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Hamiltonian Systems Close to Integrable Systems

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HAMILTONIAN SYSTEMS CLOSE TO INTEGRABLE SYSTEMS   E. ZENHDER

The stability problem of Hamiltonian systems, such as the
celestial N-body system, has been attacked again and again over
the last hundred years. Its analytical difficulties have to
do with the presence of so-called small divisors. The surprising
and famous break-through came within the last two decades
through the work of Kolmogorov, Arnold and Moser, known under
the apocryph KAM - theory. (All the references can be found in
a recent survey on this subject [1] .) The main goal of this
lecture is to report on some of this well-known progress and
on some more recent results.

In order to outline the main ideas underlying the KAM - theory
I start from a very special situation, a so-called integrable
system. It is given by an exact Hamiltonian vectorfield
$X_{h_0} = (\partial_y h_0) \partial_x - (\partial_x h_0) \partial_y$ on the symplectic manifold
$M^{2n} = T^n \times J$, $J \subset \mathbb{R}^n$ an open cube, where we
assume the Hamiltonfunction h_0 to depend only on y : $h_0 = h_0(y)$.
The Hamiltonian equations are then basically linear and the flow
is explicitly given for all times $\phi_t(x,y) = (x + t\,\partial_y h_0(y), y)$.
So, geometrically, the manifold M^{2n} is foliated into an
n-parameter family of tori $T_y := T^n \times \{y\}$, $y \in J$,
which are invariant under the flow ϕ and the restriction of
the flow onto each single torus is a Kronecker flow ϕ_t / T_y :
$x \mapsto x + t\omega$, with the frequencies $\omega := \partial_y h_0(y)$.   We assume
that these frequencies vary from torus to torus, requiring that

1)

$$\mathrm{Det}\left(\partial_y^2 h_0(y)\right) \neq 0, \quad y \in J,$$

as it turns out, this postulated nonlinearity has a stabilizing
effect under a perturbation. Integrable systems are, of course,
very rare; as examples I just mention the two-body problem
and the geodesic flow on ellipsoids with distinct axis. There
are, however, many Hamiltonian systems, such as the restricted

3-body problem, or more general, the N-body problem of celestial
mechanics, which are in a certain sense close to such an
integrable situation. Moreover, in general, every Hamiltonian
system in a very small neighbourhood of a linear stable periodic
orbit is close to an integrable system. For a long time one has,
therefore, been bothered by the following question: What happens
to the invariant foliation under a small perturbation? Or - to
put it differently - what does the flow of Hamiltonian vector-
field v_h look like for <u>all times</u>, if h is a Hamiltonfunction,
periodic in $x = (x_1, \ldots , x_n)$ and close to h_e ? Recalling
the transformation theory of Hamiltonain vectorfields, one could
try - following a first naive impulse - to find a symplectic
mapping φ on M^{2n} which would transform our perturbed system into
an integrable system:

$$h \circ \varphi (\xi , \eta) = h_1 (\eta)$$

Remember that a mapping φ is called symplectic, if
$$\varphi^* (v_h) = v_{h \circ \varphi}$$ for all functions h .
In general no such φ satisfying (2) exists. As a matter of
fact, the orbit structure of such a system is very complicated .
As it turns out, the difficulty lies in the fact that, due to
the resonances and the Hamiltonian character, solutions of
completely different behaviour over an <u>infinite</u> time interval
coexist side by side clustering on each other. In order to
separate conceptually and analytically solutions of different
type, we introduce the concept of a subsystem of a given system.
Here a pair (N, v), N a compact manifold and v a vectorfield
on it, is called a subsystem of (M , V) if there is an embedding
$f : N \longrightarrow f(N) \subset M$ carrying v onto V restricted
to $f(N)$:

$$V \circ f = Tf \circ v .$$

Geometrically this just means that $f(N)$ is an embedded sub-
manifold which is invariant under the flow $\phi_t[V]$, and on $f(N)$
this flow is described by the flow $\varphi_t[v]$ of the sub-
system : $\phi_t \cdot f = f \cdot \varphi_t$, for all $t \in \mathbb{R}$. The basic
idea of the KAM - theory is now to give up the hopeless task
of describing all solutions of h by means of solving the
initial value problem; instead one tries to fill up the
perturbed systems with well known subsystems, which, as it happens,
leads to a boundary value problem. Candidates for such systems
are easily found in the unperturbed system which is foliated into
a family of subsystems, all of which are Kronecker systems (T^n, ω).
However, according to their frequencies, these sybsystems react
completely differently to a perturbation. If the frequencies
satisfy some resonance conditions, like

