ON THE EULER-POINCARÉ CHARACTERISTICS OF FINITE DIMENSIONAL p-ADIC GALOIS REPRESENTATIONS

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CONTENTS

1. Introduction

Let p be a prime number, and let V be a finite dimensional vector space over the field $\mathbf{Q}_p$ of p-adic numbers. We write GL(V) for the group of $\mathbf{Q}_p$-linear automorphisms of V. Let G_V denote a compact subgroup of GL(V), so that G_V is a p-adic Lie group. We write $H^i(G_V, V)$ for the cohomology groups of G_V acting on V, which are defined by continuous cochains, where V is endowed with the p-adic topology. We shall say that our representation V of G_V has **vanishing** G_V**-cohomology** if $H^i(G_V, V) = 0$ for all $i \geqslant 0$. More generally, if V' is any finite dimensional continuous representation of G_V over $\mathbf{Q}_p$, we shall say that V' has vanishing G_V-cohomology if $H^k(G_V, V') = 0$ for all $k \geqslant 0$. The first interesting example of such V with vanishing G_V-cohomology which occur in arithmetic geometry is due to Serre [26], where G_V is the image of Galois in the automorphism group of the Tate module of an abelian variety defined over a finite extension of $\mathbf{Q}$. One of the aims of the present paper is to establish a broad class of new examples arising from the étale cohomology of smooth proper algebraic varieties defined over a finite extension of $\mathbf{Q}_p$, and having potential good reduction.

Throughout this paper, F will always denote a finite extension of $\mathbf{Q}_p$. Let Y be a smooth proper variety defined over such a field F. As usual, we write $Y_{\overline{\mathbf{Q}}_p}$ for the extension of scalars of Y to the algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$. For each $i \geqslant 0$, let

$$H^i_{\text{ét}}(Y_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p) = H^i_{\text{ét}}(Y_{\overline{\mathbf{Q}}_p}, \mathbf{Z}_p) \underset{\mathbf{Z}_p}{\otimes} \mathbf{Q}_p$$

denote the étale cohomology of $Y_{\overline{\mathbf{Q}}_p}$ with coefficients in $\mathbf{Q}_p$. We shall also consider the standard Tate twists of these cohomology groups by roots of unity. Put

$$T_p(\mu) = \varprojlim \mu_{p^n}, \qquad V_p(\mu) = T_p(\mu) \underset{\mathbf{Z}_p}{\otimes} \mathbf{Q}_p.$$

In general, if V is any finite dimensional vector space over $\mathbf{Q}_p$ on which the Galois group G_F acts continuously, and j is any integer, we define

$$V(j) = V \underset{\mathbf{Q}_p}{\otimes} V_p(\mu)^{\otimes j}$$

endowed with the twisted G_F action given by $\sigma(v \otimes a) = \sigma(v) \otimes \sigma(a)$. When $V = H^i_{\text{ét}}(Y_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p)$, we have a canonical isomorphism of G_F-modules

$$V(j) \simeq H^i_{\text{ét}}(Y_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p(j)) = \varprojlim \; H^i_{\text{ét}}(Y_{\overline{\mathbf{Q}}_p}, \mu_{p^n}^{\otimes j}) \underset{\mathbf{Z}_p}{\otimes} \mathbf{Q}_p.$$

These cohomology groups are finite dimensional vector spaces over $\mathbf{Q}_p$. It will be convenient to write

$$(\mathbf{1}) \qquad \rho : G_F \longrightarrow GL(H^i_{\text{ét}}(Y_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p(j)))$$

for the homomorphism giving the action of G_F on these vector spaces. Again, let V be any representation of G_F on a finite dimensional vector space over $\mathbf{Q}_p$. We recall that a **Galois subquotient** of V is a representation of G_F of the form V_1/V_2, where $V_1 \supset V_2$ are $\mathbf{Q}_p$-subspaces of V, stable under the action of G_F. If V' is such a Galois subquotient, we again write $G_{V'}$ for the corresponding image of G_F in $GL(V')$. We can then consider both the G_V-cohomology and $G_{V'}$-cohomology of V'.

Theorem **1.1.** — *Let* Y *be a smooth proper variety defined over* F *with potential good reduction. Let* $i \geqslant 0$ *and* j *be any integers such that* $i \neq 2j$. *Put*

$$(\mathbf{2}) \qquad V = H^i_{\text{ét}}(Y_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p(j)), \quad G_V = \rho(G_F).$$

Then V *has vanishing* G_V*-cohomology. Moreover, if* V' *is any Galois subquotient of* V*, then* V' *has both vanishing* G_V*-cohomology and vanishing* $G_{V'}$*-cohomology.*

It seems worthwhile to record the following corollary of Theorem 1.1, which was not known before. Let A be an abelian variety of dimension g defined over F, and let $A_{p^n} (n = 1, 2, \ldots)$ denote the group of p^n-division points on A. As usual, we let

$$T_p(A) = \varprojlim \; A_{p^n}, \qquad V_p(A) = T_p(A) \underset{\mathbf{Z}_p}{\otimes} \mathbf{Q}_p.$$

If A has good reduction over F, we write $\hat{A}$ for the formal group of A over $\mathcal{O}_F$, and $\tilde{A}$ for the reduction of A. We have the associated G_V-subspace $\hat{V} = V_p(\hat{A}) \subset V$, where

$$(\mathbf{3}) \qquad \hat{T} = T_p(\hat{A}) = \varprojlim \; \hat{A}_{p^n}, \qquad V_p(\hat{A}) = T_p(\hat{A}) \underset{\mathbf{Z}_p}{\otimes} \mathbf{Q}_p,$$

and the quotient $\tilde{V} = V/\hat{V} = V_p(\tilde{A})$.

Corollary **1.2.** — *Let* A *be an abelian variety defined over F with potential good reduction. Then* $V = V_p(A)$ *has vanishing* G_V*-cohomology, where* G_V *denotes the image of* G_F *in* GL(V). *Moreover if* A *has good reduction over* F*, then the Galois representations* $\hat{V} = V_p(\hat{A})$ *and* $\tilde{V} = V_p(\tilde{A})$ *both have vanishing* G_V*-cohomology.*

We shall prove in §5 that $V = V_p(A)$ has vanishing G_V-cohomology for all elliptic curves A over F, irrespective of whether A has potential good reduction or not. We are grateful to R. Greenberg for pointing out to us that this condition however is not true for all abelian varieties. Indeed, as Greenberg remarked, the group $H^1(G_V, V_p(A))$ is not zero, when $A = E_1 \times E_2$, where E_1 and E_2 are non-isogenous elliptic curves defined over F, with split multiplicative reduction. The non-vanishing of $H^1(G_V, V_p(A))$ is easily seen by looking at the inflation-restriction sequence for $H^1(G_V, V_p(E_i))$ relative to the kernel of the action of G_V on $V_p(E_i)$.

Again, let V′ be any finite dimensional continuous representation of G_V over $\mathbf{Q}_p$. The main theme of the present paper will be the calculation of a certain Euler characteristic, which we now define. Let T′ be any $\mathbf{Z}_p$-lattice in V′, which is stable under the action of G_V. Then V′/T′ is a discrete p-primary divisible G_V-module. Assume now that V′ has vanishing G_V-cohomology. As is well known (see [33]), it follows that $H^i(G_V, V'/T')$ is finite for all $i \geqslant 0$. Moreover, since G_V is a p-adic Lie group, it has finite p-cohomological dimension if and only if G_V has no element of order p (see [19], [24]). Assuming that G_V has no element of order p, we can therefore define

$$\chi_f(G_V, V') = \prod_{i \geqslant 0} \# \left(H^i(G_V, V'/T')\right)^{(-1)^i}. \tag{4}$$

It makes sense to write $\chi_f(G_V, V')$ in many cases which arise in arithmetic geometry, since we shall prove in §2 (see Lemma 2.1) that, provided G_V admits a quotient isomorphic to the additive group of p-adic integers $\mathbf{Z}_p$, the right hand side of (4) is independent of the choice of the G_V-invariant lattice T. It seems to be an interesting question to calculate $\chi_f(G_V, V')$ under the above hypotheses on V′ (see [30], [34], [8] for earlier work in this direction, and [7] for the original motivation, coming from Iwasawa theory). We mention here that Totaro [34] has shown that the Euler characteristic of a finite dimensional $\mathbf{Q}_p$-representation of a p-adic Lie group is usually 1, if it has vanishing cohomology. Our work proves the vanishing of the cohomology for a large class of finite dimensional representations of p-adic Lie groups which come from motivic $\mathbf{Q}_p$-representations of the Galois groups of p-adic local fields, and also that their Euler characteristic is indeed 1, by totally different methods. In general, it is not easy to compare our results with Totaro's because it is not known when the image of Galois in the automorphism group of such a motivic representation satisfies the conditions imposed by Totaro [34, Theorem 0.1]. The main local result of this paper is as follows:

Theorem **1.3.** — *Let* Y *be a smooth proper variety defined over* F *with potential good reduction. Let* V *and* G_V *be given by* (2)*, and assume that* G_V *has no element of order p. Then, for each odd integer* $i = 1, 3, 5, \ldots$, *we have*

$$\chi_f(G_V, V) = 1. \tag{5}$$

If V' *is any Galois subquotient of* V*, we also have* $\chi_f(G_V, V') = 1$.

The following corollary is a partial local analogue of the global result proven in [8]. Let A_{p^∞} denote the group of all p-power division points on an abelian variety A over F. If A has good reduction over F, we write $\widehat{A}_{p^\infty}$ and $\widetilde{A}_{p^\infty}$ for the group of all p-power division points on the formal group of A and the reduction of A, respectively.

Corollary **1.4.** — *Let* A *be an abelian variety defined over* F *with potential good reduction. Let* $V = V_p(A)$*, and let* G_V *denote the image of* G_F *in* GL(V)*. Assume that* G_V *has no element of order p. Then*

$$\chi_f(G_V, A_{p^\infty}) = 1. \tag{6}$$

Further, if A *has good reduction over* F*, then* $\chi_f(G_V, \widehat{A}_{p^\infty}) = 1$ *and* $\chi_f(G_V, \widetilde{A}_{p^\infty}) = 1$.

We prove in §5 that (6) is true for all elliptic curves A over F, irrespective of whether or not they have potential good reduction. Here, of course, $\chi_f(G_V, A_{p^\infty})$ is defined by the right hand side of (4) with $V'/T' = A_{p^\infty}$.

We prove Theorem 1.3 using the technique of [8]. We recall that this technique exploits the fact that, under the hypotheses of Theorem 1.3, the p-adic Lie group G_V has a quotient isomorphic to $\mathbf{Z}_p$. Let F_∞ denote the $\mathbf{Z}_p$-extension of F contained in the field $F(\mu_{p^\infty})$, where μ_{p^∞} denotes the group of all p-power roots of unity. We put $H_V = \rho(G_{F_\infty})$, where G_{F_∞} denotes the Galois group of $\overline{\mathbf{Q}}_p$ over F_∞. As in [8], Theorem 1.3 is an easy consequence of the following result.

Theorem **1.5.** — *Let* Y *be a smooth proper variety defined over* F *with potential good reduction. Let* V *be given by* (2)*, and let* $H_V = \rho(G_{F_\infty})$*, where* G_{F_∞} *denotes the Galois group of* $\overline{\mathbf{Q}}_p$ *over the cyclotomic* $\mathbf{Z}_p$*-extension* F_∞ *of* F*. Then, for each odd integer* $i = 1, 3, 5, \ldots$, V *has vanishing* H_V*-cohomology. If* V' *is any Galois subquotient representation of* V*, then* V' *has vanishing* H_V*-cohomology.*

We are grateful to J.-M. Fontaine for pointing out to us that the following corollary of Theorem 1.5 is a well-known consequence of Serre's description in [25] of all p-adic Hodge-Tate representations for which the image of Galois is abelian (see also Imai [17]).

Corollary **1.6.** — *Let* Y *be a smooth proper variety defined over* F, *with potential good reduction. Assume* V *is given by* (2). *Then, for all odd integers* $i = 1, 3, 5, \ldots$, *and any Galois subquotient representation* V′ *of* V, *we have* $H^0(G_{F_\infty}, V') = 0$, *where* G_{F_∞} *denotes the Galois group of* $\overline{\mathbf{Q}}_p$ *over the cyclotomic* $\mathbf{Z}_p$*-extension of* F.

The case when $V = V_p(A)$, where A is an elliptic curve with split multiplicative reduction, shows that the hypothesis of potential good reduction is necessary. Indeed, in this case, the Tate curve shows that $H^0(G_{F_\infty}, V_p(A)) = V_p(\mu)$ provided $\mu_p \subset F$ if p is odd and $\mu_4 \subset F$ if $p = 2$.

We now briefly discuss global analogues of these results that arise when we take a smooth proper algebraic variety Y which is now defined over a finite extension K of $\mathbf{Q}$. Let $Y_{\overline{\mathbf{Q}}}$ denote the extension of scalars of Y to $\overline{\mathbf{Q}}$. We take

$$V = H^i_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_p(j)), \qquad G_V = \rho(G_K),$$

where G_K is the Galois group of $\overline{\mathbf{Q}}$ over K. It has long been known (see [27, 2.4]) that the global analogue of Theorem 1.1 is true without any hypotheses on Y about good reduction at the primes dividing p. However, we stress that it is still unknown at present whether V is a semisimple G_K-module (although this result has been proven by Faltings [11] when Y is an abelian variety and $i = 1$; the H^i for an abelian variety are all semisimple Galois representations as they are exterior powers of H^1). Thus the global method of [8] cannot be applied to study the Euler characteristic $\chi_f(G_V, V)$ in the global case. Nevertheless, the proof of Theorem 1.5 leads to the following new global result.

Theorem **1.7.** — *Let* Y *be a smooth proper algebraic variety defined over a finite extension* K *of* $\mathbf{Q}$. *Assume that* Y *has potential good reduction at at least one prime* v *of* K *dividing* p. *Let*

$$V = H^i_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathbf{Q}_p(j)), \qquad G_V = \rho(G_K),$$

and assume that G_V *has no element of order* p. *Then, for each Galois subquotient* V′ *of* V, *we have* $\chi_f(G_V, V') = 1$ *for odd integers* $i = 1, 3, 5, \ldots$.

Since this paper was written, a variant of the method here has been used in [32] to prove Theorem 1.7 without the hypothesis that Y has potential good reduction at at least one prime v of K above p.

Finally, we mention a somewhat more technical result which emerges from our arguments (see §3 and §4 for a full explanation). The analogous global result when V is the étale cohomology of a smooth projective variety defined over a finite extension of $\mathbf{Q}$ is classical and due to Deligne (see [27, §2.3]).

