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SINGULAR COMPLETE INTEGRABILITY

by Laurent STOLOVITCH

ABSTRACT

We show that a holomorphic vector field in a neighbourhood of its singular point $0 \in \mathbf{C}^n$ is analytically normalizable if it has a sufficiently large number of commuting holomorphic vector fields, a sufficiently large number of formal first integrals and that a diophantine small divisors condition related to the linear parts of its centralizer is satisfied.

CONTENTS

1. Summary

Let $n \geqslant 2$ be an integer, and let $\mathfrak{g}$ be a commutative Lie algebra over $\mathbf{C}$. Let $\lambda_1, \ldots, \lambda_n$ be complex linear forms over $\mathfrak{g}$ such that the Lie morphism S from $\mathfrak{g}$ to the Lie algebra of linear vector fields of $\mathbf{C}^n$ defined by $S(g) = \sum_{i=1}^n \lambda_i(g) x_i \partial/\partial x_i$ is injective. For any $Q \in \mathbf{N}^n$ and $1 \leqslant i \leqslant n$, we define the weight $\alpha_{Q,i}(S)$ of S to be the linear form $\sum_{j=1}^n q_j \lambda_j(g) - \lambda_i(g)$. Let $\|.\|$ be a norm on the $\mathbf{C}$-vector space of linear forms on $\mathfrak{g}$. Let us define the sequence of positive real numbers:

$$\omega_k = \inf\{\|\alpha_{Q,i}\| \neq 0,\ 1 \leqslant i \leqslant n,\ 2 \leqslant |Q| \leqslant 2^k\}.$$

We define a *diophantine condition* relative to S by

$$(\omega(S)) \qquad -\sum_{k \geqslant 0} \frac{\ln \omega_k}{2^k} < +\infty.$$

Let $\mathscr{X}_n^k$ (resp. $\widehat{\mathscr{X}}_n^k$) be the Lie algebra of germs of holomorphic (resp. formal) vector fields of order k at $0 \in \mathbf{C}^n$. Let $\left(\widehat{\mathscr{X}}_n^1\right)^S$ (resp. $\left(\widehat{\mathscr{O}}_n\right)^S$) be the formal centralizer of S (resp. the ring of formal first integrals), that is, the set of formal vector fields X (resp. formal power series f) such that $[S(g), X] = 0$ (resp. $\mathscr{L}_{S(g)}(f) = 0$) for all $g \in \mathfrak{g}$. A nonlinear deformation $S + \varepsilon$ of S is a Lie morphism from $\mathfrak{g}$ to $\mathscr{X}_n^1$ such that $\varepsilon \in \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathscr{X}_n^2)$. Let $\widehat{\Phi}$ be a formal diffeomorphism of $(\mathbf{C}^n, 0)$ which is assumed to be tangent to *Id* at 0. We define $\widehat{\Phi}^*(S + \varepsilon)(g) := \widehat{\Phi}^*(S(g) + \varepsilon(g))$ to be the conjugate of $S + \varepsilon$ by $\widehat{\Phi}$. We shall define the notion of formal normal form of $S + \varepsilon$ relative to S. One of our main results is the following:

Theorem **1.0.1.** — *Let* S *be an injective diagonal morphism such that the condition* $(\omega(S))$ *holds. Let* $S+\varepsilon$ *be a nonlinear holomorphic deformation of* S. *Let us assume that it admits an element of* $\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g},\ \left(\widehat{\mathscr{O}}_n\right)^S \otimes_{\mathbf{C}} S(\mathfrak{g})\right)$ *as a formal normal form. Then there is a formal normalizing diffeomorphism* $\widehat{\Phi}$ *which is holomorphic in a neighbourhood of* 0 *in* $\mathbf{C}^n$.

In other words, if S is diophantine, then $S + \varepsilon$ is holomorphically normalizable as soon as the formal normal form of $S + \varepsilon$ belongs to $\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \left(\widehat{\mathcal{O}}_n\right)^{S} \otimes_{\mathbf{C}} S(\mathfrak{g})\right)$. Such a morphism will be called "formally completely integrable" and we shall motivate this definition later on.

We shall give another result which allows us to relax our condition on the formal normal form. We postpone this result to the "Statement" section since it requires some technical definitions.

Furthermore, we shall answer the following question:

How can we embed a formally completely integrable morphism into an higher dimensional space in order that the new nonlinear morphism is still formally completely integrable?

The precise answer will require some notation and definitions and we postpone our result to the "Statement" section.

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2. Introduction

This article is concerned with the study of holomorphic vector fields in a neighbourhood of a singular point in $\mathbf{C}^n$, that is, a point where they vanish. Let us start with a very elementary example of a similar problem. In order to study the iterates of a square complex matrix A of $\mathbf{C}^n$, that is, the orbits $\{A^k x\}_{k\in\mathbf{N}}$ for $x \in \mathbf{C}^n$ near the "singular point" 0, it is very convenient to transform, with the help of a linear change of coordinates P, the matrix A into a Jordan matrix $S + N$, with S a diagonal matrix, N an upper triangular nilpotent matrix commuting with S: $PAP^{-1} = S + N$. Using the structure of $S + N$, it is easy to study its iterates. Since $A^k = P^{-1}(S + N)^k P$, we thus obtain all the information needed to study the iterates of A.

One of the great ideas of Poincaré was to try to proceed in the same way for vector fields. Is it possible to transform a given vector field X, vanishing at the origin of $\mathbf{C}^n$, into a "simpler" one with the help of a local diffeomorphism Φ of $0 \in \mathbf{C}^n$ which maps 0 to itself? The group of germs of holomorphic (resp. formal) diffeomorphisms at $0 \in \mathbf{C}^n$ and tangent to $Id_{\mathbf{C}^n}$ at the origin, acts on the space of germs of holomorphic (resp. formal) vector fields at $0 \in \mathbf{C}^n$ by conjugacy: if X is any representative of a germ of a vector field $\mathbf{X}$, and ϕ is any representative of a germ of diffeomorphism $\mathbf{\Phi}$, then $\mathbf{\Phi}^*\mathbf{X}$ is the germ of the vector field defined by $\phi^*X(\phi(x)) = D\phi(x)X(x)$, where $D\phi(x)$ denotes the derivative of ϕ at the point x. One may first attempt to linearize X, that is find a formal change of coordinates $\hat{\Phi}$, such that $\hat{\Phi}^*X(y) = DX(0)y$. Assuming this to be the case, one would expect to understand all the dynamics of X, since the flow of the linear vector field $DX(0)y$ is easy to study. Nevertheless, this cannot be the case unless we are able to pullback this information by $\hat{\Phi}$, and this requires some "regularity" conditions on $\hat{\Phi}$. Since we are working in the analytic category, this regularity condition should be that $\hat{\Phi}$ is holomorphic in a neighbourhood of the origin. For the sake of simplicity, let us assume that $DX(0)x = \sum_{i=1}^{n} \lambda_i x_i \partial/\partial x_i$ is a diagonal vector field. If $Q = (q_1, \ldots, q_n) \in \mathbf{N}^n$, we shall write $(Q, \lambda) = \sum_{i=1}^{n} q_i \lambda_i$ and $|Q| = q_1 + \cdots + q_n$. In order to have the regularity condition, one has to assume that the collection of eigenvalues $(\lambda_1, \ldots, \lambda_n)$ satisfies a diophantine arithmetical condition: C.L. Siegel [Sie42, Arn80] showed that if there exist $C > 0$ and $\mu \geqslant 0$ such that for all $Q \in \mathbf{N}^n$, $|Q| \geqslant 2$, and all indices $1 \leqslant i \leqslant n$, $|(Q, \lambda) - \lambda_i| \geqslant C|Q|^{-\mu}$, then X is holomorphically linearizable, that is, the formal diffeomorphism $\hat{\Phi}$ (which is unique in this case) is in fact holomorphic in a neighbourhood of the origin. This arithmetical condition was improved by A. Bruno [Bru72] by a weaker sufficient condition (ω) which will be defined later on. The aim of these conditions is to control the speed at which the **small divisors** $(Q, \lambda) - \lambda_i$ accumulate at $0 \in \mathbf{C}$. In the case of 1-dimensional germs of holomorphic diffeomorphisms, J.-C. Yoccoz [Yoc88, Yoc95] showed that Bruno's condition (ω) is also a necessary one.

When we try to linearize the vector field X with a formal change of coordinates, we see what are the formal obstructions to obtain such a formal diffeomorphism: these are the numbers $(Q, \lambda) - \lambda_i$ which vanish for some $Q \in \mathbf{N}^n$ with $|Q| \geqslant 2$ and some index $1 \leqslant i \leqslant n$. They are called the **resonance relations**. This kind of relation implies that the vector field $(2x + y^2)\partial/\partial x + y\partial/\partial y$ is not C^2-conjugate in a neighbourhood of the origin to its linear part $2x\partial/\partial x + y\partial/\partial y$. This means that we cannot find a twice continuously differentiable local diffeomorphism of a neighbourhood of the origin which "transforms" the first vector field into the second one. Nevertheless, one can show [Arn80, Rou75, CS, Cha86] that there exists a formal diffeomorphism $\hat{\Phi}$ (which is not unique) such that

$$\hat{\Phi}^*X = \sum_{i=1}^{n} \lambda_i x_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{n} \left(\sum_{(Q,\lambda)=\lambda_i} a_{i,Q} x^Q \right) \frac{\partial}{\partial x_i},$$

where the sum is over the multi-integers $Q \in \mathbf{N}^n$, $|Q| \geqslant 2$, and the indices i which satisfy $(Q, \lambda) = \lambda_i$, and where the $a_{i,Q}$'s are complex numbers. As usual, if $Q = (q_1, \ldots, q_n)$, then $x^Q = x_1^{q_1} \ldots x_n^{q_n}$. This kind of formal vector field will be called a (Poincaré-Dulac) **normal form** of X. It plays the same rôle as the Jordan form does for a matrix. In fact, if we denote by S the linear part of X, then a normal form can written $\hat{\Phi}^*X = S + N$, where S is a diagonal linear vector field, N is a nilpotent vector field (that is the sum of a nilpotent linear vector field and a nonlinear one); moreover, the

Lie bracket of the vector fields [S, N] vanishes. The main drawback of this is that whenever X is analytic, the normalizing diffeomorphism $\widehat{\Phi}$ as well as the associated normal form $\widehat{\Phi}^*X$ may only be formal objects.

The holomorphy of a normalizing diffeomorphism is related, on the one hand, to the small divisors problem, and on the other hand to the presence of non-trivial formal first integrals. Indeed, for instance, the vector field $x^2\,\partial/\partial x + (x+y)\,\partial/\partial y$ does not have any non-trivial formal first integral. It has the vector field $x^2\,\partial/\partial x + y\,\partial/\partial y$ as normal form with an associated normalizing diffeomorphism $x = x_1, y = y_1 + \phi(x_1)$ defined by $\phi(x_1) = \sum_{k\geqslant 1}(k-1)!x_1^k$ which is clearly a divergent series. This example has generated a lot of papers about sectorial normalization and analytic classification [Mal82, MR82, MR83, Vor81, Eca, Eca92, Sto96]. Nevertheless, having enough formal first integrals is far from being sufficient for a holomorphic vector field to be holomorphically normalizable. In fact, J.-P. Françoise proved [Fra80] that a holomorphic volume preserving vector field X (say div $X = 0$) with a non-trivial holomorphic first integral may only be formally normalizable.

One of the striking results of Bruno [Bru72, Mar80] is the following:

Let X *be a holomorphic vector field in a neighbourhood of* $0 \in \mathbf{C}^n$. *Let us assume that its linear part* $\sum_{i=1}^n \lambda_i x_i\,\partial/\partial x_i$ *at the origin satisfies the diophantine arithmetical condition* (ω). *We assume furthermore that* X *has a formal normal form of the form:* $\widehat{a}(x)\sum_{i=1}^n \lambda_i x_i\,\partial/\partial x_i$ *for some formal power series* $\widehat{a} \in \mathbf{C}[[x_1, \ldots, x_n]]$. *Then the associated normalizing diffeomorphism is analytic in a neighbourhood of the origin.*

On the other hand, J. Vey [Vey79] proved the following result:

Let $X_1, \ldots, X_{n-1}$ *be holomorphic vector fields in a neighbourhood of a common singular point* $0 \in \mathbf{C}^n$, *and which are volume preserving (that is, the Lie derivative* $\mathscr{L}_{X_i}\omega = 0$ *for a holomorphic non singular n-differential form* ω*), "independent" and commuting with each other. Then, the vector fields* $X_1, \ldots, X_{n-1}$ *are holomorphically and simultaneously normalizable (in a sense which has to be defined).*

One of the aims of this article is to show that these results are in fact the same one. We will show that a vector field will be holomorphically normalizable if its linear part is not too wild and it has enough formal first integrals and **symmetries**. In the same spririt Bruno and Walcher proved [BW94] that in dimension 2, a vector field satisfies the assumptions of Bruno's theorem above if and only if it admits a non-trivial commuting holomorphic vector field. A much weaker result was given by the author in [Sto97]: we had assumed that the vector fields have holomorphic first integrals. We mention a manuscript of D. Cerveau and J. Ecalle in which they study pairs of holomorphic vector fields in $\mathbf{C}^3$. In the C^∞ case, the linearization problem of 2 vector fields was solved by R. Roussarie and F. Dumortier [DR80]. The linearization problem of distributions was studied by D. Cerveau [Cer79].

This problem is completely solved for semi-simple finite-dimensional Lie algebras of holomorphic vector fields: they are holomorphically linearizable [GS68, Kus67] (an elegant proof can be found in [CG97]).

2.1. *Statement*

We will be concerned not only with a holomorphic vector field in a neighbourhood of a singular point but rather with collections $(X_1, ..., X_l)$ of vector fields which commute pairwise. This kind of object will be described by a **Lie morphism** F from a complex commutative finite-dimensional Lie algebra $\mathfrak{g}$ to the Lie algebra $\mathscr{X}_n^1$ of holomorphic vector fields in a neighbourhood of the singular point $0 \in \mathbf{C}^n$ by setting $F(g_i) = X_i$, where $G = \{g_1, ..., g_l\}$ denotes a basis of $\mathfrak{g}$. We shall require that their linear parts are independent and belong to an l-dimensional vector space of linear diagonal vector fields. This leads us to define a Lie morphism ϕ from $\mathfrak{g}$ to the Lie algebra $\mathscr{P}_n^1$ of linear vector fields of $\mathbf{C}^n$. Thus, our object F will be thought of as a nonlinear **deformation** of ϕ which is still a Lie morphism. This very elementary Lie algebra setting is due to the fact that such a Lie morphism defines two natural representations: on the one hand, the map $g \mapsto [\phi(g), .]$ defines a representation of $\mathfrak{g}$ into the vector space of holomorphic (resp. formal) vector fields vanishing at the origin. On the other hand, the map $g \mapsto \mathscr{L}_{\phi(g)}(.)$ (the Lie derivative along the vector field $\phi(g)$) defines a representation of $\mathfrak{g}$ into the space of holomorphic functions (resp. formal power series) vanishing at the origin. To these representations one may associate a complex of vector spaces, namely the Chevalley-Koszul complex whose cohomology spaces (at least the 0-th and the 1-st) play an important rôle in our problem. For instance, the 0-th cohomology space associated to the first (resp. second) representation is nothing but the common holomorphic centralizer (resp. the common first integrals) of $\phi(g_1), ..., \phi(g_l)$.

