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ON THE CONVERGENCE OF WILSON'S NON-CONFORMING ELEMENT FOR SOLVING THE ELASTIC PROBLEM

Pierre LESAINT *

ABSTRACT : A non-conforming finite element, Wilson's element, for solving the elastic problem is mathematically studied. This element passes the Patch-Test. The errors on the stresses and displacements are shown to be asymptotically of order h and h^2, respectively, where h is the supremum of the elements' side lengths.

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INTRODUCTION

Conforming and non-conforming finite element methods for the plate bending problem have been extensively studied : see CIARLET /1/, CIARLET and RAVIART /2/, LASCAUX and LESAINT /3/, NITSCHE /4/. In the conforming case, finite elements of class C^1 are needed, such as the well-known 21-degrees of freedom triangle of ARGYRIS /5/. "Variational crimes" may also be commited (IRONS /6/, STRANG /7/) by using elements which are not of class C^1 and in some cases not even of class C^0, and thus defining a non-conforming method.

In the same way, we can solve the elastic problem either by conforming methods, using elements of class C^0 such as the 3-nodes or the 6-nodes triangle, or by non-conforming methods, constructed with elements which are not of class C^0.

The purpose of this paper is to present and analyse mathematically one of these non-conforming elements, Wilson's element /8/, which is pratically used by the engineers to solve the elastic problem in two (or three) dimensions.

To obtain the error estimates corresponding to non-conforming methods, the keystone is the Patch-Test of IRONS /6/. It has been already shown (IRON /6/, STRANG /7/) that Wilson's element passes the Patch-Test. In this paper, we give a mathematical proof of convergence for this element and we show that the errors on the stresses and displacements are asymptotically of order h and h^2, respectively, where h is the supremum of the elements' side lengths. One of the main difficulties consists in showing that the stiffness matrix of the problem is positive definite, independently of h (§3). For the sake of simplicity, the results are presented for problems in two dimensions, but they are also true in three dimensions.

An outline of the paper is as follows. In §1 we recall the variational formulation of an elastic problem. In §2 we define general non-conforming methods, give the corresponding error estimates and introduce the Patch-Test. The results of §2 are then applied to Wilson's element which is described and studied in §3.

I. ELASTIC PROBLEM

Let Ω be a bounded open subset of the plain x-y, with a Lipschitz-continuous ([9]) boundary Γ. We shall denote by s a curvilinear abscissa along Γ, by $\frac{\partial}{\partial n}$ the derivative along the outer normal on Γ and $\frac{\partial}{\partial s}$ the tangential derivative along Γ.

For a given integer $m \geqslant 0$, we let

(1-1) $$|v|_{m,\Omega} = \Big(\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^2 \, dxdy \Big)^{1/2}, \quad \|v\|_{m,\Omega} = \Big(\sum_{\ell=0}^{m} |v|^2_{\ell,\Omega} \Big)^{1/2},$$

where α is a multiindex such that $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \geqslant 0$, $|\alpha| = \alpha_1 + \alpha_2$ and $\partial^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdot (\frac{\partial}{\partial x_2})^{\alpha_2}$. The applications $|\cdot|_{m,\Omega}$ and $\|\cdot\|_{m,\Omega}$ are respectively a seminorm and a norm over the Sobolev Space $H^m(\Omega)$

In what follows, we shall be interested in the space

(1-2) $V = \{v=(v_i) \in (H^1(\Omega))^2 \; ; \; v_i = 0 \text{ on } \Gamma_0, \; 1 \leqslant i \leqslant 2\}$, where Γ_0 is a measurable subset of the boundary Γ.

The following inclusions hold

(1-3) $$(H_0^1(\Omega))^2 \subset V \subset (H^1(\Omega))^2$$

and the subset V of $(H^1(\Omega))^2$ is closed in $(H^1(\Omega))^2$.

For any $v = (v_1, v_2) \in V$, the expressions $(|v_1|^2_{m,\Omega} + |v_2|^2_{m,\Omega})^{1/2}$ and $(\|v_1\|^2_{m,\Omega} + \|v_2\|^2_{m,\Omega})^{1/2}$ will still be denoted by $|v|_{m,\Omega}$ and $\|v\|_{m,\Omega}$.

One can show that if the measure of Γ_0 is strictly positive, then the application : $v \in V \to |v|_{1,\Omega}$ is a norm over the space V, equivalent to $\| \; \|_{1,\Omega}$.

We want to calculate the displacements relative to an equilibrium state of an homogeneous and isotrop elastic continuum $\bar{\Omega}$, under the action of distributed body forces $f = (f_1, f_2)$ per unit volume and external loading $g = (g_1, g_2)$ per unit area, the displacements being specified and equal to zero along the subset Γ_0 of Γ.

For any $v : (v_1, v_2) \in V$, we let

(1-4) $$\varepsilon_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad 1 \leqslant i, j \leqslant 2,$$

(1-5) $$\sigma_{ij}(v) = \lambda(\operatorname{div} v)\, \delta_{ij} + 2\mu\, \varepsilon_{ij}(v), \quad 1 \leqslant i, j \leqslant 2,$$

where the constants $\lambda \geqslant 0$ and $\mu > 0$ appearing in the relationship (1-5) between the stresses σ_{ij} and the strains ε_{ij} are the coefficients of Lame of the continuum and where $\delta_{ij} = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } i = j \end{cases}$

We let the bilinear form $a(.,\ .)$ be defined on $V \times V$ by

(1-6) $$a(u,v) = \int_\Omega \sum_{i,j=1}^{2} \sigma_{ij}(u)\, \varepsilon_{ij}(v)\, dx\, dy =$$

$$= \lambda \int_\Omega \operatorname{div} u.\ \operatorname{div} v\, dxdy + 2\mu \int_\Omega \sum_{i,j=1}^{2} \varepsilon_{ij}(u).\ \varepsilon_{ij}(v)\, dx\, dy,$$

and the linear form $v \to (f,v)$ defined on V by

(1-7) $$(f,v) = \int_\Omega (f_1 v_1 + f_2 v_2)\, dx\, dy + \int_\Gamma (g_1 v_1 + g_2 v_2)\, ds, \text{ for}$$

$f_i \in L^2(\Omega)$, $g_i \in L^2(\Gamma_1)$, $i = 1, 2$, where $\Gamma_1 = \Gamma - \Gamma_0$.