Theorem **1.8.** — *Let* $\rho : G_F \to GL(V)$ *be a potentially crystalline Galois representation such that the endomorphism* Φ *of the filtered module* D(V) *attached to* V *has eigenvalues which are*

Weil numbers of some fixed weight w*, where* $w \neq 0$*. Let* $\mathrm{L}(\mathrm{G}_{\mathrm{V}}^{\mathrm{alg}})$ *be the Lie algebra of the algebraic envelope* $\mathrm{G}_{\mathrm{V}}^{\mathrm{alg}}$ *of* $\mathrm{G}_{\mathrm{V}} = \rho(\mathrm{G}_{\mathrm{F}})$*. Then* $\mathrm{L}(\mathrm{G}_{\mathrm{V}}^{\mathrm{alg}})$ *contains the homotheties.*

Corollary **1.9.** — *The Lie algebra* $\mathrm{L}(\mathrm{G}_{\mathrm{V}}^{\mathrm{alg}})$ *contains the homotheties when* $\mathrm{V} = \mathrm{V}_p(\mathrm{A})$ *for an abelian variety* A *over* F *with potential good reduction, or more generally when* V *is any Galois subquotient of* $\mathrm{H}^i_{\text{ét}}(\mathrm{Y}_{\overline{\mathbf{Q}_p}}, \mathbf{Q}_p(j))$*, where* Y *is a smooth proper variety over* F *with potential good reduction, and* $i \geqslant 0$*,* j *are integers with* $i \neq 2j$*.*

We remark that it is not true, under the hypotheses of Theorem 1.8, that the Lie algebra $\mathrm{L}(\mathrm{G}_{\mathrm{V}})$ of G_{V} itself always contains the homotheties. Indeed, consider $\mathrm{V} = \mathrm{V}_p(\mathrm{E}_1) \times \mathrm{V}_p(\mathrm{E}_2)$, where E_1 and E_2 are elliptic curves over F with good ordinary reduction. Let u_i be the p-adic unit eigenvalue giving the action of Frobenius on $\mathrm{V}_p(\widetilde{\mathrm{E}}_i)$, where $\widetilde{\mathrm{E}}_i$ denotes the reduction of E_i, $i = 1, 2$. Assume that u_1/u_2 is not a root of unity, so that the p-adic logarithms $\log(u_1)$ and $\log(u_2)$ are distinct. Then the homotheties cannot belong to $\mathrm{L}(\mathrm{G}_{\mathrm{V}})$, since the logarithm of the image of Frobenius acting on $\mathrm{V}_p(\widetilde{\mathrm{E}}_1) \times \mathrm{V}_p(\widetilde{\mathrm{E}}_2)$ is not a homothety.

The paper is organised as follows. In section 2, we state the results in an abstract setting of Lie algebra representations. The results in this section establish sufficient conditions for vanishing of cohomology and the triviality of the Euler characteristic, in terms of the existence of certain special elements in the Lie algebra and is inspired by the methods in [26]. This approach allows us to work with representations that are not necessarily semisimple. Section 3 forms the heart of the paper and uses the theory of semistable representations due to Fontaine [13], [14], to construct such special elements in the Lie algebra of the image of the Galois representation. The paper [21] of R. Pink provided the initial inspiration for this approach. These results are applied in section 4 to prove the results announced in section 1. Finally, in section 5, we consider the general case of elliptic curves with semistable reduction.

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2. Abstract setting

In this section, we describe in a general setting the arguments from Lie algebra and Lie group cohomology, which underlie the proofs of the results described in §1. These arguments are similar in spirit to those used in [8], but they have the fundamental advantage that they apply to representations of p-adic Lie groups which are not necessarily semisimple.

As in §1, let V be a finite dimensional vector space over the field $\mathbf{Q}_p$ of p-adic numbers, and let G_V denote a compact subgroup of GL(V), so that G_V is a p-adic Lie group.

Lemma **2.1.** — *Let* V' *be any finite dimensional* $\mathbf{Q}_p$*-representation of* G_V*, with vanishing* G_V*-cohomology. Assume that* G_V *has no element of order p, and that* G_V *is either pro-p, or* G_V *has a quotient isomorphic to* $\mathbf{Z}_p$*. Then the Euler characteristic given by the right hand side of* (4) *does not depend on the particular choice of* $\mathbf{Z}_p$*-lattice* T'*, which is stable under the action of* G_V.

Proof. — The essential point of the proof is the fact that, under our hypotheses on G_V, we have

$$\chi_f(G_V, M) = 1, \tag{7}$$

for all finite p-primary G_V-modules M, where, as usual $\chi_f(G_V, M)$ denotes the Euler characteristic

$$\chi_f(G_V, M) = \prod_{i \geqslant 0} \# \ (H^i(G_V, M))^{(-1)^i};$$

here the cohomology groups on the right are finite because G_V is a p-adic Lie group. Now it is well known that (7) is true when G_V is pro-p [29, Chap. I, Ex. 4.1 (e)]. To prove that (7) is true when G_V has a quotient Γ isomorphic to $\mathbf{Z}_p$, we use the argument of [8] (see the proof in [8] after Prop. 2). We give this argument in the proof of Theorem 2.4, and do not repeat it once more here. The key observation is that, writing H_V for the kernel of the homomorphism from G_V onto Γ, we again have $H^i(H_V, M)$ finite for all $i \geqslant 0$, because M is finite and H_V is a p-adic Lie group, and (7) then follows from the Hochschild-Serre spectral sequence as is explained in [8].

Here is the standard argument required to prove Lemma 2.1 from (7). Let T' denote any $\mathbf{Z}_p$-lattice in V', which is stable under the action of G_V. For each $n \neq 0$ in $\mathbf{Z}$, it is clear that multiplication by n on V' maps T' isomorphically to $T'' = nT'$, and induces a G_V-isomorphism from V'/T' onto V'/T''. Hence, if T_1 and T_2 are any two $\mathbf{Z}_p$-lattices in V' which are stable under the action of G_V, we may replace T_1 by nT_1 for a suitable non-zero integer n, and so assume that $T_1 \subset T_2$. Then we have the exact sequence of G_V-modules

$$0 \to M \to V'/T_1 \to V'/T_2 \to 0, \tag{8}$$

where $M = T_2/T_1$ is a finite G_V-module. Now, as V' has vanishing G_V-cohomology, all G_V-cohomology groups of the modules occurring in (8) are finite, and so we conclude from (7) and the multiplicativity of the Euler characteristic that $\chi_f(G_V, V'/T_1) = \chi_f(G_V, V'/T_2)$. This completes the proof of Lemma 2.1.

We now explain the basic argument from Lie algebra cohomology, due to Serre [26] and Bourbaki [3, Chap. 7, §1, Ex. 6] which we use to prove the vanishing of cohomology. Let K denote a field of characteristic 0, and U a finite dimensional vector space over K. Let $\mathfrak{G}$ be a finite dimensional Lie algebra over K. Suppose we are given a Lie algebra homomorphism

$$\tau : \mathfrak{G} \to \mathrm{End}(U).$$

As usual, we write $H^i(\mathfrak{G}, U)$ for the Lie algebra cohomology groups of U, which are K-vector spaces, and we shall say that U has **vanishing $\mathfrak{G}$-cohomology** if $H^k(\mathfrak{G}, V) = 0$ for all $k \geqslant 0$. Let $\mathrm{ad}_{\mathfrak{G}} : \mathfrak{G} \to \mathrm{End}(\mathfrak{G})$ denote the adjoint representation of $\mathfrak{G}$. Suppose m is the dimension of U over K, and n is the dimension of $\mathfrak{G}$ over K. For an element X of $\mathfrak{G}$, we write $\mu_1, \ldots, \mu_m$ for the eigenvalues of $\tau(X)$ and $\lambda_1, \ldots, \lambda_n$ for the eigenvalues of $\mathrm{ad}_{\mathfrak{G}}(X)$, repeated according to their multiplicities, in some fixed algebraic closure of K. We shall say **Serre's criterion** holds for the representation τ of $\mathfrak{G}$ if there exists an element X of $\mathfrak{G}$ with the following property: for each integer k with $0 \leqslant k \leqslant n$, we have

$$(\mathbf{9}) \qquad \mu_j - (\lambda_{i_1} + \ldots + \lambda_{i_k}) \neq 0$$

for all integers j with $1 \leqslant j \leqslant m$ and all choices of $i_1, \ldots, i_k$ with $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n$ (when $k = 0$, this should be interpreted as $\mu_j \neq 0$ for $1 \leqslant j \leqslant m$).

When the representation τ is faithful, we shall work with the original criterion used by Serre [26]. If X is in $\mathfrak{G}$, write A(X) for the set of distinct eigenvalues of $\tau(X)$ in an algebraic closure of K. We shall say that the faithful representation τ of $\mathfrak{G}$ satisfies the **strong Serre criterion** if there exists an element X of $\mathfrak{G}$ as follows: for every integer $k \geqslant 0$, and for each choice $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_{k+1}$ of $2k+1$ not necessarily distinct elements of A(X), we have

$$(\mathbf{10}) \qquad \alpha_1 + \ldots + \alpha_k \neq \beta_1 + \ldots + \beta_{k+1}$$

(when $k = 0$, this should be interpreted as meaning that every eigenvalue of $\tau(X)$ is non-zero). For faithful τ, the strong Serre criterion for $\mathfrak{G}$ does indeed imply the Serre criterion because, in this case, the eigenvalues of $\mathrm{ad}_{\mathfrak{G}}(X)$ are of the form $\mu_i - \mu_j$ (see [3, Chap. 1, §5.4, Lemma 2]) where μ_i, μ_j are elements in A(X). We refer to [3, Chap. 7, §1, Ex. 6] for the proof of the following result.

Lemma **2.2.** — *If τ satisfies the Serre criterion, then* U *has vanishing $\mathfrak{G}$-cohomology. In particular, if τ is faithful and satisfies the strong Serre criterion, then* U *has vanishing $\mathfrak{G}$-cohomology.*

Returning to our finite dimensional vector space V over $\mathbf{Q}_p$ and our p-adic Lie group $G_V \subset GL(V)$, we write $L(G_V) \subset End(V)$ for the Lie algebra of G_V. Let $\overline{\mathbf{Q}_p}$ be our fixed algebraic closure of $\mathbf{Q}_p$. For any vector space W over $\mathbf{Q}_p$, we write $W_{\overline{\mathbf{Q}_p}}$ for $W \underset{\mathbf{Q}_p}{\otimes} \overline{\mathbf{Q}_p}$.

Proposition **2.3.** — *Assume that* $L(G_V)_{\overline{\mathbf{Q}_p}} \subset End(V_{\overline{\mathbf{Q}_p}})$ *satisfies the strong Serre criterion. Then* V *has vanishing* G_V*-cohomology.*

Proof. — It is well-known and easy to see that

$$H^k(L(G_V)_{\overline{\mathbf{Q}_p}}, V_{\overline{\mathbf{Q}_p}}) = H^k(L(G_V), V)_{\overline{\mathbf{Q}_p}} \qquad (k \geqslant 0).$$

Hence Lemma 2.2 implies that $H^k(L(G_V), V) = 0$ for all $k \geqslant 0$. But, by a basic result of Lazard [19], $H^k(G_V, V)$ is a $\mathbf{Q}_p$-vector subspace of $H^k(L(G_V), V)$ for all $k \geqslant 0$. Thus we conclude that V has vanishing G_V-cohomology, as required.

We write $\det : GL(V) \to \mathbf{Q}_p^\times$ for the determinant map, and, as usual, SL(V) will denote its kernel. The following result is the basic one we use to establish the principal results of §1. It is parallel to Theorem 1 of [8], but with the basic difference that the hypothesis of semisimplicity is replaced by Serre's criterion.

Theorem **2.4.** — *Let* $H_V = SL(V) \cap G_V$*, and let* $L(H_V)$ *denote the Lie algebra of* H_V*. Assume that* $L(H_V)_{\overline{\mathbf{Q}_p}} \subset End(V_{\overline{\mathbf{Q}_p}})$ *satisfies the strong Serre criterion. If* $\det(G_V)$ *is infinite and* G_V *has no element of order p, then we have* $\chi_f(G_V, V) = 1$*, where* $\chi_f(G_V, V)$ *is defined by* (4).

Proof. — Assuming that $L(H_V)_{\overline{\mathbf{Q}_p}} \subset End(V_{\overline{\mathbf{Q}_p}})$ satisfies the strong Serre criterion, the same argument as in the proof of Proposition 2.3 shows that V has vanishing $L(H_V)$-cohomology. Hence, by Lazard's theorem, V has vanishing H_V-cohomology.

Now assume that $\det(G_V)$ is infinite. As $\det(G_V)$ is a closed subgroup of $\mathbf{Z}_p^\times$, it must therefore be isomorphic to $\mathbf{Z}_p \times \Delta$, where Δ is a finite abelian group. Hence there exists a closed normal subgroup J_V of G_V, which contains H_V as an open subgroup, such that $\Gamma = G_V/J_V$ is isomorphic to $\mathbf{Z}_p$. Since H_V is open in J_V, the Lie algebra $L(J_V)$ of J_V is equal to $L(H_V)$, and hence V has vanishing $L(J_V)$-cohomology. Again, by Lazard's theorem, it follows that V has vanishing J_V-cohomology.

Suppose now that G_V contains no element of order p, so that it has finite p-cohomological dimension. As in [8], we now complete the proof by applying the Hochschild-Serre spectral sequence. We pick any $\mathbf{Z}_p$-lattice T in V which is stable under G_V, and put $B = V/T$. Since $\Gamma = G_V/J_V$ is isomorphic to $\mathbf{Z}_p$, it has p-cohomological dimension equal to 1. Hence the Hochschild-Serre spectral sequence yields, for all $k \geqslant 1$, the exact sequence

$$0 \to H^1(\Gamma, H^{k-1}(J_V, B)) \to H^k(G_V, B) \to H^0(\Gamma, H^k(J_V, B)) \to 0. \qquad (\mathbf{11})$$

Now the fact that V has vanishing J_V-cohomology implies that $H^k(J_V, B)$ is finite for all $k \geqslant 0$. In turn, this implies that

$$\#\,(H^0(\Gamma, H^k(J_V, B))) = \#\,(H^1(\Gamma, H^k(J_V, B))) \text{ for all } k \geqslant 0.$$

Denoting the common order of the above groups by h_k, we conclude from (11) that $H^k(G_V, B)$ is finite of order $h_k \cdot h_{k-1}$ for all $k \geqslant 1$. We deduce that V has vanishing G_V-cohomology, and that

$$\chi_f(G_V, V) = h_0 \cdot (h_0 h_1)^{-1} \cdot (h_1 h_2) \cdot (h_2 h_3)^{-1} \ldots = 1.$$

This completes the proof of Theorem 2.4.

