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FACTORS OF TYPE III *

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Today, the structure of a factor of type III is described as follows:

THEOREM 1. <u>Every factor</u> $\mathfrak{M}$ <u>of type III is isomorphic to the crossed product</u> $W^*(\mathfrak{N}, \mathbb{R}, \theta)$ <u>of a uniquely associated covariant system</u> $\{\mathfrak{N}, \theta\}$ <u>of a von Neumann algebra</u> $\mathfrak{N}$ <u>of type</u> II_∞ <u>and a one parameter automorphism group</u> $\{\theta_t : t \in \mathbb{R}\}$ <u>such that the restriction of</u> θ <u>to the center</u> $\mathcal{C}$ <u>of</u> $\mathfrak{N}$ <u>is ergodic, but not isomorphic to the translations on</u> $L^\infty(\mathbb{R})$, <u>and</u> θ <u>transforms some faithful semi-finite normal trace</u> τ <u>on</u> $\mathfrak{N}$ <u>in such a way that</u> $\tau \cdot \theta_t = e^{-t}\tau$, $t \in \mathbb{R}$. <u>Here the uniqueness of</u> $\{\mathfrak{N}, \theta\}$ <u>means that if</u> $\{\mathfrak{N}_1, \theta^1\}$ <u>and</u> $\{\mathfrak{N}_2, \theta^2\}$ <u>are covariant systems satisfying the conditions for</u> $\{\mathfrak{N}, \theta\}$, <u>then</u> $W^*(\mathfrak{N}_1, \mathbb{R}, \theta^1) \cong W^*(\mathfrak{N}_2, \mathbb{R}, \theta^2)$ <u>is equivalent to the conjugacy of</u> $\{\mathfrak{N}_1, \theta^1\}$ <u>and</u> $\{\mathfrak{N}_2, \theta^2\}$ <u>in the sense that there exists an isomorphism</u> π <u>of</u> $\mathfrak{N}_1$ <u>onto</u> $\mathfrak{N}_2$ <u>such that</u> $\theta^2_t = \pi \cdot \theta^1_t \cdot \pi^{-1}$, $t \in \mathbb{R}$. cf. [2],[8],[12],[13],[28] and [29].

The aim of this paper is to present the background of the above result together with some of further development. Although it is impossible to elaborate here, I would like to emphasize that the recent interaction between mathematics and theoretical physics was indispensable in this achievement.

In 1967, ther were two very important achievements in the theory of operator algebras: R. Powers distinguished a continuum of non-isomorphic factors of type III [23] and M. Tomita showed that given a von Neumann algebra $\mathfrak{M}$ on a Hilbert space $\mathfrak{H}$ with

*) Not for formal publication.

separating cyclic vector ξ_0 there exist a conjugate linear unitary involution J and a non-singular positive self-adjoint operator Δ such that

(i) $J\Delta^{\frac{1}{2}}x\xi_0 = x^*\xi_0$, $x \in \mathfrak{M}$;

(ii) $J\mathfrak{M}J = \mathfrak{M}'$ and $\Delta^{it}\mathfrak{M}\Delta^{-it} = \mathfrak{M}$, $t \in \mathbb{R}$.

After Powers' work, a rapid progress in the classification theory of factors followed: Araki and Woods classified the factors obtained as infinite tensor product of finite factors of type I, abbreviated as ITPFI factor, by introducing algebraic invariants $r_\infty(\mathfrak{M})$ and $\rho(\mathfrak{M})$ in 1968 [3] and McDuff constructed continua of factors of type II_1 and II_∞ in 1969, [20], which was also confirmed by Sakai, [24]. The developments along this line was treated in a new book, by Anastasio and Willig. [1]

A quiet but steady development followed after Tomita's work, [30]. A serious inspecting seminar on Tomita's work took place and confirmed his result at the University of Pennsylvania for 1968/69, which was later published as lecture notes [26] by the present author. The major discovery in the seminar was that the one parameter automorphism group $\{\sigma_t\}$ of $\mathfrak{M}$ given by $\sigma_t(x) = \Delta^{it}x\Delta^{-it}$, $t \in \mathbb{R}$ and $x \in \mathfrak{M}$, and the normal functional φ given by $\varphi(x) = (x\xi_0 \mid \xi_0)$, $x \in \mathfrak{M}$, satisfy the Kubo-Martin-Schwinger condition: for any pair $x, y \in \mathfrak{M}$ there exists a continuous bounded function $F(x)$ on $0 \leq \mathrm{Im}z \leq 1$ holomorphic inside the strip such that

$$F(t) = \varphi(\sigma_t(x)y) \quad \text{and} \quad F(t+i) = \varphi(y\sigma_t(x)),$$

and that $\{\sigma_t\}$ is uniquely determined by φ subject to the KMS condition; hence it is denoted by $\{\sigma_t^\varphi\}$ and called the modular automorphism group. The notion of the KMS-condition came from physics as the name suggests. Haag, Hugenholtz and Winnink showed in 1967, [16], that the cyclic representation π_φ of a C^*-algebra A induced by a state φ satisfying the KMS-condition with respect to a given one parameter automorphism group $\{\sigma_t\}$ on A is standard: there exists a unitary involution J such that $J\,\pi_\varphi(A)''\,J = \pi_\varphi(A)'$ and $JaJ = a^*$, $a \in \pi(A)'' \cap \pi(A)'$. It is widely believed that an equilibrium state in quantum statistical mechanics is characterized by the KMS-condition.