The elastic problem described above can be formulated as follows [10]

To find the displacements $u = (u_1, u_2) \in V$ such that :

(1-8) $$a(u,v) = (f,v), \text{ for all } v = (v_1, v_2) \in V.$$

Using Korn inequality, which can be written as follows

(1-9) $$\|v\|_{1,\Omega} \leqslant c\left(\sum_{i,j=1}^{2} \|\varepsilon_{ij}(v)\|^2_{0,\Omega} + \|v\|^2_{0,\Omega}\right)^{1/2} \quad \text{for all } v \in (H^1\Omega)^2,$$

where the constant $c > 0$ depends only on the domain Ω, one can show ([2], [9]) that if the measure of Γ_0 is strictly positive, then the application

$$(1\text{-}10)\quad v \in V \rightarrow \left(\sum_{i,j=1}^{2} \|\varepsilon_{ij}(v)\|_{0,\Omega}^2\right)^{1/2}$$

is a norm over the space V, equivalent to the norm $\|\cdot\|_{1,\Omega}$.

As a consequence, we get, for a constant $c > 0$ depending only on Ω

$$(1\text{-}11)\quad a(v,v) \geq 2\mu \sum_{i,j=1}^{2} \|\varepsilon_{ij}(v)\|_{0,\Omega}^2 \geq c\|v\|_{1,\Omega}^2 \text{, for all } v \in V.$$

On the other hand, we have

$$(1\text{-}12)\quad a(u,v) \leq c|u|_{1,\Omega}\,|v|_{1,\Omega} \quad \text{for all } u,v \in V.$$

Inequalities (1-3),(1-11) and (1-12) imply (by Lax-Milgram Lemma) that problem (1-8) has a unique solution $u \in V$. ▨

We have the following Green's formula [10] :

$$(1\text{-}13)\quad a(u,v) = -\int_\Omega \sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\sigma_{ij}(u)\, v_i\, dx + \int_\Gamma \sum_{i,j=1}^{2} \sigma_{ij}(u)\, n_j\, v_i\, ds,$$

where $\vec{n} = (n_1, n_2)$ denotes the outer normal on Γ.

When the solution u of problem (1-8) is smooth enough, then one can show, using Green's formula (1-13) that u is also solution of the problem :

$$(1\text{-}14)\quad -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\sigma_{ij}(u) = f_i \quad \text{in } \Omega,\ 1 \leq i \leq 2,$$

$$(1\text{-}15)\quad u_i = 0 \quad \text{on } \Gamma_0,\ 1 \leq i \leq 2,$$

$$(1\text{-}16)\quad \sum_{j=1}^{2} \sigma_{ij}(u)\, n_j = g_i \quad \text{on } \Gamma_1,\ 1 \leq i \leq 2.$$

2. NON CONFORMING METHODS

Definition 2.1. Given an integer $k \geq 0$, we let P_k and Q_k denote the spaces of polynomials in x and y defined by

$$(2\text{-}1)\quad P_k = \{P \; ; \; P = \sum_{\ell+m \leq k} \alpha_{\ell,m}\, x^{\ell} y^{m}\},$$

$$(2\text{-}2)\quad Q_k = \{q \; ; \; q = \sum_{\ell,m \leq k} \beta_{\ell m}\, x^{\ell} y^{m}\}$$

Given a triangulation τ_h of $\overline{\Omega}$ in finite elements K, with boundary ∂K, such that $\bigcup_{K \in \tau_h} K = \Omega$, we let

$$h = \max_{K \in \tau_h} h_k, \text{ with } h_k = \text{diam }(K) \text{ for all } K \in \tau_h.$$

Over each element K, we are given a finite dimensional space P_K of shape functions such that the following inclusions hold :

$$(2\text{-}3)\qquad P_K \subset C^1(K) \quad , \quad P_K \supset P_1 \quad , \text{ for all } K \in \tau_h,$$

which implies that a first practical necessary condition of convergence is satisfied (Zienkiewicz [11, page 28]). We are also given on each $K \in \tau_h$ a set of degrees of freedom allowing to define a basis of the space P_K.

In what follows, we assume that the finite elements (K, Σ_K, P_K) are of the same type, for all $K \in \tau_h$.

We let the subspace X_h of $L^2(\Omega)$ be the space of functions defined by their degrees of freedom on the elements K of τ_h, and continuous for these degrees of freedom along each face common to two adjacent elements, and whose restriction to each K belongs to P_K.

The finite dimensional space V_h in which we look for an approximate solution u_h will be the subspace of $(X_h)^2$ of functions whose degrees of freedom along the boundary Γ_0 satisfy boundary conditions (1-15). A second practical necessary condition for convergence (Zienkiewicz [11, page 29]) would imply that the inclusion $V_h \subset V \cap C^\circ(\bar{\Omega})$ holds. On the contrary, we shall consider finite elements for which the preceding inclusion does not necessarily hold. Such elements, and also the corresponding finite element method, are called non conforming ([6], [12]).

Since the functions of V_h are smooth on each $K \in \tau_h$, according to inclusions (2-3), it is then natural to define a new bilinear form $a_h(\cdot,\cdot)$ on $V_h \times V_h$ by :

$$(2\text{-}4)\quad a_h(u_h,v_h) = \sum_{K\in\tau_h} \int_K \sum_{i,j=1}^{2} \sigma_{ij}(u_h)\, \varepsilon_{ij}(v_h)\, dx\, dy = $$
$$= \sum_{K\in\tau_h} \left(\lambda \int_K (\operatorname{div} u_h \operatorname{div} v_h) dx\, dy + 2\mu \int_K \sum_{i,j=1}^{2} \varepsilon_{ij}(u_h)\, \varepsilon_{ij}(v_h)\, dx dy \right)$$

The discrete problem will then be defined as follows

To find $u_h \in V_h$ such that

$$(2\text{-}5)\quad a_h(u_h,v_h) = (f,v_h) \quad \text{for all } v_h \in V_h.$$

We let the applications $\|\cdot\|_h$ and $|||\cdot|||_h$ from V_h into R be defined by :

$$(2\text{-}6)\quad \|v_h\|_h = \left(\sum_{K\in\tau_h} |v_h|^2_{1,K} \right)^{1/2},$$

$$(2\text{-}7)\quad |||v_h|||_h = \left(\sum_{K\in\tau_h} \sum_{i,j=1}^{2} \|\varepsilon_{ij}(v_h)\|^2_{0,K} \right)^{1/2},$$

and we make the following hypotheses :

(2-8) $|||\cdot|||_h$ is a norm on the space V_h,

There exists a constant $c > 0$ independent of h such that

$$(2\text{-}9)\quad \|v\|_h \leqslant c\,|||v_h|||_h, \quad \text{for all } v_h \in V_h.$$

If hypothesis (2-8) holds, then problem (2-5) has a unique solution $u_h \in V_h$. □

We shall now derive as in [3], [4], [7], some general estimates for the errors done on the stresses and strains (measured by the norm $\|\cdot\|_h$) and on the displacements (measured by the norm $\|\cdot\|_{o,\Omega}$). Those estimates will lead to a practical condition of convergence for non-conforming elements, called Patch Test ([6]).