By a G_V**-subquotient** of V, we shall always mean a quotient of a G_V-invariant subspace of V by another such one contained in it. We now discuss the analogues of Proposition 2.3 and Theorem 2.4 for an arbitrary G_V-subquotient V' of V. We write $G_{V'}$ for the image of the natural map of G_V in $GL(V')$. Let $H_{V'} = G_{V'} \cap SL(V')$ be the elements of determinant 1 in $G_{V'}$. Note that we can now consider both the G_V and the $G_{V'}$-cohomology of V'; of course, the representation of G_V on V' is no longer faithful in general.

Proposition **2.5.** — *Assume that* $L(G_V)_{\overline{\mathbf{Q}}_p} \subset End(V_{\overline{\mathbf{Q}}_p})$ *satisfies the strong Serre criterion. Then, for every* G_V*-subquotient* V' *of* V, *we have* $H^i(G_V, V') = H^i(G_{V'}, V') = 0$ *for all* $i \geqslant 0$.

Proof. — We claim that the hypothesis of the proposition implies that $L(G_{V'})_{\overline{\mathbf{Q}}_p} \subset End(V'_{\overline{\mathbf{Q}}_p})$ also satisfies the strong Serre criterion. Indeed, if X is an element of $L(G_V)_{\overline{\mathbf{Q}}_p} \subset End(V_{\overline{\mathbf{Q}}_p})$ satisfying (10) for all $k \geqslant 0$, then its image $X_{V'}$ in $L(G_{V'})_{\overline{\mathbf{Q}}_p} \subset End(V'_{\overline{\mathbf{Q}}_p})$ under the natural map also satisfies (10), because the eigenvalues of $X_{V'}$ form a subset of the eigenvalues of X. Hence V' has vanishing $G_{V'}$-cohomology.

Next, let

$$\tau \;:\; L(G_V)_{\overline{\mathbf{Q}}_p} \to End(V'_{\overline{\mathbf{Q}}_p})$$

be the representation obtained from the original representation. Let X be as above, so that $\tau(X) = X_{V'}$. Since the eigenvalues of $ad_{L(G_V)_{\overline{\mathbf{Q}}_p}}(X)$ are of the form $\mu_i - \mu_j$, [3, Chap. 1, §5.4, Lemma 2] where $\{\mu_h\}$ denotes the eigenvalues of X, it is clear that the strong Serre criterion for X implies that (9) is valid for $\tau(X)$. Thus $H^k(L(G_V), V') = 0$ for all $k \geqslant 0$, whence $H^k(G_V, V') = 0$ for all $k \geqslant 0$. This finishes the proof of Proposition 2.5.

Theorem **2.6.** — *Assume that* $L(H_V)_{\overline{\mathbf{Q}}_p} \subset End(V_{\overline{\mathbf{Q}}_p})$ *satisfies the strong Serre criterion. Let* V' *be any* G_V*-subquotient of* V. *Then the following assertions hold:*

(i) V' *has vanishing* H_V*-cohomology.*
(ii) *If* $\det(G_V)$ *is infinite and* G_V *has no element of order p, then* $\chi_f(G_V, V') = 1$.

Proof. — Assertion (i) follows as in the proof of Proposition 2.5. Assertion (ii) follows from (i) along the lines of the proof of Theorem 2.4, as G_V has a quotient isomorphic to $\mathbf{Z}_p$.

3. Construction of elements in the Lie algebra

Let p be a prime number, F a finite extension of $\mathbf{Q}_p$, and G_F the Galois group of $\overline{\mathbf{Q}}_p$ over F. We suppose that we are given a continuous p-adic representation

(**12**) $$\rho : G_F \to GL(V),$$

where V is a $\mathbf{Q}_p$-vector space of finite dimension d. As before, we write $G_V = \rho(G_F)$. If we are to successfully apply the Lie algebra criteria of §2 to study the G_V-cohomology of V, we must be able to construct elements in the Lie algebra

$$L(G_V)_{\overline{\mathbf{Q}}_p} = L(G_V) \underset{\mathbf{Q}_p}{\otimes} \overline{\mathbf{Q}}_p.$$

The aim of this section is to take an important first step in this direction when V is a **semistable** Galois representation in the sense of Fontaine [14].

Let F_0 denote the maximal unramified extension of $\mathbf{Q}_p$ contained in F. We recall that V is said to be semistable if

$$D(V) = (B_{st} \underset{\mathbf{Q}_p}{\otimes} V)^{G_F}$$

has dimension d over F_0, where $d = \dim_{\mathbf{Q}_p}(V)$; here B_{st} is Fontaine's ring for semistable representations (see [14]). Assume from now on that V is semistable. Then D(V) is a filtered (φ, N)-module in the sense of [13], [14]. We recall briefly the main properties of D(V). Firstly, D(V) is endowed with an F_0-linear endomorphism N, called the **monodromy operator**. Recall that the representation is said to be **crystalline** if $N = 0$. Secondly, D(V) is endowed with an isomorphism of additive groups $\varphi : D(V) \to D(V)$, which is σ-linear in the sense that $\varphi(av) = \sigma(a)\varphi(v)$ for a in F_0 and v in D(V); here σ denotes the arithmetic Frobenius in the Galois group of F_0 over $\mathbf{Q}_p$ (i.e. σ operates on the residue field of F_0 by raising to the p-th power). Let k_F denote the residue field of F, and suppose that k_F has cardinality $q = p^f$. Hence

(**13**) $$\Phi = \varphi^f$$

will be an F_0-linear automorphism of D(V), and we shall refer to it as the **Frobenius endomorphism** associated to the filtered module. Recall that we have

$$N\Phi = q\Phi N.$$

By definition, Φ is an automorphism of a vector space on which G_F acts trivially. It is therefore somewhat surprising that we shall show, using the Tannakian formalism of [9] and an elementary fact about Fontaine's theory for unramified representations, that Φ gives rise to interesting elements in the Lie algebra $L(G_V)_{\overline{\mathbf{Q}}_p}$ of $G_V = \rho(G_F)$ of our original Galois representation.

We recall the extension of the p-adic logarithm to the multiplicative group of $\overline{\mathbf{Q}}_p$. Let $\overline{\mathscr{O}}$ denote the ring of integers of $\overline{\mathbf{Q}}_p$, $\overline{\mathscr{O}}^\times$ the group of units of $\overline{\mathscr{O}}$, and $\overline{\mathfrak{m}}$ the maximal ideal of $\overline{\mathscr{O}}$. The usual series for $\log z$ converges on $1+\overline{\mathfrak{m}}$. Let μ denote the group of all roots of unity of $\overline{\mathbf{Q}}_p$ of order prime to p. Then $\overline{\mathscr{O}}^\times = \mu \times (1+\overline{\mathfrak{m}})$. We extend $\log z$ to $\overline{\mathscr{O}}^\times$ by specifying that $\log(\mu) = 0$. Now fix any non-zero element π of $\overline{\mathbf{Q}}_p$ whose absolute value is less than one. If x is any element of $\overline{\mathbf{Q}}_p^\times$, we can write $x = \pi^a y$, where $a \in \mathbf{Q}$ and $y \in \overline{\mathscr{O}}^\times$. We then define $\log_\pi(x) = \log(y)$. Note that this is well-defined as the ratio of any two such y must be a root of unity.

Theorem **3.1.** — *Assume that* V *is a semistable Galois representation, and let* D(V) *be the associated filtered* (φ, N)*-module. Let* $\lambda_1, \ldots, \lambda_d$ *denote the roots in* $\overline{\mathbf{Q}_p}$ *of the characteristic polynomial of the automorphism* $\Phi = \varphi^f$ *of* D(V). *Then there exists* X *in the Lie algebra* $L(G_V)_{\overline{\mathbf{Q}}_p}$ *of* $G_V = \rho(G_F)$ *such that the characteristic polynomial of* X *on* $V_{\overline{\mathbf{Q}}_p}$ *has roots* $\log_\pi(\lambda_1), \ldots, \log_\pi(\lambda_d)$.

We will also need an analogue of this result for the Lie algebra $L(H_V)_{\overline{\mathbf{Q}}_p}$, where, as before, $H_V = G_V \cap SL(V)$. We recall from [13], [14] that the F-vector space $D(V)_F = F \otimes_{F_0} D(V)$ is endowed with a canonical decreasing filtration $Fil^i D(V)_F$ ($i \in \mathbf{Z}$) of F-subspaces such that $Fil^i D(V)_F = D(V)_F$ for i sufficiently small and $Fil^i D(V)_F = 0$ for i sufficiently large. This filtration enables us to define the so called **Hodge-Tate weights** of D(V). These are a family of integers

$$(\mathbf{14}) \qquad i_1 \leqslant i_2 \leqslant \ldots \leqslant i_d,$$

which are defined as follows. Any integer h occurs in (14) if $Fil^h D(V)_F \neq Fil^{h+1} D(V)_F$, and when h does occur its multiplicity in (14) is the F-dimension of the quotient $Fil^h D(V)_F / Fil^{h+1} D(V)_F$. We define

$$(\mathbf{15}) \qquad t = \sum_{h=1}^{d} i_h.$$

Let $\det : G_F \to \mathbf{Z}_p^\times$ denote the character of G_F obtained by composing the representation ρ with the determinant map. We write $\xi : G_F \to \mathbf{Z}_p^\times$ for the cyclotomic character of G_F, i.e. the character giving the action of G_F on $T_p(\mu)$. As V is a semistable G_F-module, it is known (see [14, Prop. 5.4.1]) that the restriction of det to the inertial subgroup I_F is equal to the restriction of ξ^{-t} to I_F, where t is given by (15). Further, the p-adic valuation of the determinant of Φ is equal to that of q^t. If we set $\det\Phi = q^t u$, with $u \in F_0$ a unit, then $\det : G_F \to \mathbf{Z}_p^\times$ is $\eta\xi^{-t}$ with η an unramified character and, moreover, η is of finite order if and only if u is a root of unity.

Theorem **3.2.** — *Assume that* V *is a semistable Galois representation, and that its determinant character det* : $G_F \to \mathbf{Z}_p^\times$ *coincides on an open subgroup of* G_F *with* ξ^{-t} *where* ξ *is the cyclotomic character, and t is given by* (15). *Let* $\lambda_1, \ldots, \lambda_d$ *denote the roots in* $\overline{\mathbf{Q}}_p$ *of the characteristic polynomial of the automorphism* $\Phi = \varphi^f$ *of* D(V). *Then there exists* X *in the Lie algebra* $L(H_V)_{\overline{\mathbf{Q}}_p}$ *of* $H_V = \rho(G_F) \cap SL(V)$ *such that, for a suitable ordering of* $\lambda_1, \ldots, \lambda_d$, *the roots of the characteristic polynomial of* X *on* $V_{\overline{\mathbf{Q}}_p}$ *are given by* $\log_\pi(\lambda_1 q^{-i_1}), \ldots, \log_\pi(\lambda_d q^{-i_d})$, *where* $q = p^f$, *and* $i_1, \ldots, i_d$ *denote the Hodge-Tate weights* (14).

Before embarking on the proof of these two theorems, we recall some basic definitions about algebraic groups, and the p-adic logarithms of their points. Let W be any finite dimensional vector space over some finite extension K of $\mathbf{Q}_p$. We write GL_W for the general linear group of W, considered as an algebraic group over K. Thus, for each finite extension M of K contained in $\overline{\mathbf{Q}}_p$, the group $GL_W(M)$ of M-points of W is the group of M-automorphisms of $W \underset{K}{\otimes} M$. If J denotes any algebraic subgroup of GL_W, we write L(J) for its Lie algebra, which coincides with the Lie algebra L(J(K)) of the p-adic Lie subgroup J(K). Now take θ to be any element of $GL_W(K)$. We can write $\theta = su$, where s is semisimple, u is unipotent, and s and u commute. As u is unipotent, some power of $u - 1$ is zero, and so we can define $\log(u)$ by the usual series $\log(u) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(u-1)^n}{n}$. We define $\log_\pi(s)$ as follows. We can write $\overline{\mathbf{Q}}_p \otimes_K W = \oplus W_i$, where W_i is some subspace on which the semisimple element s acts via some eigenvalue α_i. We then define $\log_\pi(s)$ to be the endomorphism of $\overline{\mathbf{Q}}_p \otimes_K W$, which operates on W_i by $\log_\pi(\alpha_i)$. In fact, it is clear that if π belongs to K, $\log_\pi(s)$ belongs to the endomorphism ring of W over the original base field K. We finally define

$$\log_\pi(\theta) = \log_\pi(s) + \log(u).$$

The automorphism θ topologically generates a compact subgroup of $GL_W(K)$ if and only if its eigenvalues α_i are units. If this is the case, then $\log_\pi(\alpha_i)$, and therefore $\log_\pi(\theta)$, do not depend of the choice of the α_i. In fact, we can define $\log(\theta)$ in a more natural manner. Let r denote the cardinality of $GL_m(k)$, where m is the dimension of W over K, and k is the residue field of K, and put $\beta = \theta^r$. Our hypothesis on θ shows that θ must stabilize a lattice in W, and so it is clear that the matrix A of β relative to a K-basis of W coming from this lattice must satisfy $A \equiv 1 \bmod \pi_K$, where π_K is any local parameter of K. We can then define

$$\log(\theta) = \frac{1}{r}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(A-1)^n}{n}.$$

The following lemma is well-known (cf. [3, Chap. III, §7.6, Propositions 10 and 13]).

Lemma **3.3.** — *Let* W *be a finite dimensional vector space over* K*, where* K *is a finite extension of* $\mathbf{Q}_p$*, and let* θ *be any element of* $GL_W(K)$*. If* θ *topologically generates a compact subgroup of* $GL_W(K)$*, then* $\log_\pi(\theta) = \log(\theta)$*. If* J *is any algebraic subgroup of* GL_W*, and* θ *belongs to* J(K)*, then* $\log_\pi(\theta)$ *belongs to the Lie algebra* L(J) *of* J.