As an illustration, let us consider an example. A faithful normal positive linear functional φ on the algebra $\mathfrak{L}(\mathfrak{H})$ of all bounded operators is given by

$$\varphi(x) = \mathrm{Tr}(xh), \quad x \in \mathfrak{L}(\mathfrak{H}),$$

with some non-singular positive operator h of the trace class. If $\dim \mathfrak{H} < +\infty$ and $h = \lambda 1$, $\lambda > 0$, then we have $\varphi(xy) = \varphi(yx)$ for every $x,y \in \mathfrak{L}(\mathfrak{H})$, that is, φ is a trace. If this is the case, then the involution $x \to x^*$ in $\mathfrak{L}(\mathfrak{H})$ is a unitary involution J in the Hilbert space structure in $\mathfrak{L}(\mathfrak{H})$ induced by φ, which gives rise to a symmetry between the left multiplication representation and the right multiplication representation of $\mathfrak{L}(\mathfrak{H})$ on this Hilbert space $\mathfrak{L}(\mathfrak{H})$. In general, $\varphi(xy) \neq \varphi(yx)$ because $xh \neq hx$. However, xh and hx are homotopic under the homotopy: $t \in [0,1] \to h^t x h^{1-t}$. An analytic expression of this homotopy is nothing but the KMS-condition, that is, if we consider the one paremeter automorphism group $\sigma_t(x) = h^{it}ah^{-it}$, then the $\mathcal{L}(\mathfrak{H})$-valued function $f(t) = h\sigma_t(x)$ is extended analytically to the strip, $0 \leq \mathrm{Im}z \leq 1$; and we have

$$f(t) = h\sigma_t(x) \quad \text{and} \quad f(t + i) = \sigma_t(x)h.$$

Thus, we see that the KMS-condition or the modular automorphism group measures and compensates the non-trace like behavior of φ. As a matter of fact, we have the following characterization:

THEOREM 2. <u>A σ-finite von Neumann algebra $\mathfrak{M}$ is semi-finite if and only if the modular automorphism group $\{\sigma_t^\varphi\}$ of a faithful normal positive linear functional φ on $\mathfrak{M}$ is implemented by a one parameter unitary group $\{u(t)\}$ in $\mathfrak{M}$. If the predual $\mathfrak{M}_*$ is separable, then the innerness of each individual automorphism σ_t^φ is sufficient for $\mathfrak{M}$ to be semi-finite.</u> (cf. [22] and [26]).

This result mildly indicates some connection between the algebraic structure of the von Neumann algebra $\mathfrak{M}$ in question and the behavior of the modular automorphism group.

There was another fortunate mature development in the theory of operator algebras. In 1966, G. K. Pedersen proposed a simultaneous generalization of positive linear functionals and semi-finite traces on a C^*-algebra under the terminology C^*-integrals, which was further generalized by F. Combes to the notion of weights on a C^*-algebra. (cf. [5] and [21]). It turns out that the combination of the theory of weights and the

KMS-condition is very useful in the study of the structure of von Neumann algebras.

DEFINITION 3. A weight on a von Neumann algebra $\mathfrak{M}$ is a map φ of the positive cone $\mathfrak{M}_+$ to the extended positive reals $[0,\infty]$ such that

$$\varphi(x + y) = \varphi(x) + \varphi(y), \ x, \ y \in \mathfrak{M}_+;$$

$$\varphi(\lambda x) = \lambda\varphi(x), \ \lambda \geq 0,$$

with the usual convention $0(+\infty) = 0$. A weight φ is said to be normal if $\varphi(\sup x_i) = \sup \varphi(x_i)$ for every bounded increasing net $\{x_i\}$ in $\mathfrak{M}_+$; semi-finite if $\mathfrak{n}_\varphi = \{x : \varphi(x^* x) < +\infty\}$ is σ-weakly dense in $\mathfrak{M}$; faithful if $\varphi(x) > 0$ for every non-zero $x \in \mathfrak{M}_+$.

A weight here means, however, always a faithful semi-finite normal one. Through Tomita's theory of modular Hilbert algebras, F. Combes showed, [6], that any weight φ on $\mathfrak{M}$ gives rise to a unique one parameter automorphism group $\{\sigma_t^\varphi\}$ for which φ satisfies the KMS - condition in the sense that for any pair $x,y \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ there exists a continuous bounded function F on the strip, $0 \leq \mathrm{Im}\, z \leq 1$, holomorphic inside such that $F(t) = \varphi(\sigma_t^\varphi(x)y)$ and $F(t + i) = \varphi(y\sigma_t^\varphi(x))$ and that $\varphi \circ \sigma_t^\varphi = \varphi$, where one should note that φ is extended to a linear functional, denoted by φ again, on the linear span $\mathfrak{m}_\varphi$ of $\{x \in \mathfrak{M}_+ : \varphi(x) < +\infty\}$ which agrees with $\mathfrak{n}_\varphi^* \mathfrak{n}_\varphi = \{y^* x : x, \ y \in \mathfrak{n}_\varphi\}$. Then Theorem 2 holds for weights without the restriction of σ-finiteness.

Investigating the relation between the Araki-Woods classification of ITPFI factors and the KMS-conditions, A. Connes showed in 1971 that the asymptotic ratio set $r_\infty(\mathfrak{M})$ of an ITPFI factor $\mathfrak{M}$ is indeed the intersection of the spectrum $S_p(\Delta_\varphi)$ of the all possible modular operators Δ_φ; thus introduced a new algebraic invariant, the modular spectrum:

$$S(\mathfrak{M}) = \cap\{S_p(\Delta_\varphi) : \varphi \text{ runs all weights on } \mathfrak{M}\}.$$

He and Van Daele then showed in 1972 that $S(\mathfrak{M})\backslash\{0\}$ is a closed subgroup of the multiplications group $\mathbb{R}_+^*$ if $\mathfrak{M}$ is a factor; thus a new classification of factors of type III. A factor $\mathfrak{M}$ is said to be of type III_λ, $0 < \lambda < 1$, if $S(\mathfrak{M}) = \{\lambda^n : n + \mathbb{Z}\} \cup \{0\}$; of type III_0 if $S(\mathfrak{M}) = \{0,1\}$; of type III_1 if $S(\mathfrak{M}) = \mathbb{R}_+^*$. Therefore, the factors distinguished by R. Powers were indeed factors of type III_λ, $0 < \lambda < 1$, with $\lambda = \frac{\mu}{1-\mu}$, where μ, $0 < \mu < \frac{1}{2}$, is a number defining a state ω_μ on the 2×2 matrix algebras by

$$\omega_\mu \begin{pmatrix} x_{11}, & x_{12} \\ & \\ x_{21}, & x_{22} \end{pmatrix} = \mu x_{11} + (1-\mu)x_{22}.$$