Theorem 2.1. Assume that hypotheses (2-8) and (2-9) hold. Let $u_h = (u_{h,1}, u_{h,2}) \in V_h$ be the solution of problem (2-5) and $u \in V$ be the solution of problem (1-8). We have the estimate :

$$(2\text{-}10)\quad |||u-u_h|||_h \leqslant c\left(\inf_{v\in V_h} \|u-v\|_h + \sup_{w\in V_h} \frac{E_h(u,w)}{\|w\|_h}\right),$$

where the constant $c > 0$ is independent of h, and where

$$(2\text{-}11)\quad E_h(u,w) = -\sum_{K\in\tau_h}\left(\int_{\partial K} \sum_{i,j=1}^{2} \sigma_{ij}(u) n_{j,K} w_i \, ds\right) + \int_{\Gamma_1} \sum_{i=1}^{2} g_i w_i \, ds,$$

the $n_{j,K}$ s, $j=1,2$ being the components of the outer normal on ∂K.

Proof. We let F_h be defined by $F_h = a_h(u_h-v, u_h-v)$, for all $v=(v_1, v_2) \in V_h$. We have

$$F_h \geqslant 2\mu\, |||u_h-v|||_h^2 \geqslant c\, \|u_h-v\|_h^2 ,$$

$$F_h = (f, u_h-v) - a_h(v,u_h-v) = a_h(u-v, u_h-v) + (f,u_h-v) - a_h(u,u_h-v).$$

On the other hand, we have

$$(f,u_h-v) - a_h(u,u_h-v) =$$

$$= \sum_{K\in\tau_h} \int_K - \sum_{i,j=1}^{2}\left(\frac{\partial}{\partial x_j}\sigma_{ij}(u)\,(u_{h,i}-v_i) + \sigma_{ij}(u)\,.\,\varepsilon_{ij}(u_h-v)\right)dx\,dy + \int_{\Gamma_1}\left(\sum_{i=1}^{2} g_i w_i\right)ds$$

Applying Green's formula (1-13) on each element $K \in \tau_h$, we get

$$(f,u_h-v) - a_h(u,u_h-v) = \sum_{K\in\tau_h} \int_{\partial K}\left(-\sum_{i,j=1}^{2} \sigma_{ij}(u) n_{j,K}(u_{h,i}-v_i)\right)ds + \int_{\Gamma_1} \sum_{i=1}^{2} g_i w_i \, ds.$$

Combining the last relations with the triangular inequality, we get estimate (2-10).

The first part of the right hand side of inequality (2-10) is the same as the term of error obtained in the case of a conforming method and can be estimated by using results in approximation theory ([2], [13]) ; the second part contains only terms arising from the non continuity of the functions of V_h at the interfaces between the elements, and should converge to zero as h approaches zero, i.e. :

(2-12) $\lim_{h \to o} E_h(u,w) = 0$ for all $u \in V$ and $w \in V_h$.

Condition of convergence (2-12) is replaced in practice by the "Patch Test", which consists in showing that ([6], [7]) :

(2-13) $E_h(u,w) = 0$ for all $u \in P_1$, $w \in V_h$ and all $h > 0$.

It can be shown on most examples ([3] and § 3) that the Patch Test combined with continuity requirements at the nodes common to two (or more) elements implies convergence.

Consider now the following smoothness hypothesis for the system of elasticity.

(2-14)

$$\left\{ \begin{array}{l} \text{For all } g = (g_1, g_2) \in (L^2(\Omega))^2, \text{ the system} \\ -\sum_{j=1}^{2} \frac{\partial}{\partial x_j} \sigma_{ij}(\varphi) = g_i, \quad 1 \leq i \leq 2 \\ \varphi_i = 0 \quad \text{on } \Gamma_0, \ 1 \leq i \leq 2 \\ \sum_{j=1}^{2} \sigma_{ij}(\varphi)\, n_j = 0 \text{ on } \Gamma_1, \ 1 \leq i \leq 2 \\ \text{has a unique solution } \varphi = (\varphi_1, \varphi_2) \in (H^2(\Omega))^2 \cap V \text{ and we have} \\ \|\varphi\|_{2,\Omega} \leq \|g\|_{0,\Omega} \end{array} \right.$$

We can show the following results, the proof of which can already be found in [3], [4].

Theorem 2.2. Assume that the hypotheses (2-8), (2-9) and (2-14) hold. Let $u_h = (u_{h,1}, u_{h,2}) \in V_h$ be the solution of problem (2-5) and $u \in V$ be the solution of problem (1-8). We then have :

$$(2\text{-}15)\quad \|u-u_h\|_{0,\Omega} \leqslant c \sup_{\varphi \in (H^2(\Omega))^2} \left(\inf_{\varphi_h \in V_h} \frac{E_h(u, u_h, \varphi, \varphi_h)}{\|\varphi\|_{2,\Omega}} \right), \text{ with}$$

$$(2\text{-}16)\quad E(u, u_h, \varphi, \varphi_h) = a_h(u-u_h, \varphi-\varphi_h) - E_h(u, \varphi_h) + E_h(\varphi, u_h),$$

where the constant $c > 0$ is independent of h.

Proof. We use the following classical duality argument ([14], [15])

$$\|u-u_h\|_{0,\Omega} = \sup_{g \in (L^2(\Omega))^2} \frac{|(u-u_h, g)|}{\|g\|_{0,\Omega}}$$

For some $g = (g_1, g_2) \in (L^2(\Omega))^2$, we let $\varphi = (\varphi_1, \varphi_2) \in V$ be the solution of the system of elasticity. According to hypothesis (2-14), we have $\varphi \in (H^2(\Omega))^2 \cap V$ and $\|\varphi\|_{2,\Omega} \leqslant c\|g\|_{0,\Omega}$, so that :

$$(2\text{-}17)\quad \|u-u_h\|_{0,\Omega} \leqslant c \sup_{\varphi \in H^2(\Omega)^2} \frac{(u-u_h, g)}{\|\varphi\|_{2,\Omega}}.$$

On the other hand, using Green's formula, we may write, as in (1-13)

$$(2\text{-}18)\quad (u-u_h, g) = a_h(u-u_h, \varphi) + E_h(\varphi, u-u_h), \text{ and}$$

$$(2\text{-}19)\quad 0 = a_h(u-u_h, \varphi_h) + E_h(u, \varphi_h), \text{ for all } \varphi_h \in V_h .$$

Since we have $E_h(\varphi, u) = 0$, for all $\varphi, u \in V$, we get inequality (2-15) from inequality (2-17) and equalities (2-18) and (2-19).