The first part of this paper is devoted to the study of these representations of an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$ (not necessarily commutative) defined by a morphism ϕ from $\mathfrak{g}$ to the space of linear vector fields. This study will be followed by the definition of a **formal normal form** of a nonlinear deformation $\phi + \varepsilon$ of ϕ, that is, a normal form relative to the Lie subalgebra $\phi(\mathfrak{g})$. The formal obstructions for such a deformation to be linearizable are closely related to the first cohomology space of the associated Chevalley-Koszul complex. After choosing a supplementary space $\tilde{V}$(resp. V) of the 1-cocycles space in the 1-cochains space with values in the space of formal vector fields (resp. 1-coboundaries space in the 1-cocycles space), there is a formal diffeomorphism $\hat{\Phi}$ such that $\hat{\Phi}^*(\phi + \varepsilon) - \phi \in \tilde{V} \oplus V$. We shall call this a **formal normal form**. The morphism $\phi + \varepsilon$ is said to be **compatible** whenever its normal form belongs to V; in this case, a normal form is given by an element of the first cohomology space $H^1_\phi\left(\mathfrak{g}, \widehat{\mathscr{X}}_n^2\right)$ of the Chevalley-Koszul complex ($\widehat{\mathscr{X}}_n^2$ denotes the space of formal vector fields of order $\geqslant 2$ at $0 \in \mathbf{C}^n$). We shall call these elements the **resonant vector fields**. After choosing a suitable supplementary space K of $H^0_\phi\left(\mathfrak{g}, \widehat{\mathscr{X}}^2_n\right)$ in $\widehat{\mathscr{X}}^2_n$, a formal normalizing diffeomorphism is uniquely determined if it belongs to the exponential of K.

In order to describe the space of normal forms, one has to compute the first cohomology space. This is the goal of section "Fundamental structures". We shall study the case where $\mathfrak{g}$ is **nilpotent**. We shall show that the linear morphism ϕ, after a linear change of variable, can be written as $S + N$ where both S and N are linear morphisms, for all $g \in \mathfrak{g}$, $S(g)$ is a linear diagonal vector field and $N(g)$ is a nilpotent vector field commuting with $S(g)$; that is $[S(g), N(g)] = 0$. Let us set $\widehat{\mathscr{O}}^\phi_n := H^0_\phi(\mathfrak{g}, \widehat{\mathscr{O}}_n)$ (resp. $\left(\widehat{\mathscr{X}}^1_n\right)^\phi := H^0_\phi\left(\mathfrak{g}, \widehat{\mathscr{X}}^1_n\right)$), the space of formal first integrals of ϕ (resp. the space of formal centralizers of ϕ vanishing at the origin). We shall show that

$$H^1_S\left(\mathfrak{g}, \widehat{\mathscr{X}}^1_n\right) = \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], \left(\widehat{\mathscr{X}}^1_n\right)^S\right);$$

thus we need some information about $\left(\widehat{\mathscr{X}}^1_n\right)^S$. We shall show that $\widehat{\mathscr{O}}^S_n$ is a **formal algebra of finite type**; it can be written as $\widehat{\mathscr{O}}^S_n = \mathbf{C}[[u_1, \ldots, u_p]]$ for some homogeneous polynomials $u_1, \ldots, u_p$. Furthermore, $\left(\widehat{\mathscr{X}}^1_n\right)^S$ is an $\widehat{\mathscr{O}}^S_n$**-module of finite type**. One of the main objects introduced in this section is the notion of **weight** of the representation. A weight α for S is a **C**-linear form on $\mathfrak{g}$ vanishing on $[\mathfrak{g}, \mathfrak{g}]$ and such that $\{X \in \widehat{\mathscr{X}}^1_n \mid \forall g \in \mathfrak{g},\ [S(g), X] = \alpha(g)X\}$ does not reduce to zero. The latter space is called the **weight space** associated to the weight α. We may think of the weights as belonging to $\mathbf{R}^{2l'}$ (equipped with a suitable norm), l' being the dimension of $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. For each positive integer k, we define the real numbers $\omega_k(S)$ to be the smallest value of the norms of the non-zero weights of S into the space of nonlinear polynomial vector fields of degree $\leqslant 2^k$. We shall say that the diagonal morphism S **doesn't have small divisors** if the sequence of $\omega_k(S)$ is bounded away from zero. If it is not the case, we shall say that S is **diophantine** if

$$-\sum_{k\geqslant 0} \frac{\ln \omega_k(S)}{2^k} < +\infty$$

and this condition doesn't depend on the chosen norm.

The next section is devoted to the notion of complete integrability. We shall assume that $\mathfrak{g}$ is **commutative** and that $S := \phi$ is a **linear diagonal** morphism. The **isoresonant hull** of S, *IsoRes*(S), is the largest Lie subalgebra of the Lie algebra of diagonal vector fields which has the same invariants as S. By this, we mean that there is a commutative Lie algebra $\tilde{\mathfrak{g}}$ together with an injection $j\colon \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$, an injective diagonal Lie morphism $\tilde{S} : \tilde{\mathfrak{g}} \rightarrow \mathscr{P}^1_n$ which satisfies:

1. $\tilde{\phi}\circ j=\phi$,
2. $\left(\mathcal{X}_n^1\right)^{\tilde{S}} = \left(\mathcal{X}_n^1\right)^{S}$ and $\widehat{\mathcal{O}}_n^{\tilde{S}}=\widehat{\mathcal{O}}_n^{S}$.

With this notation, we set $IsoRes(S)=\tilde{S}(\tilde{\mathfrak{g}})$. A **diophantine hull** of S is a Lie subalgebra of linear diagonal vector fields of $\mathbf{C}^n$ which contains $S(\mathfrak{g})$, which has the same invariants and for which all the quotients of the norm of one of its weights by the norm of the same weight restricted to $\mathfrak{g}$ is universally bounded (this means: there is an injection i: $\mathfrak{g}\hookrightarrow\overline{\mathfrak{g}}$ into a commutative Lie algebra $\overline{\mathfrak{g}}$, a linear diagonal injective morphism $\overline{S}$ with $\overline{S}\circ i=S$, there is $c>0$ such that for all weights $\overline{\alpha}$ of $\overline{S}$, $\|\overline{\alpha}\|\leqslant c\|\overline{\alpha}\circ i\|$). Any diophantine hull of S is included in the isoresonant hull of S.

Let us first assume that S is **injective and has small divisors**. We shall say that a compatible nonlinear deformation $S+\varepsilon$ is **formally completely integrable** if its formal normal form belongs to the $\widehat{\mathcal{O}}_n^{S}$-module generated by one of the diophantine hulls *Dioph*(S) of S. We shall prove the following:

Theorem **2.1.1.** — *Under the above assumptions, if* S *is diophantine, then any formally completely integrable nonlinear deformation* $S+\varepsilon$ *of* S *is holomorphically normalizable.*

In other words, if the nonlinear holomorphic and commuting vector fields $X_1,\dots,X_l$ whose linear parts belong to $S(\mathfrak{g})$ have enough formal common first integrals, then these are in fact holomorphic first integrals as long as $\phi(\mathfrak{g})$ is not too wild. This result is very similar, in the case of vector fields, to the Malgrange singular Frobenius theorems in the case of holomorphic forms [Mal76, Mal77, Ram79] (see also [MM80] for 1-forms).

The next question that can be asked is the following: under what assumptions can a formally completely integrable nonlinear deformation $S+\varepsilon$ of S be extended in a higher dimensional space into another formally completely integrable nonlinear deformation $\widehat{S}+\widehat{\varepsilon}$ of $\widehat{S}$, both being morphisms from the **same** Lie algebra $\mathfrak{g}$ as S? We should point out to the reader that in general, it is completely wrong that the **same** number of commuting vector fields would ensure that the extension of a morphism is holomorphically normalizable; *a priori*, more vector fields will be needed.

Nevertheless, under certain hypotheses, this kind of statement is true. First of all, we shall define a good extension of S in $\mathbf{C}^{n+m}$ as $\widehat{S}:=S\oplus S''$, where S'' is a diagonal linear morphism from $\mathfrak{g}$ to $\mathcal{P}_m^1$. Of course, we want the properties of $\widehat{S}$ to be derived from those of S; that is, we want $\widehat{S}$ to be diophantine if S is and we require $\mathcal{O}_{n+m}^{\widehat{S}}=\mathcal{O}_n^{S}$. One way to achieve this is to assume that S'' is a **Poincaré morphism relative to** S: we require that the weights of S all belong to a real linear hyperplane of $\mathbf{R}^{2l}$, but all but a finite number of weights of S'', belong to one and the same side of the hyperplane. Such an extension will be called proper if the only weight of S'' which belongs to the hyperplane is the zero weight.

A **trivial deformation** of 0 over $\widehat{\mathscr{O}}_n^{S}$ relative to S″ is a formal nonlinear deformation

$$X = D'' + \mathcal{N}il'' + R'' \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \widehat{\mathscr{O}}_n^{S} \otimes \left(\widehat{\mathscr{X}}_m^1\right)^{S''}\right),$$

where $D'' \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \widehat{\mathscr{O}}_n^{S} \otimes_{\mathbf{C}} \mathit{IsoRes}(S'')\right)$ is diagonal, $\mathcal{N}il''$ is nilpotent, R″ is nonlinear (as vector fields of $\mathbf{C}^m$) and $[D'', \mathcal{N}il'' + R''] = 0$. Now we can define the notion of complete integrability for an extended morphism. Let $\widehat{S} = S \oplus S''$ be a proper Poincaré extension of S which is assumed to be diophantine and injective. A nonlinear deformation of the proper **Poincaré extension** $\widehat{S}$ will be said to be **formally completely integrable** if its formal normal form is the sum of an element of $\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \widehat{\mathscr{O}}_n^{S} \otimes_{\mathbf{C}} \mathit{Dioph}(\widehat{S})\right)$ and a trivial deformation of 0 over $\widehat{\mathscr{O}}_n^{S}$ relative to S″. Therefore, the restriction of such a completely integrable deformation of $\widehat{S}$ to $x_{n+1} = \ldots = x_{n+m} = 0$ is a completely integrable deformation of S. We shall prove the following:

Theorem **2.1.2.** — *Let* S *be a diophantine, injective diagonal linear morphism from* $\mathfrak{g}$ *to* $\mathscr{P}_n^1$. *We assume that* $\widehat{S} = S \oplus S''$ *is a proper Poincaré extension of* S *in* $\mathbf{C}^{n+m}$ *by* S″. *Then any nonlinear deformation of* $\widehat{S}$ *which is formally completely integrable is holomorphically normalizable.*

We shall be able to give a similar result if the Poincaré extension of S by S″ is not proper. Let h be the set a weights of S″ which belong to the hyperplane. If the extension is proper then h reduces to 0; otherwise, let $\left(\widehat{\mathscr{X}}_m^1\right)_h(S'')$ be the direct sum of the weight spaces of S″ corresponding to the weights of h. Let $\mathscr{L}_h(S'')$ be the largest Lie algebra of the Lie algebra of diagonal linear vector fields of $\mathbf{C}^m$ such that $\left[\mathscr{L}_h(S''), \left(\widehat{\mathscr{X}}_m^1\right)_h(S'')\right] = 0$. This means that the vector fields of $\left(\widehat{\mathscr{X}}_m^1\right)_h(S'')$ belong to the centralizer of $\mathscr{L}_h(S'')$. In this case, we have to restrict the notion of "trivial deformation" to the following notion. A **good deformation** of 0 relative to (S, S″) is a formal nonlinear deformation

$$X = D'' + \mathcal{N}il'' + R'' \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \left(\widehat{\mathscr{O}}_n \otimes \left(\widehat{\mathscr{X}}_m^1\right)_h(S'')\right) \cap \left(\widehat{\mathscr{X}}_{n+m}^1\right)^{S\oplus S''}\right)$$

such that

$$D'' \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \left(\widehat{\mathscr{O}}_n^{S} \otimes_{\mathbf{C}} \mathscr{L}_h(S'')\right) \cap \left(\widehat{\mathscr{X}}_{n+m}^1\right)^{S\oplus S''}\right) \text{ is diagonal,}$$

$$\mathcal{N}il'' \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \widehat{\mathscr{O}}_n^{S} \otimes \left(\mathscr{P}_m^1\right)^{S''}\right) \text{ is nilpotent, and}$$

$$R'' \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \left(\widehat{\mathscr{O}}_n \otimes \left(\widehat{\mathscr{X}}_m^2\right)_h(S'')\right) \cap \left(\widehat{\mathscr{X}}_{n+m}^1\right)^{S\oplus S''}\right) \text{ is nonlinear}$$

(as vector fields of $\mathbf{C}^m$). Furthermore, we will have $[D'', \mathcal{N}il'' + R''] = 0$. A compatible nonlinear deformation of the **Poincaré extension** $\widehat{S}$ will be said **formally completely integrable** if its formal normal form is the sum of an element of $\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \widehat{\mathcal{O}}_n^S \otimes_{\mathbf{C}} \mathit{Dioph}(\widehat{S})\right)$ and a good deformation of 0 relative to (S, S'').

Theorem **2.1.3.** — *Let* S *be a diophantine, injective diagonal linear morphism from* $\mathfrak{g}$ *to* $\mathcal{P}_n^1$. *We assume that* $\widehat{S} = S \oplus S''$ *is a Poincaré extension of* S *in* $\mathbf{C}^{n+m}$ *by* S''. *Then, any nonlinear deformation of* $\widehat{S}$ *which is formally completely integrable is holomorphically normalizable.*

Both of the above theorems are the most difficult ones. Now, we wonder what can happen when S **doesn't have small divisors**. A compatible nonlinear deformation of a Poincaré extension $\widehat{S}$ of S will be said **formally completely integrable** if its formal normal form is the direct sum of a formally completely integrable deformation of S and a nonlinear deformation of S''; that is, no assumption is required on the projection onto $\partial/\partial x_{n+1}, \ldots, \partial/\partial x_{n+m}$ of the formal normal form.