In 1971, A. Connes further proved that the Araki-Woods invariant $\rho(\mathfrak{M})$ for an ITPFI factor $\mathfrak{M}$ is given under a trivial change of scale by the modular period group:

$$T(\mathfrak{M}) = \{t \in \mathbb{R} : \sigma_t^\varphi = \iota \text{ for some weight } \varphi\},$$

and that $T(\mathfrak{M})$ is a subgroup of the additive group $\mathbb{R}$. The formula between $\rho(\mathfrak{M})$ and $T(\mathfrak{M})$ for an ITPFI factor $\mathfrak{M}$ is given by

$$\rho(\mathfrak{M}) = \{e^{t/2\pi} : t \in T(\mathfrak{M})\}.$$

By definition, $T(\mathfrak{M})$ is an algebraic invariant for a factor $\mathfrak{M}$. If $\mathfrak{M}_*$ is separable, the semi-finiteness of $\mathfrak{M}$ is equivalent to $T(\mathfrak{M}) = \mathbb{R}$.

Besides these algebraic invariants, he showed the following:

THEOREM 4. [8] If φ and ψ are weights on a von Neumann algebra $\mathfrak{M}$, then there exists a unique σ-weakly continuous one parameter family $\{u_s\}$ of unitaries in $\mathfrak{M}$ such that

$$u_{s+t} = u_s \sigma_s^{\varphi}(u_t);$$

$$\sigma_t^{\psi} = \mathrm{Ad}\, u_t \circ \sigma_t^{\varphi},\ t \in \mathbb{R};$$

for any $x \in \mathfrak{n}_{\varphi}^* \cap \mathfrak{n}_{\psi}$ and $y \in \mathfrak{n}_{\psi}^* \cap \mathfrak{n}_{\varphi}$ there is a bounded continuous function F on the strip, $0 \leq \mathrm{Im}\, z \leq 1$, holomolophic inside the strip such that

$$F(t) = \varphi(\sigma_t^{\varphi}(x)u_t y),\ F(t+i) = \psi(y\sigma_t^{\varphi}(x)u_t).$$

The construction of $\{u_t\}$ is surprisingly simple. Consider the weight χ on the tensor product $\mathcal{P} = \mathfrak{M} \otimes \mathfrak{B}_2$ of $\mathfrak{M}$ and the 2×2 matrix algebra $\mathfrak{B}_2$ defined by:

$$\chi\begin{pmatrix} x_{11}, & x_{12} \\ x_{21}, & x_{22} \end{pmatrix} = \varphi(x_{11}) + \psi(x_{22}).$$

It is then shown that

$$\sigma_t^{\chi}\begin{pmatrix} 0, & 0 \\ 1, & 0 \end{pmatrix} = \begin{pmatrix} 0, & 0 \\ u_t, & 0 \end{pmatrix}, \ t \in \mathbb{R}.$$

If $\mathfrak{M}$ is abelian, then φ and ψ are given by measures μ and ν on a Borel space Ω, and mutually absolutely continuous with respect to each other. Let $h = \frac{d\nu}{d\mu}$ be the Radon derivative of ν with respect to μ. Then $\{u_t\}$ is nothing but $\{h^{it}\}$. With this evidence, $\{u_t\}$ is called the <u>cocycle Radon-Nikodym derivative</u> of ψ with respect to φ and denoted by

$$u_t = (D\psi : D\varphi)_t, \ t \in \mathbb{R}.$$

Considering the 3×3-matrix algebra over $\mathfrak{M}$, he showed the chain rule:

$$(D\psi : D\varphi)_t = (D\psi : D\omega)_t (D\omega : D\varphi)_t, \quad t \in \mathbb{R},$$

for any three weights φ, ω and ψ.

It is clear from Connes' Radon-Nikodym theorem that

$$T(\mathfrak{M}) = \{t \in \mathbb{R} : \sigma_t^{\varphi} \in \mathrm{Int}(\mathfrak{M})\},$$

where $\mathrm{Int}(\mathfrak{M})$ denotes the group of inner automorphisms; hence

$T(\mathfrak{M})$ is a subgroup of $\mathbb{R}$. He then showed that for any fixed weight φ on a factor $\mathfrak{M}$,

$$S(\mathfrak{M}) = \bigcap S_p(\Delta_{\varphi_e}),$$

where e runs over the central projections of the fixed point subalgebra $\mathfrak{M}_\varphi$ of $\mathfrak{M}$ under σ^φ and φ_e means the restriction of φ to $e\mathfrak{M}e$. We call $\mathfrak{M}_\varphi$ the <u>centralizer</u> of φ.

In order to get some idea about the structure of a factor of type III, let us consider a very special case. Suppose that a factor $\mathfrak{M}$ admits a faithful normal state φ such that $\mathfrak{M}_\varphi$ is a factor and $\sigma_t^\varphi = \iota$ for some $T > 0$. The smallest such $T > 0$ is called the <u>period</u> of φ. Connes proved, however, that every factor of type III_λ, $0 < \lambda < 1$, with separable predual admits such a state with $T = -2\pi/\log \lambda$. [8]). Let $\lambda = e^{-2\pi/T}$ and

$$\mathfrak{M}_n = \{x \in \mathfrak{M} : \sigma_t^\varphi(x) = \lambda^{int} x\}.$$

Of course, $\mathfrak{M}_0 = \mathfrak{M}_\varphi$. Clearly, we have

$$\mathfrak{M}_n \mathfrak{M}_m \subset \mathfrak{M}_{n+m}, \ \mathfrak{M}_n^* = \mathfrak{M}_{-n}, \ n, m \in \mathbb{Z}.$$