Proof of Theorem 3.1. — If H is any subgroup of $GL_V(\mathbf{Q}_p)$, we define H^{alg} to be the Zariski closure of H in GL_V i.e. the intersection of all algebraic subgroups J of GL_V such that J is defined over $\mathbf{Q}_p$ and $J(\mathbf{Q}_p)$ contains H. As earlier, let I_F denote the inertial subgroup of the Galois group G_F. We put

$$G_V = \rho(G_F), \;\; I_V = \rho(I_F).$$

We then have the algebraic groups G_V^{alg} and I_V^{alg} in GL_V, and we can consider the four Lie algebras $L(G_V)$, $L(I_V)$, $L(G_V^{alg})$, $L(I_V^{alg})$. By a basic result of Serre and Sen, [23], we have

$$L(I_V) = L(I_V^{alg}). \tag{16}$$

Hence we have the inclusions

$$L(I_V^{alg}) \subset L(G_V) \subset L(G_V^{alg}). \tag{17}$$

By definition, the image of the Galois representation ρ of G_F is contained in $G_V^{alg}(\mathbf{Q}_p)$. Thus, for each representation α of the algebraic group G_V^{alg} in a finite dimensional $\mathbf{Q}_p$-vector space V_α, we obtain a new Galois representation:

$$\rho_\alpha : G_F \to G_V^{alg}(\mathbf{Q}_p) \to GL_{V_\alpha}(\mathbf{Q}_p) = GL(V_\alpha). \tag{18}$$

The main idea of the proof of Theorem 3.1 is to work with this new Galois representation ρ_α for a suitable choice of α. We first note that, for every such α, the Galois representation ρ_α is also semistable. Indeed, it is known [9, Proposition 2.20] that the representation α of G_V^{alg} is a subquotient of a finite direct sum of copies of tensor products of the tautological representation of G_V^{alg} and its dual. But it is also known that the category of semistable representations of G_F is stable under the Tannakian operations in the category of finite dimensional p-adic representations of G_F and so ρ_α must be semistable, because ρ is semistable. For simplicity, we write D_α

for the filtered (φ, N)-module associated with ρ_α. In general, we write a subscript α on each object associated with the original representation ρ to denote the corresponding object attached to ρ_α.

We put $G_{V_\alpha} = \rho_\alpha(G_F)$. The algebraic groups I_V^{alg}, G_V^{alg} clearly act on V_α, and we denote the images of these groups in GL_{V_α} by $I_{V_\alpha}^{alg}$, $G_{V_\alpha}^{alg}$, respectively.

For the rest of the proof, we fix the following representation of G_V^{alg}. As I_V^{alg} is clearly normal in G_V^{alg}, there exists a representation α_0 of G_V^{alg} in $GL_{V_{\alpha_0}}$ whose kernel is precisely I_V^{alg}. For simplicity, we write V_0 instead of V_{α_0}, ρ_0 instead of ρ_{α_0}, and so on for all objects attached to the representation ρ_{α_0}. We write Fr for the arithmetic Frobenius in G_F/I_F. Thus $\rho_0(\mathrm{Fr})$ topologically generates $G_{V_0} = \rho_0(G_F)$. Hence the Lie algebra $L(G_{V_0})$ of G_{V_0} is the one dimensional $\mathbf{Q}_p$-subspace of End (V_0) generated by $\log(\rho_0(\mathrm{Fr}))$; note that $\log(\rho_0(\mathrm{Fr}))$ is defined as explained earlier because $\rho_0(\mathrm{Fr})$ generates a compact group. Clearly $G_{V_0}^{alg}$ is the smallest algebraic subgroup of GL_{V_0} containing $\rho_0(\mathrm{Fr})$, and is abelian. Let $\pi : L(G_V^{alg}) \to L(G_{V_0}^{alg})$ denote the natural surjection. Since $L(I_V^{alg}) = L(I_V)$, we evidently have

$$L(G_V) = \pi^{-1}(L(G_{V_0})). \tag{19}$$

There are two basic steps in the proof of Theorem 3.1. The first uses all the representations α of G_V^{alg} to construct, via the Tannakian formalism, an element in the Lie algebra $L(G_V^{alg})_{\overline{\mathbf{Q}}_p}$ having the same eigenvalues as the endomorphism $\log_\pi(\Phi)$ of $D(V)$. The second step exploits the unramified Galois representation ρ_0 arising from α_0 to show, using Fontaine's theory in the modest case of unramified Galois representations, together with the key fact (19), the existence of the desired element X in $L(G_V)_{\overline{\mathbf{Q}}_p}$.

We now give the first step. Denote by $\mathrm{Rep}(G_V^{alg})$ (respectively, $\mathrm{Rep}(G_{V_\alpha}^{alg})$) the category of all finite dimensional $\mathbf{Q}_p$-representations of the algebraic group G_V^{alg} (respectively, $G_{V_\alpha}^{alg}$). Since α gives rise to a homomorphism from G_V^{alg} to $G_{V_\alpha}^{alg}$, we can identify $\mathrm{Rep}(G_{V_\alpha}^{alg})$ with a sub-category of $\mathrm{Rep}(G_V^{alg})$. We refer to [9] for the following basic facts about Tannakian categories and their associated formalism. We note that $\mathrm{Rep}(G_V^{alg})$ is a Tannakian category, and that $\mathrm{Rep}(G_{V_\alpha}^{alg})$ is a sub-Tannakian category. Let Vec_{F_0} denote the category of all finite dimensional vector spaces over the field F_0. We have two fibre functors over F_0,

$$\omega_G : \mathrm{Rep}(G_V^{alg}) \to \mathrm{Vec}_{F_0}, \qquad \omega_D : \mathrm{Rep}(G_V^{alg}) \to \mathrm{Vec}_{F_0}$$

which are defined by

$$\omega_G(\alpha) = V_\alpha \underset{\mathbf{Q}_p}{\otimes} F_0, \qquad \omega_D(\alpha) = D_\alpha;$$

we recall that D_α is the filtered (φ, N)-module attached by Fontaine's theory to the semistable Galois representation ρ_α. It is proven in [9, Proposition 2.8] that we can identify $G^{alg}_{V_\alpha}$ over F_0 with the group of $\otimes$-automorphisms of ω_G restricted to the sub-category $\mathrm{Rep}(G^{alg}_{V_\alpha})$ of $\mathrm{Rep}(G^{alg}_V)$. We define G^{alg}_D to be the algebraic group of $\otimes$-automorphisms of the fibre functor ω_D. Let $G^{alg}_{D_\alpha}$ denote the algebraic group of $\otimes$-automorphisms of ω_D restricted to the sub-category $\mathrm{Rep}(G^{alg}_{V_\alpha})$. Both G^{alg}_D and $G^{alg}_{D_\alpha}$ are defined over F_0, and $G^{alg}_{D_\alpha}$ is the image of G^{alg}_D in GL_{D_α}.

Write $\mathfrak{Is}$ for the affine algebraic variety of $\otimes$-isomorphisms from ω_D to ω_G, and $\mathfrak{Is}_\alpha$ for the variety of $\otimes$-isomorphisms from ω_D and ω_G restricted to $\mathrm{Rep}(G^{alg}_{V_\alpha})$. The variety $\mathfrak{Is}_\alpha$ is a right torsor under $G^{alg}_{D_\alpha}$, and a left torsor under $G^{alg}_{V_\alpha}$. Now the choice of a point i in $\mathfrak{Is}(\overline{\mathbf{Q}}_p)$ gives, for each α, a point i_α in $\mathfrak{Is}_\alpha(\overline{\mathbf{Q}}_p)$. This point i_α gives rise to an isomorphism

$$(\mathbf{20}) \qquad \eta_{i_\alpha} : D_\alpha \underset{F_0}{\otimes} \overline{\mathbf{Q}}_p \simeq (V_\alpha)_{\overline{\mathbf{Q}}_p} = V_\alpha \underset{\mathbf{Q}_p}{\otimes} \overline{\mathbf{Q}}_p$$

which induces an isomorphism of algebraic groups

$$G^{alg}_{D_\alpha} \underset{F_0}{\times} \overline{\mathbf{Q}}_p \simeq G^{alg}_{V_\alpha} \underset{\mathbf{Q}_p}{\times} \overline{\mathbf{Q}}_p$$

and an isomorphism of Lie algebras

$$L(G^{alg}_{D_\alpha}) \underset{F_0}{\otimes} \overline{\mathbf{Q}}_p \simeq L(G^{alg}_{V_\alpha})_{\overline{\mathbf{Q}}_p}.$$

Moreover, a different choice of the point i has the effect of changing these two isomorphisms by an inner automorphism of $G^{alg}_{D_\alpha}(\overline{\mathbf{Q}}_p)$ or an inner automorphism of $G^{alg}_{V_\alpha}(\overline{\mathbf{Q}}_p)$.

For each α in $\mathrm{Rep}(G^{alg}_V)$, we recall that the associated Galois representation ρ_α is semistable, and that the filtered (φ, N)-module D_α attached to ρ_α has the F_0-automorphism $\Phi_\alpha = \varphi^f_\alpha$. Thus the family of all Φ_α define an automorphism of the functor ω_D, and so we obtain a canonical element of $G^{alg}_D(F_0)$, which we again denote by Φ. Recalling the definition of $\log_\pi(\Phi)$ given earlier, it follows that it belongs to the Lie algebra $L(G^{alg}_D)$ of the algebraic group G^{alg}_D. Now fix any point i in $\mathfrak{Is}(\overline{\mathbf{Q}}_p)$, and let η_i denote the isomorphism (20) for the tautological representation of G^{alg}_V in GL_V. We define the endomorphism X_i of $V_{\overline{\mathbf{Q}}_p}$ by

$$(\mathbf{21}) \qquad X_i = \eta_i \circ \log_\pi(\Phi) \circ \eta_i^{-1}.$$

As $\log_\pi(\Phi)$ belongs to $L(G^{alg}_D)$, it follows that X_i belongs to $L(G^{alg}_D)_{\overline{\mathbf{Q}}_p}$. Moreover, if $\lambda_1, \ldots, \lambda_d$ denote the eigenvalues of Φ with multiplicities, it is clear that the eigenvalues of X_i with multiplicities are $\log_\pi(\lambda_1), \dots, \log_\pi(\lambda_d)$. This construction of X_i completes the first basic step in the proof of Theorem 3.1.

To finish the proof of Theorem 3.1, we must show that X_i belongs to the Lie subalgebra $L(G_V)_{\overline{\mathbf{Q}}_p}$ of $L(G_V^{\mathrm{alg}})_{\overline{\mathbf{Q}}_p}$. Here we make use of the representation α_0 of G_V^{alg}. Let $L(\alpha_0)$ be the map from $L(G_V^{\mathrm{alg}})_{\overline{\mathbf{Q}}_p}$ to $L(GL_{V_0})_{\overline{\mathbf{Q}}_p}$ induced by α_0. In view of (19), it suffices to show that

$$L(\alpha_0)(X_i) \in L(G_{V_0})_{\overline{\mathbf{Q}}_p}. \tag{22}$$

Let i_0 denote the point in $\mathfrak{I}s_{\alpha_0}(\overline{\mathbf{Q}}_p)$ arising from i. We have

$$L(\alpha_0)(X_i) = \eta_{i_0} \circ \log_\pi(\Phi_0) \circ \eta_{i_0}^{-1}, \tag{23}$$

where $\Phi_0 = \varphi_0^f$ comes from the filtered (φ, N)-module D_0. Since the representation ρ_0 of G_F is unramified, it is well-known that Φ_0 fixes a lattice (see [13]). Hence Φ_0 generates a compact subgroup of $GL_{D_0}(F_0)$. Thus $\log_\pi(\Phi_0) = \log(\Phi_0)$ is independent of the choice of π. Moreover, G_{V_0} is topologically generated by $\rho_0(\mathrm{Fr})$, and so $G_{V_0}^{\mathrm{alg}}$ is abelian, whence $G_{D_0}^{\mathrm{alg}} \underset{F_0}{\times} \overline{\mathbf{Q}}_p$ is also abelian. In view of the remark made earlier, we see that the right hand side of (23) does not depend on the choice of i in $\mathfrak{I}s(\overline{\mathbf{Q}}_p)$ nor on the choice of π. In fact, we see that the right hand side of (23) is the same for any choice of i_0 in $\mathfrak{I}s_{\alpha_0}(M)$, where M is any extension field of F_0. We now make a suitable choice of M and i_0, which will enable us to compute explicitly the right hand side of (23).

We will use the following lemma about Fontaine's theory for arbitrary unramified Galois representations $\psi : G_F \to GL(W)$, where W is a $\mathbf{Q}_p$-vector space of finite dimension. Let K denote the maximal unramified extension of $\mathbf{Q}_p$, and write $\widehat{K}$ for the completion of K. In the following, G_F acts on $\widehat{K} \underset{\mathbf{Q}_p}{\otimes} W$ by $\tau(a \otimes w) = \tau(a) \otimes \tau(w)$ for a in $\widehat{K}$ and w in W.

Lemma **3.4.** — *Assume that the Galois representation* ψ *is unramified. Consider the* F_0-*subspace of* $\widehat{K} \underset{\mathbf{Q}_p}{\otimes} W$ *given by* $U = (\widehat{K} \underset{\mathbf{Q}_p}{\otimes} W)^{G_F}$. *Then* W *is crystalline in the sense of Fontaine* [13], [14] *and the associated filtered* (φ, N)-*module is* $D(W) = U$. *Moreover, the* F_0-*automorphism* $\Phi_W = \varphi_W^f$ *of* $D(W)$ *is given by*

$$\beta \circ \Phi_W \circ \beta^{-1} = \psi(\mathrm{Fr}^{-1}), \tag{24}$$

where Fr *is the arithmetic Frobenius in* G_F/I_F, *and* $\beta : \widehat{K} \underset{F_0}{\otimes} U \simeq \widehat{K} \underset{\mathbf{Q}_p}{\otimes} W$ *is the isomorphism obtained by extending scalars on the* F_0-*subspace* U *of* W.

Proof. — The fact that β is an isomorphism is, of course, not obvious, and we refer to the Appendix of Chapter 3 of [25] for a proof. Also, $\widehat{K}$ is naturally included in Fontaine's ring B_{cris}, and so U is an F_0-subspace of D(W) (see [13], [14]). But the fact that β is an isomorphism shows that the F_0-dimension of U must be equal to the $\mathbf{Q}_p$-dimension of W, and hence $D(W) = U$. Let φ_B denote the action of Frobenius on the Fontaine ring B_{cris} (see [13], [14]). We write σ for the arithmetic Frobenius in the Galois group of K over $\mathbf{Q}_p$. The restriction of φ_B to $\widehat{K}$ is the arithmetic Frobenius σ. Let 1_W denote the identity map of W. By definition, φ_W is the restriction to U of the map $\sigma \underset{\mathbf{Q}_p}{\otimes} 1_W$. Hence the σ-linear extension $\sigma \underset{\mathbf{Q}_p}{\otimes} \varphi_W$ of φ_W to $\widehat{K} \underset{F_0}{\otimes} U$ satisfies

$$\beta \circ (\sigma \underset{F_0}{\otimes} \varphi_W) \circ \beta^{-1} = \sigma \underset{\mathbf{Q}_p}{\otimes} 1_W.$$

Raising both sides to the power f and extending Φ_W linearly to the whole of $\widehat{K} \underset{F_0}{\otimes} U$, we conclude that

$$\beta \circ (\sigma^f \underset{F_0}{\otimes} 1_U) \circ \Phi_W \circ \beta^{-1} = \sigma^f \underset{\mathbf{Q}_p}{\otimes} 1_W. \tag{25}$$

As the restriction to $\widehat{K}$ of the action of G_F on B_{cris} is the usual action, U is the F_0-subspace of $\widehat{K} \underset{\mathbf{Q}_p}{\otimes} W$ fixed by the σ^f-linear extension of $\psi(\mathrm{Fr})$ to $\widehat{K} \underset{\mathbf{Q}_p}{\otimes} W$. Extending $\psi(\mathrm{Fr})$ linearly to $\widehat{K} \underset{\mathbf{Q}_p}{\otimes} W$, it follows that

$$\beta \circ (\sigma^f \underset{F_0}{\otimes} 1_U) \circ \beta^{-1} = (\sigma^f \underset{\mathbf{Q}_p}{\otimes} 1_W) \circ \psi(\mathrm{Fr}). \tag{26}$$

The equation (24) follows on comparing (25) and (26). This completes the proof of Lemma 3.4.