$$ <\omega, \lambda> \; = \; 0 \; , \qquad \lambda \in \mathbb{Z}^n \setminus \{0\}, $$

then, the invarinat torus will break apart under a perturbation.
The KAM - theory states on the other hand, that all those
Kroneckersystems of the integrable system will survive under
a perturbation which have frequencies ω which are not only
rationally independent but also satisfy the following set of
inequalities

$$ |<\omega, k>| \; \geq \; \nu \; |k|^{-\tau} $$

for all $k \in \mathbb{Z}^n \setminus \{0\}$, and for some $\nu > 0$ and $\tau \geq (n-1)$
where $|k| = \sum |k_i|$. Every solution on such a torus covers the
torus densly, as is, of course, well known. In other words: tori
of the integrable systems with frequencies (5) have, under a
Hamiltonian perturbation, a continuation as parameterized
invariant manifold on which the flow is conjugate to the
unperturbed Kronecker flow. Now, since the set of real numbers
for which no such ν and τ exist, is a set of measure zero, and
since by assumption $y \longmapsto \partial_y h_*(y) = \omega$ is a local diffeo-
morphism, such frequencies occur in abundance in our perturbed
system.

(5)

In order to formulate this KAM-result more precisely and to
outline the basic idea of the proof we pick one single Kronecker-
system of the integrable system, say $\left(T^n \times \{ \overset{o}{y} \}, \omega = \partial_y h_0(\overset{o}{y}) \right)$
whose frequencies satisfy (5), we may assume $\overset{o}{y} = 0$. Hence,
as unperturbed Hamiltonfunction h_0 in the neighbourhood of this
invariant torus we consider the following function which we let
now also depend on x :

6) $$ h_0(x,y) = c_0 + <\omega, y> + \frac{1}{2} <Q_0(x)y, y> + O_3(|y|). $$

Generalizing (1), we require that $\mathcal{D}et\left([Q_0] \right) \neq 0$, where $[\ ]$
stands for the meanvalue of the symmetric matrixvalued function Q_0
on T^n. Let us denote for our fixed ω the set of functions
which are in such a normalform (6) by N_ω ; we therefore write
for the particular h_0, $h_0 \in N_\omega(c_0, Q_0)$. The KAM-
statement then can be rephrased by saying that this set N_ω
is in the set of all Hamiltonfunctions on M structurally stable
under a subgroup of symplectic diffeomorphisms of M, that is to
say, if h is any function close to $h_0 \in N_\omega(c_0, Q_0)$, then
there is a <u>symplectic</u> diffeomorphism φ, such that

7) $$ h \circ \varphi \in N_\omega(c, Q), $$

for a (c, Q) close to (c_0, Q_0). So, in contrast to (2), not the
whole system is structurally stable but only the Kronecker-
system we singled out with fixed frequencies, chosen in advance.
The second and higher order terms in y (6) depend also on x.
Geometrically (7) says in particular that (T_0, ω) is a sub-
system of v_h, O refers to $z = 0$. Indeed, using the
fact that φ is symplectic we have for the flows for all $t \in \mathbb{R}$:

8) $$ \phi_t[v_h] \circ \varphi \mid T_0 = \varphi \circ \phi_t[v_{h \circ \varphi}] \mid T_0 . $$

Reformulating (7) as a functional equation for functions on the
torus this existence statement can presently by derived from
a generalized implicit function theorem of Moser-Nash type, [4],
abstracting the new methods and techniques introduced by the
KAM-theory in order to deal with the small divisor problem.
Such theorems are designed for nonlinear functional equations,
where the linearized problem has an approximate solution losing
moreover some derivatives, however, with quantitative estimates.
This loss of derivatives in the problem at hand is due to the
small divisors and depends on τ in (5). The basic idea of such
theorems is a modifications of Newton's rapidly converging
iteration algorithm in the framework of families of linear
spaces. One has to replace the right inverse of the linearized
which does not exist with an approximate right inverse, (the
existence of an approximate right inverse is based on the
algebraic structure of the problem (7)), combined with an analytic
smoothing technique with very quantitative estimates; in order
to catch up the loss of derivatives one has at each iteration step.
So, the existence of the above subsystems is indeed very much
related to excessive smoothness requirements on the Hamiltonian
function h, roughly h has to have a little more than $2n$
derivatives, $2n$ the dimension of the symplectic manifold.
In order to continue the Kronecker system (T, ω) we kept the fre-
quencies fixed. It is sufficient to merely keep the frequency ratio
fixed. One can make use of this extra freedom to ensure that the
continued invariant torus lies on an energy surface ~~which~~ with
fixed energy value, say $\varepsilon \in \mathbb{R}$. Here one singles out for fixed ~~&~~ ω
satisfying (5) and fixed ε, the following subset $N_{\omega, \varepsilon}$ of
normalforms: $h \in N_{\omega, \varepsilon}(\mu, Q)$, iff:

$$ h(x, y) = \varepsilon + \mu \langle \omega, y \rangle + \frac{1}{2} \langle Q(x) y, y \rangle + O_3(|y|), $$

where $\mu \neq 0$, and where the nondegeneracy assumption is different,
namely one requires that $Det[Q_\omega] \neq 0$, where

$$Q_\omega(x) = \begin{pmatrix} 0 & \omega_1 & & \omega_n \\ \omega_1 & & & \\ & \ddots & & Q(x) \\ & & & \\ \omega_n & & & \end{pmatrix}$$
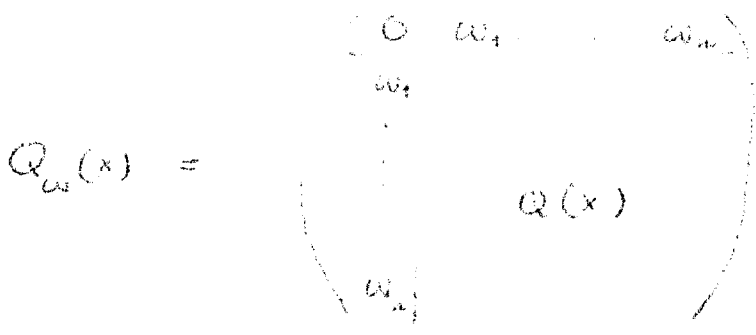
The fact that this subset $N_{\omega,\varepsilon}$ is structurally stable means
that the Kroneckersystem survives with the same frequency ratio
and lies on a energy surface with fixed energy value ε.
From the normalforms (6) and (8) one easily gains some more
information on the orbit structure in a neighbourhood of the
invariant torus T_o. Namely an easy application of an old
fixed point theorem by Birkhoff yields that T_o is contained
in the closure of the set of periodic orbits of v_h: $T_o \subset \overline{Per(v_h)}$.
In this sense all the solutions on the invariant torus T_o, which
are by definition quasi periodic, are limits of periodic solutions.

The KAM-theory of course provides not only one single invariant
torus but gives a whole set of such Kroneckersystems according
to different choices of ω satisfying (5). Since the complement
of these frequencies is a set of measure zero, one might at first
expect that the invariant tori so found leave out a set
of measure zero only. However, this is not the case, since the
smallness condition on the perturbation in (7) in order to
continue the subsystem with a chosen frequency (5), depends
heavily on v and τ , $|h - h_o| \leq \varepsilon(v, \tau)$,
τ measuring the loss of derivatives. Yet the following
assertion can be proved, if we denote with S the set in the
phase space, covered by the continued invariant tori, then for
any $\varepsilon > o$,

$$m(M \setminus S) < \varepsilon \, m(M)$$

if, loosely speaking, $|h - h_o| < \delta(\varepsilon)$. Here m
denotes the Lionville (Lebesque) measure on M. By the same
token one gets the curious statement that $m(\overline{Per(v_h)}) > 0$.

The KAM-theory extends the above ideas to the equilibrium problem of any Hamiltonian vectorfield v_h. In what sense is a linear stable equilibrium point which I assume to be 0,

(12)
$$v_h (0) = 0$$

actually stable? Since the vectorfield is Hamiltonian, a necessary condition for the stability is that all the eigenvalues of the, in 0, linearized system are purely imaginary; we assume them to be distinct and label them such that α_1, α_n, $-\alpha_1$, $-\alpha_n$, are all the eigenvalues. If we now exclude the following low order resonances:

(13)
$$\sum_{k=1}^{n} j_k \alpha_k \neq 0 \quad , \quad 0 < |j| \leq 4,$$

then we can go close to an integrable situation. For there exists an analytic symplectic transformation such that the Hamiltonfunction h in 0 has the following Birkhoff normalform:

(14)
$$h(\xi, \eta) = \sum_{k=1}^{n} \alpha_k p_k + <Q p, p> + O_5$$

where $p_k = \frac{1}{2}(\xi_k^2 + \eta_k^2)$, O_5 is a power series in ξ, η containing only terms of order ≥ 5 in ξ and η. The KAM-theory then says that if

(15)
$$Det (Q) \neq 0,$$

and if h is sufficiently smooth ($h \in C^\ell$, $\ell > 2n+2$), then for any $\varepsilon > 0$ there is an open neighbourhood $U(\varepsilon)$ of 0 containing an invariant set S, consisting of Kronecker subsystems, such that $S \subset Per(v_h)$ and

$$m \left(U(\varepsilon) \setminus S \right) \; < \; \varepsilon \; m \left(U(\varepsilon) \right) .$$

The important point of this statement is that one has under only
finitely many conditions, namely (13),(15), on the low order
coefficients of the Hamiltonfunction a weakened type of stability,
a kind of stability in measure; aside from a small proportion all
solutions in the open small neighbourhood $U(\varepsilon)$ remain in
it for all times. This is in particular not just a generic statement
As one sees, if the system is not linear, that is to say, if
$Det (Q) \neq 0$, then only low order resonances are harmful. One
concludes of course that a Hamiltonian flow, possessing one
single such equilibrium point is certainly not ergodic. If the
degree of freedom is only two, $n = 2$, then the invariant tori
bound open regions in the 3-dimensional energy surface, hence
the KAM-theory yields in this case a criterion for topological
stability. Indeed if (13) holds true, and if

$$Q_{11} x_2^2 - 2 Q_{12} x_1 x_2 + Q_{22} x_1^2 \neq 0 ,$$

then the equilibrium point is topologically stable. The condition
(17) is precisely the condition we imposed in order to continue
the Kroneckersystems on fixed energy surfaces (10). However, if
the degree of freedom is greater than 2, $n > 2$, then the
complement of the set of invariant tori on the energy surfaces is
connected and examples show that the set of escaping solutions
may not be empty, if the system has no positive definite integral,
of course. This leads us to the still wide open problem for $n > 2$,
namely, what is really going on in the complement of the Kron-
eckersystems, in the so-called zone of instability, where
the resonances (4) play a crucial role and give, as is to
be expected, rise to a very unstable and unpredictalbe orbit -
structure which can be described by topological embeddings of
Bernoulli flows, Markov flows and so on, with hyperbolic structures.

So far we have considered invariant tori of dimension precisely
half of that of the phase space $\mathbb{R}^{2n}$ and the question arises
whether one can extend such a perturbation theory to other dimensions

and other subsystems. Indeed, a perturbation theory of tori of
all dimensions $r < n$ has recently been established by S. Graff
By a different method this theory has been generalized [4]
to the case of exact symplectic diffeomorphisms instead of
Hamiltonian vectorfields. The theory, however, applies only to
unstable tori, so called whiskered tori. This theory yields in
particular unstable quasiperiodic solutions in every neighbourhood
of a particular hyperbolic equilibrium point. To outline the
idea, we consider on the symplectic manifold M^{2n}, =
$T^r \times J^r \times \mathbb{R}^s \times \mathbb{R}^s$, $r + s = n$, the following set of normalforms
$h_0 \in \widehat{N}_\omega (c_0, Q_0, \Omega_0)$:

$$h_0(x, y, p, q) = c_0 + <\omega, y> + \tfrac{1}{2} <Q_0(x)y, y>$$
$$+ <\Omega_0(x)p, q> + O_3 (|y| + |p| + |q|)$$

where $\omega = (\omega_1, \ldots \omega_r)$ <u>fixed</u> and satisfying conditions like (5),
Det $[Q_0] \neq 0$, and where Ω_0 is a $(s \times s)$-matrixvalued function
on the torus T^r, such that

$$Re \; (\Omega_0(x) z, z) \; \geqslant \; k \; |z|^2 ,$$

for some $k > 0$ and all $z \in \mathbb{C}^s$. The condition (19) just says that
the eigenvalues of the symmetric part of $\Omega_0(x)$ lie in the right
half plane. Note that we do not impose any further conditions
on the eigenvalues. These normalforms (18) describe symplectic
hyperbolic Kronecker-subsystems illustrated by the following picture
in which we do not indicate the unpleasant neutral directions.
(To get the picture just drop for simplicity the O_3 terms in (18),
and assume Q_0 and Ω_0 to be independent of x):

