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Exposé de théorie ergodique

par

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GENERAL ERGODIC THEOREMS FOR SEMIGROUPS OF LINEAR OPERATORS

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0. The first operator theoretical generalization of G. D. Birkhoff's pointwise ergodic theorem was given by J. L. Doob in 1938 and then extended by S. Kakutani in 1940. However one can say that a really essential contribution to the theory was done by Eberhard Hopf in 1954. Hopf's theorem was extended into two directions. One extension is that of N. Dunford-J. T. Schwartz in 1956 and the other is that of R. V. Chacon-D. S. Ornstein in 1960. In 1961 Chacon succeeded in proving a general ergodic theorem beeing an extension of both of Dunford-Schwartz's theorem and Chacon-Ornstein's theorem. At that time he introduced a notion of admissible sequence.

Let $(X,\mathcal{B},\mu)$ be a σ-finite measure space . Consider a linear contraction T on $L_1 = L_1(X)$ and a sequence $\{p_n\}_{n\geq 0}$ of nonnegative-measurable (not necessarily integrable) functions on X . Then, $\{p_n\}_{n\geq 0}$ is called T-admissible if $f\in L_1$ and $|f|\leq p_n$ for some n imply $|Tf|\leq p_{n+1}$. Chacon's general ergodic theorem is

Theorem 1. [1] . Let T be a linear contraction on L_1 and let $\{p_n\}_{n\geq 0}$ be T-admissible. Then, for every $f\in L_1$,

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} T^k f}{\sum_{k=0}^{n} p_k}$$

exists and is finite a.e. in $\{x : \sum_{k=0}^{\infty} p_k(x) > 0\}$.

1. I will state a general ergodic theorem for a semigroup of linear contractions which is the continuous case corresponding to theorem 1.

Let $\{T_t\}_{t\geq 0}$ be a semigroup of linear operators on L_1 . This means that

(t. 1) each T_t is a linear (necessarily bounded) operator on L_1 ;

(t. 2) $T_{s+t} = T_t T_s$ for every s, t ;

(t. 3) $\lim_{t\downarrow 0} ||T_t f - f||_1 = 0$ for every $f\in L_1$.

Concerning a semigroup of linear contractions, we have

Proposition 2. [3] . Let $\{T_t\}_{t\geq 0}$ be a semigroup of linear contractions on L_1. Then there exists a semigroup $\{S_t\}_{t\geq 0}$ of positive linear contractions on L_1 such that

$$|T_t f| \leq S_t |f| \qquad \text{for every } f\in L_1 .$$

Next, consider a family $\{p_t\}_{t\geq 0}$ of nonnegative measurable (not necessarily integrable) functions on X . $\{p_t\}_{t\geq 0}$ is called $\{T_t\}$-admissible if it has the properties :

(i) $f\in L_1$ and $|f|\leq p_s$ for some s imply $|T_t f|\leq p_{s+t}$ for every t ;

(ii) there exists a strictly positive function $p\in L_1$ such that $\lim_{t\downarrow s} ||\, |p_t - p_s| \wedge p\, ||_1 = 0$ for every s , where $a\wedge b$ stands for min (a,b) .

We have two typical examples of $\{T_t\}$-admissible family.

Example. 1. Consider the case where each T_t is positive. Let $0\leq g\in L_1$ and define $p_t = T_t g$. Then $\{p_t\}_{t\geq 0}$ is $\{T_t\}$-admissible.

Example. 2. Consider the case where each T_t is also a contraction on L_∞ in the sense of that $|| T_t f ||_\infty \leq ||f||_\infty$ for every

$f \in L_1 \cap L_\infty$. Define $p_t = 1$. Then $\{p_t\}_{t>0}$ is $\{T_t\}$-admissible.

From now on, we have to deal with integrals $\int_0^\alpha T_t f(x)\, dt$ and $\int_0^\alpha p_t(x)\, dt$. However, when x is fixed, each of $T_t f(x)$ and $p_t(x)$ may not be Lebesgue measurable as a function of a variable t . Hence we have to consider a suitable measurable version of each of them.