Theorem **2.1.4.** — *Let* S *be an injective diagonal linear morphism from* $\mathfrak{g}$ *to* $\mathcal{P}_n^1$. *We assume that* S *doesn't have small divisors and that* $\widehat{S} = S \oplus S''$ *is a Poincaré extension of* S *in* $\mathbf{C}^{n+m}$. *Then any nonlinear deformation of* $\widehat{S}$ *which is formally completely integrable is holomorphically normalizable.*

We will show that both Bruno's and Vey's theorems are direct corollaries of these results.

2.2. *Geometric interpretation*

In order to illustrate our result, let us first recall the Liouville theorem [Arn76]. Let $H_1, \ldots, H_n$ be smooth functions on a smooth symplectic manifold M^{2n}; let $\pi : M^{2n} \to \mathbf{R}^n$ denotes the map $\pi(x) = (H_1(x), \ldots, H_n(x))$. We assume that for any $1 \leqslant i, j \leqslant n$, the Poisson bracket $\{H_i, H_j\}$ vanishes. Let $c \in \mathbf{R}^n$ be a regular value of π; we assume that $\pi^{-1}(c)$ is compact and connected. Then there exists a neighbourhood U of $\pi^{-1}(c)$ and a symplectomorphism Φ from U to $\pi(U) \times \mathbf{T}^n$ such that, in this new coordinate system, each symplectic vector field X_{H_i} associated to H_i is tangent to the fibre $\{d\} \times \mathbf{T}^n$. It is constant on it and the constant depends only on the fibre. Such a family of hamiltonian vector fields is called a completely integrable system.

Let us come back to our problem and let S be a diophantine injective diagonal linear morphism from $\mathfrak{g}$ to $\mathcal{P}_n^1$. Let $\widehat{\mathcal{O}}_n^S$ be its ring of formal first integrals. If $\widehat{\mathcal{O}}_n^S \neq \mathbf{C}$, it is a $\mathbf{C}$-algebra of finite type and there are homogeneous polynomials $u_1, \ldots, u_p$ such that $\widehat{\mathcal{O}}_n^S = \mathbf{C}[[u_1, \ldots, u_p]]$. Let $\pi : \mathbf{C}^n \to \mathbf{C}^p$ be defined by $\pi(x) = (u_1(x), \ldots, u_p(x))$. Let s be the degree of transcendence of the field of fractions of $\mathbf{C}[u_1, \ldots, u_p]$; it is the maximal number of algebraically-independent polynomials among $u_1, \ldots, u_p$. The algebraic relations among $u_1, \ldots, u_p$ define an s-dimensional algebraic variety $\mathscr{C}_S$ in $\mathbf{C}^p$. Hence, π defines a **singular** fibration over $\mathscr{C}_S$. The linear vector fields $S(g_1), \ldots, S(g_l)$ ($\{g_1, \ldots, g_l\}$ denotes a basis of $\mathfrak{g}$) are tangent and independent on each fibre of π. Note that we must have $l \leqslant n - s$. Now we come to the nonlinear deformation. Let $S + \varepsilon$ be a nonlinear deformation of S. Let us assume that it is formally completely integrable. Then according to our result, there exist a neighbourhood U of 0 in $\mathbf{C}^n$ and a holomorphic diffeomorphism Φ on U such that, in the new coordinate system, the vector fields $\Phi^*(S + \varepsilon)(g_1), \ldots, \Phi^*(S + \varepsilon)(g_l)$ are linear diagonal vector fields on each fibre restricted to U, they commute to each other and their eigenvalues depend only on the fibre. Indeed, in these new coordinates, we have $\Phi^*(S + \varepsilon)(g_i) = \sum_{j=1}^{r+l} a_{i,j} S(g_j)$ where $a_{i,j} \in \mathscr{O}_n^S$; here $\{S(g_1), \ldots, S(g_{l+r})\}$ denotes a linearly-independent set of some diophantine hull of S (r may be equal to 0). By definition, these vector fields are all tangent to the fibres of π (therefore, we must have $r + l \leqslant n - s$). As a consequence the $\Phi^*(S + \varepsilon)(g_i)$'s are all tangent to the fibres of π. On each fibre, the functions $a_{i,j}$ are constant so that each $\Phi^*(S + \varepsilon)(g_i)$ may be written as a linear diagonal vector field.

Let us give the geometric interpretation of a formally completely integrable deformation of a proper Poincaré extension $\tilde{S} = S \oplus S''$ of S in $\mathbf{C}^{n+m}$. Let $\tilde{\pi} : \mathbf{C}^{n+m} \to \mathbf{C}^p$ be the map defined by $\tilde{\pi}(x, y) = \pi(x)$, where $(x, y) \in \mathbf{C}^n \times \mathbf{C}^m$. Let $c \in \mathscr{C}_S$, then $\tilde{\pi}^{-1}(c) = \pi^{-1}(c) \times \mathbf{C}^m$. Let $\tilde{S} + \tilde{\varepsilon}$ be a nonlinear deformation of $\tilde{S}$. Let us assume that it is formally completely integrable. Then there exists a neighbourhood $\tilde{U}$ of 0 in $\mathbf{C}^{n+m}$ and a holomorphic diffeomorphism $\tilde{\Phi}$ on $\tilde{U}$ such that, in the new coordinate system, the vector fields $\tilde{\Phi}^*(\tilde{S} + \tilde{\varepsilon})(g_1), \ldots, \tilde{\Phi}^*(\tilde{S} + \tilde{\varepsilon})(g_l)$ are commuting vector fields tangent to each fibre of $\tilde{\pi}$ restricted to $\tilde{U}$. On such a fibre $\tilde{\pi}^{-1}(c) \cap \tilde{U}$, each of the $\tilde{\Phi}^*(\tilde{S} + \tilde{\varepsilon})(g_i)$'s is the sum of a vector field V'_i tangent $\pi^{-1}(c)$ and a local vector field V''_i of $\mathbf{C}^m$. As above, V'_i is a linear diagonal vector field whose eigenvalues depend only on c. Moreover, V''_i is a Poincaré normal form (polynomial vector field) whose coefficients depend only on the fiber and are holomorphic; that is, V''_i may be written as $\tilde{S}'' + \tilde{N}'' + R''$ where $\tilde{S}''$ is a linear diagonal vector field of $\mathbf{C}^m$, $\tilde{N}''_i$ is a linear nilpotent vector field of $\mathbf{C}^m$ and R''_i is a nonlinear polynomial vector field of $\mathbf{C}^m$ such that $[\tilde{S}''_i, \tilde{N}''_i + \tilde{R}''_i] = 0$ and $[S'', \tilde{N}''_i + \tilde{R}''_i] = 0$.

2.3. *Sketch of the proof*

Let us give a sketch of the proof. In order to normalize the nonlinear deformation $S + \varepsilon$ of S, we shall proceed using a classical Newton method, that is a Nash-Moser induction type.

Let $1 \leqslant k \leqslant p$ be integers, and let $\mathscr{P}_n^{k,p}$ be the space of polynomial vector fields of $\mathbf{C}^n$ of degree $\leqslant p$ and of order $\geqslant k$. Let $\mathfrak{g}$ be a complex commutative Lie algebra of dimension l, and let S be a Lie morphism from $\mathfrak{g}$ to $\mathscr{P}_n^{1,1}$ such that $S(g)$ is a diagonal linear vector field. Let us consider the map $\rho : \mathfrak{g} \to \mathrm{Hom}_{\mathbf{C}}\left(\mathscr{P}_n^{k,p}, \mathscr{P}_n^{k,p}\right)$ defined by $\rho(g)(X) = [S(g), X]$, where $g \in \mathfrak{g}$, $X \in \mathscr{P}_n^{k,p}$($[.,.]$ denotes the Lie bracket of vector fields of $\mathbf{C}^n$). It is well defined and it is a **representation** of S into $\mathscr{P}_n^{k,p}$. If α is a complex linear form on $\mathfrak{g}$, we define $\mathscr{P}_{n,\alpha}^{k,p} = \{X \in \mathscr{P}_n^{k,p} \mid \forall g \in \mathfrak{g},\ [S(g), X] = \alpha(g)X\}$. If $\mathscr{P}_{n,\alpha}^{k,p} \neq 0$ then α is called a **weight** of S and $\mathscr{P}_{n,\alpha}^{k,p}$ is called the associated **weight space**, and we have the Fitting decomposition $\mathscr{P}_n^{k,p} = \left(\mathscr{P}_n^{k,p}\right)^* \oplus \mathscr{P}_{n,0}^{k,p}$, where $\left(\mathscr{P}_n^{k,p}\right)^*$ is the sum of the weight spaces associated to the nonzero weights of S.

Let us assume that the nonlinear deformation $S + \varepsilon$ is normalized up to order m; we will build a diffeomorphism Φ_m which normalizes the deformation up to order $2m$; it is tangent to Id up to order m. Let us show how this works. First of all, we can write the deformation $S + \varepsilon = NF^m + B + R \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{X}_n^1\right)$, where NF^m is a normal form of degree m, B is a polynomial of degree $\leqslant 2m$ and of order $\geqslant m+1$, and R is of order $\geqslant 2m+1$ (we mean that B and R are linear maps from $\mathfrak{g}$ to the corresponding spaces). Let us denote by B^* (resp. B_0) the projection of B onto the sum of the weight spaces associated to a nonzero weight (resp. zero weight) of S into $\mathscr{P}_n^{m+1,2m}$. The compatibility condition (i.e $S + \varepsilon$ a Lie morphism) shows that, for all $(g_1, g_2) \in \mathfrak{g}^2$, the $2m$-jet

$$\textbf{(2.3.1)} \qquad J^{2m}\left([NF^m(g_1), B^*(g_2)] - [NF^m(g_2), B^*(g_1)] \right) = 0.$$

On the other hand, if we conjugate $S+\varepsilon$ by a diffeomorphism of the form $\exp(U)$ for some polynomial vector field $U \in \mathscr{P}_n^{m+1,2m}$ and write $\exp(U)^*(S+\varepsilon) = NF^m + B' + R'$ as above, we find that ·

$$J^{2m}\left(B' - B + [NF^m, U]\right) = 0.$$

The algebraic properties of the weight spaces of S show that we have

$$J^{2m}\left((B')^* - B^* + [NF^m, U^*]\right) = 0.$$

If we assume that the diffeomorphism $\exp(U)$ normalizes $S+\varepsilon$ up to order $2m$ then we must have $(B')^* = 0$ (this is a consequence of the description of the Chevalley-Koszul cohomology associated to S); hence, we have

$$\textbf{(2.3.2)} \qquad J^{2m}\left(-B^* + [NF^m, U^*]\right) = 0.$$

Let us denote by $\left(\mathscr{P}_n^{m+1,\,2m}\right)^*$ the projection of $\mathscr{P}_n^{m+1,\,2m}$ onto the direct sum of weight spaces associated to a nonzero weight of ρ. Let us define the linear map

$$\bar{\rho} : \mathfrak{g} \to \mathrm{Hom}_{\mathbf{C}}\left(\left(\mathscr{P}_n^{m+1,\,2m}\right)^*, \left(\mathscr{P}_n^{m+1,\,2m}\right)^*\right)$$

by $\bar{\rho}(g)(\mathrm{X}) = \mathrm{J}^{2m}([\mathrm{NF}^m(g), \mathrm{X}])$. It is well defined and it is a representation of $\mathfrak{g}$ into $\left(\mathscr{P}_n^{m+1,\,2m}\right)^*$. To this representation one may associate a complex of finite-dimensional complex vector spaces; it is the Chevalley-Kozsul complex of this representation. Let us write d_{N}^i for the i-th differential of this complex. Then equation (2.3.1) reads $d_{\mathrm{N}}^1(\mathrm{B}^*) = 0$, that is B^* is a 1-cocycle for this complex; equation (2.3.2) reads $d_{\mathrm{N}}^0(\mathrm{U}) = \mathrm{B}^*$, that is B^* is the 0-coboundary of U: it is a **cohomological equation.**

Hence, the Chevalley-Koszul complex of the representation $\bar{\rho}$ plays an important rôle in our problem. We shall call it the **Newton complex** of order m. Its study is a large part of our work. According to the discussion above, the first important problem to study is its cohomology. We shall show that the 0-th cohomology as well as the 1-st cohomology spaces are zero. This is a general fact which holds even for nilpotent Lie algebras and nondiagonal linear morphisms. It is not very difficult, but rather technical. It leads to the important consequence that if B^* is given as above, there exists a unique $\mathrm{U} \in \left(\mathscr{P}_n^{m+1,\,2m}\right)^*$ such that, for all $g \in \mathfrak{g}$, $\mathrm{J}^{2m}([\mathrm{NF}^m(g), \mathrm{U}]) = \mathrm{B}^*(g)$; hence, conjugating $\mathrm{S} + \varepsilon$ by $\exp(\mathrm{U})$ normalizes $\mathrm{S} + \varepsilon$ up to order $2m$.

We find that the formal diffeomorphism defined by $\widehat{\Phi} := \lim_{k\to+\infty} \Phi_{2^k} \circ \ldots \circ \Phi_2$ normalizes $\mathrm{S} + \varepsilon$, where the Φ_i's are built as above. In order to prove that $\widehat{\Phi}$ is holomorphic in a neighbourhood of $0 \in \mathbf{C}^n$, one has to estimate the behaviour of each Φ_i. Here comes the analysis and the major part of this article. To get an estimate for $\Phi_m = \exp(\mathrm{U})$, we have to estimate U. Hence, we are led naturally to give bounds of the cohomology of the Newton complex: let $r > 0$, the spaces of the Newton complex are equipped with norms (depending on a real positive number r) which turn it into a topological complex of vector spaces. By the above algebraic properties, the 0-differential has a right inverse s on the space of 1-cocycles; if Z is a 1-cocycle of the Newton complex, then $s(\mathrm{Z})$ is the unique element of $\left(\mathscr{P}_n^{m+1,\,2m}\right)^*$ such that $d_{\mathrm{N}}^0(s(\mathrm{Z})) = \mathrm{Z}$. The main assumptions are as follows: if $\mathrm{S} + \varepsilon$ is **completely integrable** then we shall show that the map s is **continuous** and we shall give bounds for its norm. More precisely, we shall show that there exist constants $d, \eta_1, c(\eta_1)$, such that if $m = 2^k$ and if the r-norms of $\mathrm{NF}^m - \mathrm{S}$ and $\mathrm{D}(\mathrm{NF}^m - \mathrm{S})$ are sufficiently small, say $< \eta_1$ (for some $1/2 < r \leqslant 1$) then

$$\textbf{(2.3.3)} \qquad |s(\mathrm{Z})|_r \leqslant \frac{c(\eta_1)}{\omega_{k+1,\,\mathrm{G}}^d} |\mathrm{Z}|_r;$$

the constant d doesn't depend on η_1 (we recall that $\omega_{k,G}$ is the smallest norm of the nonzero weights of S into $\mathscr{P}_n^{2,2^k}$).