3. WILSON'S ELEMENT

Assume now that the domain Ω is the square $]0,1[\times]0,1[$. For the sake of simplicity, we consider a triangulation of Ω in equal squares with sides equal to $h = \frac{1}{I}$, for some integer I, but the following results are still valid when the elements are non-equal rectangles. We let

$$x_k = k\,h,\ y_\ell = \ell\,h,\ A_{k\ell} = (x_k, y_\ell),\ \text{for } 0 \leqslant k,\ \ell \leqslant I,$$

$$G_{k\ell} = \left((k + \tfrac{1}{2})h, (\ell + \tfrac{1}{2})h\right),\ K_{k\ell} = [x_k, x_{k+1}] \times [x_\ell, x_{\ell+1}],\ \text{for } 0 \leqslant k,\ \ell \leqslant I-1$$

For $0 \leqslant k, \ell \leqslant I-1$, we let $F_{k\ell} \in (P_1)^2$ be the affine transformation mapping the reference square $\hat{K} = [-1,+1] \times [-1,+1]$ on the square $K_{k\ell}$, with $F_{k\ell} : (\xi, \eta) \in \hat{K} \to (x,y) \in K_{k\ell}$,

(3-1) $x = \frac{1+\xi}{2} x_{k+1} + \frac{1-\xi}{2} x_k$

(3-2) $y = \frac{1+\eta}{2} y_{\ell+1} + \frac{1-\eta}{2} y_\ell$.

Definition 3.1. Wilson's "Brick" [8] can be defined on the reference square $\hat{K}$ as follows (figure 1) :

(i) The space of shape functions is $\hat{P} = P_2$,

(ii) The degrees of freedom $\hat{\Sigma}$ are the values of the functions $\hat{p}$ at the four vertices of the square and the values of $\frac{\partial^2 \hat{p}}{\partial \xi^2}$ and $\frac{\partial^2 \hat{p}}{\partial \eta^2}$ on the square $\hat{K}$.

The function $\hat{p} \in \hat{P}$ such that

(3-3) $\hat{p}(\hat{a}_i) = p_i,\ 1 \leqslant i \leqslant 4,\ \frac{\partial^2 \hat{p}}{\partial \xi^2} = p_\xi,\ \frac{\partial^2 \hat{p}}{\partial \eta^2} = p_\eta$

can be written as follows

$$(3\text{-}4)\quad \hat{p} = \frac{(1+\xi)(1+\eta)}{4} p_1 + \frac{(1-\xi)(1+\eta)}{4} p_2 + \frac{(1-\xi)(1-\eta)}{4} p_3 + \frac{(1+\xi)(1-\eta)}{4} p_4 + $$

$$+ \frac{1}{2}(\xi^2 - 1)\, p_\xi + \frac{1}{2}(\eta^2 - 1)\, p_\eta \quad .$$

The finite elements $(K, \Sigma, P)_{k,\ell}$ will be the images by the transformations $F_{k,\ell}$ of the element of reference $(\hat{K}, \hat{\Sigma}, \hat{P})$, with

(3-5) $P_{k,\ell} = \{p = \hat{p} \circ F_{k\ell}^{-1} ; \forall \hat{p} \subset \hat{P}\}$, $0 \leqslant k, \ell < J-1$.

The finite dimensional subspace X_h of $L^2(\Omega)$ will be the space of functions defined by their values at the vertices of the elements $K_{k\ell}$, and by the values of their second derivatives $\frac{\partial}{\partial x^2}$ and $\frac{\partial}{\partial y^2}$ on each element $K_{k\ell}$, and whose restriction to each element $K_{k\ell}$ belongs to $P_{k\ell}$, $0 \leqslant k, \ell \leqslant I-1$. In the general case, the inclusion $X_h \subset C^0(\overline{\Omega})$ does not hold.

We shall also need the space Y_h of continuous functions defined by their values at the vertices of the elements and whose restriction to each element $K_{k\ell}$, $0 \leqslant k, \ell \leqslant I-1$, is a polynomial of Q_1. The following inclusion holds :

(3-6) $Y_h \subset H^1(\Omega) \cap C^0(\overline{\Omega})$.

<u>Definition 3.2</u>. For any function $\varphi \in H^2(\hat{K})$, its <u>interpolate</u> $\Pi\varphi$ will be the unique function of $\hat{P}$, equal to φ at the vertices of $\hat{K}$ and such that

$$\int_{\hat{K}} \frac{\partial^2}{\partial \xi^2} (\varphi - \Pi\varphi)\, d\xi d\eta = \int_{\hat{K}} \frac{\partial^2}{\partial \eta^2} (\varphi - \Pi\varphi)\, d\xi d\eta = 0 .$$

The following equality is then satisfied:

(3-7) $\varphi - \Pi\varphi = 0$ for all $\varphi \in P_2$.

Now for all $u = (u_1, u_2) \in \left(H^2(\Omega)\right)^2$, we let its $(X_h)^2$-<u>interpolate</u> $\Pi_h u$ be the unique function of $(X_h)^2$ whose restriction to each element K of τ_h has its components respectively equal to Πu_1 and Πu_2.

We shall need the following hypothesis on the triangulation τ_h.

(3-8) Assume that $\Gamma_0 = \bigcup_{1 \leqslant i \leqslant i_0} \Gamma_{0,i}$, where the $\Gamma_{0,i}$'s are subsets of Γ, then the end points of $\Gamma_{0,i}$, $1 \leqslant i \leqslant i_0$, are nodes of the triangulation, The space V_h will be the subspace of $(X_h)^2$ of functions equal to zero at the vertices belonging to Γ_0. In the same way, we define the space W_h as the subspace of $(Y_h)^2$ of functions equal to zero at the vertices belonging to Γ_0. We then have

(3-9) $W_h \subset V \cap C^0(\bar{\Omega})$.