Hence each $\mathfrak{M}_n$ is a two-sided module over $\mathfrak{M}_0$. It is not hard to see that $\mathfrak{M}_1 \neq \{0\}$. Let $a = uh$ be the polar decomposition of an $a \in \mathfrak{M}_1$. Then we have $u^*u = e \in \mathfrak{M}_0$ and $uu^* = f \in \mathfrak{M}_0$ and

$$\varphi(uxu^*) = \lambda\varphi(xu^*u), \ x \in \mathfrak{M},$$

by the KMS-condition. Consider the tensor product $\overline{\mathfrak{M}} = \mathfrak{M} \otimes \mathcal{B}$, $\bar{\varphi} = \varphi \otimes \mathrm{Tr}$, $\bar{u} = u \otimes 1$, $\bar{e} = e \otimes 1$ and $\bar{f} = f \otimes 1$ where $\mathcal{B}$ denotes a factor of type I_∞. The centralizer $\overline{\mathfrak{M}}_{\bar{\varphi}}$ of $\bar{\varphi}$ is $\mathfrak{M}_\varphi \otimes \mathcal{B}$, and hence infinite. The projections $\bar{e}$ and $\bar{f}$ are both infinite in $\overline{\mathfrak{M}}_{\bar{\varphi}}$, so that there are partial isometries v and w in $\overline{\mathfrak{M}}_{\bar{\varphi}}$ such that $v^*v = \bar{e}$, $vv^* = 1$, $ww^* = \bar{f}$ and $w^*w = 1$. Put

$$U = v\bar{u}w.$$

It follows that U is a unitary in $\overline{\mathfrak{M}}$ such that $\sigma_t^{\bar{\varphi}}(U) = \lambda^{it}U$ and $U\,\overline{\mathfrak{M}}_0\,U^* = \overline{\mathfrak{M}}_0$. Thus $\mathrm{Ad}\,U$ gives rise to an automorphism θ of $\overline{\mathfrak{M}}_0$ and

$$\bar{\varphi} \circ \theta(x) = \lambda\bar{\varphi}(x), \quad x \in \overline{\mathfrak{M}}_0.$$

It is then easily shown that $\overline{\mathfrak{M}} \cong W^*(\overline{\mathfrak{M}}_0, \mathbb{Z}, \theta)$. Hence in this case, $\mathfrak{M}$ is isomorphic to the crossed product of a factor $\overline{\mathfrak{M}}_0$ by a single automorphism θ multiplying the trace by λ. The existence of such an automorphism implies that $\overline{\mathfrak{M}}_0$ must be of type II_∞. Apart from the uniqueness, this is, in essence, the decomposition theorem for some factors of type III, at some earlier stage of the development in the structures theory in 1972. (cf. [2], [8] and [28]). The uniqueness requires similar Fourier analysis of the cocycle Radon-Nikodym derivatives $(D\psi : D\varphi)$. Instead of doing this, we will, however, go on to the general case.

Let $\mathcal{B}$ denote the algebra $\mathfrak{L}(L^2(\mathbb{R}))$ of all bounded operators on the Hilbert space $L^2(\mathbb{R})$ of square integrable functions on the real line $\mathbb{R}$ with respect to the Lebesgue measure. We define then one parameter unitary groups $\{U(t)\}$ and $\{V(s)\}$ in $\mathcal{B}$ by the following:

$$U(t)\,\xi(s) = \xi(s + t);$$

$$V(t)\,\xi(s) = e^{ist}\,\xi(s),\ \xi \in L^2(\mathbb{R}),\ s,\ t \in \mathbb{R}.$$

It follows then that

$$U(s)\,V(t)\,U(s)^*\,V(t)^* = e^{ist}\,1,\ s,\ t \in \mathbb{R}.$$

Hence the one parameter automorphism groups $\{\mathrm{Ad}\,U(s)\}$ and $\{\mathrm{Ad}\,V(t)\}$ of $\mathcal{B}$ commute. Now, let $\mathfrak{M}$ be a properly infinite von Neumann algebra. It is easily seen almost by definition that $\mathfrak{M} \cong \mathfrak{M} \otimes \mathcal{B}$. For a weight φ on $\mathfrak{M}$, we consider the one parameter automorphism groups $\{\sigma_t\}$ and $\{\theta_t\}$ of $\mathfrak{M} \otimes \mathcal{B}$ given by:

$$\begin{cases} \sigma_t = \sigma_t^\varphi \otimes \mathrm{Ad}\,U(t),\ t \in \mathbb{R}; \\ \theta_t = \iota \otimes \mathrm{Ad}\,V(t). \end{cases}$$

Clearly $\{\sigma_t\}$ and $\{\theta_s\}$ commute, so that $\{\theta_s\}$ gives rise to a one parameter automorphism group, denoted by $\{\theta_s\}$ again, of the fixed point algebra $\mathfrak{N}$ of $\{\sigma_t\}$. It is not hard to see that $\mathfrak{N}$ is generated by $1 \otimes U(t)$ and the operators:

$$\pi^\varphi(x)\,\xi(t) = \sigma_{-t}^\varphi(x)\,\xi(t),\ t \in \mathbb{R},\ x \in \mathfrak{M},\ \xi \in L^2(\mathfrak{H}; \mathbb{R})$$

where $\mathfrak{H}$ denotes a Hilbert space on which $\mathfrak{M}$ acts; hence $\mathfrak{N} \cong W^*(\mathfrak{M}, \mathbb{R}, \sigma^{\varphi})$. We have the following:

LEMMA 5. The von Neumann algebra $\mathfrak{N}$ admits a faithful semi-finite normal trace τ such that $\tau \circ \theta_s = e^{-s}\tau$, $s \in \mathbb{R}$. The von Neumann algebra $\mathfrak{M} \otimes \mathfrak{B}$, hence $\mathfrak{M}$, is isomorphic to the crossed product $W^*(\mathfrak{N}, \mathbb{R}, \theta)$ of $\mathfrak{N}$ by $\mathbb{R}$ with respect to the action θ of $\mathbb{R}$.