Let us describe the way in which we obtain this estimate. Let $\{g_1, \ldots, g_l\}$ be a fixed basis of $\mathfrak{g}$. In order to solve the cohomological equation associated to the 1-cocycle Z, it is necessary and sufficient to solve the system of l equations $J^{2m}([NF^m(g_i), U]) = Z(g_i)$, $i = 1, \ldots, l$. Here Z is assumed to belong to a weight space of S for some nonzero weight α. This set of equations can be written in the following matrix form

$$A(x)\begin{pmatrix} \alpha(g_1)U \\ \vdots \\ \alpha(g_l)U \end{pmatrix} + \begin{pmatrix} D_1(U) \\ \vdots \\ D_l(U) \end{pmatrix} = \begin{pmatrix} Z_1 + \mathfrak{Z}_1 \\ \vdots \\ Z_l + \mathfrak{Z}_l \end{pmatrix},$$

where A is a square $l \times l$ matrix with coefficients in the $\mathbf{C}$-algebra $\mathscr{O}_n^S$ of holomorphic first integrals of the linear part S (in fact, they are polynomials); $A(0) = Id$; the operators $D_1, \ldots, D_l$ are $\mathscr{O}_n^S$-linear; $\mathfrak{Z}_1, \ldots, \mathfrak{Z}_l$ have order $\geqslant 2m + 1$ and Z_i stands for $Z(g_i)$. After inverting the matrix A, we obtain l equations $(\alpha(g_i)Id + \tilde{D}_i)(U) = \tilde{Z}_i + \tilde{\mathfrak{Z}}_i$, $i = 1, \ldots, l$. The $\tilde{D}_i$'s (resp. $\tilde{Z}_i$, $\tilde{\mathfrak{Z}}_i$) are still $\mathscr{O}_n^S$-linear operators and they are linear combinations of the D_i's (resp. Z_i, $\mathfrak{Z}_i$) with coefficients in $\mathscr{O}_n^S$. Let us set $\|\alpha\| = \max_{1 \leqslant j \leqslant l} |\alpha(g_j)|$, and let i be such that $|\alpha(g_i)| = \|\alpha\| \neq 0$. Let us look at the i-th equation; we find that, at least formally, its solution U is given by

$$U = \frac{1}{\alpha(g_i)} \sum_{k \geqslant 0} \left(\frac{-1}{\alpha(g_i)}\right)^k \tilde{D}_i^k(\tilde{Z}_i + \tilde{\mathfrak{Z}}_i).$$

This expression is not too helpful since it involves *a priori* infinitely large powers of $\alpha(g_i)$ which can be very small. Thus, instead of using this expression, we shall split the i-th equation in an appropriate way. First of all, we shall split the linear diagonal morphism S into two parts S′ and S″ corresponding to the splitting of $\mathbf{C}^n$ as $\mathbf{C}^{n'} \times \mathbf{C}^{n-n'}$; that is, for all $g \in \mathfrak{g}$,

$$S(g) = \underbrace{\sum_{k=1}^{n'} \lambda_k(g) x_k \frac{\partial}{\partial x_k}}_{S'(g)} + \underbrace{\sum_{k=n'+1}^{n} \lambda_k(g) x_k \frac{\partial}{\partial x_k}}_{S''(g)}.$$

The integer n' is chosen such that the linear forms $\{\lambda_k\}_{1 \leqslant k \leqslant n'}$ all belong to a real hyperplane H of $\mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathbf{C})$ whereas all the linear forms $\{\lambda_k\}_{n'+1 \leqslant k \leqslant n}$ all belong (strictly) to one and the same side of H. The integer n' is taken to be the smallest possible; it may be equal to 0 as well as equal to n. We shall call this splitting the **analytic splitting** of S. It has been chosen in such a way that the **small divisors** as well as the **first integrals** only depend on S′. We show that there is a **separating**

constant $Sep(S) > 0$ such that if α is a weight of S whose norm is smaller than $Sep(S)$ then it must belong to H (if $n' = n$ we shall set $Sep(S) = +\infty$ in order to have a single proof for the theorems). Let X be a vector field of $\mathbf{C}^n$, we shall denote by X′ (resp. X″) its projection onto $\partial/\partial x_1, ..., \partial/\partial x_{n'}$ (resp. $\partial/\partial x_{n'+1}, ..., \partial/\partial x_n$). This being said, let us go back to the study of our equation $(\alpha(g_i)Id + \tilde{\mathrm{D}}_i)(\mathrm{U}) = \tilde{\mathrm{Z}}_i + \mathfrak{Z}_i$. Using the analytic splitting of S as well as the structure of the operator D_i, we show that this equation can be written in the following form:

$$\text{(2.3.4)} \qquad \mathrm{U}' - \frac{1}{\alpha(g_i)}(\mathrm{P}_i(\mathrm{U}'))' = \frac{1}{\alpha(g_i)}(\tilde{\mathrm{Z}}_i' + \mathfrak{Z}_i' + (\mathrm{Q}_i(\mathrm{U}'))')$$

$$\text{(2.3.5)} \qquad \mathrm{U}'' - \frac{1}{\alpha(g_i)}(\mathrm{Q}_i(\mathrm{U}''))'' = \frac{1}{\alpha(g_i)}(\tilde{\mathrm{Z}}_i'' + \mathfrak{Z}_i'' + (\mathrm{P}_i(\mathrm{U}'))'' + (\mathrm{Q}_i(\mathrm{U}'))'');$$

both P_i and Q_i are $\mathcal{O}_n^{\mathrm{S}}$-linear operators. Let us assume that the weight α is of small norm, that is smaller than $Sep(S)$. Then, we show that $(\mathrm{Q}_i(\mathrm{U}'))' = 0$ and that, according to complete integrability, $\mathrm{P}_i' \circ \mathrm{P}_i' = 0$. Therefore the solution of equation (2.3.4) is given by

$$\mathrm{U}' = \left(Id + \frac{1}{\alpha(g_i)}\mathrm{P}_i'\right)\left(\frac{1}{\alpha(g_i)}(\tilde{\mathrm{Z}}_i' + \mathfrak{Z}_i')\right);$$

since U′ is a polynomial of order $\leqslant 2m$, then in fact we have

$$|\mathrm{U}'|_r \leqslant \left| \left(Id + \frac{1}{\alpha(g_i)}\mathrm{P}_i'\right)\left(\frac{1}{\alpha(g_i)}(\tilde{\mathrm{Z}}_i')\right) \right|_r.$$

An estimate of the operator P_i' will provide the desired estimate of U′. Now let us study equation (2.3.5); if we denote by $\frac{1}{\alpha(g_i)}\mathfrak{w}_i$ the left-hand side of this equation, then, at least formally, we have

$$\mathrm{U}'' = \sum_{k\geqslant 0}\left(\frac{1}{\alpha(g_i)}\right)^k \mathrm{Q}_i^k\left(\frac{\mathfrak{w}_i}{\alpha(g_i)}\right).$$

By assumption, NF^m is the m-jet of a completely integrable normal form. Therefore, its projection $(\mathrm{NF}^m)''$ is the m-jet of a **good deformation** of S″. The point is that there exists an integer k_0 which **does not depend on** m and such that $\mathrm{J}^{2m}\left(\mathrm{Q}_i^k\left(\frac{\mathfrak{w}_i}{\alpha(g_i)}\right)\right) = 0$ for all $k \geqslant k_0$. The important consequence for the estimates is that the above sum which gives U″ is **finite**. Using the estimate of U′ which were found above, we can give an estimate for $\mathfrak{w}_i$; then using the estimate of Q_i, we obtain an estimate of U″. The last case deals with weight α such that $\|\alpha\| \geqslant Sep(S)$; it is the easiest case.

Now let us give an idea of the induction argument. Let $1/2 < r \leqslant 1$, and let us assume the the nonlinear morphism $S + \varepsilon = NF^m + R_{m+1}$ is normalized up to order $m = 2^k$. Let us assume that the norms $|NF^m - S|_r$ and $|D(NF^m - S)|_r$ are small enough, say smaller than η_1, and that $|R|_r < 1$. The solution of the cohomological equation allows us to normalize the nonlinear morphism up to order $2m$: $\Phi_m^*(S + \varepsilon) = NF^{2m} + R_{2m+1}$. Using the estimate of this solution, we show that $|NF^{2m} - S|_R$ and $|D(NF^{2m} - S)|_R$ are **still less** than η_1, where

$$R = \left(\frac{c(\eta_1)}{\omega_{k+1}^d}\right)^{-1/m} m^{-2/m} r < r,$$

and that $|R_{2m+1}|_R < 1$. After a preliminary renormalization, we show that, at each stage, our new objets still satisfy the required assumptions in order to have again the estimate for the solution of the new cohomological equation. Thus, we may repeat the process... Now, because of the diophantine condition, the product of these R is bounded below by some positive constant *Rad*. Therefore, in the limit, we have a holomorphic diffeomorphism in the polydisc of radius *Rad* centred at $0 \in \mathbf{C}^n$ which normalizes our nonlinear deformation $S + \varepsilon$.

As the reader will see, this work is inspired by the work of Bruno [Bru72]. For one vector field with a diophantine linear part, the "complete integrability condition" of Bruno is not only sufficient but necessary for holomorphic normalization. Our conditions are not in general necessary as can be seen in my recent work [Sto00a, Sto00b]. From the algebraic point of view, we point out the article of Walcher [Wal91] which has been of some help for this work. This work is partially contained in [Sto98a, Sto98b].

3. Notation

Let $R = (r_1, \ldots, r_n) \in (\mathbf{R}_+^*)^n$ and $a \in \mathbf{C}^n$. The open polydisc centered at a and of polyradius R will be denoted by $D_R(a) = \{z \in \mathbf{C}^n \mid |z_i - a_i| < r_i\}$. When $a = 0$, it will be denoted by D_R. If $r > 0$ then $D_r(a)$ denotes the polydisc $D_{(r,\ldots,r)}(a)$. We shall denote $\mathscr{C}_R$ the *distinguished boundary* of the polydisc D_R, that is the set $\mathscr{C}_R = \{z \in \mathbf{C}^n \mid \forall\, 1 \leqslant i \leqslant n, |z_i| = R_i\}$.

Let f be a holomorphic function in a neighbourhood of the closed polydisc $\overline{D_R}$, then we define the norm $\|f\|_R = \sup_{x \in \overline{D_R}} |f(x)|$.

3.1. *Norms*

Let $f \in \mathbf{C}[[x_1, \ldots, x_n]]$ be a formal power series : $f = \sum_{Q \in \mathbf{N}^n} f_Q x^Q$. We define $\bar{f}$ to be the formal power series $\bar{f} = \sum_{Q \in \mathbf{N}^n} |f_Q| x^Q$. We shall say that a formal power series g dominates a formal power series f, if $\forall Q \in \mathbf{N}^n$, $|f_Q| \leqslant |g_Q|$. In that case, we shall write $f \prec g$. More generally, let $q \geqslant 1$ be an integer and let $F=(f_1,...,f_q)$ and $G=(g_1,...,g_q)$ be elements of $(\mathbf{C}[[x_1,...,x_n]])^q$; we shall say that G dominates F, and we shall write $F \prec G$, if $f_i \prec g_i$ for all $1 \leqslant i \leqslant q$. We shall write $\overline{F}=(\overline{f}_1,...,\overline{f}_q)$. We shall say that F is of order $\geqslant m$ (resp. polynomial of degree $\leqslant m$), if each of its components is of order $\geqslant m$ (resp. polynomial of degree $\leqslant m$).

Let r be a positive number and $(f, F, G) \in \mathbf{C}[[x_1,...,x_n]] \times (\mathbf{C}[[x_1,...,x_n]])^q \times (\mathbf{C}[[x_1,...,x_n]])^q$, we define $|f|_r = \sum_{Q\in\mathbf{N}^n} |f_Q| r^{|Q|} = \overline{f}(r,...,r)$ and $|G|_r = \max_i |g_i|_r$; these may not be finite. We have the following properties:

$$\begin{array}{ccc} \overline{fG} & \prec & \overline{f}\,\overline{G}, \\ \text{if } F \prec G & \text{then} & |F|_r \leqslant |G|_r, \\ \overline{\dfrac{\partial F}{\partial x_k}} & = & \dfrac{\partial \overline{F}}{\partial x_k}. \end{array}$$

Let us define $\mathscr{H}_n^q(r) = \{F \in (\mathbf{C}[[x_1,...,x_n]])^q \mid |F|_r < +\infty\}$; $|.|_r$ is a norm on this space. Equipped with this norm $|.|_r$, this space is a Banach space (see [GR71]). Let $F = \sum_{Q\in\mathbf{N}^n} F_Q x^Q$ an element of $\mathscr{H}_n^q(r)$, then we have the following inequalities

(3.1.1) $$\|F\|_r \leqslant |F|_r,$$

(3.1.2) $$|F|_R \leqslant \left(\frac{R}{r}\right)^m |F|_r \quad \text{if} \quad \operatorname{ord}(F) \geqslant m,\ R \leqslant r,$$

(3.1.3) $$|DF|_r \leqslant \frac{d}{r}|F|_r \quad \text{if F is a polynomial of degree} \leqslant d.$$

We shall often use the estimate $|(DG).\,F|_r \leqslant n|DG|_r|F|_r$ whenever $(F, G) \in \mathscr{H}_n^n(r)$.

Lemma **3.1.1.** — *Let $r > 0$, $a \in \mathbf{C}^*$ and $g \in \mathscr{H}_n^1(r)$. We assume that $|g|_r < |a|$. Then*

$$\left|\frac{1}{a+g}\right|_r \leqslant \frac{1}{|a| - |g|_r}.$$

Proof. — We have

$$\frac{1}{a+g} = \frac{1}{a}\frac{1}{1+g/a} = \frac{1}{a}\sum_{k\geqslant 0}(-1)^k\left(\frac{g}{a}\right)^k,$$

thus,

$$\frac{1}{a+g} \prec |a|^{-1}\sum_{k\geqslant 0}|a|^{-k}\overline{g}^k = |a|^{-1}\frac{1}{1-|a|^{-1}\overline{g}}.$$

So,

$$\left|\frac{1}{a+g}\right|_r \leqslant \frac{1}{|a| - |g|_r} \quad \square$$

3.2. *Spaces of vector fields and spaces of functions*

Let us set some notation which will be used all along this article. Let $k \geqslant 1$ be integers:

- $\mathscr{P}_n^k$ denotes the $\mathbf{C}$-space of homogeneous polynomial vector fields on $\mathbf{C}^n$ and of degree k,
- $\mathscr{P}_n^{m,k}$ denotes the $\mathbf{C}$-space of polynomial vector fields on $\mathbf{C}^n$, of order $\geqslant m$ and of degree $\leqslant k$ ($m \leqslant k$),
- $\widehat{\mathscr{X}}_n^k$ denotes the $\mathbf{C}$-space of formal vector fields on $\mathbf{C}^n$ and of order $\geqslant k$ at 0,
- $\mathscr{X}_n^k$ denotes the $\mathbf{C}$-space of germs of holomorphic vector fields on $(\mathbf{C}^n, 0)$ and of order $\geqslant k$ at 0,
- p_n^k denotes the $\mathbf{C}$-space of homogeneous polynomials on $\mathbf{C}^n$ and of degree k,
- $\widehat{\mathscr{M}}_n^k$ denotes the $\mathbf{C}$-space of formal power series on $\mathbf{C}^n$ and of order $\geqslant k$ at 0,
- $\mathscr{M}_n^k$ denotes the $\mathbf{C}$-space of germs of holomorphic functions on $(\mathbf{C}^n, 0)$ and of order $\geqslant k$ at 0,
- $\widehat{\mathscr{O}}_n$ denotes the ring of formal power series in $\mathbf{C}^n$,
- $\mathscr{O}_n$ denotes the ring of germs at 0 of holomorphic functions in $\mathbf{C}^n$.