For any $v_h = (v_{h,1}, v_{h,2}) \in (X_h)^2$, we let

$$v_{i,x}(G_{k\ell}) = h^2(\frac{\partial^2}{\partial x^2} v_{h,i})(G_{k\ell}) , \quad i = 1, 2,$$

$$v_{i,y}(G_{k\ell}) = h^2(\frac{\partial^2}{\partial y^2} v_{h,i})(G_{k\ell}) , \quad i = 1,2 , \quad 0 \leq k, \ \ell \leq I-1.$$

For any $V_h = (v_{h,1}, v_{h,2}) \in (X_h)^2$, we let

$$v^i_{h,\ell} = v_{h,i}(A_{k\ell}) , \quad i=1,2 , \quad 0 \leq k, \ \ell \leq I,$$

(3-10) $$B_{k\ell}(v_h) = \sum_{i=1}^{2} (v^i_{k,\ell+1} - v^i_{h,\ell})^2 + (v^i_{k+1,\ell+1} - v^i_{k+1,\ell})^2 + (v^i_{k+1,\ell+1} - v^i_{k,\ell+1})^2 + (v^i_{k+1,\ell} - v^i_{k,\ell})^2$$

(3-11) $$D_{k\ell}(v_h) = (v^2_{k,\ell+1} - v^2_{k,\ell})^2 + (v^2_{k+1,\ell+1} - v^2_{k+1,\ell})^2 + (v^1_{k+1,\ell+1} - v^1_{k,\ell+1})^2 + (v^1_{k+1,\ell} - v^1_{k,\ell})^2$$
$$+ \Big((v^2_{k+1,\ell} + v^2_{k+1,\ell+1} - v^2_{k,\ell} - v^2_{k,\ell+1}) + (v^1_{k+1,\ell+1} + v^1_{k,\ell+1} - v^1_{k+1,\ell} - v^1_{k,\ell})\Big)^2,$$

for $0 \leq k, \ \ell \leq I-1$.

We shall show that hypothesese (2-8) and (2-9) are satisfied.

<u>Lemma 3.1. Assume that hypothesis 3-8 holds. Then there exist two constants c and C, with $0 < c < C$, independent of h, such that</u>

(3-12) $$c \sum_{i,j=1}^{2} \|\varepsilon_{ij}(v_h)\|^2_{o,K_{k\ell}} \leq D_{k\ell}(v_h) + \sum_{i=1}^{2}\Big((v_{i,x}(G_{k\ell}))^2 + (v_{i,y}(G_{k\ell}))^2\Big)$$
$$\leq C \sum_{i,j=1}^{2} \|\varepsilon_{ij}(v_h)\|^2_{o,K_{k\ell}} ,$$

<u>for all</u> $v_h = (v_{h,1}, v_{h,2}) \in (X_h)^2$ <u>and for</u> $o \leq k, \ell \leq I-1$.

<u>Proof</u>. On the square of reference $\hat{K}$, we let

$$\varphi(\xi,\eta) = v_{h,1}(x,y) \ , \ \psi(\xi,\eta) = v_{h,2}(x,y) \ ,$$

with $(x,y) = F_{k,\ell}(\xi,\eta)$. We have

$$(3\text{-}13) \quad \sum_{i,j=1}^{2} \|\varepsilon_{ij}(v_h)\|^2_{o,K_{k\ell}} = \int_{\hat{K}} \left(\left(\frac{\partial\varphi}{\partial\xi}\right)^2 + \left(\frac{\partial\psi}{\partial\eta}\right)^2 + \left(\frac{\partial\varphi}{\partial\eta} + \frac{\partial\psi}{\partial\xi}\right)^2\right) d\xi d\eta$$

According to equality (3-4), we may write :

$$\frac{\partial\varphi}{\partial\xi} = \frac{1}{4}(\varphi_1 - \varphi_2 - \varphi_3 + \varphi_4) + \frac{\eta}{4}(\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4) + \xi\,\varphi_\xi$$

$$\frac{\partial\psi}{\partial\eta} = \frac{1}{4}(\psi_1 + \psi_2 - \psi_3 - \psi_4) + \frac{\xi}{4}(\psi_1 - \psi_2 + \psi_3 - \psi_4) + \eta\,\psi_\eta$$

$$\frac{\partial\varphi}{\partial\eta} + \frac{\partial\psi}{\partial\xi} = \frac{1}{4}(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4 + \psi_1 - \psi_2 - \psi_3 + \psi_4) +$$

$$+ \frac{\xi}{4}(\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4 + 4\psi_\xi) + \frac{\eta}{4}(\psi_1 - \psi_2 + \psi_3 - \psi_4 + 4\varphi_\eta).$$

If the expression (3-13) is equal to zero, we then have :

$$\frac{\partial\varphi}{\partial\xi} = \frac{\partial\psi}{\partial\eta} = \frac{\partial\varphi}{\partial\eta} + \frac{\partial\psi}{\partial\xi} = 0 \ , \text{ for all } \xi,\eta \in \hat{K},$$

and then

$$\varphi_1 - \varphi_2 = \varphi_3 - \varphi_4 = \psi_1 - \psi_4 = \psi_2 - \psi_3 = \varphi_\xi = \varphi_\eta = \psi_\xi = \psi_\eta = 0 \ ,$$

$$(\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4) + (\psi_1 - \psi_2 - \psi_3 + \psi_4) = 0,$$

so that there exists two constants c and C, with $o < c < C$, depending only on $\hat{K}$ such that

$$c \sum_{i,j=1}^{2} \|\varepsilon_{ij}(v_h)\|^2_{K_{k\ell}} \leqslant (\varphi_1 - \varphi_2)^2 + (\varphi_3 - \varphi_4)^2 + (\psi_1 - \psi_4)^2 + (\psi_2 - \psi_3)^2 +$$

$$(\varphi_\xi)^2 + (\varphi_\eta)^2 + (\psi_\xi)^2 + (\psi_\eta)^2 + (\varphi_1 + \varphi_2 - \varphi_3 - \varphi_4 + \psi_1 - \psi_2 - \psi_3 + \psi_4)^2$$

$$\leqslant C \sum_{i,j=1}^{2} \|\varepsilon_{ij}(v_h)\|^2_{o,K_{k\ell}},$$

which is exactly inequality (3-12).

In the same way, one can easily show

Lemma 3.2. Assume that hypothesis 3-8 holds. Then there exist two constants c and C, with $o \leqslant c \leqslant C$, independent of h, such that :

$$(3\text{-}14)\quad c|v_h|^2_{1,K_{k\ell}} \leqslant B_{k\ell}(v_h) + \sum_{i=1}^{2}\left(\left(v_{i,x}(G_{k\ell})\right)^2 + \left(v_{i,y}(G_{k\ell})\right)^2\right) \leqslant C|v_h|_{1,K_{k\ell}} ,$$

For all $v_h = (v_{h,1}, v_{h,2}) \in (X_h)^2$ and for $0 \leqslant k, \ell \leqslant I-1$.

