D. J. RUDOLPH

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What follows here is a sketch of the basic ideas involved in representing an *ergodic, measurable, measure preserving* flow as a flow built under a two valued function and an application of this result to a problem due to Sinai in the theory of k-flows. I will write the flow as a one-parameter group $\{T_t\}_{t\in\mathbb{R}}$, where each T_t is a measure preserving transformation of a probability space.

Rather than discuss the meaning, for flows, of the underlined words above, let me simply say that, from a theorem due to Ambrose [2] , such a flow can be represented as a flow built under a function. What this means is indicated schematically in fig. 1 .

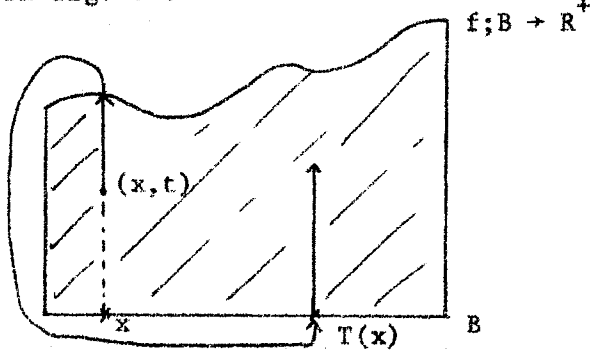

figure 1

B is a measure space with finite measure m . T is an invertible m-preserving, ergodic map of B to itself. f is an m-measurable function, bounded away from zero, of B to $\mathbb{R}^+$ with $\int_B f\, dm = 1$. The flow moves a point up the set lying below the graph of f , at unit speed, as the arrows indicate, and returns to B by the map T . The measurable sets in the picture are those in the completed product algebra of the base with Lebesgue measurable sets on the fibers.

The representation which is constructed here is one possible direction in which this picture could be simplified. The picture indicates there are two possible ways to go in such a simplification. One would be to try to select a nice T for the base map. This is equivalent to finding a nice representative in the Kakutani equivalence class of T. The nature of Kakutani equivalence classes still remains a very difficult problem. A more tractable direction to go would be to try to make the function f has as nice a form as possible. Techniques already exist in this direction, see E. Eberlein's paper [4], and if one could choose a suitably simple f, then one could, in some sense, reduce the problem of ergodic flows to that of ergodic transformations.

The simplest possibly for f would be a constant function. This could not possibly be general as such can be characterized as those for which some term T_α is non ergodic. The next best would be f taking on only two values p and q where $p/_q$ is irrational. Such a representation has some chance of being general as Ornstein has shown that a Bernoulli flow has such a form [6].

The first part of the representation is that indeed any measurable, measure-preserving ergodic flow can be represented as a flow built under a two valued step function. The values p and q need only be irrationally related. The following is a short sketch of that argument.

As the flow is ergodic, the base transformation T is ergodic. Applying Rohlin's theorem, we can break most of the base up into disjoint sets $B_0, B_1, B_2 \ldots B_N$ for some chosen N, with $T(B_i) = B_{i+1}$, $i < N$.

Filling in between B_i and B_{i+1} the sets of the space lying above each B_i, we get a picture of most of the flow as a very tall thin flow under a function. This picture can now be cut up however we please into pieces of lengths p and q and unstacked (see fig. 2).