4. Normal forms relatively to a Lie algebra of linear vector fields

Let $\mathfrak{g}$ a finite-dimensional Lie algebra over $\mathbf{C}$ and let l be its dimension. Let $\phi : \mathfrak{g} \rightarrow \mathscr{P}_n^1$ be a Lie morphism from $\mathfrak{g}$ to $\mathscr{P}_n^1$, the Lie algebra of linear vector fields on $\mathbf{C}^n$. It is a $\mathbf{C}$-linear mapping which satisfies $\phi([g_1, g_1]) = [\phi(g_1), \phi(g_2)]$ for all $(g_1, g_2) \in \mathfrak{g}^2$. If ϕ is injective, then the family of vector fields $\{\phi(g_1), \ldots, \phi(g_l)\}$ ($\mathrm{G} = \{g_1, \ldots, g_l\}$ denotes a basis of $\mathfrak{g}$) are linearly independent. If we assume furthermore that the vector fields $\phi(g_i)$ are diagonal linear vector fields, then $\phi(\mathfrak{g})$ is a commutative Lie algebra and by injectivity of ϕ, $\mathfrak{g}$ is commutative too. In that case, ϕ is just a $\mathbf{C}$-linear map.

4.1. *Lie algebras of linear vector fields and their associated representations and Chevalley-Koszul complex*

Let M_k denotes one of the spaces of vector fields or functions introduced in the notation section (k denotes a positive integer). Let $\phi : \mathfrak{g} \rightarrow \mathscr{P}_n^1$ be a Lie morphism from $\mathfrak{g}$ to the Lie algebra $\mathscr{P}_n^1$ of linear vector fields of $\mathbf{C}^n$. This morphism induces on M a structure of $\mathfrak{g}$-module in the following way:

- if M_k is a function space then, the **C**-linear map

$$\mathfrak{g} \to \mathrm{Hom}_{\mathbf{C}}(M_k, M_k)$$
$$g \mapsto \mathscr{L}_{\phi(g)}(.), \quad (\mathscr{L}_{\phi(g)} \text{ is the Lie derivative along } \phi(g))$$

defines a representation of $\mathfrak{g}$ into M_k,

- if M_k is a vector field space then, the **C**-linear map

$$\mathfrak{g} \to \mathrm{Hom}_{\mathbf{C}}(M_k, M_k)$$
$$g \mapsto [\phi(g), .]$$

defines a representation of $\mathfrak{g}$ into M_k.

These representations will be denoted by ρ_k. It should be clear from the context what its target is. We will omit the index k as soon as the context will permit it. It satisfies $\rho_k([g_1, g_2])X = \rho_k(g_1)(\rho_k(g_2)X) - \rho_k(g_2)(\rho_k(g_1)X)$ for all $(g_1, g_2) \in \mathfrak{g}^2$ and all $X \in M_k$.

To each representation is associated a complex of **C**-vector spaces, namely the Chevalley-Koszul complex:

(4.1.1) $$0 \to M_k \xrightarrow{d_0} \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, M_k) \xrightarrow{d_1} \mathrm{Hom}_{\mathbf{C}}\left(\wedge^2\mathfrak{g}, M_k\right) \xrightarrow{d_2} \dots \xrightarrow{d_{l-1}} \mathrm{Hom}_{\mathbf{C}}\left(\wedge^l\mathfrak{g}, M_k\right) \to 0,$$

where the differentials d_i are defined in the following way: if $\omega \in \mathrm{Hom}_{\mathbf{C}}(\wedge^p\mathfrak{g}, M_k)$ and $(g_1, \dots, g_{p+1}) \in \mathfrak{g}^{p+1}$, then

(4.1.2) $$d_p(\omega)(g_1, \dots, g_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1}\rho(g_i)\left(\omega(g_1, \dots, \widehat{g}_i, \dots, g_{p+1})\right) + \sum_{1 \leqslant i < j \leqslant p+1} (-1)^{i+j}\omega([g_i, g_j], g_1, \dots, \widehat{g}_i, \dots, \widehat{g}_j, \dots, g_{p+1}).$$

As usual, we write $(g_1, \dots, \widehat{g}_i, \dots, g_{p+1}) \in \mathfrak{g}^p$ for the vector $(g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_{p+1})$. The differentials d_0 and d_1 will be particularly useful:

if $U \in \mathscr{P}_n^k$, then $\forall g \in \mathfrak{g}$, $d_0(U)(g) = [\phi(g), U]$;

if $u \in p_n^k$, then $\forall g \in \mathfrak{g}$, $d_0(u)(g) = \mathscr{L}_{\phi(g)}(u)$;

if $F \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{P}_n^k\right)$, then $\forall (g_1, g_2) \in \mathfrak{g}^2$,

$$d_1(F)(g_1, g_2) = [\phi(g_1), F(g_2)] - [\phi(g_2), F(g_1)] - F([g_1, g_2]);$$

if $f \in \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, p_n^k)$, then $\forall (g_1, g_2) \in \mathfrak{g}^2$,

$$d_1(f)(g_1, g_2) = \mathscr{L}_{\phi(g_1)}(f(g_2)) - \mathscr{L}_{\phi(g_2)}(f(g_1)) - f([g_1, g_2]).$$

We shall denote by $H^i_\phi(\mathfrak{g}, M_k)$ the i-th cohomology space of this complex, that is $H^i_\phi(\mathfrak{g}, M_k) = Z^i_\phi(\mathfrak{g}, M_k)/B^i_\phi(\mathfrak{g}, M_k)$ where $Z^i_\phi(\mathfrak{g}, M_k) = \mathrm{Ker}\ d_i$, $B^i_\phi(\mathfrak{g}, M_k) = \mathrm{Im}\ d_{i-1}$. We define also the "noncocycle space" to be $\widetilde{H}^i_\phi(\mathfrak{g}, M_k) := \mathrm{Hom}_{\mathbf{C}}(\wedge^i \mathfrak{g}, M_k)/Z^i_\phi(\mathfrak{g}, M_k)$. We shall denote by M_k^ϕ the space of invariants of $\mathfrak{g}$-module M_k, that is $M_k^\phi = H^0_\phi(\mathfrak{g}, M_k)$. The following lemma will give a link between the cohomology spaces of the various $\mathfrak{g}$-modules defined above.

Lemma **4.1.1.** — *If* M_k *stands for* $\mathcal{M}_n^k$ *(resp.* $\mathcal{X}_n^k$*),* $\widehat{M}_k$ *for* $\widehat{\mathcal{M}}_n^k$ *(resp.* $\widehat{\mathcal{X}}_n^k$*) and, for any positive integer* i*,* m_i *stands for* p_n^i *(resp.* $\mathcal{P}_n^i$*) then, for any integer* $1 \leqslant p \leqslant l$ *we have* $H^p_\phi\left(\mathfrak{g}, \widehat{M}_k\right) \cong \prod_{i\geqslant k} H^p_\phi(\mathfrak{g}, m_i)$ *as well as* $\widetilde{H}^p_\phi\left(\mathfrak{g}, \widehat{M}_k\right) \cong \prod_{i\geqslant k} \widetilde{H}^p_\phi(\mathfrak{g}, m_i)$.

Proof. — Let $z \in \mathrm{Hom}_{\mathbf{C}}\left(\wedge^p \mathfrak{g}, \widehat{M}_k\right)$. It can be written $z = \sum_{i\geqslant k} z_i$ with $z_i \in Hom_{\mathbf{C}}(\wedge^p \mathfrak{g}, m_i)$. Let us consider the following $\mathbf{C}$-map:

$$p : Hom_{\mathbf{C}}\left(\wedge^p \mathfrak{g}, \widehat{M}_k\right) \to \prod_{i\geqslant k} Hom_{\mathbf{C}}\left(\wedge^p \mathfrak{g}, m_i\right)$$
$$z \mapsto (z_i)_{i\geqslant k}.$$

If $d_k z = 0$ then, since the m_i's are $\mathfrak{g}$-modules, $d_k z_i = 0$ for all i. In the other hand, if $z = d_{k-1}u$ with $u \in Hom_{\mathbf{C}}(\wedge^{p-1}\mathfrak{g}, \widehat{M}_k)$ then, by writing $u = \sum_{i\geqslant k} u_i$ with $u_i \in Hom_{\mathbf{C}}(\wedge^{p-1}\mathfrak{g}, m_i)$, we obtain $z_i = d_{p-1}u_i$. Thus, the map p can be quotiented and provides two $\mathbf{C}$-linear maps:

$$p_1 : H^p\left(\mathfrak{g}, \widehat{M}_k\right) \to \prod_{i\geqslant k} H^p(\mathfrak{g}, m_i)$$
$$[z] \mapsto ([z_i])_{i\in\mathbf{N}};$$
$$p_2 : \widetilde{H}^p\left(\mathfrak{g}, \widehat{M}_k\right) \to \prod_{i\geqslant k} \widetilde{H}^p(\mathfrak{g}, m_i)$$
$$\widetilde{[z]} \mapsto \left(\widetilde{[z_i]}\right)_{i\geqslant k}.$$

These maps are one-to-one. In fact, let $z \in Hom_{\mathbf{C}}\left(\wedge^p \mathfrak{g}, \widehat{M}_k\right)$ such that $p_1([z]) = 0$. Then, $p(z) = (z_i = d_{k-1}u_i)_{i\geqslant k}$ for some $u_i \in Hom_{\mathbf{C}}(\wedge^{p-1}\mathfrak{g}, m_i)$, it follows that $z = d_{p-1}(\sum_{i\geqslant k} u_i)$ so that $[z] = 0$. Now, let us assume that $p_2(\widetilde{[z]}) = 0$, then $d_p z_i = 0$ for all $i \geqslant k$, thus, $d_p(\sum_{i\geqslant k} z_i) = 0$ so that $\widetilde{\left[\sum_{i\geqslant k} z_i\right]} = \widetilde{[z]} = 0$. These maps are onto. In fact, let $([z_i])_{i\geqslant k} \in \prod_{j\geqslant k} H^p(\mathfrak{g}, m_j)$ (resp. $(\widetilde{[z_i]})_{i\geqslant k} \in \prod_{j\geqslant k} \widetilde{H}^p(\mathfrak{g}, m_j)$), if z_i denotes a representative of $[z_i]$ (resp. $\widetilde{[z_i]}$) in $Hom_{\mathbf{C}}(\wedge^p \mathfrak{g}, m_i)$, then $p_1(\sum_{i\geqslant k} z_i) = (\widetilde{[z_i]})_{i\geqslant k}$ and $p_2(\sum_{i\geqslant k} z_i) = ([z_i])_{i\geqslant k}$. □

4.2. *Normal form of a nonlinear deformation of a linear morphism*

We will consider non-linear deformations of order $m > 1$ of a linear morphism ϕ from $\mathfrak{g}$ to $\mathcal{P}_n^1$, that is, a Lie morphism $\phi + \varepsilon : \mathfrak{g} \to \widehat{\mathcal{X}}_n^1$ where $\varepsilon \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \widehat{\mathcal{X}}_n^m\right)$.

Thus, for all $(g_1, g_2) \in \mathfrak{g}^2$, we have

$$\begin{aligned}(\phi+\varepsilon)([g_1,g_2]) &= [\phi(g_1)+\varepsilon(g_1), \phi(g_2)+\varepsilon(g_2)]\\ &= [\phi(g_1),\phi(g_2)] + [\phi(g_1),\varepsilon(g_2)] - [\phi(g_2),\varepsilon(g_1)] + [\varepsilon(g_1),\varepsilon(g_2)].\end{aligned}$$

Since ϕ is a Lie morphism, we obtain the following equality in $\mathrm{Hom}_{\mathbf{C}}\left(\wedge^2\mathfrak{g}, \widehat{\mathcal{X}}_n^m\right)$:

$$\textbf{(4.2.1)} \qquad \forall (g_1, g_2) \in \mathfrak{g}^2 \quad d_1\varepsilon(g_1, g_2) = -[\varepsilon(g_1), \varepsilon(g_2)].$$

Let us denote ε^i the homogeneous component of degree i of ε. Since, for all $(i, j) \in (\mathbf{N}^*)^2$, we have $[\mathcal{P}_n^i, \mathcal{P}_n^j] \subset \mathcal{P}_n^{i+j-1}$, then we can consider, for all positive integer k, the non-linear map

$$\begin{aligned}\mathrm{C}_k : \mathit{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \widehat{\mathcal{X}}_n^m\right) &\to \mathit{Hom}_{\mathbf{C}}\left(\wedge^2\mathfrak{g}, \mathcal{P}_n^k\right)\\ \varepsilon &\mapsto -\sum_{i+j-1=k} [\varepsilon^i, \varepsilon^j].\end{aligned}$$

It is clear that $\mathrm{C}_k(\varepsilon) = \mathrm{C}_k(\mathrm{J}^{k-m+1}(\varepsilon))$ and that $\mathrm{C}_k(\varepsilon) = 0$ for all integer $k \leqslant 2m-1$. The fundamental equation (4.2.1) can thus be written

$$d_1(\varepsilon_k) = \mathrm{C}_k(\mathrm{J}^{k-m+1}(\varepsilon)) \text{ for all } k \in \mathbf{N}^*.$$

We will denote $\mathrm{Def}\left(\mathfrak{g}, \phi, \widehat{\mathcal{X}}_n^m\right)$ the set of such deformations. Let us denote $\widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)$ the group of formal diffeomorphisms which leave $0 \in \mathbf{C}^n$ invariant and which are tangent to the identity up to order m at the origin. Let us denote $\left(\widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)\right)^\phi$ the subgroup whose elements leave invariant the morphism ϕ, that is $\Phi^*\phi = \phi$ whenever $\Phi \in \left(\widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)\right)^\phi$. Let us denote by $\widehat{\mathrm{Diff}}_{m,\phi}(\mathbf{C}^n, 0)$ the quotient group $\widehat{\mathrm{Diff}}(\mathbf{C}^n, 0) / \left(\widehat{\mathrm{Diff}}(\mathbf{C}^n, 0)\right)^\phi$. These definitions come from the commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \left(\widehat{\mathcal{X}}_n^m\right)^\phi & \longrightarrow & \widehat{\mathcal{X}}_n^m & \longrightarrow & \widehat{\mathcal{X}}_n^m / \left(\widehat{\mathcal{X}}_n^m\right)^\phi & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \left(\widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)\right)^\phi & \longrightarrow & \widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0) & \longrightarrow & \widehat{\mathrm{Diff}}_{m,\phi}(\mathbf{C}^n, 0) & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

where the vertical map is the exponential.