Collorary 3.1. The application $\|\cdot\|_h$ (resp. $|||\cdot|||_h$) is a norm on the subspace of functions of $(X_h)^2$ equal to zero at one (resp. two) vertices belonging to Γ. The applications $\|\cdot\|_h$ and $|||\cdot|||_h$ are then norms on the space V_h.

We have the following result

Lemma 3.3. The two norms $\|\cdot\|_h$ and $|||\cdot|||_h$ on V_h are uniformly equivalent with respect to h, that is to say, there exist two constants c and C, with $o < c < C$, independent of h, such that

$$(3\text{-}15)\quad c\|v_h\|_h \leqslant |||v_h|||_h \leqslant C\|v_h\|_h, \text{ for all } v_h \in V_h.$$

Proof. The proof of inequality $|||v_h|||_h \leqslant C\|v_h\|_h$ is straightforward.. We shall show the other one. For any $v_h = (v_{h,1}, v_{h,2}) \in V_h$, we let $w_h = (w_{h,1}, w_{h,2})$ be the function of $(Y_h)^2$, taking the same values as v_h at the vertices of the elements. We then have

$$w_h \in V \cap C^0(\overline{\Omega}).$$

Using inequalities (1-10) and (1-11) (Korn inequality) we may write :

$$|w_h|_{1,\Omega} \leqslant c' \sum_{i,j=1}^{2} \|\varepsilon_{ij}(w_h)\|^2_{0,\Omega} ,$$

where the constant $c' > 0$ is independent of h.

Now, applying Lemmas 3.1. and 3.2. to the functions $w_h \in (X_h)^2$ such that $w_{i,x}(G_{k\ell}) = w_{i,y}(G_{k\ell}) = 0$, $o \leqslant k, \ell \leqslant I-1$, $i = 1,2$, we have :

$$B_{k\ell}(v_h) = B_{k\ell}(w_h) \leqslant C|w_h|^2_{1,K_{k\ell}} ,$$

$$c \sum_{i,j=1}^{2} \|\varepsilon_{ij}(w_h)\|^2_{0,K_{k\ell}} \leq D_{k\ell}(w_h) = D_{k\ell}(v_h) \ , \ 0 \leq k, \ \ell \leq I-1.$$

Combining the last three inequalities, we get

(3-16) $$\sum_{0\leq k,\ell\leq I-1} B_{k\ell}(v_h) \leq C \sum_{0\leq k,\ell\leq I-1} D_{k\ell}(v_h) \quad ,$$

where the constant $C > 0$ is independent of h.

Inequality (3-15) is then a consequence of inequality (3-16) and Lemmas (3.1) and (3.2).

We let $V_{h,o} = \{v_h \in V_h \ ; \ v_h = 0 \text{ on } \Gamma\}$.

Lemma 3.4. Patch Test. Assume that hypothesis (3-6) holds and that $g_i = 0$, $i = 1,2$. Then :

(3-17) $E_h(u, v_h) = 0$ for all $u \in (P_1)^2$, $v_h \in V_{h,o}$

Proof. For any $v_h = (v_{h,1}, v_{h,2}) \in V_{h,o}$, we let w_h be the function of $(X_h)^2$ equal to v_h at the vertices of the elements. The function w_h belongs to the space $\left(H^1_0(\Omega)\right)^2 \cap \left(C^0(\bar{\Omega})\right)^2$ and we have

(3-18) $E_h(u, v_h) = E_h(u, v_h - w_h)$.

We let, for any $K \in \tau_h$

(3-19) $$E_{j,K}(u, v_h - w_h) = \int_{\partial K} \left(\sum_{i=1}^{2} \sigma_{ij}(u) n_{j,K}(v_{h,i} - w_{h,i}) \right) ds \ , \quad j = 1,2$$

$$\varphi(\xi,\eta) = v_{h,1}(x,y) = \hat{v}_{h,1}(\xi,\eta) \ ,$$

$$\psi(\xi,\eta) = v_{h,2}(x,y) = \hat{v}_{h,2}(\xi,\eta) \ ,$$

with $x,y = F_K(\xi,\eta)$.

We then have

$$v_{h,1} - w_{h,1} = \frac{1}{2}(\xi^2 - 1)\varphi_\xi + \frac{1}{2}(\eta^2 - 1)\varphi_\eta,$$

$$v_{h,2} - w_{h,2} = \frac{1}{2}(\xi^2 - 1)\psi_\xi + \frac{1}{2}(\eta^2 - 1)\psi_\eta.$$

If $u \in (P_1)^2$, then $\sigma_{ij}(u)$ is a constant for $1 \leqslant i, j \leqslant 2$ and we may write :

$$E_{j,K}(u,v_h-w_h) = \frac{h}{2}\sum_{i=1}^{2}\sigma_{ij}(u)\int_{-1}^{+1}\left((\hat{v}_{h,i}-\hat{w}_{h,i})(1,\eta)-(\hat{v}_{h,i}-\hat{w}_{h,i})(-1,\eta)\right)d\eta$$

Since $(\hat{v}_{h,1} - \hat{w}_{h,1})(1,\eta) = \frac{1}{2}(\eta^2 - 1)\varphi_\eta = (\hat{v}_{h,1} - \hat{w}_{h,1})(-1,\eta)$,

$(\hat{v}_{h,2} - \hat{w}_{h,2})(1,\eta) = \frac{1}{2}(\eta^2 - 1)\psi_\eta = (\hat{v}_{h,2} - \hat{w}_{h,2})(-1,\eta)$,

we get

$$E_{j,K}(u, v_h-w_h) = 0 \ , \ 1 \leqslant j \leqslant 2.$$

Summing up on all the elements $K \in \tau_h$ leads us to equality (3-17) .

Remark 3.1. Equality (3-17) is not true for all $v_h \in V_h$, because of the boundary conditions. To derive the estimates, we shall in fact use the equalities : $\sum_{j=1}^{2}\sigma_{ij}(u)\, n_j = g_i$, for $i = 1,2$, on Γ_1 , and equality (3-18) is then still valid.

We shall need the following generalization of Bramble and Hilbert Lemma [16] to bilinear forms [1] :

Lemma 3.5. Let Ω be an open bounded subset of R^2 with a sufficiently smooth boundary, let r and m be two integers and let W be a space of functions satisfying the inclusions $P_m \subset W \subset H^{m+1}(\Omega)$: the space W is considered as being equipped with the norm $\|\cdot\|_{m+1,\Omega}$. Finally, let $A : H^{r+1}(\Omega) \times W \to R$ be a continuous bilinear form such that

(3-20) $A(u,v) = 0$ for all $u \in P_r$, $v \in W$,

(3-21) $A(u,v) = 0$ for all $u \in H^{r+1}(\Omega)$, $v \in P_m$.