THEOREM 6. [29]. If $\mathfrak{N}$ is a von Neumann algebra equipped with a one parameter automorphism group $\{\theta_s\}$ and a faithful semi-finite normal trace τ such that $\tau \circ \theta_s = e^{-s}\tau$, then (i) the crossed product $\mathfrak{M} = W^*(\mathfrak{N}, \mathbb{R}, \theta)$ is properly infinite; (ii) the center $C_{\mathfrak{M}}$ of $\mathfrak{M}$ is precisely the fixed point subalgebra $C_{\mathfrak{N}}^{\theta}$ of the center $C_{\mathfrak{N}}$ of $\mathfrak{N}$; hence $\mathfrak{M}$ is a factor if and only if θ is ergodic on the center $C_{\mathfrak{N}}$ of $\mathfrak{N}$; (iii) $\mathfrak{M}$ is semi-finite if and only if $C_{\mathfrak{N}}$ contains a continuous one paremeter unitary group $\{v(t)\}$ such that $\theta_s(v(t)) = e^{ist} v(t)$, s, $t \in \mathbb{R}$; (iv) if $\mathfrak{M}$ is of type III, then $\mathfrak{N}$ is of type II_{∞} and $\{C_{\mathfrak{N}}, \theta\}$ has no direct summand isomorphic to a multiple of $L^{\infty}(\mathbb{R})$ equipped with the translation; (v) if $\{C_{\mathfrak{N}}, \theta\}$ is ergodic, then

$$S(\mathfrak{M}) = \{e^t : \theta_t |_{C_{\mathfrak{N}}} = \iota\} \cup \{0\};$$

$$T(\mathfrak{M}) = \{t \in \mathbb{R};\ \text{there exists a unitary } v \in C_{\mathfrak{N}} \text{ with } \theta_s(v) = e^{ist}v,\ s \in \mathbb{R}\}.$$

As a direct consequence of Theorem 4, we have the following:

LEMMA 7. Suppose that $\{\mathfrak{N}_1, \theta^1\}$ and $\{\mathfrak{N}_2, \theta^2\}$ are properly infinite von Neumann algebras equipped with one parameter automorphism groups and faithful semi-finite traces τ_1 and τ_2 respectively such that $\tau_1 \circ \theta_s^1 = e^{-s}\tau_1$ and $\tau_2 \circ \theta_s^2 = e^{-s}\tau_2$, $s \in \mathbb{R}$. Then the following two statements are equivalent:

(i) $W^*(\mathfrak{n}_1, \mathbb{R}, \theta^1) \cong W^*(\mathfrak{n}_2, \mathbb{R}, \theta^2)$;

(ii) There exist an isomorphism π of $\mathfrak{n}_1$ onto $\mathfrak{n}_2$ and a continuous one parameter family $\{v_s\}$ in $\mathfrak{n}_1$ such that

$$v_{s+t} = v_s \ \theta_s^1(v_t); \ s, t \in \mathbb{R};$$

$$\theta_s^2 = \pi \circ \mathrm{Ad}(v_s) \circ \theta_s^1 \circ \pi^{-1}.$$

However, the next result, together with the above lemma, yields the uniqueness criteria in Theorem 1:

THEOREM 8. [13]. (i) If θ is an automorphism of $\mathfrak{n}$ such that $\tau \circ \theta \leq \lambda\tau$ for some $\lambda > 0$ and a faithful semi-finite normal trace τ. Then every unitary $u \in \mathfrak{n}$ is of the form $v^*\theta(v)$ for some unitary $v \in \mathfrak{n}$;

(ii) If $\{\theta_s\}$ is a one parameter automorphism group of $\mathfrak{n}$ such that $\tau \circ \theta_s = \lambda^s\tau$ for some $\lambda \neq 1$ and a faithful semi-finite normal trace τ, then for every θ-one cocycle $\{u_s\}$, that is, a continuous one parameter family of unitaries in $\mathfrak{n}$ with $u_{s+t} = u_s \ \theta_s(u_t)$, there exists a unitary $v \in \mathfrak{n}$ such that $u_s = v^*\theta_s(v)$, $s \in \mathbb{R}$.

Here a natural question is how this structure theorem and the discrete decomposition described above are related. The answer is quite simple: If $\mathfrak{M}$ is of type III_λ, $0 < \lambda < 1$, then

$\{C_{\mathfrak{n}}, \theta\}$ is periodic with the period $T = -2\pi/\mathrm{Log}\lambda$. Hence, considering the central decomposition

$$\mathfrak{n} = \int_{\Gamma}^{\oplus} \mathfrak{n}(\gamma) d\mu(\gamma),$$

θ_T induces an antomorphism of each fibre algebra $\mathfrak{n}(\gamma)$. The covariant systems $\{\mathfrak{n}(\gamma), \theta_T\}$ are equivalent to that appearing in the above discrete decomposition. In the type III_0 case, A. Connes proved the following:

THEOREM 9. [8]. If $\mathfrak{n}$ is a von Neumann algebra with non-atomic center and equipped with an automorphism θ and a faithful semi-finite normal trace τ such that $\tau \circ \theta \leq \lambda\tau$ for some $0 < \lambda < 1$ and θ is ergodic on the center $C_{\mathfrak{n}}$ of $\mathfrak{n}$, then the crossed product $\mathfrak{M} = W^*(\mathfrak{n}, \theta)$ of $\mathfrak{n}$ by θ is a factor of type III_0. Every factor of type III_0 is of this form for some $\{\mathfrak{n}, \theta\}$.

The uniqueness criteria for factors of type III_0 requires more preparations; so we omit the detail. But he did give the uniqueness of this decomposition within some equivalence.