The group $\widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)$ acts on $\mathrm{Def}(\mathfrak{g}, \phi, \widehat{\mathscr{X}}_n^m)$ by conjugacy: if $\hat{\Phi} \in \widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)$ and $X \in \mathrm{Def}\left(\mathfrak{g}, \phi, \widehat{\mathscr{X}}_n^m\right)$ then

$$\forall g \in \mathfrak{g},\ (\hat{\Phi}^*X)(g) \circ \hat{\Phi} = \hat{\Phi}^*(X(g)) \circ \hat{\Phi} = D(\hat{\Phi})X(g)$$

where $D(\hat{\Phi})$ denotes the derivative of $\hat{\Phi}$; $\mathrm{Def}\left(\mathfrak{g}, \phi, \widehat{\mathscr{X}}_n^m\right)$ is invariant under the action of $\widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)$. Indeed, we have

$$\hat{\Phi}^*X([g_1, g_2]) = \hat{\Phi}^*(X([g_1, g_2])) = \hat{\Phi}^*[X(g_1), X(g_2)] = [\hat{\Phi}^*X(g_1), \hat{\Phi}^*X(g_2)].$$

Let $m \geqslant 2$ be an integer. For any integer $k \geqslant m$, we have the following short exact sequences of finite-dimensional $\mathbf{C}$-vector spaces:

$$\begin{array}{lccc}
0 \to Z^1_\phi(\mathfrak{g}, \mathscr{P}_n^k) \to & \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathscr{P}_n^k) & \overset{p_1^k}{\to} \widetilde{H}^1_\phi(\mathfrak{g}, \mathscr{P}_n^k) \to 0 \\
0 \to B^1_\phi(\mathfrak{g}, \mathscr{P}_n^k) \to & Z^1_\phi(\mathfrak{g}, \mathscr{P}_n^k) & \overset{p_2^k}{\to} H^1_\phi(\mathfrak{g}, \mathscr{P}_n^k) \to 0 \\
0 \to \left(\mathscr{P}_n^k\right)^\phi \to & \mathscr{P}_n^k & \overset{p_3^k}{\to} \mathscr{P}_n^k / \left(\mathscr{P}_n^k\right)^\phi \to 0.
\end{array}$$

These sequences are split : we may choose $\mathbf{C}$-linear maps

$$s_1^k : \widetilde{H}^1_\phi(\mathfrak{g}, \mathscr{P}_n^k) \to \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathscr{P}_n^k),\quad s_2^k : H^1_\phi(\mathfrak{g}, \mathscr{P}_n^k) \to Z^1_\phi(\mathfrak{g}, \mathscr{P}_n^k)$$
$$\text{and } s_3^k : \mathscr{P}_n^k / \left(\mathscr{P}_n^k\right)^\phi \to \mathscr{P}_n^k$$

such that $p_i^k \circ s_i^k = Id$. This choice is equivalent to the choice of a supplementary subspace $\tilde{v}_k$ (resp. v_k, resp. p_k) of $Z^1_\phi\left(\mathfrak{g}, \mathscr{P}_n^k\right)$ (resp. of $B^1_\phi\left(\mathfrak{g}, \mathscr{P}_n^k\right)$, resp. of $\left(\mathscr{P}_n^k\right)^\phi$) in $\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{P}_n^k\right)$ (resp. in $Z^1_\phi\left(\mathfrak{g}, \mathscr{P}_n^k\right)$, resp. $\mathscr{P}_n^k$):

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{P}_n^k\right) = \tilde{v}_k \oplus Z^1_\phi\left(\mathfrak{g}, \mathscr{P}_n^k\right), \\
&Z^1_\phi\left(\mathfrak{g}, \mathscr{P}_n^k\right) = v_k \oplus B^1_\phi\left(\mathfrak{g}, \mathscr{P}_n^k\right), \\
&\mathscr{P}_n^k = p_k \oplus \left(\mathscr{P}_n^k\right)^\phi.
\end{aligned}$$

Let us define, for $i = 1, 2, 3$,

$$\begin{aligned}
&s_{i,m} := \oplus_{k \geqslant m} s_i^k,\quad \widetilde{V}_m := \oplus_{k \geqslant m} \tilde{v}_k = \mathrm{Range}\ s_{1,m}, \\
&V_m = \oplus_{k \geqslant m} v_k = \mathrm{Range}\ s_{2,m} \text{ and } P_m := \oplus_{k \geqslant m} p_k = \mathrm{Range}\ s_{3,m}.
\end{aligned}$$

Let us write $\widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)(s_{3,m})$ for the subgroup of $\widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)$ defined by $\exp P_m$. According to lemma (4.1.1), we have $\widetilde{V}_m \cong \widetilde{H}^1_\phi\left(\mathfrak{g}, \widehat{\mathscr{X}}_n^m\right)$ as well as $V_m \cong H^1_\phi\left(\mathfrak{g}, \widehat{\mathscr{X}}_n^m\right)$.

Proposition **4.2.1** *(Formal normal form). — Let $m \geqslant 2$ be an integer and let $\{s_{i,m}\}_{i=1,2,3}$ be fixed sections as above. Let $\phi + \varepsilon \in \mathrm{Def}(\mathfrak{g}, \phi, \widehat{\mathscr{X}}_n^m)$ be a nonlinear deformation of ϕ of order m.*

Then there exists a unique formal diffeomorphism $\widehat{\Phi} \in \widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)(s_{3,m})$ *such that*

$$\widehat{\Phi}^*(\phi + \varepsilon) - \phi \in \widetilde{\mathrm{V}}_m \oplus \mathrm{V}_m.$$

We will say that $\widehat{\Phi}^*(\phi+\varepsilon)$ *is the* **normal form relative** *to* ϕ *and to the sections* $\{s_{i,m}\}_{i=1,2,3}$. *We will say that* $\widehat{\Phi}$ *is the* **formal normalizing diffeomorphism.**

V. V. Lychagin has developped a more general definition of normal forms [Lyc88] but we shall not use his result.

Proof. — We shall prove by induction on $k \geqslant m$, that there exists $\mathrm{U}_k \in \oplus_{p=m}^{k} p_k$, $\widetilde{\mathrm{R}}_k \in \oplus_{p=m}^{k} \widetilde{v}_p$ and $\mathrm{R}_k \in \oplus_{p=m}^{k} v_p$ uniquely determined, such that

$$\mathrm{J}^k(\exp \mathrm{U}_k^*(\phi + \varepsilon)) = \phi + \widetilde{\mathrm{R}}_k + \mathrm{R}_k.$$

Let us recall that, if $\mathrm{U} \in \widehat{\mathscr{X}}_n^m$ then

$$\exp \mathrm{U}^*(\phi + \varepsilon) = \sum_{i \geqslant 0} \frac{\mathrm{ad}_{\mathrm{U}}^i(\phi + \varepsilon)}{i!}$$
$$= \phi + \varepsilon + [\mathrm{U}, \phi + \varepsilon] + \sum_{i \geqslant 2} \frac{\mathrm{ad}_{\mathrm{U}}^i(\phi + \varepsilon)}{i!},$$

where ad_{U} denotes the linear mapping $[\mathrm{U}, .]$ and ad_{U}^i its i-th iterate. For $k = m$ and since we have the decomposition

$$\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{P}_n^m\right) = \widetilde{v}_m \oplus v_m \oplus \mathrm{B}_\phi^1\left(\mathfrak{g}, \mathscr{P}_n^m\right),$$

and the isomorphism $\mathrm{B}_\phi^1(\mathfrak{g}, \mathscr{P}_n^m) \cong \mathscr{P}_n^m / \left(\mathscr{P}_n^m\right)^\phi$, we can write $\varepsilon^m = \widetilde{r}_m + r_m + d_0 \mathrm{U}_m$ with $\widetilde{r}_m \in \widetilde{v}_m$, $r_m \in v_m$ and $\mathrm{U}_m \in p_m$ uniquely determined. Thus, we have

$$\exp \mathrm{U}_m^*(\phi + \varepsilon) = \phi + \widetilde{r}_m + r_m + \varepsilon_1$$

where, since $\mathrm{U}_m \in \mathscr{P}_n^m$,

$$\varepsilon_1 = \varepsilon - \mathrm{J}^m(\varepsilon) + [\mathrm{U}_m, \varepsilon] + \sum_{i \geqslant 2} \frac{\mathrm{ad}_{\mathrm{U}_m}^i(\phi + \varepsilon)}{i!} \in \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \widehat{\mathscr{X}}_n^{m+1}).$$

Let us assume that the result holds for any integer $p \leqslant k - 1$. Thus there exists unique $\mathrm{U}_{k-1} \in \oplus_{p=m}^{k-1} p_p$, $\widetilde{\mathrm{R}}_{k-1} \in \oplus_{p=m}^{k-1} \widetilde{v}_p$, $\mathrm{R}_{k-1} \in \oplus_{p=m}^{k-1} v_p$ and $\varepsilon' \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \widehat{\mathscr{X}}_n^k\right)$ such that $\exp \mathrm{U}_{k-1}^*(\phi + \varepsilon) = \phi + \widetilde{\mathrm{R}}_{k-1} + \mathrm{R}_{k-1} + \varepsilon'$. We may decompose $\mathrm{J}^k(\varepsilon')$ as follows:

$J^k(\varepsilon') = \widetilde{r}_k + r_k + d_0 u_k$ where $\widetilde{r}_k \in \widetilde{v}_k$, $r_k \in v_k$ and $u_k \in p_k$ are uniquely determined. It follows that

$$\exp(u_k + U_{k-1})^*(\phi + \varepsilon) = \phi + \widetilde{R}_{k-1} + R_{k-1} + \widetilde{r}_k + r_k + \varepsilon_2,$$

where

$$\varepsilon_2 = \varepsilon' - J^k(\varepsilon') + [U, \widetilde{R}_{k-1} + R_{k-1} + \varepsilon'] + \sum_{i\geqslant 2} \frac{\mathrm{ad}^i_{u_k}(\exp U_{k-1})^*(\phi + \varepsilon)}{i!} \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \widehat{\mathcal{X}}_n^{k+1}\right).$$

We just have to set $U_k = u_k + U_{k-1}$, $R_k = r_k + R_{k-1}$ and $\widetilde{R}_k = \widetilde{r}_k + \widetilde{R}_{k-1}$ to conclude the proof. □

Remark **4.2.2.** — *Under the assumptions of the above proposition, let* $U \in P_m$ *be such that* $\exp U^*(\phi + \varepsilon) = \phi + \widetilde{R} + R$ *is a normal form. If* $\widetilde{r}_k$ *denotes the homogeneous polynomial of degree* k *of* $\widetilde{R}$, *then* $d_1(\widetilde{r}_k) = C_k(J^{k-m+1}(\exp(J^{k-m+1}(U)) * (\phi + \varepsilon) - \phi))$ *if* $k \geqslant 2m - 1$ *and* $d_1(\widetilde{r}_k) = 0$ *if* $m \leqslant k < 2m - 1$.

Let us give some examples.

1. Let us first consider the case where $\dim_{\mathbf{C}} \mathfrak{g} = 1$. A non-trivial semi-simple morphism ϕ is determined by a nontrivial semisimple linear vector field S. Let $m \geqslant 2$ be an integer, then we have $\mathrm{Def}\left(\mathfrak{g}, \phi, \widehat{\mathcal{X}}_n^m\right) = S + \widehat{\mathcal{X}}_n^m$. Indeed, since $\mathrm{Hom}_{\mathbf{C}}\left(\wedge^2\mathfrak{g}, \widehat{\mathcal{X}}_n^m\right) = \{0\}$, then $d_1 = 0$ so that $d_1\varepsilon = -[\varepsilon, \varepsilon]$ for all $\varepsilon \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \widehat{\mathcal{X}}_n^m\right)$. Let $V = \mathrm{Ker}\ ad_{S|\widehat{\mathcal{X}}_n^m}$ be a subspace of $\widehat{\mathcal{X}}_n^m$. Since S is semi-simple, so is ad_S and we have $\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \widehat{\mathcal{X}}_n^m\right) \cong \widehat{\mathcal{X}}_n^m = V \oplus ad_S\left(\widehat{\mathcal{X}}_n^m\right)$. According the above proposition, for any formal vector field $\varepsilon \in \widehat{\mathcal{X}}_n^m$, there exists a formal diffeomorphism $\hat{\Phi}$ such that $\hat{\Phi}^*(\phi + \varepsilon) - \phi \in \mathrm{Ker}\ ad_{S|\widehat{\mathcal{X}}_n^m}$, that is $[S, \hat{\Phi}^*(\phi + \varepsilon)] = 0$. This is the formal Poincaré-Dulac theorem [Arn80].

2. Let $\mathfrak{g}$ be a semi-simple Lie algebra. We will only use the following result, known as the first Whitehead lemma, which states that if M is any finite-dimensional $\mathfrak{g}$-module then $H^1(\mathfrak{g}, M) = 0$. It follows that if ϕ is a Lie morphism from $\mathfrak{g}$ to $\mathscr{P}_n^1$, then for any integer $i \geqslant 1$, $H^1_\phi\left(\mathfrak{g}, \mathscr{P}_n^i\right) = 0$. Let $\phi + \varepsilon$ be a nonlinear deformation of ϕ of order $m \geqslant 2$. According to the Whitehead lemma and to remark (4.2.2), we can show by induction on $k \geqslant m$, that there exists a polynomial diffeomorphism Φ_k such that $J^k(\Phi_k^*(\phi + \varepsilon)) = \phi$. In fact, with the above notations, we have $v_k = 0$ for all integer $k \geqslant 1$. Moreover, if the result holds for $p \leqslant k - 1$, then $J^k(\Phi_{k-1}^*(\phi + \varepsilon)) - \phi$ is a 1-cocycle according to (4.2.2). It follows that $\widetilde{r}_k = r_k = 0$.

It follows that any nonlinear deformation of ϕ is *formally linearizable*. This is a result of Hermann [Her68].