M_-^{r+s}

T^r

M_+^{r+s}
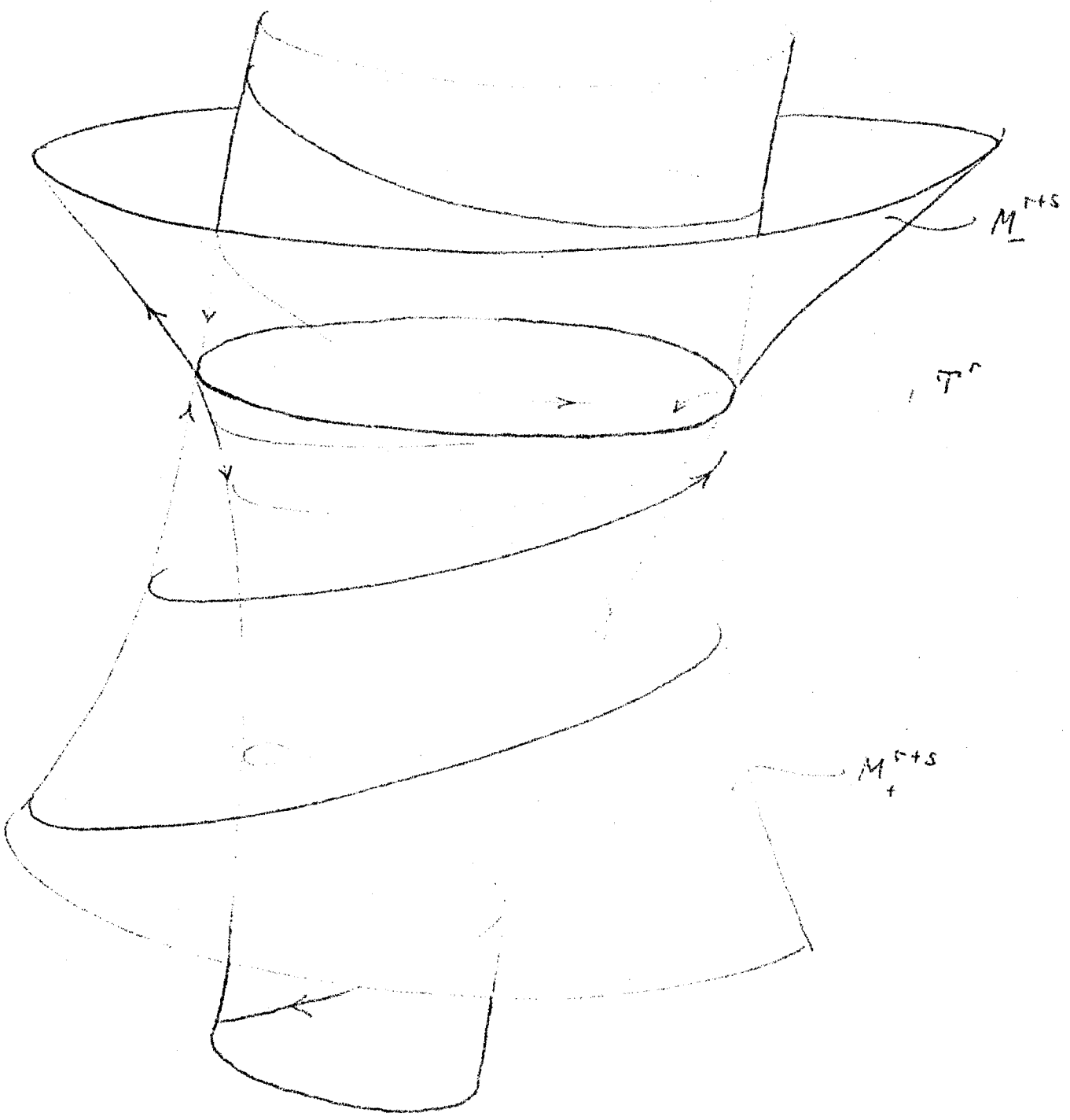
From the invariant torus T^r $(y = p = q = c)$ on which we have
a Kroneckerflow with r frequencies ω, there issue two in-
variant manifolds M_+^{r+s} and M_-^{r+s} consisting of solutions which
approach the invariant torus T^r at an exponential rate as
$t \to \pm \infty$, $T^r = M_+ \cap M_-$. In addition, an orbit on these
whiskers which are Lagrangian submanifolds, not only tends to the
invariant torus T^r but approximates a particular orbit on the
invariant torus. If now h is real analytic (which is, from
the mathematical point of view, the interesting case, requiring
techniques that go beyond the ones used previously), and close to h_o,
then there is a symplectic diffeomorphism φ on M^{2n} such
that $h \circ \varphi \in \hat{N}_\omega (c, Q, \Omega)$. Hence the whole
geometric, symplectic and analytic structure of the subsystem
survives under a perturbation. The fact that the invariant whiskers
are analytic is related to the fact that on the invariant torus
the flow is linear. This statement is one of the main ingredients
one needs to prove the following, now merely generic statement
which sheds some light on what is going on in between the invariant
tori of dimension n in M^{2n}.