Then there exists a constant $C = C(\Omega)$ such that

(3-22) $|A(u,v)| \leqslant C\, \|A\|\, |u|_{r+1,\Omega}|v|_{m+1,\Omega}$ for all $u \in H^{r+1}(\Omega)$, $v \in W$.

The classical inverse inequality holds :

Lemma 3.6. For all $v_h \in (X_h)^2$, we have

(3-23) $|v_h|_{m+1,K} \leq c\, h^{-1} |v_h|_{m,K}$, $0 \leq m \leq 1$,

for all $K \subset \tau_h$, the constant $c > 0$ being independent of h.

Using equality (3-7) and results in approximation theory [1], we get Lemma 3.7. Let s be an integer with $2 \leq s \leq 3$, let $u \in \left((H^s(\Omega))^2 \cap V\right.$, and $\Pi_h u \in V_h$, the V_h interpolate of u. We have

(3-24) $|u - \Pi_h u|_{m,K} \leq c\; h^{s-m} |u|_{s,K}$, $0 \leq m \leq s$,

for all $u \in H^s(\Omega)$, all $K \in \tau_h$, the constant $c > 0$ being independent of h.

We are now able to show the following fundamental result.

Lemma 3.8. Assume that hypothesis (3-8) holds, then we have

(3-25) $E_h(u, v_h) \leq c\; h^2 |u|_{2,\Omega} \left(\sum_{K \in \tau_h} |v_h|^2_{2,K} \right)^{1/2}$,

(3-26) $E_h(u, v_h) \leq c\; h\, |u|_{2,\Omega} \|v_h\|_h$,

For all $u \in V$, and $v_h \in V_h$, the constant $c > 0$ being independent of h .

Proof. Consider expressions (3-18) and (3-19). We may write

$$E_{j,K}(u, v_h - w_h) = \frac{h}{2} \int_{-1}^{+1} \sum_{i=1}^{2} \{ (\widehat{\sigma_{ij}(u)} (\widehat{v_{h,i}} - \widehat{w_{h,i}}))(1,\eta) - (\widehat{\sigma_{ij}(u)} (\widehat{v_{h,i}} - \widehat{w_{h,i}}))(-1,\eta) \}$$

$$= \frac{h}{2} \hat{E} (\hat{\sigma}_j, \hat{v}_h - \hat{w}_h) , \quad \text{with } \hat{\sigma}_j = (\hat{\sigma}_{1j}, \hat{\sigma}_{2j}).$$

The mapping : $(\hat{\sigma}_j, \hat{v}_h) \to \hat{E}(\hat{\sigma}_j, \hat{v}_h - \hat{w}_h)$ is linear and continuous from $\left(H^2(\hat{K})\right)^2 \times \left(H^2(\hat{K})\right)^2$ into R and we have

$$\hat{E}\,(\hat{\sigma}_j,\ \hat{v}_h-\hat{w}_h) = 0 \quad \text{for all } \hat{\sigma}_j \in (P_0)^2,\ \hat{v}_h \in \left(H^2(\hat{K})\right)^2$$

$$\hat{E}\,(\hat{\sigma}_j,\ \hat{v}_h-\hat{w}_h) = 0 \quad \text{for all } \hat{\sigma}_j \in \left(H^1(\hat{K})\right)^2,\ \hat{v}_h \in (Q_1)^2$$

A consequence of Lemma 3.5 is then

$$|\hat{E}\,(\hat{\sigma}_j,\ \hat{v}_h-\hat{w}_h)| \leqslant C\,|\hat{\sigma}_j|_{1,\hat{K}}\,|\hat{v}_h|_{2,\hat{K}} \quad .$$

Using the inverse of the transformation F_K, we get

$$|E_{j,K}(u,\ v_h-w_h)| \leqslant C\,h^2\,|u|_{2,K}\,|v_h|_{2,K} \quad , \text{ for all } K \in \tau_h .$$

Summing up on all the elements $K \in \tau_h$ and on the indices $j = 1,2$, we get inequality (3-25). Inequality (3-26) is a direct consequence of inequalities (3-25) and (3-23).

We have the error estimates

Theorem 3.1. Let $u \in \left(H^2(\Omega)\right)^2 \cap V$ be the solution of problem (1-8) and $u_h \in V_h$ be the solution of problem (2-5), the space V_h being constructed by using Wilson's brick. Assume that hypothesis (3-8) holds. Then we have :

(3-27) $\quad \|u-u_h\|_h \leqslant c\,h\,|u|_{2,\Omega}$,

where the constant $c > 0$ is independent of h.

Moreover, if hypothesis (2-15) holds, then we have :

(3-28) $\quad \|u-u_h\|_{0,\Omega} \leqslant c\,h^2\,|u|_{2,\Omega}$.

Proof. Since hypotheses (2-8) and (2-9) are satisfied, for the space V_h constructed above, we can apply Theorem 2.1 and we have

$$\|u-u_h\| \leqslant c\left(\|u-\Pi_h\,u\|_h + \sup_{w \in V_h} \frac{|E_h(u,w)|}{\|w\|_h}\right),$$

where the function $\Pi_h\,u$ is the V_h - interpolate of u.

According to Lemmas 3.7 and 3.8, we have respectively the estimates

$$E_h(u, w) \leq c \; h \; |u|_{2,\Omega} \; \|w\|_h \quad ,$$

$$\|u-\Pi_h u\|_h \leq c \; h \; |u|_{2,\Omega} \; .$$

The last three inequalities lead to estimate (3-27), and also to the following inequality

$$\|u_h - \Pi_h \, u\| \leq c \, h \; |u|_{2,\Omega} \; .$$

Using Lemmas 3.6 and 3.7, we get

$$\text{(3-29)} \quad \sum_{K\in\tau_h} |u_h|^2_{2,K} \leq \sum_{K\in\tau_h} \left(|u_h - \Pi_h u|^2_{2,K} + |\Pi_h \, u|^2_{2,K} \right) \leq c \; |u|^2_{2,\Omega}$$

Assume now that hypothesis (2-15) holds ; then we may apply Theorem 2.2. :

$$\|u-u_h\|_{0,\Omega} \leq c \sup_{\varphi\in(H^2(\Omega))^2} \frac{|E_h(u,u_h,\varphi,\Pi_h\varphi)|}{\|\varphi\|_{2,\Omega}} \quad .$$