Once again examining the way the II_∞-von Neumann algebra $\mathfrak{n}$ was constructed, one realizes that the algebra $\mathfrak{n}$ is the centralizer of the weight $\bar{\omega} = \varphi \otimes \omega$ on $\mathfrak{M} \otimes \mathfrak{B}$ where the weight ω on $\mathfrak{B}$ is given by

$$\omega(x) = \mathrm{Tr}(hx), \ x \in \mathfrak{B}_+;$$

$$h = \exp(\frac{d}{idt}),$$

i.e. $(D\omega : DTr)_t = U(t)$, $t \in \mathbb{R}$. A. Connes proved indeed, in the course of proving the converse of the cocycle Radon-Nikodym theorem, that for any one-cocycle $\{u_t\}$ in $\mathfrak{M}$, there exists a unitary $v \in \mathfrak{M} \otimes \mathfrak{B}$ such that

$$u_t \otimes U(t) = v^* \sigma_t(v), \ t \in \mathbb{R}.$$

In other words, for any weights φ and ψ on $\mathfrak{M}$, $\psi \otimes \omega$ and $\varphi \otimes \omega$ are conjugate under the inner automorphism group $\mathrm{Int}(\mathfrak{M} \otimes \mathfrak{B})$. This means then that on a properly infinite von Neumann algebra there is a unique class of weights which describes the structure of the algebra. The weights of this class is characterized by the following:

THEOREM 10. [13]. _Let $\mathfrak{M}$ be an infinite factor with separable predual. For a weight $\bar{\omega}$ on $\mathfrak{M}$ with properly infinite centralizer, the following two conditions are equivalent:_

(i) _For any $\lambda > 0$, there exists a unitary $u \in \mathfrak{M}$ such that_ $\lambda\bar{\omega}(x) = \bar{\omega}(uxu^*)$, $x \in \mathfrak{M}_+$;

(ii) _For any weight φ on $\mathfrak{M}$, there exists an isomorphism π of $\mathfrak{M}$ onto $\mathfrak{M} \otimes \mathfrak{B}$ such that_

$$\bar{\omega}(x) = (\varphi \otimes \omega) \circ \pi(x), \ x \in \mathfrak{M}_+,$$

where ω is the weight on $\mathfrak{B}$ defined above.

DEFINITION 11. [13]. The weight $\bar{\omega}$ satisfying the condition in the above theorem is called _dominant_.

In other words, a dominant weight is characterized as one fixed, within unitary equivalence, under the multiplication by positive scalars.

Let $\mathfrak{W}_{\mathfrak{M}}$ denote the space of all weights on $\mathfrak{M}$ and $\mathfrak{W}_{\mathfrak{M}}^{0}$ the space of all weights with properly infinite centralizer. For a pair φ, ψ of weights on $\mathfrak{M}$, we write $\varphi \prec \psi$ if there exists an isometry $u \in \mathfrak{M}$ with $uu^{*} \in \mathfrak{M}_{\psi}$ such that $\varphi(x) = \psi(uxu^{*})$, $x \in \mathfrak{M}_{+}$. If the above u is unitary, then we write $\varphi \sim \psi$. We see then that "$\sim$" is equivalence relation associated with the partial ordering "$\prec$". The space $\mathfrak{W}_{\mathfrak{M}}^{0}/$"$\sim$" is then a σ-complete Boolean lattice which is isomorphic to the lattice of all σ-finite projections of a unique abelian von Neumann algebra $\underset{\sim}{\mathcal{P}}(\mathfrak{M})$. For each $\varphi \in \mathfrak{W}_{\mathfrak{M}}^{0}$, there corresponds a unique projection $p(\varphi)$ of $\underset{\sim}{\mathcal{P}}(\mathfrak{M})$ such that

$$\varphi \prec \psi \Longleftrightarrow p(\varphi) \leq p(\psi).$$

Since the multiplication by a positive scalar preserves the ordering, to each $\lambda > 0$ there corresponds a unique automorphism $\mathcal{F}_{\lambda}^{\mathfrak{M}}$ of $\underset{\sim}{\mathcal{P}}(\mathfrak{M})$ such that

$$\mathcal{F}_{\lambda}^{\mathfrak{M}} p(\varphi) = p(\lambda\varphi), \quad \varphi \in \mathfrak{W}_{\mathfrak{M}}^{0}.$$

We call $\{\underset{\sim}{\mathcal{P}}(\mathfrak{M}), p, \mathcal{F}_{\lambda}\}$ the <u>global flow of weights</u>. Theorem 10 means then that there exists the only one σ-finite projection $d \in \underset{\sim}{\mathcal{P}}(\mathfrak{M})$ invariant under $\mathcal{F}_{\lambda}^{\mathfrak{M}}$, which is given by $d = p(\bar{\omega})$. Putting $p(\varphi) = p(\varphi \otimes \mathrm{Tr})$ for the general $\varphi \in \mathfrak{W}_{\mathfrak{M}}$, we have the following:

THEOREM 12. [13]. Let $\mathfrak{M}$ be an infinite factor with separable predual. For any $\varphi \in \mathfrak{W}_{\mathfrak{M}}$, the following conditions are equivalent:

i) $\varphi < \tilde{\omega}$;

ii) The map: $\lambda \in \mathbb{R}_+^* \to \mathcal{F}_\lambda p(\varphi) \in \underset{\sim}{\mathcal{P}}(\mathfrak{M})$ is σ-strongly continuous;

iii) The integral $\int_{-\infty}^{\infty} \sigma_t^\varphi(x)dt = E_\varphi(x)$, $x \in \mathfrak{M}_+$, exists for σ-weakly dense x's in $\mathfrak{M}_+$.

DEFINITION 13. A weight φ is said to be integrable if φ satisfies any of the above conditions.