<u>Proposition 3.</u> [5]. Let $\{g_t\}_{t \geq 0}$ be a family of measurable functions on X . Suppose that there exists a strictly positive function $g \in L_1$ such that

$\lim_{t \downarrow s} \| \, |g_t - g_s| \wedge g \|_1 = 0$ for every s . Then there exists

a function $g(t,x)$ on $[0,\infty) \otimes X$ such that

a) $g(t,x)$ is $\mathcal{M} \otimes \mathcal{F}$-measurable, where $\mathcal{M}$ is the σ-algebra of all Lebesgue measurable subsets of $[0,\infty)$;

b) for every t , $g(t,x) = g_t(x)$ for a.a.x.

Such a function $g(t,x)$ is uniquely determined up to on a set of $m \otimes \mu$-measure zero, where m is Lebesgue measure on $\mathcal{M}$.

$g(t,x)$ is called the good version of $g_t(x)$ and denoted by $g_t(x)$ itself.

By virtue of Proposition 3 each of $T_t f(x)$ and $p_t(x)$ has its good version denoted by itself. Then, by Fubini's theorem, for a.a.x., each of the good versions is Lebesgue measurable as a function of t , and moreover $T_t f(x)$ is Lebesgue integrable on any bounded subintervals of $[0,\infty)$.

Our general ergodic theorem is

<u>Theorem 4.</u> [5] . Let $\{T_t\}_{t \geq 0}$ be a semigroup of linear contractions on L_1 and let $\{p_t\}_{t \geq 0}$ be $\{T_t\}$-admissible. Then, for every $f \in L_1$,

$$\lim_{\alpha \to \infty} \frac{\int_0^\alpha T_t f(x)\, dt}{\int_0^\alpha p_t(x)\, dt}$$

exists and is finite a.e. in $\{x : \int_0^\infty p_t(x)\, dt > 0\}$.

The proof of the theorem is based on the reduction to Theorem 1 by making use of Proposition 2.

<u>2</u>. Next, I will talk about local ergodic theorems. A local ergodic theorem for a semiflow of measure preserving transformations was worked out firstly by N. Wiener in 1939. Wiener's theorem was extended to the case of semigroups of linear contractions by U. Krengel in 1969 and also by D. S. Ornstein in 1970. After that, M. A. Akcoglu-R. V. Chacon, T. R. Terrell and H. Fong-L. Sucheston gave several generalizations of Krengel-Ornstein's theorem. Now we shall state the most general local ergodic theorem obtained by Y. Kubokawa.

<u>Theorem 5.</u> [2] , [3] . Let $\{T_t\}_{t>0}$ be

(i) a semigroup of positive linear operators (not necessarily contractions) on L_1 ,

or

(ii) a semigroup of linear (not necessarily positive) contractions on L_1 .

Then, for every $f \in L_1$,

$$\lim_{\alpha \downarrow 0} \frac{1}{\alpha} \int_0^\alpha T_t f(x)\, dt = f(x) \quad \text{a.e.}$$

The proof for the case (i) is based on the following maximal ergodic inequality.

<u>Proposition 6.</u> [2] . Let $\{T_t\}_{t>0}$ be a semigroup of positive linear operators on L_1 and let $f \in L_1$. Then

$$\limsup_{\alpha \downarrow 0} \frac{1}{\alpha} \int_0^\alpha T_t f(x)\, dt > 0 \quad \text{on a measurable set } E$$

implies

$$\int_E f^- \, d\mu \leq \int_X f^+ \, d\mu \quad ,$$

where f^- and f^+ are the negative and positive part of f, respectively.

On the other hand, the case (ii) of Theorem 5 is reduced to the case (i) by virtue of Proposition 2 .

Now, Kubokawa and I would like to propose an open question.

<u>Question 1.</u> Let $\{T_t\}_{t\geq 0}$ be a semigroup of linear operators on L_1 and let $f \in L_1$. Does

$$\lim_{\alpha\downarrow 0} \frac{1}{\alpha} \int_0^\alpha T_t f(x)\, dt = f(x)$$

a. e. ?