4.3. *Compatible nonlinear deformations*

Definition **4.3.1.** — *Let $\phi+\varepsilon$ be a nonlinear deformation of order m. We shall say that $\phi+\varepsilon$ is* **compatible** *if there exists sections $\{s_{i,m}\}_{i=1,2,3}$ as above, such that its normal form relative to these sections is a 1-cocycle relative to ϕ : $d_{1,\phi}(\hat{\Phi}^*(\phi+\varepsilon)-\phi)=0$ where $\hat{\Phi}$ is the normalizing diffeomorphism.*

Lemma **4.3.2.** — *The compatibility condition of a nonlinear deformation $\phi+\varepsilon$ of order m does not depend on the choice of the sections $\{s_{i,m}\}_{i=1,2,3}$.*

Proof. — Let us assume that $\phi+\varepsilon$ is compatible relative to a given set of sections $\{s_{i,m}\}_{i=1,2,3}$. Let $\hat{\Phi}$ be its normalizing diffeomorphism. Then, according to the definition, $R=\hat{\Phi}^*(\phi+\varepsilon)$ is a 1-cocycle relative to ϕ. This property does not depend on the choice of the section $s_{1,m}$ nor on $s_{2,m}$. Let us show that this property does not depend on $s_{3,m}$. Let $\hat{\Phi}$ be a formal normalizing diffeomorphism. It is sufficient to prove that $d_{1,\phi}((\Psi\circ\hat{\Phi})^*(\phi+\varepsilon))=d_{1,\phi}(\Psi^*\left(\hat{\Phi}^*(\phi+\varepsilon)\right)=d_{1,\phi}\phi$ for all Ψ belonging to $\left(\widehat{\mathrm{Diff}}_m(\mathbf{C}^n,0)\right)^\phi$. Let us write X for $\hat{\Phi}^*(\phi+\varepsilon)$; then, for all $(g_1,g_2)\in\mathfrak{g}^2$, we have

$$\begin{aligned}
d_{1,\phi}\phi(g_1,g_2) &= \phi([g_1,g_2])=\Psi^*\phi([g_1,g_2])\\
&= \Psi^*d_{1,\phi}\phi(g_1,g_2)=\Psi^*(d_{1,\phi}X(g_1,g_2))\\
&= \Psi^*([\phi(g_1),X(g_2)]-[\phi(g_2),X(g_1)]-X([g_1,g_2]))\\
&= [\Psi^*\phi(g_1),\Psi^*X(g_2)]-[\Psi^*\phi(g_2),\Psi^*X(g_1)]-\Psi^*X([g_1,g_2])\\
&= [\phi(g_1),\Psi^*X(g_2)]-[\phi(g_2),\Psi^*X(g_1)]-\Psi^*X([g_1,g_2])\\
&\quad (\Psi \text{ leaves invariant } \phi),\\
&= d_{1,\phi}(\Psi^*X)(g_1,g_2).
\end{aligned}$$

This proves the lemma. □

Remark **4.3.3.** — *If $\mathfrak{g}$ is 1-dimensional then any non-linear deformation is compatible.*

Let us define $\mathrm{Compat}(\mathfrak{g},\phi,\widehat{\mathcal{X}}_n^m)$ the set of compatible formal deformations of ϕ of order m.

Corollary **4.3.4.** — *For each choice of the sections $s_{2,m}$, $s_{3,m}$, there is a well defined map*

$$\widehat{\mathrm{NF}}_{s_{2,m},s_{3,m}}:\mathrm{Compat}\left(\mathfrak{g},\phi,\widehat{\mathcal{X}}_n^m\right)/\widehat{\mathrm{Diff}}_m(\mathbf{C}^n,0)(s_{3,m})\to \mathrm{H}^1_\phi\left(\mathfrak{g},\widehat{\mathcal{X}}_n^m\right)$$

defined by $\widehat{\mathrm{NF}}_{s_{2,m}, s_{3,m}}\left(\widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)(s_{3,m})^*(\phi+\varepsilon)\right) = [\hat{\Phi}^*(\phi+\varepsilon) - \phi]$ *which is the cohomology class of* $\hat{\Phi}^*(\phi+\varepsilon) - \phi$ *where* $\hat{\Phi}$ *is any normalizing diffeomorphism relative to* $s_{2,m}, s_{3,m}$.

Proof. — Let $\phi+\varepsilon$ be a compatible formal deformation of ϕ of order m. As we have seen, for any choice of the section $\{s_{i,m}\}_{i=1,2,3}$, its relative normal form is a 1-cocycle relative to ϕ; this define a cohomology class. We will prove that, the sections $s_{2,m}, s_{3,m}$ beeing chosen, this class is constant along the orbit $\widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)(s_{3,m})^*(\phi+\varepsilon)$. Its value does not depend on $s_{1,m}$ and will be denoted by $\widehat{\mathrm{NF}}_{s_{3,m}}\left(\widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)(s_{3,m})^*(\phi+\varepsilon)\right)$.

Let $\phi+\eta$ be a formal deformation of ϕ of order m such that $\hat{\Psi}^*(\phi+\eta) = \phi+\varepsilon$ with $\hat{\Psi} \in \widehat{\mathrm{Diff}}_m(\mathbf{C}^n, 0)(s_{3,m})$. If $\hat{\Phi}$ denotes the normalizing diffeomorphism of $\phi+\varepsilon$ relative to $\{s_{i,m}\}_{i=1,2,3}$ then, $(\hat{\Phi} \circ \hat{\Psi})^*(\phi+\eta) = \hat{\Phi}^*(\phi+\varepsilon)$ is the normal form of $\phi+\eta$; thus it defines the same cohomology class. Moreover, this cohomology class does not depend on $s_{1,m}$ since $\phi+\varepsilon$ is compatible. □

5. The fundamental structures

As we have seen, the first cohomology space has an important rôle in the study of normal forms of compatible nonlinear deformations. In this section, we shall give a more precise description of this space under some additionnal hypothesis on the Lie algebra $\mathfrak{g}$. We shall also define "natural" sections for the normalization. The contain of this section is purely linear algebra. Our main source is [Bou90].

5.1. *Nilpotent Lie algebra of linear vector fields*

Let us first recall some useful basic facts on linear algebra. Let V be a finite-dimensional $\mathbf{C}$-vector space and let u be an endomorphism of V. We have the decomposition $V = \bigoplus_{a \in \mathbf{C}} V^a(u)$ where $V^a(u) = \{v \in V \mid \exists n\ (u - aId)^n v = 0\}$ denotes the characteristic space relative to the complex number a. Of course, $V^a(u) \neq 0$ if and only if a belongs to the set of eigenvalues of u. The associated eigenspace will be denoted by $V_a(u)$.

Let *Mat* be the Lie isomorphism:

$$\begin{aligned} Mat : \mathscr{P}_n^1 &\to \mathfrak{gl}_n(\mathbf{C}) \\ X &\mapsto Mat(X) = DX(0) \end{aligned}$$

If $P \in \mathfrak{gl}_n(\mathbf{C})$ is invertible, then $Mat\,(Mat^{-1}(P)_* X) = P_* Mat(X) = P Mat(X) P^{-1}$. Since ϕ is a Lie morphism from $\mathfrak{g}$ to $\mathscr{P}_n^1$, then $Mat \circ \phi$ defines a representation of $\mathfrak{g}$ in $\mathbf{C}^n$.

Let assume that $\mathfrak{g}$ is a **solvable** Lie algebra over $\mathbf{C}$ of dimension l. Let us recall the definition of the derived series: it is the decreasing sequence of ideals $(\mathscr{D}^k\mathfrak{g})_{n \geq 1}$

defined by $\mathscr{D}^1\mathfrak{g} = \mathfrak{g}$, $\mathscr{D}^{p+1}\mathfrak{g} = [\mathscr{D}^p\mathfrak{g}; \mathscr{D}^p\mathfrak{g}]$. By definition, the Lie algebra $\mathfrak{g}$ is solvable if there exists an integer $n \geqslant 1$ such that $\mathscr{D}^n\mathfrak{g} = 0$.

By a theorem of Lie [Ser92] [p. 36], since **C** is an algebraic closed field of characteristic 0, there exists a basis in $\mathbf{C}^n$ in which each matrix of $Mat(\phi(g))$ is upper triangular; it follows that there exists an invertible matrix $P \in \mathfrak{gl}_n(\mathbf{C})$ such that, for all $g \in \mathfrak{g}$, $Mat(Mat^{-1}(P)_*\phi(g)) = D(g) + SUT(g)$ where $D(g)$ is a diagonal matrix and $SUT(g)$ a strictly upper triangular matrix. Despite the fact that, in general, $D(g)$ and $SUT(g)$ do not commute with each other, it remains that the eigenvalues of $Mat(Mat^{-1}(P)_*\phi(g))$ are those of $D(g)$. We shall denote by $\lambda(g) = (\lambda_1(g), \ldots, \lambda_n(g))$ the vector of eigenvalues of $Mat(\phi(g))$. It is a **C**-linear map on $\mathfrak{g}$. Let us set $V = \mathbf{C}^n$.

In order to have more informations, we shall make the following natural assumption: for all $(g_1, g_2) \in \mathfrak{g}^2$ and for all $a \in \mathbf{C}$, $V^a(Mat(\phi(g_1)))$ is invariant under $Mat(\phi(g_2))$. According to Bourbaki [Bou90] [chap. 7, 1,1, lemma 1], this implies that for all $(g_1, g_2) \in \mathfrak{g}^2$, there exists an integer n such that $ad^n_{Mat(\phi(g_1))}Mat(\phi(g_2)) = 0$. In other words, the Lie algebra $Mat(\phi(\mathfrak{g}))$ is a nilpotent Lie algebra (and so $\phi(\mathfrak{g})$ is). If the morphism ϕ is injective, then $\mathfrak{g}$ is also nilpotent (this means that, for all $g \in \mathfrak{g}$, $ad_g := [g, .] \in \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathfrak{g})$ is a nilpotent linear map). Thus, let us assume that $\mathfrak{g}$ is a **nilpotent Lie algebra.** Let us set $V = \mathbf{C}^n$, then we have the following decomposition into $Mat(\phi(\mathfrak{g}))$-stable subspaces

$$V = \bigoplus_{\alpha \in K} V^{\alpha}(\phi)$$

$$\text{where } V^{\alpha}(\phi) = \{v \in V \mid \forall g \in \mathfrak{g},\ \exists k \in \mathbf{N},\ (\rho_k(\phi(g)) - \alpha(g)Id)^k v = 0\};$$

here K denotes the space of **C**-linear forms on $\mathfrak{g}$. Let $\alpha \in K$ such that $V^{\alpha}(\phi) \neq 0$; it vanishes on $[\mathfrak{g}, \mathfrak{g}]$ which denotes the linear span of $\{[g_1, g_2], (g_1, g_2) \in \mathfrak{g}^2\}$. The restriction $Mat(\phi(\mathfrak{g}))_{|V^{\alpha}}$ of $Mat(\phi(\mathfrak{g}))$ to $V^{\alpha}(\phi)$ is still a nilpotent Lie subalgebra of $\mathfrak{gl}(V^{\alpha}(\phi))$, thus it is solvable. It follows that, according to the Lie theorem quoted above, there exists a basis of $V^{\alpha}(\phi)$ in which each element of $Mat(\phi(\mathfrak{g}))_{|V^{\alpha}}$ is an upper triangular matrix. Such an element can be written as $\alpha(g)Id_{V^{\alpha}} + T(g)$ with $T(g)$ a strictly upper triangular matrix, which obviously commutes with $\alpha(g)Id_{V^{\alpha}}$.

As a summary, we can say that there exists an invertible matrix $P \in \mathfrak{gl}(V)$ such that, for all $g \in \mathfrak{g}$, $Mat(Mat^{-1}(P)_*\phi(g)) = S(g) + N(g)$ where $S(g)$ is a diagonal matrix, $N(g)$ a strictly upper triangular matrix **commuting** with $S(g)$. Moreover for any couple $(g_1, g_2) \in \mathfrak{g}^2$, $[S(g_1), N(g_2)] = 0$. It is to be noticed that both S and N are Lie morphisms from $\mathfrak{g}$ to the Lie algebras of diagonal matrices and of strictly upper triangular matrices respectively. We shall say that the morphism ϕ is **nilpotent** if $S = 0$.

In the sequel, we shall write $S(g) = \sum_{i=1}^{n} \lambda_i(g) x_i \partial/\partial x_i$ where the λ_i's are linear forms on $\mathfrak{g}$. We shall call them the "eigenvalues" of S.

5.2. *Weight spaces and computations of the cohomology spaces of the Chevalley-Koszul complex*

From now on, we shall assume that $Mat(\phi(g))$ has the form just described. We recall that $\mathcal{P}_n^k$ and p_n^k $(k \geqslant 1)$ are both $\mathfrak{g}$-modules relative to ϕ; we will denote m_k one of them. The associated linear space of representations $\rho_k(\phi(\mathfrak{g}))$ is a nilpotent subalgebra of $\mathrm{Hom}_{\mathbf{C}}(m_k, m_k)$ (if $\mathfrak{g}$ is commutative then, by Jacobi identity, $\rho_k(\phi(\mathfrak{g}))$ is commutative). Therefore, if K denotes the space of $\mathbf{C}$-linear forms on $\mathfrak{g}$, we have the decomposition into weight spaces:

$$m_k = \bigoplus_{\alpha \in \mathrm{K}} m_k^{\alpha}(\phi)$$

$$\text{where } m_k^{\alpha}(\phi) = \{x \in m_k \mid \forall g \in \mathfrak{g},\ \exists p \in \mathrm{N},\ (\rho_k(\phi(g)) - \alpha(g)Id)^p x = 0\}.$$

The weights are those linear forms α for which the vector space m_k^{α} does not reduce to 0. They vanish on $[\mathfrak{g}, \mathfrak{g}]$ which denotes the linear span of $\{[g_1, g_2], (g_1, g_2) \in \mathfrak{g}^2\}$. We will also denote by

$$m_{k,\alpha}(\phi) = \{x \in m_k \mid \forall g \in \mathfrak{g},\ \rho_k(\phi(g))x = \alpha(g)x\} \subset m_k^{\alpha}(\phi),$$

the α-eigenspace. We can also write the **Fitting decomposition** relative to ϕ: $m_k = \overset{+}{m_k}(\phi) \oplus \overset{0}{m_k}(\phi)$ where $\overset{+}{m_k} = \bigoplus_{\alpha \in \mathrm{K}, \alpha \neq 0} m_k^{\alpha}(\phi)$. Both $\overset{+}{m_k}(\phi)$ and $\overset{0}{m_k}(\phi)$ are left invariant by $\rho_k(\phi)$ and thus are $\mathfrak{g}$-modules relative to the restriction of the representation. When $m_k = \mathcal{P}_n^k$, this decomposition defines a section $s_3^k : \mathcal{P}_n^k/(\mathcal{P}_n^k)^{\phi} \to \mathcal{P}_n^k$.