Therefore, $(\underset{\sim}{\mathcal{P}}_{\mathfrak{M}})_d$ is the continuous part of the flow $\mathcal{F}_\lambda^{\mathfrak{M}}$. The restriction of $\{\mathcal{F}_\lambda^{\mathfrak{M}}\}$ to $(\underset{\sim}{\mathcal{P}}_{\mathfrak{M}})_d = \mathcal{P}_{\mathfrak{M}}$ is called the smooth flow of weights on $\mathfrak{M}$, and denoted by $\{F_\lambda^{\mathfrak{M}}\}$. Since there is no non-trivial invariant projection properly majorized by d, the smooth flow of weights is ergodic. By construction, the association: $\mathfrak{M} \rightsquigarrow F^{\mathfrak{M}}$ of the smooth flow of weights to each infinite factor $\mathfrak{M}$ is a functor. The relation between this function $F^{\mathfrak{M}}$ and the structure theorem, Theorem 6, is described as follows:

THEOREM 14. [13]. Let $\mathfrak{M}$ be an infinite factor with separable predual and $\{\mathfrak{N}, \theta\}$ be the coveriant system over $\mathbb{R}$ in Theorem 6 such that $\mathfrak{M} \cong W^*(\mathfrak{N}, \mathbb{R}, \theta)$.

(i) $\{C_{\mathfrak{N}}, \theta_{-\mathrm{Log}\lambda}\} \cong \{\mathcal{P}(\mathfrak{M}), F_\lambda^{\mathfrak{M}}\}$;

(ii) $S(\mathfrak{M})\setminus\{0\} = \{\lambda \in \mathbb{R}_+^* : F_\lambda^{\mathfrak{M}} = \iota\}$.

Therefore, the algebraic invariant $S(\mathfrak{M})$, the modular spectrum, of $\mathfrak{M}$ is essentially the kernel of the smooth flow

$F^{\mathfrak{M}}$ of weights. One should note here that the smooth flow $F^{\mathfrak{M}}$ of weights is defined directly, hence functionally, from $\mathfrak{M}$. We then determine this flow for a factor given by the so-called group measure space construction.

Let $\mathcal{A}$ be an abelian von Neumann algebra with separable predual equipped with a continuous action α of a separable locally compact group G. This is equivalent to having a standard measure space $\{\Gamma,\mu\}$ equipped with a Borel action of G, and $\mathcal{A} = L^\infty(\Gamma,\mu)$, $\alpha_g(a)(\gamma) = a(g^{-1}\gamma)$, $a \in \mathcal{A}$, $g \in G$, $\gamma \in \Gamma$. For simplicity, we assume that the action of G is free in the sense that $N_g = \{\gamma : g\gamma = \gamma\}$ is a null set for every $g \neq e$, although this restriction is not necessary, cf [17]. Let $\mathfrak{M} = W^*(\mathcal{A},G,\alpha)$. If the action of G is ergodic, then $\mathfrak{M}$ is a factor. We have then the following:

(i) $\mathfrak{M}$ is of type I $\iff$ The action of G on Γ is transitive;

(ii) $\mathfrak{M}$ is of type II_1 $\iff$ The action of G on Γ is not transitive, and admits a finite invariant measure;

(iii) $\mathfrak{M}$ is of type II_∞ $\iff$ The action is not transitive and admits an infinite invariant measure;

(iv) $\mathfrak{M}$ is of type III $\iff$ The action does not admit any invariant measure,

where the measures here are absolutely continuous with respect to the original measure μ. Let ρ be a positive Borel function on $G \times \Gamma$ such that

$$\int f(g\gamma)\rho(g,\gamma)d\mu(\gamma) = \int f(\gamma)d\mu(\gamma);$$

$$\rho(g_1g_2,\gamma) = \rho(g_1,g_2\gamma)\rho(g_2,\gamma),$$

namely $\rho(g,\cdot) = \frac{d\mu\circ g}{d\mu}(\gamma)$. Consider the product measure space $\Gamma \times \mathbb{R}_+^*$, where $\mathbb{R}_+^*$ is equipped with the Lebesgue measure m. By setting

$$\begin{cases} T_g(\gamma,\lambda) = (g\gamma, \rho(g,\gamma)\lambda), \ \gamma \in \Gamma, \ \lambda > 0; \\ \phi_\mu(\gamma,\lambda) = (\gamma, \lambda\mu), \end{cases}$$

G and $\mathbb{R}_+^*$ act on $\Gamma \times \mathbb{R}_+^*$ and commute. Hence we get a abelian von Neumann algebra $L^\infty(\Gamma \times \mathbb{R}_+^*, \mu \times m)$ on which G and $\mathbb{R}_+^*$ act.

THEOREM 15. [13]. <u>In the above situation, the smooth flow $F^{\mathfrak{M}}$ of weights on $\mathfrak{M}$ is isomorphic to the action of $\mathbb{R}_+^*$ on the fixed point subalgebra $L^\infty(\Gamma \times \mathbb{R}_+^*)^G$ induced naturally by $\{\Phi_\lambda\}$.</u>

This construction is known as the <u>Anzai skew product</u>, or the closure of the range of the module ρ by G. W. Mackey [18]. A recent result of W. Krieger, [17], can be interpreted in the following way:

THEOREM 16. [17]. <u>In the same situation as above, if G is the additive integer group $\mathbb{Z}$, or equivalently if the action is given by a single ergodic transformation, then the smooth flow $F^{\mathfrak{M}}$ of weights on $\mathfrak{M}$ is a complete invariant for the algebraic structure of $\mathfrak{M}$.</u>

Thus, we have the following equivalence in different problems:

"The weak equivalence classification of the ergodic transformations"

~ "The classification of the factors given by the group measure space construction from an ergodic transformation"

~ "The conjugacy classification of the ergodic flows".

The weak equivalence classification of ergodic transformation groups was first introduced by H. A. Dye [15], and he proved in fact that all countable abelian ergodic transformation groups with finite invariant measure are weakly equivalent and give rise to hyperfinite II_1-factors. This classification was later reformulated by G. W. Mackey as the isomorphism classification of virtual subgroups. The relation between the weak equivalence classification of ergodic transformation groups and the isomorphism classification of the associated factors has been puzzled since Dye's work. In fact, H. Choda showed that if an isomorphism of the two factors associated with ergodic transformation groups preserves the maximal abelian subalgebras canonically attached to the constructions, then the groups are indeed weakly equivalent. [4].