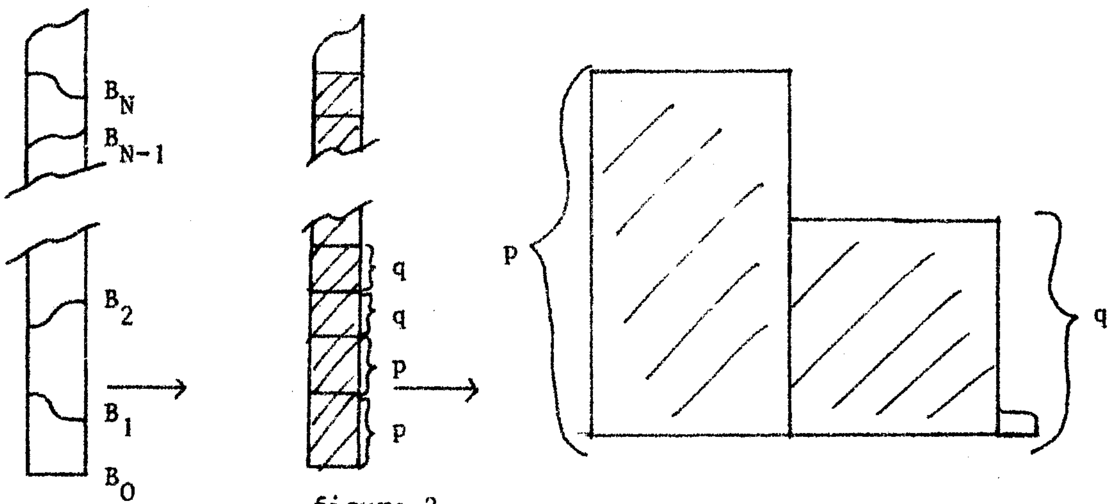

figure 2

We now must see how to eliminate, in a convergent manner, the small bad set. The idea is to build a much taller picture, with much smaller left over. In such a tower the sections already labeled will appear as bands across the tower with unlabeled sections in between. Focus on the lowest unlabeled section. If the block already labeled is long enough, ie the N above large enough, by dipping only a tiny percent into the labeled block below, we can relabel a segment with a string of p's and q's and get very near the bottom of the next good block. Do so, and then slide this next block down the little space and move up to the next gap (see fig. 3).

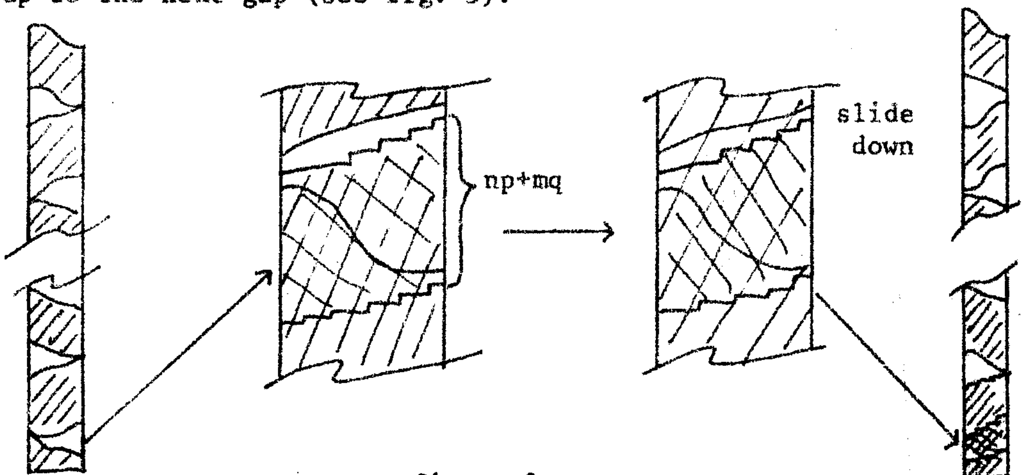

figure 3

When the top of the tower is reached, by making a measure theoretically small change we have decreased the size of the unlabeled section and increased the length of the labeled section. Continue this process so that the size of the changes sum and the result will follow.

To motivate the next step of the construction, consider the following. Break the space into two sets, P lying below the function value p, and Q lying below the function value q. Now consider the process generated by this partition. A name in the process has a very simple form. Some ergodic process gives us the letter P or Q, if P, then the process prints out a P for time p, if a Q, then the process prints out Q for time q, and then we get another letter from the base process.

The difficulty our next step overcomes is that the process described above is not necessarily isomorphic to the original flow. That it be isomorphic is to asking that (P,Q) be a generating partition for the flow, which, because of the form of the partition, is equivalent to asking that the base sets $\bar{P}, \bar{Q}$ which lie below P and Q generate for the base transformation.