Generically a linear, stable equilibrium point $(n \geqslant 2)$ is a cluster
point of whiskered tori T^r for all dimesnions $1 \leq r < n$.
Moreover, each such torus T^r lies in the closure of the set of
unstable periodic orbits : $T^r \subset \overline{UPer(v_h)}$. [5].

Analogously, one has for the exact symplectic diffeomorphisms
on M^{2n} : Generically a linear stable fixed point $(n \geqslant 2)$ is
a cluster point of invariant whiskered tori T^r, $1 \leq r < n$,
and each such hyperbolic torus T^r lies in the closure of the set
of unstable periodic points of the mapping [5]. The proof is based
on the study of high order resonance tori T^n on which one has
$(n - r)$ independent resonance conditions (4). These statements
can be considered as a first little step towards proving the
conjecture that generically a linear stable equilibrium point, which,
as one knows, is stable in measure, is actually topologically unstable.
Here, of course, one excludes again the trivial case of definite
Hamiltonian functions where one has topological stability.
A prove would follow an idea by Arnold, according to which one
would construct a transition chain of such whiskered tori, intersecting

transversally and would so establish an escaping solution on the
energy surface, sneaking out between all the invariant n-dimen-
sional tori. Such a process would be very slow, as a recent
estimate by Nekhoroshev [3] indicates. Roughly speaking, he
proves that if

$$h(x,y) = h_0(y) + \varepsilon h_1(x,y)$$

is real analytic on $M = T^n \times J$, and if h_0 is in a precise sense
generic, (the Taylorcoefficients of h_0 do not satisfy certain
algebraic equations), then there are constants $C > o$, $o < a,b < 1$,
depending on h_0 such that for <u>all</u> initial conditions $x(o)$, $y(o)$,
one has the following strong estimate for the solutions

$$|y(t) - y(o)| < \varepsilon^b, \quad t \in [o, \exp(C \varepsilon^{-a})]$$

The generic conditions on h_0 exclude of course resonance strips on
which the solution runs away.

CONCLUSION:

From our discussion now emerges the following concept of stability
for systems close to integrable systems. Before the KAM - theory
the conventional view was that in the phase space there are open
regions in which the solutions remain stable and bounded, while
outside of such regions the solutions behave unboundedly or escape.
The KAM - theory replaced this rather crude picture with the
following intricate modell: In the phase space one finds complicated
cantor sets which can be compared to a sponge. The solid part of
this sponge, of positive measure, consists of Kronecker subsystems
on which the solutions are bounded for all times, while the
solutions lying in the tiny holes of the sponge behave in an
unpredictable and unstable way. Here, in this zone one would
expect to find whiskered tori and Anosov subsystems which for
would give rise to a very slow diffusion through the sponge.

Such a strange modell emerges, of course, only, if one studies the
motion for <u>all times</u> and not merely for a reasonable but finite
time interval.

Still open remains the problem of what happens if the perturbation
gets large. Here numerical calculations indicate that the KAM - re-
sults hold for perturbations which are several magnitudes larger.
This phenomenon is not understood. It may be that the KAM - theory,
a highly sophisticated existence theory, is, in an way, too fine
and too precise to give an insight into this question. One expects
here that under larger perturbations the solid part of the sponge
will shrink, while the region of instability blows up, showing up
regions of positive measure on which the system behaves like a
Bernoulli system. Also not yet clear is the precise and crucial role
of the smoothness requirements one needs to establish the stable
phenomena in this picture. Also the dependence of this modell on the
number of degrees of freedom has not yet been investigated.

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