We can at last complete the proof of Theorem 3.1. We apply Lemma 3.4 to the unramified Galois representation ρ_0 of G_F in V_0. We have the isomorphism

$$\beta_0 : \widehat{K} \underset{F_0}{\otimes} D_0 \simeq \widehat{K} \underset{\mathbf{Q}_p}{\otimes} V_0, \tag{27}$$

and the analogous isomorphisms for all the unramified Galois representations in the Tannakian category generated by ρ_0. Thus these isomorphisms define a point i_0 in $\mathfrak{I}s_0(\widehat{K})$. On applying log to both sides of (24), we deduce from (23) that

$$L(\alpha_0)(X_i) = \log \rho_0(\mathrm{Fr}^{-1}). \tag{28}$$

As the right hand side of (28) is clearly an element of $L(G_{V_0})$, the proof of Theorem 3.1 now follows from (19).

We shall need the following lemma for the proof of Theorem 3.2. Recall that $\xi : G_F \to \mathbf{Z}_p^\times$ is the cyclotomic character.

Lemma **3.5.** — *Assume that* V *is a semistable representation, and let* det : $G_F \to \mathbf{Z}_p^\times$ *be its determinant character. Then the following statements are equivalent:*

(i) *The map* det *coincides on an open subgroup of* G_F *with* ξ^{-t}, *where t is given by* (15).

(ii) *If* $\delta = \delta(\Phi)$ *denotes the determinant of the endomorphism* $\Phi = \varphi^f$ *of* D(V), *then*

$$\log_\pi(\delta) = \log_\pi(q^t), \quad \textit{where } q = p^f. \tag{29}$$

In particular, when $t \neq 0$, *these equivalent assertions imply that the image of* det *is infinite.*

Proof. — Recall that d is the $\mathbf{Q}_p$-dimension of V, and put $W = (\Lambda^d V)(t)$. As the restriction of the determinant character of G_F to I_F is equal to ξ^{-t}, we see that W is an unramified representation of G_F of dimension 1 over $\mathbf{Q}_p$. Let $\psi : G_F \to \mathbf{Q}_p^\times$ denote the character giving the action of G_F on W. We can compute Φ_W in terms of δ as follows. We have that $D(W) = Z[-t]$, where $Z = \Lambda^d D(V)$, and $Z[-t]$ means that the automorphism Φ_Z of Z is replaced by the automorphism $q^{-t}\Phi_Z$. Hence Φ_W is multiplication by $\delta.q^{-t}$, and we recall that $\delta.q^{-t}$ is a p-adic unit. Now assertion (i) is equivalent to saying that G_F/I_F acts on W via a finite quotient. Hence the arithmetic Frobenius Fr in G_F/I_F must act on W via a root of unity, and therefore we have $\psi(\mathrm{Fr})^n = 1$ for some integer n. Applying Lemma 3.4 to W, we conclude that $\Phi_W^n = 1$. Hence (i) is equivalent to

$$\delta.q^{-t} = \zeta,$$

for some *n-th* root of unity ζ. The lemma is now obvious as $\log_\pi(\zeta) = 0$.

Proof of Theorem 3.2. — Assume now that the hypotheses of Theorem 3.2 hold for V. Then $\lambda_1, \ldots, \lambda_d = \delta$, where δ is the determinant of Φ. By Theorem 3.1, there exists an element X in $L(G_V)_{\overline{\mathbf{Q}}_p}$, whose characteristic polynomial has roots $\log_\pi(\lambda_1), \ldots, \log_\pi(\lambda_d)$. Recall that $H_V = G_V \cap SL(V)$, so that $L(H_V)_{\overline{\mathbf{Q}}_p}$ consists of the elements of $L(G_V)_{\overline{\mathbf{Q}}_p}$ of trace zero. But, by Lemma 3.5,

$$\mathrm{tr}(X) = \sum_{i=1}^{d} \log_\pi(\lambda_i) = \log_\pi(q^t). \tag{30}$$

Hence, as $t = \sum_{i=1}^{d} i_j$, it suffices to find another element Y in $L(G_V)_{\overline{\mathbf{Q}}_p}$, whose characteristic polynomial has roots

$$\log_\pi(\lambda_1 q^{-i_1}), \ldots, \log_\pi(\lambda_d q^{-i_d}), \tag{31}$$

for a suitable ordering of $\lambda_1, ..., \lambda_d$. Since V is semistable, it is of Hodge-Tate type (see [28, §2]), and we now exploit this fact. Let C denote the completion of $\overline{\mathbf{Q}}_p$. We consider $V_C = C \otimes_{\mathbf{Q}_p} V$, endowed with the semilinear action of G_F given by $\tau(c \otimes v) = \tau(c) \otimes \tau(v)$ for all τ in G_F, c in C, and v in V. For each m in $\mathbf{Z}$, we write $V_C\{m\}$ for the F-subspace of V_C consisting of all v such that $\tau(v) = \xi^m(\tau)v$ for τ in G_F. Put $V_C(m) = C \otimes_F V_C\{m\}$, again endowed with the semilinear action of G_F. To say that V is of Hodge-Tate type means that we have a direct sum decomposition

(32) $$V_C = \bigoplus_{m \in J} V_C(m),$$

where J is some finite set of integers. It is well-known [13], [14, Theorem 3.8] that the integers in J are the negatives of the distinct integers occuring in the sequence of Hodge-Tate weights (14), and that the C-dimension of $V_C(m)$ is equal to the F-dimension of $Fil^{-m}D(V)_F/Fil^{-m+1}D(V)_F$. Moreover, the direct sum decomposition (32) allows us to define a homomorphism

(33) $$\mu \ : \ \mathbf{G}_m \to I_V^{alg} \underset{\mathbf{Q}_p}{\times} C$$

of algebraic groups over C, where, for $c \in C^\times$, $\mu(c)$ is the automorphism of V_C given by the formula

$$\mu(c)(x) = c^m x \text{ for all } x \text{ in } V_C(m) \qquad (m \in J).$$

As is explained in [28, §1.5], the image of μ is contained in $I_V^{alg} \times_{\mathbf{Q}_p} C$.

As in the proof of Theorem 3.1, let $\Im s$ denote the affine algebraic variety of $\otimes$-isomorphisms from the fibre functor ω_D to the fibre functor ω_G. Again, we fix a point i in $\Im s(\overline{\mathbf{Q}}_p)$, and we write η_i for the isomorphism (20), when α is taken to be the tautological representation of G_V^{alg} in GL_V. Put

(34) $$\Omega = \eta_i \circ \Phi \circ \eta_i^{-1},$$

where Φ is given by (13). Thus Ω belongs to $G_V^{alg}(\overline{\mathbf{Q}}_p)$. Now it is well-known [2, Theorem 4.4] that there then exist commuting elements s and u in $G_V^{alg}(\overline{\mathbf{Q}}_p)$ such that s is semisimple, u is unipotent, and

(35) $$\Omega = su = us.$$

Let Θ be the smallest algebraic subgroup (over $\overline{\mathbf{Q}}_p$) which contains s. As s is semisimple, Θ is a multiplicative group (see [10, Chap. IV, §3]). Let Θ^0 be the connected component of Θ so that $\Theta = \Theta^0 \times P$, where P is a finite group. Suppose that F' is a finite extension of F such that the residue field extension has degree a multiple of #P. Passing to the extension F' and working with the Φ and s associated to F', we may clearly assume that the multiplicative group Θ is a torus.

Let T denote a maximal torus in $G_V^{alg} \times_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$ containing the torus Θ. Returning to our homomorphism μ coming from p-adic Hodge theory, we choose a maximal torus in $G_V^{alg} \times_{\mathbf{Q}_p} C$ containing the image of μ. But all maximal tori in $G_V^{alg} \times_{\mathbf{Q}_p} C$ are conjugate [B, Prop. 11.3], and so there exists g in $G_V^{alg}(C)$ such that $\mu' = g\mu g^{-1}$ has image in T. Moreover, the induced map $\mu' : \mathbf{G}_m \to T$ is necessarily defined over $\overline{\mathbf{Q}}_p$ (see [2, Proposition 8.11]). Hence $\mu'(q)$ belongs to $T(\overline{\mathbf{Q}}_p)$.

We recall that we are seeking to construct an element in the Lie algebra $L(G_V)_{\overline{\mathbf{Q}}_p}$, whose characteristic polynomial has the roots (31). To complete the proof, we use once more the representation α_0 of G_V^{alg} in GL_{V_0}, whose kernel is precisely I_V^{alg}. The torus T acts on $G_V^{alg} \times_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$ and its normal subgroup $I_V^{alg} \times_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$ by inner automorphisms, and so also on the quotient $G_{V_0}^{alg} \times_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$. As $G_{V_0}^{alg} \times_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$ is abelian, T acts trivially on it by inner automorphisms. Given an algebraic group M over $\mathbf{Q}_p$, let $M_{\overline{\mathbf{Q}}_p} = M \times_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$ denote its extension to $\overline{\mathbf{Q}}_p$. For an algebraic group H defined over $\overline{\mathbf{Q}}_p$ we write H_u for its unipotent radical [2, §11.21] and $L(H)_u$ for the corresponding Lie algebra. Now we have the exact sequence of Lie algebras over $\overline{\mathbf{Q}}_p$

$$0 \to L((I_V^{alg})_{\overline{\mathbf{Q}}_p})_u \to L((G_V^{alg})_{\overline{\mathbf{Q}}_p})_u \to L((G_{V_0}^{alg})_{\overline{\mathbf{Q}}_p})_u \to 0$$

[2, Cor. 14.11]. The action of T on the algebraic groups induces the adjoint action of T on the Lie algebras. We denote with the superscript T the maximal Lie subalgebras on which T acts trivially. Then, since representations of a torus are semisimple, we have the exact sequence

$$0 \to L((I_V^{alg})_{\overline{\mathbf{Q}}_p})_u^T \to L((G_V^{alg})_{\overline{\mathbf{Q}}_p})_u^T \to L((G_{V_0}^{alg})_{\overline{\mathbf{Q}}_p})_u \to 0. \tag{36}$$

Here we have used the fact that T acts trivially on the Lie algebra $L((G_{V_0}^{alg})_{\overline{\mathbf{Q}}_p})_u$, as G_{V_0} is abelian. Let u_0 be the image of u in $(G_{V_0}^{alg})_{\overline{\mathbf{Q}}_p}$ and let n' be an element in $L((G_V^{alg})_{\overline{\mathbf{Q}}_p})_u^T$ which is a lift of $\log(u_0)$. Consider the element $u' = \exp(n')$; clearly u' commutes with elements of T. In particular, we have now arranged that the elements $\mu'(q)$, s and u' commute. It is plain that the roots of the characteristic polynomial of

$$Y = \log_\pi(s.\mu'(q).u')$$

are as in (31). As X and Y have the same image in $L(G_{V_0}^{alg})_{\overline{\mathbf{Q}}_p}$, by Theorem 3.1 and (19), we conclude that Y lies in $L(H_V)_{\overline{\mathbf{Q}}_p}$, as required. This completes the proof of Theorem 3.2.

4. Applications

The main aim of this section is to show that the results of §3 apply to a large class of motivic Galois representations V, thereby enabling us to show that V has vanishing G_V-cohomology. We continue with the notation of the previous sections, so that $\rho : G_F \to GL_V$ is a p-adic Galois representation of F, with image G_V; here, as always, F denotes a finite extension of $\mathbf{Q}_p$. Further, $H_V = G_V \cap SL(V)$ and $q = p^f$ is the cardinality of the residue field k_F of F. Let w be an integer. Recall that a **Weil number of weight** w (relative to q) is an algebraic number all of whose archimedean absolute values are $q^{w/2}$, and the v-adic absolute value is one for any non-archimedean prime v, which does not divide p.

Suppose $\rho : G_F \to GL(V)$ potentially crystalline and let w be an integer. Let us define **purity condition** (P_w) for ρ. If ρ is crystalline, ρ satisfies (P_w) if the associated endomorphism Φ of the filtered module D(V) (see (13)) has eigenvalues which are Weil numbers of weight w. In general, we say that ρ satifies (P_w) if its restriction to $G_{F'}$ does, where F' is a finite extension of F such that the restriction of ρ to $G_{F'}$ is crystalline (this definition does not depend on the choice of F').

Proposition **4.1.** — *Assume that* $\rho : G_F \to GL(V)$ *is a potentially crystalline Galois representation and suppose that* (P_w) *holds for the representation* V *where* w *is a non-zero integer. Then, for every* G_V*-subquotient* V' *of* V*, we have* $H^i(G_V, V') = H^i(G_{V'}, V') = 0$ *for all* $i \geqslant 0$.

Proof. — We choose the number "π" used to define $\log_\pi$ to be transcendental over the rational field $\mathbf{Q}$. The beauty of such a choice is that it guarantees that $\log_\pi(z)$ is non-zero for every element z of $\overline{\mathbf{Q}}_p$ which is algebraic over $\mathbf{Q}$, and which is not a root of unity. We check that the strong Serre criterion holds for the representation $G_V \subset GL(V)$. We can replace F by a finite extension and suppose that ρ is crystalline. For any set

$$\lambda_1, ..., \lambda_{i+1}, \mu_1, ..., \mu_{i+1}$$

of $(2i+1)$ eigenvalues of Φ, the product

$$\lambda_1, ..., \lambda_i.\mu_1^{-1}, ..., \mu_{i+1}^{-1}$$

is a Weil number κ of weight $w \neq 0$. Therefore κ is an algebraic number which is not a root of unity, hence $\log_\pi(\kappa) \neq 0$. This implies, by Theorem 3.1, that there exists X in the Lie algebra $L(G_V)_{\overline{\mathbf{Q}}_p}$ which satisfies the strong Serre criterion. The proposition now follows from Proposition 2.5.

