obtained because $\theta^s 0$ and $\theta^s 1$ always appear in $w = w^{10}$ at even places, so that one can " recognize " by looking at a finite number of successive symbols of a point $x \in X(\theta)$ whether it is a limit of even or odd translates of w . The $\mathbb{Z}(3)$-part arises from the fact that for $s \geq 2$, the lengths $\ell_0^{(s)}$ and $\ell_1^{(s)}$ of $\theta^s 0$ and $\theta^s 1$ are multiples of 3^{s-1} , which allows the " recognition " of a point x as a limit of translates n_k of w with n_k mod 3^{s-1} fixed.

The proof of the following theorem is simple in comparison to theorem 10 of II.

Theorem 19

θ_1 and θ_2 have discrete spectrum.

Proof : Let $a = 0\ 1$ and $b = 1\ 0$. Then

$$\theta_1\ a = 0\ 1\ 1\ 0\ 0\ 1 = a\ b\ a$$
$$\theta_1\ b = 1\ 0\ 0\ 1\ 0\ 1 = b\ a\ a$$
$$\theta_2\ a = 0\ 1\ 1\ 0\ 1\ 0 = a\ b\ b$$
$$\theta_2\ b = 1\ 0\ 1\ 0\ 0\ 1 = b\ b\ a.$$

The substitutions

$$\eta_1 : \begin{array}{l} a \longrightarrow a\ b\ a \\ b \longrightarrow b\ a\ a \end{array}$$

and

$$\eta_2 : \begin{array}{l} a \longrightarrow a\ b\ b \\ b \longrightarrow b\ b\ a \end{array}$$

obtained in this manner have discrete spectrum (see [1]) with equicontinuous factor $\mathbb{Z}(3)$, and the homomorphism $X(\theta_i) \longrightarrow X(\eta_i)$, $i = 1, 2$, yields the additional factor $\mathbb{Z}_2$ in an obvious manner.

We turn now to the substitution θ_3 . As above, let $a = 01$, $b = 11$ and $c = 00$. Then

$$\theta_3 a = 0\ 1\ 1\ 1\ 0\ 0 = a\ b\ c$$
$$\theta_3 b = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0 = b\ c\ b\ c$$
$$\theta_3 c = 0\ 1\ 0\ 1 = a\ a.$$

This leads us to consider the substitution of constant length

$$\eta_3 : \begin{array}{l} a \longrightarrow a\ b\ c \\ b \longrightarrow b\ c\ b \\ c \longrightarrow c\ a\ a \end{array}$$

Lemma 20.

There exists a continuous map

$$\pi : X(\theta_3) \longrightarrow X(\eta_3)$$

such that $\pi\, T^2 = T\, \pi$.

Proof : If $x = (\dots, x_{-1}, x_o, x_1, \dots) \in X(\theta_3)$, the block 1 1 0 occurs at least once in x at a place k. According to whether k is even or odd, we group symbols of x as

$$\dots, (x_{-2}\ x_{-1}), (x_o\ x_1), (x_2\ x_3), \dots$$

or

$$\dots, (x_{-1}\ x_o), (x_1\ x_2), (x_3\ x_4), \dots,$$

and replace each group by its corresponding symbol a, b or c. (Note that in this " canonical decomposition " of any point of $X(\theta_3)$, the block b = 1 1 is always followed by c = 0 0.)
This yields a well-defined continuous map π from $X(\theta_3)$ to $X(\eta_3)$ and it follows immediately that $\pi T^2 = T \pi$.

Lemma 21

The substitution η_3 has partly continuous spectrum.

Proof : The structure group of η_3 is $Z(3)$ (see [15]).
Let $\sigma : X(\eta_3) \longrightarrow \mathbb{Z}(3)$ be the corresponding projection.
Then the subspace $H = \{ f \circ \sigma : f \in \mathbb{L}^2(\mathbb{Z}(3)) \}$ of $\mathbb{L}^2(X(\eta_3))$ is the subspace spanned by the continuous eigenfunctions of T . Moreover, since σ is almost everywhere 3 to 1, $H^\perp$ is not $\{ 0 \}$.
Let $0 \neq h \in H^\perp$ and suppose that for some complex ξ with $|\xi| = 1$ we have $T h = \xi h$. By ergodicity of $(X(\eta_3),T)$, $|h|$ is a non-zero constant, and it follows that $h^3 \in H$ and $T h^3 = \xi^3 h^3$.
Thus ξ^3 is a 3^k-root of unity for some k , and so is ξ . Therefore there exists another eigenfunction $h_o \in H$ with eigenvalue ξ , and this contradicts the ergodicity of T .