From entropy theory, this requires $H(T) \leq H(\bar{Q},\bar{P})$, $= H(\bar{Q}\,\bar{P})$ iff T acts independantly on $(\bar{Q},\bar{P})$. From a result due to Abramow [1] we know

$$H(T) = \frac{H(T_t)}{t}\ m(B) = \frac{H(T_1)\ m(B)}{pm(\bar{P})+qm(\bar{Q})}\ .$$

Hence we must have $(\bar{Q},\bar{P})$ dividing B nearly enough in half, $H(T_1)$ finite, and p and q large enough to allow T to have a two set generator with the distribution of $(\bar{P},\bar{Q})$.

The second part of the representation says, modulo these restrictions $\bar{P}$ and $\bar{Q}$ can be modified to $\hat{P}$, $\hat{Q}$ so that they generate for T, and the original flow is still isomorphic to the flow built over T with the function $p\chi_{\hat{P}} + q\chi_{\hat{Q}}$. Again I will indicate the idea at the core of this modification. The reason $\bar{P},\bar{Q}$ may not generate for T is that it may fail to separate points, ie different points of B may have the same $\bar{P}$, $\bar{Q}$ name under T. Suppose we consider two such points. If , as one would do in general, we took any modification of some finite block of one of the names, in order to make them differ, and then put the two valued function over it, we might lose the isomorphism. If the modification altered the number of $\bar{P}$'s and $\bar{Q}$'s in the block, this would alter the block's flow length, and the change would propagate down the entire name, a non measurable map. But if the modification was simply a permutation of the $\bar{P}$'s and $\bar{Q}$'s of one of the names, then the flow isomorphism would be preserved. Thus the problem becomes one of building Rohlin towers, and modifying $\bar{P}$, $\bar{Q}$ names by permutations of blocks in order to separate points. The conditions on the size of P and Q and the entropy of T give bounds on how many names we need and how many are available.

I want to state one other fact about this representation. The partition P, Q is a generator for the transformation T_{t_o}, if T_{t_o} is ergodic and t_o is small enough. That is to say

$$\bigvee_{t=-\infty}^{\infty} T_t\,(P,Q) = \bigvee_{n=-\infty}^{\infty} T_{t_o}^n\,(P,Q)\ .$$

What can be verified is the stronger one-sided result ;

$$(*)\qquad \bigvee_{t=0}^{-\infty} T_t\,(P,Q) = \bigvee_{n=0}^{-\infty} T_{t_o}^n\,(P,Q)\ .$$

What this says is that reading off the past of a point at intervals t_o apart allows one to reconstruct its continuous past. This is a nice standard argument which I leave to the reader, only requiring the existence of strings of P's and Q's of all lengths sufficiently large.

It is (*) that we can now apply to the following problem. One says a transformation T is a K-automorphism if for a generating partition P , $\bigcap_{N=0}^{-\infty} \bigvee_{n=N}^{-\infty} T^n(P)$ is trivial, ie the tail field of the process is trivial. For a transformation, this implies the tail field of any partition is trivial. The corresponding definition of a K-flow is that $\bigcap_{N=0}^{-\infty} \bigvee_{t=N}^{-\infty} T_t(P)$ is trivial for *some* generating partition P . A difficulty arises here in that this does not force all tails to be trivial, in fact it is easy to make a partition for which the tail is everything. Hence, the question, if some T_{t_o} is K , is the flow K ? First, note that if any T_{t_o} is K , then all are. Pick t_o small enough, and (*) gives

$$\bigvee_{t=-Nt_o}^{-\infty} T_t(P,Q) = \bigvee_{n=-N}^{-\infty} T_{t_o}^n (P,Q) \quad \text{for all } N .$$

Intersecting over N , the right side is trivial, hence the left, and the flow is K . Similar results were already known if K in this question was replaced with ergodic, weak-mixing, mixing or Bernoulli. This fills in the one missing term in that hierarchy. A complete text of the proofs of these arguments will appear soon.

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