Proposition **4.2.** — *Suppose that* $\rho : G_F \to GL(V)$ *is a potentially crystalline Galois representation and that* (P_w) *holds with* w *an odd integer. Suppose we are given a finite extension* F' *of* F *such that the restriction* $\rho_{|F'}$ *of* ρ *to* $G_{F'}$ *is crystalline. Let* Φ *be the associated endomorphism of* *the filtered module associated to* $\rho_{|F'}$. *Suppose further that the determinant* $\delta(\Phi)$ *of the endomorphism* Φ *is a rational number. Then* $L(H_V) \subset End(V)$ *satisfies the strong Serre criterion. In particular, for each subquotient* V' *of* V, *we have* $\chi_f(G_V, V') = 1$ *if* G_V *has no element of order* p.

Proof. — As before, we can suppose $F = F'$ and choose π transcendental. The rational number $\delta(\Phi)$ is a Weil number whose p-adic absolute value is the same as that of q^t, where t is given by (15). Therefore it must be equal to $\pm q^t$. Let $Fr \in G_F/I_F$ be the arithmetic Frobenius. It follows from Lemma 3.4 that Fr acts on the Tate twist $\Lambda^d(V)(t)$ via multiplication by ± 1. Hence by Lemma 3.5, there exists an open subgroup of G_F on which the determinant character coincides with ξ^{-t}, where ξ is the cyclotomic character. Now assume (P_w) holds, with w odd. We have $t = dw/2$ and hence $t \neq 0$. It follows that the image of $\det(G_V)$ is infinite. By Theorem 3.2, there exists an element X_1 in $L(H_V)$ whose eigenvalues are as in (31) where the integers i_j are the Hodge-Tate weights (cf. (14)). If $(\lambda_1, ..., \lambda_{j+1}, \mu_1, ..., \mu_j)$ is a family of $2j + 1$ eigenvalues of X_1, then it is easily checked that $\lambda_1 + ... + \lambda_{j+1} - \mu_1 - ... - \mu_j$ is equal to $\log_\pi(\kappa)$, for κ a Weil number of weight $m/2$, with m an odd integer. In particular $\log_\pi(\kappa) \neq 0$, and hence the strong Serre criterion holds for the representation $L(H_V) \subset GL(V)$. The proposition now follows from Theorem 2.4.

Remark **4.3.** — If w is an integer and V is a potentially semistable representation such that the eigenvalues of Φ are Weil numbers of weight w, then V is potentially crystalline and therefore satisfies (P_w). We give a proof of this well-known statement. Obviously the proof reduces to the case where V is semistable. We must show that the endomorphism N of D(V) is equal to 0. We can clearly extend scalars on D(V) to $\overline{\mathbf{Q}}_p$ and write $W = D(V)_{\overline{\mathbf{Q}}_p}$. For each eigenvalue λ of Φ, and each integer $n \geqslant 1$, we put

$$E(\lambda, n) = Ker(\Phi - \lambda)^n.$$

Since W is a direct sum of such spaces $E(\lambda, n)$ for suitable n, it suffices to show that N vanishes on $E(\lambda, n)$, and we now proceed to prove this by induction on n. If v belongs to $E(\lambda, 1)$, then v is an eigenvector for Φ with eigenvalue λ. As $N\Phi = q\Phi N$, it follows immediately that $u = Nv$ satisfies $\Phi(u) = ru$, where $r = \lambda/q$. But r does not have complex absolute value equal to $q^{w/2}$, and so we see that $u = 0$. Assuming that we have already shown that N annihilates $E(\lambda, n)$, take v to be any element of $E(\lambda, n+1)$. Thus $z = (\Phi - \lambda)(v)$ belongs to $E(\lambda, n)$, and so $N(z) = 0$. But, again using that $N\Phi = q\Phi N$, we obtain

$$0 = N(z) = q\Phi N(v) - \lambda N(v)$$

and so we see that $N(v) = 0$, completing the argument.

The above propositions can be applied to a wide class of motivic Galois representations. We list a few of them below; these will cover Theorems 1.1-1.7 stated in the introduction. By the above remark, the representations occuring in the examples are necessarily potentially crystalline.

Examples **4.4**

1) Let X be a smooth and proper scheme over F with potential good reduction. Let $i \geqslant 0$ and j be integers. If $i \neq 2j$, Proposition 4.1 applies to $V = H^i_{\acute{e}t}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p(j))$. To see this, we may suppose that X has good reduction. Let $\mathfrak{X}$ be a smooth proper scheme over the ring of integers $\mathcal{O}_F$ of F whose generic fibre is X. Denote by $\mathfrak{X}_0$ the special fibre of $\mathfrak{X}$. Let F_0 be the maximal unramified extension of $\mathbf{Q}_p$ in F; it is the fraction field of the ring of Witt vectors W of the residue field k_F. The crystalline cohomology groups $H^i_{cris}(\mathfrak{X}_0/W) \otimes_W F_0$ [1] possess a natural filtered φ-module structure, which we call D. More precisely, it has an action of the Frobenius φ. One has an isomorphism of $H^i_{cris}(\mathfrak{X}_0/W) \otimes_W F$ with the de Rham cohomology of X and the Hodge filtration on the de Rham cohomology gives the filtration. By the C_{cris} conjecture proved by Fontaine-Messing and Faltings (cf. [16]), the representation V is crystalline and the associated filtered module $D(V)_F$ is canonically isomorphic to D.

It is known that crystalline cohomology is a Weil cohomology theory, hence the purity results for crystalline cohomology imply that the eigenvalues of the Frobenius automorphism Φ are Weil numbers with complex absolute value $q^{i/2}$ (the eigenvalues are the same as for l-adic étale cohomology groups, $l \neq p$). When $\mathfrak{X}$ is projective, this follows from results of Katz-Messing [18] and for $\mathfrak{X}$ proper it is a consequence of results of Chiarellotto-Le Stum [5]. Furthermore, as the characteristic polynomial of Φ has rational coefficients, we can apply Proposition 4.2 when i is odd and G_V has no element of order p.

Let V' be a subrepresentation of V. If i is odd and $G_{V'}$ has no element of order p, we can apply Proposition 4.2 to V' and prove that $\chi_f(G_{V'}, V') = 1$ provided we know that the determinant of the Frobenius Φ' on $D(V')$ is rational. We mention one particular case, as it is closely related to motives arising from modular forms. Let z be a correspondence of degree zero on X, i.e. z is a $\mathbf{Q}$-linear combination of cycles of dimension equal to the dimension of X in $X \times_F X$ modulo rational equivalence. Suppose that the p-adic étale class $c_{\acute{e}t}(z)$ induces a projector on $V = H^i_{\acute{e}t}(X_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_p(j))$ and let $V' = c_{\acute{e}t}(z)(V)$. Let us prove that the determinant of the Frobenius Φ' is rational. The correspondence z comes from a pullback of a correspondence $\underline{z}$ on $\mathfrak{X}$, whose pullback z_0 to the special fibre $\mathfrak{X}_0$ is the specialisation of $\underline{z}$. Let $c_{cris}(z_0)$ be the crystalline class of z_0 [15]. It is known that $c_{cris}(z_0)$ corresponds to $c_{\acute{e}t}(z)$ in the p-adic comparison theorem [12, Lemma 5.1]. Thus we have $D(V') = c_{cris}(z_0)(H^i_{cris}((\mathfrak{X}_0/W) \otimes_W F)(j))$. As in [18, §3], the purity results of [18] and [5] imply that the projection from the whole crystalline cohomology $H^*_{cris}(\mathfrak{X}_0/W) \otimes_W F$ of $\mathfrak{X}_0$ to $H^i_{cris}(\mathfrak{X}_0/W) \otimes_W F$ is given by the class of a correspondence on $\mathfrak{X}_0$, which is in fact a polynomial in the Frobenius Φ with coefficients in $\mathbf{Q}$. Furthermore, one knows that crystalline cohomology is a Weil cohomology theory in the strong sense [15]. Therefore the Lefschetz trace formula implies that the characteristic polynomial of Φ' on $D(V')$ has rational coefficients (which are in fact integral if $j=0$), and hence the determinant $\delta(\Phi')$ is a rational number. Thus our claim that $\chi_f(G_{V'}, V')=1$ is established.

2) Let A/F be an abelian variety of dimension g and

$$T_p(A)=\varprojlim A_{p^n}, \qquad V_p(A)=T_p(A)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p.$$

Then $V=V_p(A)$ is a $\mathbf{Q}_p$-vector space of dimension $2g$. There is the natural Galois representation $\rho : G_F \to GL(V)$ and we let G_V denote the image, as before. The representation V is dual to the representation induced by $H^1_{\acute{e}t}(A\times_F \overline{F}, \mathbf{Q}_p)$. It is potentially semistable always and potentially crystalline if A has potential good reduction. Assume that A has potential good reduction over F. Then V has vanishing G_V-cohomology and $\chi_f(G_V, V)=1$ whenever G_V has no element of order p. There is a "natural" G_F-invariant subspace W of $V=V_p(A)$ defined as follows: W is the G_F-invariant $\mathbf{Q}_p$-subspace of V of minimal dimension such that some open subgroup of finite index of the inertia group I_F, acts trivially on V/W. It can be checked that W exists and is unique (cf. [6, p. 150]). Let C be the image of W in A_{p^∞}. If A has good reduction over F, W is none other than $\hat{V}=V_p(\hat{A})$, where $\hat{A}$ is the formal group over $\mathcal{O}_F$, $C=\hat{A}_{p^\infty}$, and the quotient V/W is the Galois module $\tilde{V}=V_p(\tilde{A})$ associated to the reduced abelian variety $\tilde{A}$. Applying Theorem 2.6, we can conclude that both W and V/W have vanishing G_V-cohomology and $\chi_f(G_V, C)=1=\chi_f(G_V, A_{p^\infty}/C)$ whenever G_V has no element of order p and A has potential good reduction.

3) Let K be a finite extension of $\mathbf{Q}$. As for the proof of Theorem 1.7, let D_V be the image of the Galois representation restricted to the decomposition group. Then by the above results, the representations restricted to D_V satisfy the strong Serre criterion. It suffices to observe that therefore the original representation of G_V satisfies the strong Serre criterion. Further, the image of the determinant character is clearly infinite, and H_V satisifes the strong Serre criterion too. Hence Theorem 1.7 follows from Theorem 2.6.

Remark **4.5** — Observe that in Example 4.4 1), we can show that $\chi_f(G_V, V)$, when defined, is equal to 1 only for the cohomology representations V with odd degree i. When i is an even positive integer, a similar result would hold by our result on the vanishing of cohomology in conjunction with a theorem of Totaro (see [34, Theorem 0.1]), if one knew that the dimension of the centraliser of every element in the Lie algebra $L(G_V)$ has dimension at least 2.

We state a more general theorem below. As before let $\rho : G_F \to GL(V)$ be a potentially semistable representation with image G_V and let G_V^{alg} be its Zariski closure in GL_V. Given a linear representation α of the algebraic group G_V^{alg} in a finite dimensional $\mathbf{Q}_p$-vector space V', and composing ρ with the morphism

$$G_V^{alg} \to \mathrm{GL}(V')$$

induced by α, we obtain a p-adic Galois representation $\rho' : G_F \to \mathrm{GL}(V')$, which is again potentially semistable (resp. potentially crystalline if V is potentially crystalline). As before, the proof of the theorem below reduces to the case when the representation is crystalline.

Theorem **4.6.** *Let w, w' be two integers. Suppose that the representations ρ and ρ' satisfy condition* (P_w) *and* $(P_{w'})$ *respectively, where, as explained above ρ' comes from any representation of the algebraic group G_V^{alg} in a finite dimensional $\mathbf{Q}_p$-vector space V'. Then:*

(i) *If $w' \neq 0$, V' has vanishing G_V-cohomology.*

(ii) *Suppose that w' is odd and the determinant $\delta(\Phi)$ of the endomorphism Φ of* $D(V)$ *is equal to $q^t.\varepsilon$, where t is given by* (15) *and ε is a root of unity. If G_V has no element of order p, and $T' \subset V'$ is a $\mathbf{Z}_p$-lattice stable under G_V, then $\chi_f(G_V, V'/T') = 1$.*

Proof. — Let Φ' be the associated endomorphism of the filtered module $D(V')$. We claim that the eigenvalues of Φ' are products of eigenvalues of Φ. Indeed, let M be a torus over an algebraically closed field, W a faithful representation of M and W' another representation of M. As W is faithful, the characters χ_i of M in W generate the character group of M. Given a character χ' of M in W' therefore, χ' can be written as a finite product $\prod_i \chi_i^{n_i}$, $n_i \in \mathbf{Z}$. Hence, given an element m of M, we have $\chi'(m) = \prod_i \chi_i^{n_i}(m)$, and thus the eigenvalues of m in W' are products of the eigenvalues of m in W. Let s and s' denote respectively the semisimple components of Φ and Φ'. The claim now follows on considering, as in the proof of Theorem 3.2, an element which is a power of Φ and a torus M in G_V^{alg} containing s, and noting that s' is the image of s under the composite

$$M \subset G_V^{alg} \to \mathrm{GL}(V').$$

Let $G_{V'}$ be the image of ρ', $G_{V'}$ is a quotient of G_V. Applying Theorem 3.1, we therefore obtain elements X and X' respectively in the Lie algebras $L(G_V)_{\overline{\mathbf{Q}}_p}$ and $L(G_{V'})_{\overline{\mathbf{Q}}_p}$ whose eigenvalues are logarithms of the eigenvalues of Φ and Φ' respectively. Now, the discussion above allows us to deduce that the eigenvalues of Φ in V' are products of eigenvalues of Φ' in V'. Thus if w' is not zero, w is also non-zero and hence one checks that the element X satisifes Serre's criterion in the representation $L(G_V) \to \mathrm{End}(V')$. Assertion (i) now follows from Lemma 2.2.

As before, let $H_V = G_V \cap SL(V)$. We have the natural representation

$$(\mathbf{37}) \qquad L(H_V) \to \mathrm{End}(V')$$

induced by ρ'. The hypothesis of (ii) along with Lemma 3.5 imply that Theorem 3.2 can be applied to V and V′. We argue as in the proof of (i), noting that w' odd implies w odd. One then checks (cf. proof of Proposition 4.2) that the representation (37) satisfies Serre's criterion and therefore V′ has vanishing H_V-cohomology. Using the Hochschild-Serre spectral sequence as in the proof of Theorem 2.4, assertion (ii) follows. This completes the proof of Theorem 4.6.