The conjugacy classification of ergodic transformations and flows is, of course, one of the central problems in ergodic theory. Apparently, the weak equivalence classification looks much coarser than the conjugacy classification. But the above mentioned fact says that they are indeed the same problem.

Unlike the discrete crossed product, the relative commutant of the original algebra in the crossed product behaves mysteriously in general. We do have, however, the following:

THEOREM 17. [13]. <u>If</u> φ <u>is an integrable weight on a factor</u> $\mathfrak{M}$ <u>with separable predual, then the relative commutant</u> $\mathfrak{M}'_\varphi \cap \mathfrak{M}$ <u>of the centeralizer</u> $\mathfrak{M}_\varphi$ <u>of</u> φ <u>is contained in</u> $\mathfrak{M}_\varphi$ <u>as the center</u> C_φ.

This result, together with the construction of antomorphisms similar to that of [25], enables us to prove the following:

THEOREM 18. [13]. Let φ be an integrable weight on a factor $\mathfrak{M}$ with separable predual. There exists an isomorphism $\overline{\sigma}^{\varphi}$ of the multiplicative group $Z^1(F^{\mathfrak{M}})$ of unitary one co-cycles of the smooth flow of weights on $\mathfrak{M}$ onto the group of all automorphisms leaving the centeralizer $\mathfrak{M}_{\varphi}$ elementwise fixed, such that $\overline{\sigma}^{\varphi}_{\overline{t}} = \sigma^{\varphi}_t$, where $\overline{t} \in Z^1(F^{\mathfrak{M}})$ means the cocycle given $\overline{t}(\lambda) = \lambda^{it}$, $\lambda > 0$, and $\overline{\sigma}^{\varphi}_c$, $c \in Z^1(F^{\mathfrak{M}})$, is inner if and only if c is cobundar, i.e. there exists a unitary $v \in \mathcal{C}(\mathfrak{M})$ such that $c_\lambda = v^* F^{\mathfrak{M}}_\lambda(v)$, $\lambda > 0$.

Therefore, this extended modular automorphism group $\{\overline{\sigma}^{\varphi}_c : c \in Z^1(F^{\mathfrak{M}})\}$ can be viewed as the Galois group of $\mathfrak{M}$ relative to $\mathfrak{M}_{\varphi}$. Furthermore, the co-cycle Radon-Nikodym derivative $\{(D\psi : D\varphi)_t : t \in \mathbb{R}\}$ is extended to $\{(D\psi : D\varphi)_c : c \in Z^1(F^{\mathfrak{M}})\}$, which behaves in the obvious way with respect to $\{\overline{\sigma}^{\varphi}_c\}$ and $\{\overline{\sigma}^{\psi}_c\}$. Hence there exists an isomorphism $\overline{\delta}_{\mathfrak{M}}$ independent of φ, of $H^1(F^{\mathfrak{M}}) = Z^1(F^{\mathfrak{M}})/B^1(F^{\mathfrak{M}})$ into $\mathrm{Out}(\mathfrak{M}) = \mathrm{Aut}(\mathfrak{M})/\mathrm{Iut}(\mathfrak{M})$. Fixing the decomposition $\mathfrak{M} = W^*(\mathfrak{N}, \mathbb{R}, \theta)$ in Theorem 6, we can obtain an exact sequence:

$$\{1\} \to H^1(F^{\mathfrak{M}}) \xrightarrow{\overline{\delta}_{\mathfrak{M}}} \mathrm{Out}(\mathfrak{M}) \xrightarrow{\overline{\gamma}} \mathrm{Out}_{\tau,\theta}(\mathfrak{N}) \to \{1\},$$

where $\mathrm{Out}_{\tau,\theta}(\mathfrak{N}) = \{\varepsilon(\alpha) : \alpha \in \mathrm{Aut}(\mathfrak{N}),\ \tau \circ \alpha = \tau,\ \alpha\theta_s = \theta_s\alpha\}$ and ε means the canonical homomorphism of $\mathrm{Aut}(\mathfrak{N})$ onto $\mathrm{Out}(\mathfrak{N})/\mathrm{Iut}(\mathfrak{N})$.

We should note here that the extended modular automorphism $\overline{\sigma}^{\varphi}_c$ is, in some sense, "functional calculus" of the "generator" of the modular automorphism group $\{\sigma^{\varphi}_t\}$. The evidence for this is the following: If $\mathfrak{M}$ is a semi-finite factor then $F^{\mathfrak{M}}$ is isomorphic to $L^{\infty}(\mathbb{R}^*)$ with translations; hence every $c \in Z^1(F^{\mathfrak{M}})$ is of the form $c_\lambda = f\, F^{\mathfrak{M}}_\lambda(f^*)$, $f \in L^{\infty}(\mathbb{R}^*)$ and if $\varphi = \mathrm{Tr}(h\cdot)$, then

$\bar{\sigma}^{\varphi}_{c} = \mathrm{Ad}(f(h))$.

The smooth flow $F^{\mathfrak{M}}$ of weights being a functor, each $\alpha \in \mathrm{Aut}(\mathfrak{M})$ gives rise to an automorphism $\mathrm{mod}(\alpha)$ of the flow $F^{\mathfrak{M}}$ by

$$\mathrm{mod}(\alpha)p(\varphi) = p(\varphi \circ \alpha^{-1}),$$

which corresponds in the semi-finite case to the translation by λ determined by $\tau \circ \alpha = \lambda\tau$. Thus we call mod the <u>fundamental homomorphism</u> after Murray and von Neumann. We leave the detail to the original paper [13].

After all, the problem in understanding the structure of von Neumann algebras is reduced to the von Neumann algebras of type II_1 and type II_∞. Here, A. Connes has been making some substantial progress especially in the analysis of automorphism groups. cf [9] and [10]. The author believes that we will be able to understand much better the structure of von Neumann algebras in the near future.

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