If it were possible to extend Proposition 2 to the case where each T_t is not necessarily a contraction, then the question can be reduced to the case (i) of Theorem 5. However, I think Proposition 2 is not extended to the general case, because the assumption of contraction property of each T_t in Proposition 2 is really essential. Thus it might be impossible to reduce the question to the case (i) of Theorem 5. But, I think the question will be answered affirmatively.

<u>3</u>. Now, I will talk about random ergodic theorems. A random ergodic theorem for measure preserving transformations was proved by H. R. Pitt in 1942 and also by S. M. Ulam-J. von Neumann in 1945. After that, S. Kakutani, C. Ryll-Nardzewski, S. Gladysz, A. Beck-J. T. Schwartz, E. Kin and I extended the theory. We shall state now one of the most general random ergodic theorems.

Consider two σ-finite measure spaces $(X,\mathcal{F},\mu)$ and $(Y,\mathcal{Q},\nu)$. Let $\{\varphi_t\}_{t\geq 0}$ be a semiflow of measure preserving transformations on Y . This means that

(φ. 1) each φ_t is a measure preserving transformation on Y ;

(φ. 2) $\varphi_{s+t} = \varphi_t \varphi_s$ for every s, t ;

(φ. 3) the mapping : $(t,y) \in [0,\infty) \otimes Y \longrightarrow \varphi_t y \in Y$ is $\mathcal{M}\otimes \mathcal{Q}$-measurable.

Let $\{T_t(y)\}_{t\geq 0,\, y\in Y}$ be a quasi-semigroup of linear contractions on $L_1(X)$ associated with a semiflow $\{\varphi_t\}_{t\geq 0}$ on Y . This means that

(T. 1) each $T_t(y)$ is a linear contraction on $L_1(X)$;

(T. 2) $T_{s+t}(y) = T_t(y)\, T_s(\varphi_t y)$ for every s, t and a.a.y ;

(T. 3) for every t , $T_t(y)$ is strongly $\mathcal{Q}$-measurable, that is, for every t and every $f \in L_1(X)$, $T_t(y)f$ is a $L_1(X)$-valued, $\mathcal{Q}$-measurable function on Y ;

(T. 4) for a.a.y , $T_t(y)$ is strongly t-continuous, that is, for a.a.y and every $f \in L_1(X)$,

$$\lim_{t\downarrow 0} \| T_t(y)f - f \|_{L_1(X)} = 0 .$$

Then, given $f \in L_1(X \otimes Y)$ and given t , $f(\cdot,\varphi_t y) \in L_1(X)$ for a.a.y , and so we can define $T_t(y)f(\cdot,\varphi_t y)$ for a.a.y . Note that in general, when x and y are fixed, $T_t(y)\, f\, (x,\varphi_t y)$ is not Lebesgue measurable as a function of a variable t . However we have

<u>Proposition 7.</u> [4] . There exists a function $g(t,x,y)$ on $[0,\infty) \otimes X \otimes Y$ having the properties :

a) $g(t,x,y)$ is $\mathcal{M}\otimes \mathcal{F} \otimes \mathcal{Q}$-measurable ;

b) for every t , there exists a subset N_t of Y with ν-measure zero such that, for every $y \notin N_t$,

$$g(t,x,y) = T_t(y)\, f\, (x,\varphi_t y) \quad \text{for a.a.} x .$$

Such a function $g(t,x,y)$ is uniquely determined up to on a set of $m \otimes \mu \otimes \nu$-measure zero.

$g(t,x,y)$ is called the good version of $T_t(y)\, f\, (x,\varphi_t y)$ and denoted by $T_t(y)\, f\, (x,\varphi_t y)$ itself.

Next, we consider a family $\{p_t\}_{t\geq 0}$ of nonnegative measurable (not necessarily integrable) functions on $X \otimes Y$. $\{p_t\}_{t\geq 0}$ is called $\{T_t(y)\}$-admissible if it has the properties :

(i) $f \in L_1(X \otimes Y)$ and $|f(x,y)| \leq p_s(x,y)$ a.e. for some s imply $|T_t(y)\ f\ (x,\varphi_t y)| \leq p_{s+t}(x,y)$ a.e. for every t ;

(ii) there exists a strictly positive function $p \in L_1(X \otimes Y)$ such that

$$\lim_{t \downarrow s} \|\ |p_t - p_s| \wedge p\ \|_{L_1(X \otimes Y)} = 0 \quad \text{for every } s .$$

Note that $p_t(x,y)$ has its good version by virtue of Proposition 3. We have two typical examples of $\{T_t(y)\}$-admissible family.