Theorem 22.

The substitution θ_3 has partly continuous spectrum.

Proof : If not, then by lemma 20, η_3 would have discrete spectrum, and this contradicts lemma 21.

We remark that the article [2] of M. Dekking contains a systematic development of substitutions of the type considered in this last paragraph.

REFERENCES

[1] COVEN - KEANE
The structure of substitution minimal sets.
Trans. Amer. Math. Soc. vol. 162 (1971), 89-102

[2] DEKKING
The spectrum of dynamical systems arising from substitutions of constant length (1977) Preprint

[3] DEKKING-KEANE-MICHEL
Substitutions. Seminar. Rennes (1977) . Unpublished manuscript

[4] GOTTSCHALK
Substitution minimal sets. Trans. Amer. Math. Soc. vol. 109, (1963),467-491

[5] GOTTSCHALK-HEDLUND
Topological dynamics. Amer. Math. Soc. Colloq. Publ. vol. 36, (1955)

[6] JACOBS-KEANE
0-1 sequences of Toeplitz type
Wahrscheinlichkeitstheorie und Verw. Geb. 13 (1969)

[7] KAKUTANI
Ergodic theory of shift transformations.
Proceedings of the fifth Berkeley symposium on Mathematical Statistics and Probability. Berkeley and Los Angeles University of California Press

[8] KAKUTANI
Strictly ergodic symbolic dynamical systems.
Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability. Berkeley and Los Angeles of California Press (1972)

[9] KAMAE
A topological invariant of substitution minimal sets.
J. Math. Soc. Japan 24 (1972), 285-306

[10] KEANE
Generalized Morse sequences.
Wahrscheinlichkeitstheorie und Verw. Geb. 10 (1968), 335-353

[11] KEANE
Strongly mixing g-measures. Inventiones Math. 16, Springer Verlag (1972), 309-324

[12] KEANE
Sur les mesures quasi ergodiques des translations irrationnelles.
CRAS 272 (4/1/1971), 54-55

[13] KEANE
Irrational rotations and quasi-ergodic measures
Publ. Sem. Math. Rennes, vol. 1 (1971), 17-26

[14] KLEIN
Homomorphisms of symbolic dynamical systems
Math. Systems Theory, vol. 6, n° 2 (1972), 107-122

[15] MARTIN
Substitution minimal flows. Amer. J. Math. 93, (1971), 503-526

[16] MARTIN
Minimal flows arising from substitutions of non constant length.
Math. Systems Theory, vol. 7, n° 1, (1973), 73-82

[17] MICHEL
Stricte ergodicité d'ensembles minimaux de substitution.
CRAS 278 (1974), 811-813

[18] MICHEL
Stricte ergodicité d'ensembles minimaux de substitution.
Lecture Notes in Math. 532, Springer Verlag (1976), 189-201

[19] MICHEL
Sur les ensembles minimaux engendrés par les substitutions de longueur non constante
Thèse.Université de Rennes (1975)

[20] MICHEL
Minimal sets generated by a substitution of non constant length.
Astérisque 40 (1976) 143-146

[21] MORSE
Recurrent geodesics on a surface of negative curvature.
Trans. AMer. Math. Soc. vol. 22 (1921)

[22] MORSE-HEDLUND
Symbolic dynamics II.
Sturmian trajectories. Amer. J. of Math. 62 (1940), 1-42

[23] NEVEU
Sur les suites de Toeplitz.
Wahrscheinlichkeitstheorie und Verw. Geb. 13 (1969), 132-134

SUMMARY

Measure - theoretic properties of substitution dynamical systems generated by some substitutions of non constant length on two symbols 0 and 1 are studied. For this, a new concept is introduced. The coincidence density of a substitution θ is defined as the density of the set of integers k for which the sequences $\theta^{\infty}0$ and $\theta^{\infty}1$ take the same value in the place k . This coincidence density does not always exist. A class of substitutions for which this coincidence density takes the value 1 is given and it is proved that these substitutions have discrete spectrum and that all their eigenfunctions are continuous. An example of a substitution dynamical system with partly continuous spectrum is also given.

Key words : Substitution, dynamical system .
AMS (MOS) Subject classification : 54H20, 28A65, 10L10.

Author ' s address : MICHEL Pierre
Département de Mathématiques
Université de Bretagne Occidentale
6, avenue Le Gorgeu
29283 BREST CEDEX (France)