Lemma 3.7 and inequality (3-27) imply that

$$a_h(u-u_h, \varphi-\Pi_h\varphi) \leq c \; \|u-u_h\|_h \; \|\varphi-\Pi_h \varphi\|_h \leq c \; h^2 \; |u|_{2,\Omega} \; |\varphi|_{2,\Omega} \; .$$

According to Lemma 3.8 and inequality (3-29), we get

$$E_h(\varphi, u_h) \leq c \; h^2 \; |\varphi|_{2,\Omega} \left(\sum_{K\in\tau_h} |u_h|^2_{2,K} \right)^{1/2} \leq c \; h^2 \; |u|_{2,\Omega} \; |\varphi|_{2,\Omega} \; .$$

Finally, with Lemmas 3.7 and 3.8 we can write

$$E_h(u,\Pi_h\varphi) \leq c \; h^2 \; |u|_{2,\Omega} \left(\sum_{K\in\tau_h} |\Pi_h \varphi|^2_{2,K} \right)^{1/2} \leq c \; h^2 |u|_{2,\Omega} \; |\varphi|_{2,\Omega} \; .$$

Estimate (3-28) is a consequence of the last four inequalities.

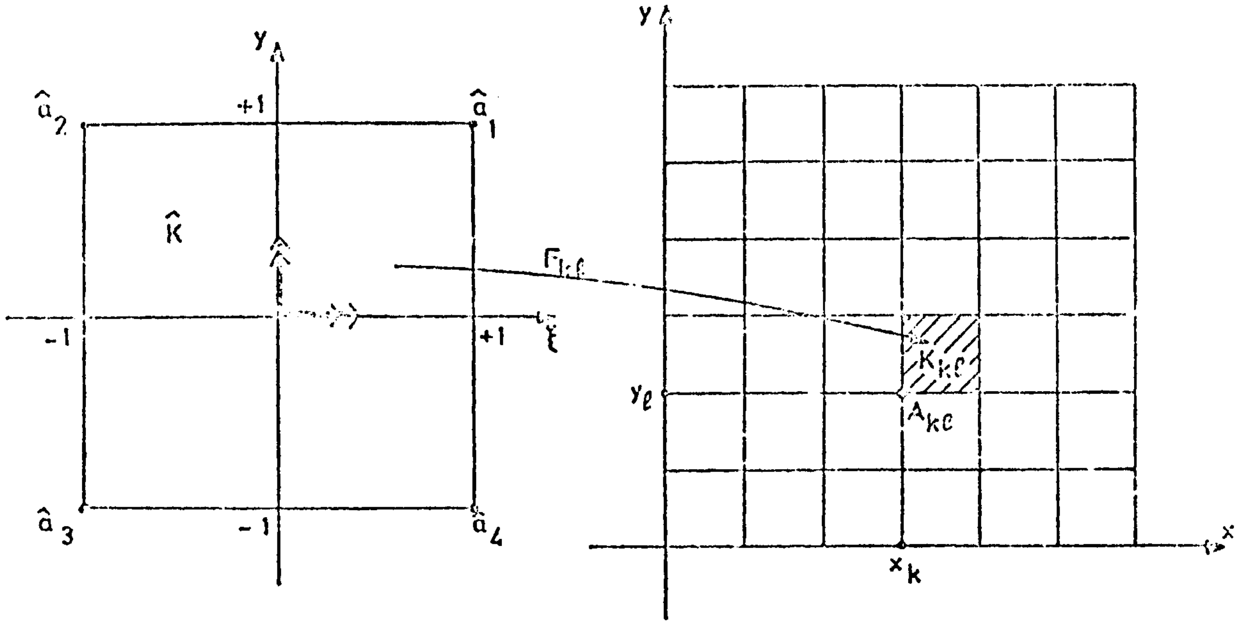

Figure 1. Wilson's element

REFERENCES

/1/ CIARLET, P.G. : Conforming and non-conforming finite element methods for solving the plate problem, Conference on the Numerical Solution of Differential Equations, University of Dundee, July 03-06, 1973.

/2/ CIARLET, P.G. ; RAVIART, P.-A. : La Méthode des éléments finis pour les problèmes aux limites elliptiques. (To appear).

/3/ LASCAUX, P. ; LESAINT, P. : Some Non-conforming Finite Elements for the Plate Bending Problem. To appear in R.A.I.R.O.

/4/ NITSCHE, J. : Convergence of Non-conforming Elements. Paper presented at the Symposium on Mathematical Aspects of Finite Elements in Partial Differential Equations. Madison, Wisconsin, April 1-3, 1974.

/5/ ARGYRIS, J.H. ; FRIED, I. ; SCHARPF, D.W. : The TUBA family of plate elements for the matrix displacement method, the Aeronautical J.R. Ac. S. 72 (1968), 701-709.

/6/ IRON, B.M. ; RAZZAQUE, A. : Experience with the patch test for convergence of finite elements, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (A.K. Aziz, Editor), pp. 557-587, Academic Press, New York, 1972.

/7/ STRANG, G. : Variational Crimes in the finite element method, The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations (A.K. Aziz, Editor) pp. 689-710, Academic Press, New York, 1972.

/8/ WILSON, E.L. ; TAYLOR, R.L. : DOHERTY W.P. ; GHABOUSSI J. : Incompatible Displacement Models. Symposium on Numerical and Computer Methods in Structural Engineering, O.N.R. University of Illinois, 1971.

/9/ NECAS, J. : Les Méthodes Directes en Théorie des Equations Elliptiques, Masson, Paris, 1967.

/10/ LANDAU, L. ; LIFCHITZ, E. : Théorie de l'Elasticité, Mir, Moscou, 1967.

/11/ ZIENKIEWICZ, O.C. : The Finite Element Method in Engineering Science, Mc Graw-Hill, London, 1971.

/12/ STRANG, G. ; FIX, G. : An analysis of the Finite Element Method, Prentice-Hall, Englewood Cliffs, 1973.

/13/ CIARLET, P.G. ; RAVIART, P.-A. : General Lagrange and Hermite interpolation in R^n with applications to finite element methods, Arch. Rational Mech. Anal. 46 (1972), 177-199.

/14/ AUBIN, J.P. : Behavior of the error of the approximate solutions, of boundary value problems for linear elliptic operators by Galerkin's and finite difference methods, Ann. Scuola Norm. Sup. Pisa 21 (1967), 599-637.

/15/ NITSCHE, J. : Ein Kriterium für die quasi-optimalität des Ritzchen Verfahrens, Numer. Math. 11 (1968), 346-348.

/16/ BRAMBLE, J.H. ; HILBERT, S.H. : Bounds for a class of linear functionals with applications to Hermite interpolation, Numer. Math. 16 (1971), 362-369.