We give an interesting application of the above theorem. The examples considered above in 4.4 are all of algebraic varieties with potential good reduction. In the case of an arbitrary abelian variety A over F, it is known that $V = V_p(A)$ is potentially semistable. But even in the case of elliptic curves with potential multiplicative reduction, any element in the Lie algebra $L(G_V)$ has at least one eigenvalue zero [25, Appendix]. Thus the methods of this paper fail as Serre's criterion cannot be applied. Indeed, as mentioned in the introduction, it is not true that V has vanishing G_V-cohomology for arbitrary abelian varieties A over F. Nevertheless, we now show that our methods give a partial description of the Lie algebra $L(G_V)$ as a semidirect product of certain Lie subalgebras. Recall that, for every abelian variety A over F, there is a canonical filtration of V by subrepresentations [22, Exposé 1]

$$(\mathbf{38}) \qquad W_{-2}(V) \subset W_{-1}(V) \subset V;$$

here there is a finite unramified extension F′ of F such that the absolute Galois group $G_{F'}$ acts on $W_{-2}(V)$ via the cyclotomic character, and on the quotient $V/W_{-1}(V)$ via the trivial character. Moreover, the one remaining quotient $W_{-1}(V)/W_{-2}(V)$ can be explained as follows [22, Exposé 9, §7]. There exists a second abelian variety A_r over F with good reduction such that there is a canonical Galois isomorphism from $W_{-1}(V)/W_{-2}(V)$ to $V_p(A_r)$. Let G_r denote the image of the Galois representation

$$\rho_r \;:\; G_F \to GL(V_p(A_r)).$$

We write $L(G_V)$ and $L(G_r)$ respectively for the Lie algebras of G_V and G_r. The natural surjection of G_V onto G_r induces a surjection

$$\pi_r : L(G_V) \to L(G_r)$$

and we denote its kernel by $\mathfrak{N}$. As $\mathfrak{N} = \mathfrak{N}^{alg} \cap L(G_V)$, and $\mathfrak{N}^{alg}$ is the kernel of the natural map $L(G_V^{alg}) \to L(G_r^{alg})$, $\mathfrak{N}$ is an ideal in $L(G_V^{alg})$. Recall that a **special automorphism** of a Lie algebra is an automorphism of the form exp(adX) [3, Chap. 1, §6.8], where X is in the nilpotent radical of the Lie algebra.

Proposition **4.7.** — *Let* A *be an arbitrary abelian variety defined over a finite extension* F *of* $\mathbf{Q}_p$, *and let* $V = V_p(A)$. *Let* A_r *be the abelian variety over* F *with good reduction, which is attached as explained above, to the canonical filtration* (38) *of the Galois module* V. *Let* G_V *and* G_r *denote the respective images of* G_F *in* GL(V) *and* $GL(V_p(A_r))$. *Then the natural surjection* π_r *from* $L(G_V)$ *onto* $L(G_r)$ *has a section, and* $L(G_V)$ *is the semidirect product of* $L(G_r)$ *and the kernel* $\mathfrak{N}$ *of* π_r. *Moreover, the section of* π_r *is unique up to a special automorphism of* $L(G_V)$.

Proof. — The filtration (38) induces a filtration on $\mathfrak{N}$ which we denote by

$$W_{-2}(\mathfrak{N}) \subset W_{-1}(\mathfrak{N}) = \mathfrak{N}, \tag{39}$$

where $W_{-2}(\mathfrak{N})$ consists of all elements X in $L(G_V)$ such that $X(V) \subset W_{-2}(V)$ and $X(W_{-1}(V)) = 0$. This filtration is stable under the adjoint representation of $L(G_V)$ and $L(G_V^{alg})$. For U equal to either V or $\mathfrak{N}$, we define

$$\mathrm{gr}^i(U) = W_i(U)/W_{(i-1)}(U) \qquad i = 0, -1, -2,$$

where it is understood that $W_{-3}(U) = 0$. Now the representation $\mathrm{gr}^i(V)$ satisfies the purity condition (P_w) with $w = i$ for $i = 0, -1, -2$. Since we have inclusions

$$\mathrm{gr}^i(\mathfrak{N}) \subset \bigoplus_j \mathrm{Hom}(\mathrm{gr}^j(V), \mathrm{gr}^{i+j}(V)),$$

it follows that the representations $\mathrm{gr}^i(\mathfrak{N})$ satisfy (P_w) for $w = i$, with $i = -1$ and -2. Hence we can apply Theorem 4.6 to the adjoint representation of $L(G_r)$ on $\mathfrak{N}/W_{-2}(\mathfrak{N})$ and $W_{-2}(\mathfrak{N})$. As the filtration (39) is stable under the adjoint representation of $L(G_V^{alg})$, we can conclude from assertion (i) of Theorem 4.6 that $H^k(L(G_r), \mathfrak{N}/W_{-2}(\mathfrak{N}))$ and $H^k(L(G_r), W_{-2}(\mathfrak{N}))$ are trivial for all $k \geqslant 0$. In particular, it is trivial for $k = 2$. But $H^2(L(G_r), \mathfrak{N}/W_{-2}(\mathfrak{N}))$ classifies the set of equivalence classes of extensions of $L(G_r)$ by $\mathfrak{N}/W_{-2}(\mathfrak{N})$ (cf. [4, Theorem 26.1]). Therefore, we see that $L(G_V)/W_{-2}(\mathfrak{N})$ is a semidirect product of $L(G_r)$ and $\mathfrak{N}/W_{-2}(\mathfrak{N})$. Denote by $\mathfrak{N}_0$ the inverse image in $L(G_V)$ of $s_0(L(G_r))$ where s_0 is a section in the above semidirect product. Then as $H^2(L(G_r), W_{-2}(\mathfrak{N})) = 0$, the extension

$$0 \to W_{-2}(\mathfrak{N}) \to \mathfrak{N}_0 \to L(G_r) \to 0$$

is a semidirect product. This implies that $L(G_V)$ is a semidirect product of $\mathfrak{N}$ and $L(G_r)$. The uniqueness of the section up to special automorphisms now follows from the vanishing of the corresponding H^1-cohomology groups and this completes the proof of the proposition.

We close this section by proving Theorem 1.8.

Proof of Theorem 1.8. — The proof is entirely analogous to the classical case. We first observe that the hypotheses along with Theorem 3.1 gives an element in

$L(G_V)$ whose eigenvalues are the logarithms of the eigenvalues of the Frobenius Φ. As the representation is potentially crystalline, the purity condition (P_w) holds with $w \neq 0$. Arguing as in [27, 2.3], we see that this is sufficient to guarantee the inclusion of the homotheties in the algebraic envelope $L(G_V^{alg})$.

5. Elliptic curves

In this section, we complete the proof that $\chi_f(G_V, A_{p^\infty}) = 1$ for all elliptic curves A over F, where, as always, F denotes a finite extension of $\mathbf{Q}_p$. We also discuss some interesting results about the Euler characteristic $\chi_f(G_V, A_{p^\infty}(n))$, when A is an elliptic curve over F with split multiplicative reduction and n is any integer not equal to ± 1, which were first remarked to us by B. Totaro.

We assume for the rest of this section that F is a finite extension of $\mathbf{Q}_p$, and that A is an elliptic curve defined over F with non-integral j-invariant (in other words, A does not have potential good reduction over F). We consider the Galois representation $V = V_p(A)$, and again write G_V for the image of the Galois group G_F in the automorphism group of V. We observe that if $p \neq 2$, the group G_V has no p-torsion. This follows from the fact that G_V can then be represented as a subgroup of the upper triangular matrices in $GL_2(\mathbf{Z}_p)$. We shall prove the following two results.

Theorem **5.1.** — *Let* A *be an elliptic curve defined over* F*, with non-integral j-invariant. Then* $V = V_p(A)$ *has vanishing* G_V*-cohomology. Moreover, if* G_V *has no element of order p, then* $\chi_f(G_V, A_{p^\infty}) = 1$.

If M is a $\mathbf{Z}_p$-module on which G_F acts, and n is any integer, we define the Tate twist as usual, by

$$M(n) = M \otimes_{\mathbf{Z}_p} \mathbf{Z}_p(n),$$

endowed with the natural G_F-action. Put $\Omega = \mathbf{Q}_p/\mathbf{Z}_p$, endowed with the trivial action of G_F, and denote its *n-th* Tate twist $\mathbf{Q}_p/\mathbf{Z}_p(n)$ by $\Omega(n)$.

Theorem **5.2.** — *Let* A *be an elliptic curve which has split multiplicative reduction over* F*. Let n be any integer not equal to* 1 *or* -1*. Then, if* $V = V_p(A)$*,* $V(n)$ *has vanishing* G_V*-cohomology. Moreover, if* G_V *has no element of order p, then*

$$\chi_f(G_V, A_{p^\infty}(n)) = \#\ \Omega(n+1)(F)/\#\ \Omega(n-1)(F). \qquad (\mathbf{40})$$

We are very grateful to B. Totaro for pointing out Theorem 5.2 to us when $n \neq 0$. Indeed, the representation V in Theorem 5.2 is one of the simplest cases in which G_V is a p-adic Lie group of dimension 2, [25, Appendix A.1] but in which there exists an element in the Lie algebra $L(G_V)$ whose centralizer has dimension 1.

This is precisely the situation in which Totaro's general method [34] for proving the Euler characteristic is 1 breaks down. But, as Totaro explained to us, his arguments in [34, Proof of Theorem 7.4] work beautifully to calculate the Euler characteristic in this exceptional case, and give precisely the result (40) when F is replaced by a suitably large finite extension. We do not use Totaro's method here, but instead prove Theorem 5.2 by direct arguments with Tate curves, as in the proof of Theorem 5.1.

Assume that $\mu_p \subset \mathrm{F}$ if p is odd, and μ_4 is contained in F if $p=2$. Then $\mathrm{F}(\mu_{p^r})$ is a cyclic extension of F of p-power order for all $r \geqslant 1$, and it follows easily that

$$\# \ \Omega(n)(\mathrm{F}) = \# \ \Omega(1)(\mathrm{F}) p^{\mathrm{ord}_p(n)} \qquad (n \neq 0). \tag{41}$$

Hence we obtain the following corollary of Theorem 5.2.

Corollary **5.3.** — *In addition to the hypotheses of Theorem 5.2, assume that* $\mu_p \subset \mathrm{F}$ *if p is odd, and* $\mu_4 \subset \mathrm{F}$ *if* $p=2$. *Then, for all integers* $n \neq \pm 1$, *we have*

$$\chi_{\mathrm{f}}(\mathrm{G}_{\mathrm{V}}, \mathrm{A}_{p^\infty}(n)) = p^{\mathrm{ord}_p(n+1) - \mathrm{ord}_p(n-1)}. \tag{42}$$

In particular, by choosing n appropriately, we obtain examples of representations where the Euler characteristic is either a strictly positive or a strictly negative power of p, even though the associated motives have odd weight. This illustrates the completely new phenomena which arise if one seeks to extend Theorem 1.3 to algebraic varieties which do not have potential good reduction.

We now turn to the proofs of Theorem 5.1 and 5.2, beginning with a well-known lemma.

Lemma **5.4.** — *Assume that* G_{V} *has no element of order p. If r is any integer such that* $\mathrm{H}^2(\mathrm{G}_{\mathrm{V}}, \mathrm{A}_{p^\infty}(r))$ *is finite, then* $\mathrm{H}^2(\mathrm{G}_{\mathrm{V}}, \mathrm{A}_{p^\infty}(r)) = 0$.

Proof. — We choose n so large that p^n kills $\mathrm{H}^2(\mathrm{G}_{\mathrm{V}}, \mathrm{A}_{p^\infty}(r))$. Taking G_{V}-cohomology of the exact sequence

$$0 \to \mathrm{A}_{p^n}(r) \to \mathrm{A}_{p^\infty}(r) \xrightarrow{p^n} \mathrm{A}_{p^\infty}(r) \to 0,$$

we obtain the exact sequence

$$\mathrm{H}^2(\mathrm{G}_{\mathrm{V}}, \mathrm{A}_{p^\infty}(r)) \xrightarrow{p^n} \mathrm{H}^2(\mathrm{G}_{\mathrm{V}}, \mathrm{A}_{p^\infty}(r)) \to \mathrm{H}^3(\mathrm{G}_{\mathrm{V}}, \mathrm{A}_{p^n}(r)). \tag{43}$$

Now the group on the right of (43) is zero because G_{V} has p-cohomological dimension equal to 2 as dimension G_{V} is 2 and we are assuming that G_{V} has no elements of order p. The image of the map on the left of (43) is zero by our choice of n. Hence (43) gives $\mathrm{H}^2(\mathrm{G}_{\mathrm{V}}, \mathrm{A}_{p^\infty}(r)) = 0$, as required.

We now prove Theorems 5.1 and 5.2, beginning first with the proof of Theorem 5.2.

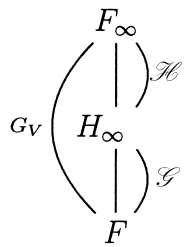
Proof of Theorem 5.2. — We assume for this proof that A has split multiplicative reduction over F, and so is isomorphic over F to a Tate curve with p-adic period q_A [31, Chap. V, §5]. Put

$$F_\infty = F(A_{p^\infty}), \qquad H_\infty = F(\mu_{p^\infty}).$$

We can then identify G_V with the Galois group of F_∞ over F, and we define

$$\mathscr{H} = G(F_\infty/H_\infty), \qquad \mathscr{G} = G(H_\infty/F).$$

We have the diagram of fields:



Now the theory of the Tate curve shows that F_∞ is obtained by adjoining to H_∞ the p^n-th roots ($n = 1, 2, \ldots$) of q_A. Hence, by multiplicative Kummer theory, we have an isomorphism of $\mathscr{G}$-modules

$$(\mathbf{44}) \qquad \mathscr{H} \simeq \mathrm{Hom}(J, \mu_{p^\infty}),$$

where J is the image in $H_\infty^\times \otimes \mathbf{Q}_p/\mathbf{Z}_p$ of the tensor product with $\mathbf{Q}_p/\mathbf{Z}_p$ of the subgroup $q_A^{\mathbf{Z}}$ of $H_\infty^\times$. Since q_A lies in F, it follows immediately from (44) that we have an isomorphism of $\mathscr{G}$-modules

$$(\mathbf{45}) \qquad \mathscr{H} \simeq \mathbf{Z}_p(1).$$

Now, as A is a Tate curve over F, we have the exact sequence of G_F-modules

$$0 \to \mu_{p^\infty} \to A_{p^\infty} \to \Omega \to 0.$$

For each integer n, this gives rise to the exact sequence

$$(\mathbf{46}) \qquad 0 \to \Omega(n+1) \to A_{p^\infty}(n) \to \Omega(n) \to 0.$$

Now $\mathscr{H}$ is isomorphic to $\mathbf{Z}_p$ as an abelian group, and hence $\mathscr{H}$ has p-cohomological dimension equal to 1. Taking $\mathscr{H}$-cohomology of (46), and recalling that from multiplicative Kummer theory, the connecting map from Ω to $H^1(\mathscr{H}, \mu_{p^\infty})$, whence also the connecting map from $\Omega(n)$ to $H^1(\mathscr{H}, \Omega(n+1))$, are isomorphisms, we conclude that

(**47**) $$H^0(\mathscr{H}, A_{p^\infty}(n)) = \Omega(n+1), \qquad H^1(\mathscr{H}, A_{p^\infty}(n)) = H^1(\mathscr{H}, \Omega(n)),$$

and that $H^i(\mathscr{H}, A_{p^\infty}(n)) = 0$ for $i \geqslant 2$. Since $\mathscr{H}$ acts trivially on $\Omega(n)$, it follows from (45) that

(**48**) $$H^1(\mathscr{H}, A_{p^\infty}(n)) = \mathrm{Hom}(\mathscr{H}, \Omega(n)) = \Omega(n-1).$$

We need the following folkloric lemma, whose proof we include for completeness.