Example. 1. Consider the case where each $T_t(y)$ is positive. Let $0 \leq g \in L_1(X \otimes Y)$ and define $p_t(x,y) = T_t(y)\ g(x,\varphi_t y)$. Then $\{p_t\}_{t\geq 0}$ is $\{T_t(y)\}$-admissible.

2. Consider the case where each $T_t(y)$ is also a contraction on $L_\infty(X)$. Define $p_t(x,y) = 1$. Then $\{p_t\}_{t\geq 0}$ is $\{T_t(y)\}$-admissible .

We are now in a position to state a general random ergodic theorem which is more general than that in [4] .

Theorem 8. Let $\{T_t(y)\}_{t\geq 0,\ y\in Y}$ be a quasi-semigroup of linear contractions on $L_1(X)$ associated with a semiflow $\{\varphi_t\}_{t\geq 0}$ of measure preserving transformations on Y . Let $\{p_t\}_{t\geq 0}$ be $\{T_t(y)\}$-admissible. Then, for every $f \in L_1(X \otimes Y)$,

$$\lim_{\alpha\to\infty} \frac{\int_0^\alpha T_t(y)\ f\ (x,\varphi_t y)\ dt}{\int_0^\alpha p_t(x,y)\ dt}$$

exists and is finite a.e. in $\{(x,y) : \int_0^\infty p_t(x,y)\ dt > 0\}$.

The theorem is proved in the same way as in [4] by making use of Theorem 4 .

Concerning Theorem 8 , I would like to propose one more open question. Our formulation of quasi-semigroup of linear contractions comes from quasi-semiflow of measure preserving transformations. Let $\{\psi_t(y)\}_{t\geq 0,\ y\in Y}$ be a quasi-semiflow of measure preserving transformations on X associated with a semiflow $\{\varphi_t\}_{t>0}$ on Y , that is,

(ψ. 1) each $\psi_t(y)$ is a measure preserving transformation on X ;

(ψ. 2) $\psi_{s+t}\ (y) = \psi_s(\varphi_t y)\ \psi_t(y)$ for every s, t and a.a.y ;

(ψ. 3) the mapping : $(t,x,y) \in [0,\infty) \otimes X \otimes Y \to \psi_t(y)x \in X$ is $\mathcal{N} \otimes \mathcal{F} \otimes \mathcal{G}$ - measurable .

Now, if we consider an operator $T_t(y)$ on $L_1(X)$ defined by

$$T_t(y)\ f\ (x) = f(\psi_t(y)x) \quad \text{for } f \in L_1(X) ,$$

then $\{T_t(y)\}_{t\geq 0,\ y\in Y}$ is an example of our quasi-semigroup. Observe that (ψ. 2) induces (T. 2) . The property (ψ. 2) is natural. However, in view point of functional analysis, the property (T. 2) is not satisfactory for us, because the second passage time t appears in the first operator $T_s(\varphi_t y)$. Thus, in the formulation of quasi-semigroup, (T. 2) should be replaced by

(T. 2*) $T_{s+t}(y) = T_t(\varphi_s y)\ T_s(y)$ for every s, t and a.a.y .

Then a questions arises.

Question 2. Consider a quasi-semigroup $\{T_t(y)\}_{t\geq 0,\ y\in Y}$ satisfying (T. 2*) instead of (T. 2) . Let $f \in L_1(X \otimes Y)$. Does

$$\lim_{\alpha\to\infty} \frac{\int_0^\alpha T_t(y)\ f\ (x,\varphi_t y)\ dt}{\int_0^\alpha p_t(x,y)\ dt}$$

exist a.e. in $\{(x,y) : \int_0^\infty p_t(x,y)\ dt > 0\}$?

I have still nothing to tell you about the answer.

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