Lemma **5.5.** — *Let* F *be a finite extension of* $\mathbf{Q}_p$, *and put* $H_\infty = F(\mu_{p^\infty})$. *Write* $\mathscr{G} = G(H_\infty/F)$. *Let r be a non-zero integer. Then* $H^i(\mathscr{G}, \Omega(r))$ *is finite for all* $i \geqslant 0$, *and*

$$\# H^i(\mathscr{G}, \Omega(r)) = \# H^{i+1}(\mathscr{G}, \Omega(r))$$

for all integers $i \geqslant 1$.

Proof. — We have $\mathscr{G} = \Delta \times \Gamma$, where Γ is isomorphic to $\mathbf{Z}_p$, and Δ is a cyclic group (in fact, Δ has order prime to p, unless $p = 2$ and μ_4 is not contained in F). Since $r \neq 0$, we claim that

(**49**) $$H^i(\Gamma, \Omega(r)) = 0 \quad \text{for all } i \geqslant 1.$$

This is automatically true for $i \geqslant 2$, since Γ has p-cohomological dimension equal to 1. For $i = 1$, we have

$$H^1(\Gamma, \Omega(r)) = \Omega(r)/(\gamma - 1)\Omega(r),$$

where γ denotes any topological generator of Γ. But, as $r \neq 0$, $\gamma - 1$ is surjective on $\mathbf{Q}_p(r)$, and therefore also surjective on $\Omega(r) = \mathbf{Q}_p(r)/\mathbf{Z}_p(r)$, proving (49) for $i = 1$. In view of (49), and the Hochschild-Serre spectral sequence where we view Δ as a quotient of $\mathscr{G}$, we immediately get

(**50**) $$H^i(\mathscr{G}, \Omega(r)) = H^i(\Delta, B) \qquad (i \geqslant 0),$$

where $B = H^0(\Gamma, \Omega(r))$. But B is clearly finite because $r \neq 0$, and so we deduce that the $H^i(\mathscr{G}, \Omega(r))$ are finite for all $r \geqslant 0$. Moreover, as Δ is cyclic and B is finite, we know that $H^i(\Delta, B)$ and $H^{i+1}(\Delta, B)$ have the same order for all $i \geqslant 1$. In view of (50), this proves the second assertion of Lemma 5.5.

We can now complete the proof of Theorem 5.2. We claim that Lemma 5.5 implies that $H^i(G_V, A_{p^\infty}(n))$ is finite for all $n \neq \pm 1$. Indeed, since $H^i(\mathscr{H}, A_{p^\infty}(n)) = 0$ for $i \geqslant 2$, the Hochschild-Serre spectral sequence gives, in view of (47) and (48), the exact sequence

(**51**) $$H^i(\mathscr{G}, \Omega(n+1)) \to H^i(G_V, A_{p^\infty}(n)) \to H^{i-1}(\mathscr{G}, \Omega(n-1)) \qquad (i \geqslant 2).$$

When $i = 1$, (47) and (48) show that (51) remains exact by the usual inflation-restriction sequence. Hence Lemma 5.5 proves that the $H^i(G_V, A_{p^\infty}(n))$ are finite for all $i \geqslant 0$.

Suppose now that G_V has no element of order p, so that G_V has p-cohomological dimension equal to 2. Since $H^2(G_V, A_{p^\infty}(n))$ is finite, Lemma 5.4 shows that $H^2(G_V, A_{p^\infty}(n)) = 0$. Thus the usual inflation-restriction exact sequence, together with (47) and (48), yields the exact sequence

$$
\begin{aligned}
&0 \to H^1(\mathscr{G}, \Omega(n+1)) \to H^1(G_V, A_{p^\infty}(n)) \to \\
&\quad H^0(\mathscr{G}, \Omega(n-1)) \to H^2(\mathscr{G}, \Omega(n+1)) \to 0.
\end{aligned}
\tag{52}
$$

In view of Lemma 5.5, we conclude that

$$\# \ H^1(G_V, A_{p^\infty}(n)) = \# \ H^0(\mathscr{G}, \Omega(n-1)). \tag{53}$$

Combining (53) with the fact that (47) gives

$$\# \ H^0(G_V, A_{p^\infty}(n)) = \# \ H^0(\mathscr{G}, \Omega(n+1)), \tag{54}$$

and the proof of Theorem 5.2 is now complete.

Proof of Theorem 5.1. — We shall use the following standard notation. Let M be a G_F-module. If L is an algebraic extension of F, we write

$$M(L) = H^0(G(\overline{\mathbf{Q}}_p/L), M).$$

Suppose that $\varpi : G_F \to \mu_2$ is a continuous homomorphism. Then $M(\varpi)$ will denote the twist of M by ϖ, i.e. $M(\varpi)$ is the same underlying abelian group as M, but with the new action of G_F given by $\sigma \circ m = \varpi(\sigma)\sigma(m)$, where $\sigma(m)$ denotes the original action of σ in G_F on m. Suppose first that A has split multiplicative reduction over F. Then Theorem 5.2 with $n = 0$ implies Theorem 5.1 because, in this case, $\Omega(1)(F)$ and $\Omega(-1)(F)$ are clearly dual finite abelian groups. Hence they have the same order, and so $\chi_f(G_V, A_{p^\infty}) = 1$. Assume for the rest of this section that A has potential multiplicative reduction over F. Then, by the theory of the Tate curve, (see [31, Chap. V, Lemma 5.2]), there exists a quadratic extension K of F such that A is isomorphic over K to a Tate curve. Let ϖ be the homomorphism from G_F to μ_2 corresponding to the quadratic extension K over F. The theory of the Tate curve shows that we then have the exact sequence of G_F-modules

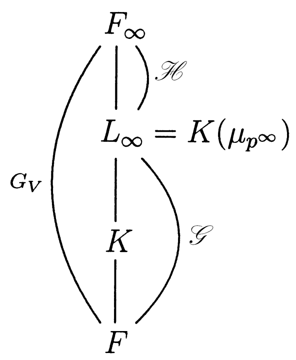
$$0 \to \Omega(1)(\varpi) \to A_{p^\infty} \to \Omega(\varpi) \to 0. \tag{55}$$

In particular, (55) shows that K is contained in $F_\infty = F(A_{p^\infty})$, because $\varpi(G(\overline{\mathbf{Q}}_p/F_\infty)) = 1$ since $\Omega(\varpi)$ is a quotient of A_{p^∞}.

We now define

$$L_\infty = K(\mu_{p^\infty}), \qquad \mathscr{H} = G(F_\infty/L_\infty), \qquad \mathscr{G} = G(L_\infty/F).$$

We have the diagram of fields:



Since A is defined over F, its j-invariant lies inside F, and thus the Tate period q_{A} of A over K actually lies in F rather than K. Hence, just as in the proof of (45), multiplicative Kummer theory shows that again $\mathscr{H} \simeq \mathbf{Z}_p(1)$ as a $\mathscr{G}$-module, where we stress that $\mathscr{G}$ now denotes the Galois group of $\mathrm{L}_\infty = \mathrm{K}(\mu_{p^\infty})$ over F. Taking $\mathscr{H}$-cohomology of the exact sequence (55), an entirely similar argument as before shows that

$$\text{(56)}\qquad \begin{array}{ll} \mathrm{H}^0(\mathscr{H}, \mathrm{A}_{p^\infty}) = \Omega(1)(\varpi), & \mathrm{H}^1(\mathscr{H}, \mathrm{A}_{p^\infty}) = \Omega(-1)(\varpi), \\ \mathrm{H}^i(\mathscr{H}, \mathrm{A}_{p^\infty}) = 0 \quad (i \geqslant 2). & \end{array}$$

Lemma **5.6.** — *Let* $\mathrm{L}_\infty = \mathrm{K}(\mu_{p^\infty})$*, and put* $\mathscr{G} = \mathrm{G}(\mathrm{L}_\infty/\mathrm{F})$*. Let r be a non-zero integer. Then* $\mathrm{H}^i(\mathscr{G}, \Omega(r)(\varpi))$ *is finite for all integers* $i \geqslant 0$.

Proof. — We now have $\mathscr{G} = \mathrm{D} \times \Gamma$, where $\Gamma = \mathrm{D} \times \Gamma$, where Γ is isomorphic to $\mathbf{Z}_p$, and $\mathrm{D} = \mathrm{G}(\mathrm{K}(\mu_{2p})/\mathrm{F})$. Unlike the proof of Lemma 5.5, D will not now generally be a cyclic group. However, since $r \neq 0$, exactly the same argument as in the proof of Lemma 5.5 shows that

$$\text{(57)}\qquad \mathrm{H}^i(\Gamma, \Omega(r)(\varpi)) = 0 \qquad (i \geqslant 1).$$

Hence we again obtain from the Hochschild-Serre spectral sequence that

$$\text{(58)}\qquad \mathrm{H}^i(\mathscr{G}, \Omega(r)(\varpi)) = \mathrm{H}^i(\mathrm{D}, \mathrm{B}),$$

where now $\mathrm{B} = \mathrm{H}^0(\Gamma, \Omega(r)(\varpi))$. But B is finite, and the assertion of the lemma is now clear.

We continue with the proof of the theorem. Assume from now on that $\mathrm{G_V}$ has no element of order p, so that $\mathrm{G_V}$ has p-cohomological dimension equal to 2. Since $\mathrm{H}^2(\mathrm{G_V}, \mathrm{A}_{p^\infty})$ is finite, Lemma 5.4 shows that $\mathrm{H}^2(\mathrm{G_V}, \mathrm{A}_{p^\infty}) = 0$. The argument now breaks up into two cases.

Case 1. — Assume that either p is odd or $p = 2$ and $K = F(\mu_4)$. We claim that in this case, we have

$$\# \ H^1(\mathscr{G}, \Omega(1)(\varpi)) = \# \ H^2(\mathscr{G}, \Omega(1)(\varpi)). \tag{59}$$

If p is odd, this is immediate from (58), since D is of order prime to p, and so both groups have order 1. If $p = 2$ and $K = F(\mu_4)$, then D is a cyclic group of order 2. But, as D is cyclic and B is finite, we have that $H^1(D, B)$ and $H^2(D, B)$ have the same order, and so again (58) implies (59). Once we have (59), we can complete the proof of Theorem 5.1 in exactly the same way as that of Theorem 5.2. Again, using (56) and the fact that $H^2(G_V, A_{p^\infty}) = 0$, we obtain the exact sequence

$$\begin{aligned} 0 \to H^1(\mathscr{G}, \Omega(1)(\varpi)) \to H^1(G_V, A_{p^\infty}) \to \\ H^0(\mathscr{G}, \Omega(-1)(\varpi)) \to H^2(\mathscr{G}, \Omega(1)(\varpi)) \to 0. \end{aligned}$$

Hence, in view of (59),

$$\# \ H^1(G_V, A_{p^\infty}) = \# \ H^0(\mathscr{G}, \Omega(-1)(\varpi)).$$

Also, (56) gives

$$\# \ H^0(G_V, A_{p^\infty}) = \# \ H^0(\mathscr{G}, \Omega(1)(\varpi)).$$

But since $\varpi^{-1} = \varpi$, $\Omega(-1)(\varpi)(F)$ and $\Omega(1)(\varpi)(F)$ are dual finite abelian groups, and hence they have the same order. Thus $\chi_f(G_V, A_{p^\infty}) = 1$ in this case.

Case 2. Assume that $p = 2$ and $K \neq F(\mu_4)$, so that $D = G(K(\mu_4)/F)$ is a product of two cyclic groups of order 2. Our earlier argument to prove (59) breaks down in this case, and we proceed as follows. Since $\Omega(1)(\varpi)(F) = \mathbf{Z}/2$, it follows from the first equation in (56) that

$$H^0(G_V, A_{2^\infty}) = \mathbf{Z}/2. \tag{60}$$

To prove the analogous statement for $H^1(G_V, A_{2^\infty})$, we first consider

$$G'_V = G(F_\infty/K), \qquad \mathscr{G}' = G(K(\mu_{2^\infty})/K).$$

By the definition of K, A has split multiplicative reduction over K, and hence (52) holds for A over K. In particular, when $n = 0$, this gives the exact sequence

$$\begin{aligned} &0 \to H^1(\mathscr{G}', \Omega(1)(\varpi)) \to H^1(G'_V, A_{2^\infty}) \\ &\to H^0(\mathscr{G}', \Omega(-1)(\varpi)) \to H^2(\mathscr{G}', \Omega(1)(\varpi)) \to 0. \end{aligned} \tag{61}$$

Put $\Delta' = G(K(\mu_4)/K)$, so that $\mathscr{G}' = \Delta' \times \Gamma$. Applying Lemma 5.5 to the extension $K(\mu_{2^\infty})/K$, (50) gives, for $r = \pm 1$, that the non-trivial element of the cyclic group Δ'

of order 2 acts on B by -1. Hence it follows that

(**62**) $$H^i(\mathscr{G}', \Omega(r)(\varpi)) = \mathbf{Z}/2 \qquad (i \geqslant 0, \quad r = \pm 1).$$

It follows immediately from (61) that

(**63**) $$H^1(G'_V, A_{2^\infty}) = \mathbf{Z}/2.$$

But, as $H^2(G_V, A_{2^\infty}) = 0$, we have the inflation-restriction sequence

(**64**) $$\begin{aligned} 0 \to H^1(\Delta, A_{2^\infty}(F')) &\to H^1(G_V, A_{2^\infty}) \\ \to H^1(G'_V, A_{2^\infty}) &\to H^2(\Delta, A_{2^\infty}(F')) \to 0, \end{aligned}$$

where $\Delta = G(K/F)$. Now $A_{2^\infty}(F') = \mathbf{Z}/2$ and Δ is of order 2, whence we conclude from (63) and (64) that

(**65**) $$H^1(G_V, A_{2^\infty}) = \mathbf{Z}/2.$$

Hence (60) and (65) show that $\chi_f(G_V, A_{2^\infty}) = 1$ in this case. The proof of Theorem 5.1 is now complete.

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