Moreover, there exists a $g_0 \in \mathfrak{g}$, such that $\overset{+}{m_k}(\phi) = \overset{+}{m_k}(\phi(g_0))$ and $\overset{0}{m_k}(\phi) = \overset{0}{m_k}(\phi(g_0))$, the spaces of the Fitting decomposition of the endomorphism $\phi(g_0)$ ([Bou90] [VII, 1, 2, prop. 7]). We will say that g_0 **realizes the Fitting decomposition**.

Lemma **5.2.1.** — *Let $\mathfrak{g}$ be a* **nilpotent** *Lie algebra over* $\mathbf{C}$ *of finite dimension and ϕ a morphism from $\mathfrak{g}$ to $\mathcal{P}_n^1$. Then for any positive integer k, we have*

1. $\mathrm{H}_{\phi}^1(\mathfrak{g}, m_k) = \mathrm{H}_{\phi}^1\left(\mathfrak{g}, \overset{0}{m_k}\right)$;
2. $\mathrm{H}_{\mathrm{S}}^1(\mathfrak{g}, m_k) \cong \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], \overset{\mathrm{S}}{m_k}\right)$; *this isomorphism defines a section*

$$s_2^k : \mathrm{H}_{\mathrm{S}}^1(\mathfrak{g}, m_k) \to \{f \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \overset{\mathrm{S}}{m_k}\right) \mid f_{|[\mathfrak{g},\mathfrak{g}]} = 0\}.$$

Proof. — The first point follows readily from [Bou90] [VII, 1, 3, cor.] which states that, since $\overset{+}{m_k}$ is a $\mathfrak{g}$-module with $\left(\overset{+}{m_k}\right)^0 = 0$, then for any $f \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \overset{+}{m_k}\right)$ such that $\forall (g_1, g_2) \in \mathfrak{g}^2$, $[\phi(g_1), f(g_2)] - [\phi(g_2), f(g_1)] = f([g_1, g_2])$, there exists an $e \in \overset{+}{m_k}$ such that $\forall g \in \mathfrak{g}, f(g) = [\phi(g), e]$. In other words, $\mathrm{H}_{\phi}^1\left(\mathfrak{g}, \overset{+}{m_k}\right) = 0$ and the results follows.

Since S is morphism from $\mathfrak{g}$ to the space of diagonal vector fields, we have $\overset{0}{m_k} = m_{k,0} = \overset{\mathrm{S}}{m_k}$ as well as $0 = [\mathrm{S}(g_1), \mathrm{S}(g_2)] = \mathrm{S}([g_1, g_2])$. Since $d_{0,\mathrm{S}|m_k}\mathrm{S} = 0$, we have $H^1_S(\mathfrak{g}, m_k) = Z^1_S\left(\mathfrak{g}, m_k^S\right) = \{ f \in \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, m_k^S) \mid f_{|[\mathfrak{g},\mathfrak{g}]} = 0\}$. Thus, $H^1_S(\mathfrak{g}, m_k) \cong \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}], m_k^S\right)$. □

Remark **5.2.2.**

- *Since* $m_k^0(\phi) = m_k^S$, *then whatever the section* s_2 *is, we have* $H^1_\phi(\mathfrak{g}, m_k) \cong H^1_\phi\left(\mathfrak{g}, m_k^S\right)$; *this means that the normal form of a compatible nonlinear deformation of* ϕ *belongs to* $\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \left(\mathscr{X}_n^1\right)^S\right)$;
- *if* $\mathfrak{g}$ *is abelian, then* $H^1_S(\mathfrak{g}, m_k) \cong \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, m_k^S\right)$.

Lemma **5.2.3.** — *With the hypothesis of the previous lemma,*

- *the set of weights of* $\mathfrak{g}$ *into* p_n^k *is*

$$\mathscr{W}^k_{f,n}(\phi) = \{\beta_Q(g) := (Q, \lambda(g)),\ Q \in \mathbf{N}^n,\ |Q| = k\}$$

($\mathscr{W}_f$ *stands for Weights on Functions);*

- *the set of weights of* $\mathfrak{g}$ *into* $\mathscr{P}_n^k$ *is*

$$\mathscr{W}^k_{v,n}(\phi) = \{\alpha_{Q,j}(g) := (Q, \lambda(g)) - \lambda_j(g), Q \in \mathbf{N}^n,\ |Q| = k,\ 1 \leqslant j \leqslant n\}$$

($\mathscr{W}_v$ *stands for Weights on Vector fields).*

Proof. — As we have seen, for all $g \in \mathfrak{g}$, $\phi(g) = S(g) + N(g)$ where $S(g)$ is a diagonal vector field, $N(g)$ a strictly upper triangular vector field commuting with $S(g)$. Let us write $S(g) = \sum_{i=1}^n \lambda_i(g) x_i \partial/\partial x_i$. An easy computation shows that, for all $Q \in \mathbf{N}^n$ with $|Q| = k$ and all $1 \leqslant i \leqslant n$, we have

$$[S(g), x^Q \frac{\partial}{\partial x_i}] = \left((Q, \lambda(g)) - \lambda_i(g)\right) x^Q \frac{\partial}{\partial x_i}.$$

It follows that the eigenvalues of $\rho_k(S(g))$ are the $\alpha_{Q,i}(g)$'s. But, for any $a \in \mathbf{C}$, we have $\left(\mathscr{P}_n^k\right)^a(\phi(g)) = \left(\mathscr{P}_n^k\right)^a(S(g)) = \left(\mathscr{P}_n^k\right)_a(S(g))$. Moreover, for $\alpha \in \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathbf{C})$, we have

$$\left(\mathscr{P}_n^k\right)^\alpha(\phi) = \bigcap_{g \in \mathfrak{g}} \left(\mathscr{P}_n^k\right)^{\alpha(g)}(\phi(g)) = \bigcap_{g \in \mathfrak{g}} \left(\mathscr{P}_n^k\right)_{\alpha(g)}(S(g)).$$

This space is zero unless it contains $x^Q \partial/\partial x_i$, so that $\alpha(g) = \left((Q, \lambda(g)) - \lambda_i(g)\right)$ for all $g \in \mathfrak{g}$. The same proof holds for the representation on functions using the fact that $\mathscr{L}_{S(g)}(x^Q) = (Q, \lambda(g)) x^Q$. □

Let $p \leqslant k$ be positive integers, we shall denote by $\mathscr{W}^{p,k}_{v,n}(\phi)$ the set of weights of ϕ into $\mathscr{P}_n^{p,k}$. We shall denote by $\mathscr{W}^{p,k}_{f,n}(\phi)$ the set of weights of ϕ into $p_n^p \oplus \cdots \oplus p_n^k$.

From now on, we choose the sections s_2^k and s_3^k for the normalization as defined above in this section.

Remark **5.2.4.** — *The set of weights* $\oplus_{p\geq 2} \mathscr{W}_{f,n}^{p}(\phi)$ *has the semi-group property; indeed, let* β_Q *and* $\beta_{Q'}$ *be two weights, then* $\beta_Q + \beta_{Q'} = \beta_{Q+Q'}$ *is a weight. Moreover, if* $\alpha \in \oplus_{p\geq 2} \mathscr{W}_{v,n}^{p}(\phi)$ *then it can be written as* $\alpha = \beta - \beta_{E_j}$ *for some* $\beta \in \oplus_{p\geq 2} \mathscr{W}_{f,n}^{p}(\phi)$ *and some index* $1 \leqslant j \leqslant n$.

5.3. *Fundamental structures of the weight spaces*

Lemma **5.3.1.** — *The* **C**-*vector space* $\widehat{\mathscr{O}}_n^{S}$ *is a* **C**-*algebra. Let* M *denotes one of the two spaces* $\widehat{\mathscr{X}}_n^1$ *or* $\widehat{\mathscr{O}}_n$. *Let* α *be any weight of* ϕ *into* M; *then its associated weight space* $(M)^{\alpha}(\phi)$ *has a canonical strucure of* $\widehat{\mathscr{O}}_n^{S}$-*module.*

Proof. — We assume that ϕ is not nilpotent otherwise the result is trivial. Let us prove the first point. For any $g \in \mathfrak{g}$ and any weight α of ϕ into $\widehat{\mathscr{O}}_n$, we set $N_{g,\alpha} = \mathscr{L}_{\phi(g)} - \alpha(g)Id$. We have to prove that if $a \in \left(\widehat{\mathscr{O}}_n\right)^0(\phi)$ and if $f \in \left(\widehat{\mathscr{O}}_n\right)^{\alpha}(\phi)$, then $af \in \left(\widehat{\mathscr{O}}_n\right)^{\alpha}(\phi)$. Let us prove by induction on a positive integer k, that $N_{g,\alpha}(af)$ is a linear combination of $N_{g,0}(a)^{k-p}N_{g,\alpha}^{p}(f)$, $0 \leqslant p \leqslant k$. Since $\mathscr{L}_{\phi(g)}$ is a derivation of $\widehat{\mathscr{O}}_n$, we have $N_{g,\alpha}(af) = aN_{g,\alpha}(f) + N_{g,0}(a)f$, so the results holds for $k=1$. Let us assume it is so for $k-1$, so that $N_{g,\alpha}^{k-1}(av) = \sum_{p=0}^{k-1} C_p^{k-1} N_{g,0}^{k-1-p}(a) N_{g,\alpha}^{p}(f)$. Thus we have

$$\begin{aligned} N_{g,\alpha}^{k}(av) &= \sum_{p=0}^{k-1} C_p^{k-1} N_{g,\alpha}\left(N_{g,0}^{k-1-p}(a)N_{g,\alpha}^{p}(f)\right) \\ &= \sum_{p=0}^{k-1} C_p^{k-1}\left(N_{g,0}^{k-1-p}(a)N_{g,\alpha}^{p+1}(f) + N_{g,0}^{k-p}(a)N_{g,\alpha}^{p}(f)\right). \end{aligned}$$

Since $N_{g,0}^{k-1-p}(a) = N_{g,0}^{k-(p+1)}(a)$, the result follows. Since both operators $N_{g,\alpha}$ and $N_{g,0}$ are nilpotent on $\left(\widehat{\mathscr{O}}_n\right)^{\alpha}(\phi)$ and $\left(\widehat{\mathscr{O}}_n\right)^{0}(\phi)$ respectively, then for all $g \in \mathfrak{g}$ there exists an integer m such that $N_{g,\alpha}^{m}(af) = 0$. By definition, this means that af belongs to $\left(\widehat{\mathscr{O}}_n\right)^{\alpha}(\phi)$. This shows that $\left(\widehat{\mathscr{O}}_n\right)^{0}(\phi)$ is a **C**-algebra (if we take $\alpha = 0$) and $\left(\widehat{\mathscr{O}}_n\right)^{\alpha}(\phi)$ an $\left(\widehat{\mathscr{O}}_n\right)^{0}(\phi)$-module.

In the case where $M = \widehat{\mathscr{X}}_n^1$; for any $g \in \mathfrak{g}$ and any weight α of ϕ into M, we set $M_{g,\alpha} = \rho(\phi(g)) - \alpha(g)Id$, and $N_{g,0} = \mathscr{L}_{\phi(g)}$. Let a be an element of $\left(\widehat{\mathscr{O}}_n\right)^{0}(\phi)$ and let v be an element of $\left(\widehat{\mathscr{X}}_n^1\right)^{\alpha}(\phi)$; according to the basic property of the Lie bracket, we have $M_{g,\alpha}(av) = aM_{g,\alpha}(v) - N_{g,0}(a)v$. As above, we claim that, for any positive integer k, $M_{g,\alpha}(av)$ is a linear combination of $N_{g,0}^{k-p}(a)M_g^{p,\alpha}(v)$, $0 \leqslant p \leqslant k$. The proof is the same as above and we leave it to the reader. Since both operators $M_{g,\alpha}$ and $N_{g,0}$ are nilpotent on $\left(\widehat{\mathscr{X}}_n^1\right)^\alpha(\phi)$ and $\left(\widehat{\mathscr{O}}_n\right)^0(\phi)$ respectively, then for all $g \in \mathfrak{g}$ there exists an integer m such that $M_{g,\alpha}^m(av) = 0$. By definition, this means that $av \in \left(\widehat{\mathscr{X}}_n^1\right)^\alpha(\phi)$. □

The next statement is due to Walcher [Wal91] [prop. 1.6] in the case of a single linear vector field; but we shall give a different proof.

Proposition **5.3.2.** — *With the notation above, $\widehat{\mathscr{O}}_n^{\mathrm{S}}$ is a formal* **C**-*algebra of finite type; $\widehat{\mathscr{X}}_n^{\mathrm{S}}$ is a $\widehat{\mathscr{O}}_n^{\mathrm{S}}$-module of finite type.*

Proof. — The result on the finiteness of $\widehat{\mathscr{O}}_n^{\mathrm{S}}$ is almost classical in Invariant Theory [Bri96] [p. 42].

We shall show that $\widehat{\mathscr{O}}_n^{\mathrm{S}} = \mathbf{C}[[u_1, \ldots, u_p]]$ where each u_i is a monomial. It follows that $\widehat{\mathscr{O}}_n^{\mathrm{S}} \otimes x_i\partial/\partial x_i \hookrightarrow \widehat{\mathscr{X}}_n^{\mathrm{S}}$. Therefore, it remains to show that there is a finite number of vector fields Y_j, not belonging to $\sum \widehat{\mathscr{O}}_n^{\mathrm{S}} \otimes x_i\partial/\partial x_i$, such that $\widehat{\mathscr{X}}_n^{\mathrm{S}} = \sum \widehat{\mathscr{O}}_n^{\mathrm{S}} \otimes x_i\partial/\partial x_i \oplus \sum \widehat{\mathscr{O}}_n^{\mathrm{S}} \otimes Y_j$. In order to prove the finiteness property in both cases, we shall construct a noetherian ring with an action of $\mathfrak{g}$ and use the noetherian property of an ideal.

Let us recall some basic facts about derivations of algebras over a commutative ring. Let A be a commutative ring and B be a commutative A-algebra that is B is a commutative ring together with a ring morphism $\phi : A \to B$. This morphism induces a A-module structure over B. A A-derivation of B is a morphism of A-modules $D : B \to B$ such that $D(bb') = bD(b') + b'D(b)$. Let $\mathscr{I}$ be an ideal of B; it is a A-submodule of B. Let $p : B \to B/\mathscr{I}$ be the quotient map; it is a ring morphism and the ring morphism $p \circ \phi$ provides $B/\mathscr{I}$ with an A-algebra structure. If the derivation D leaves $\mathscr{I}$ invariant, then it induces an A-derivation $\overline{D}$ of $B/\mathscr{I}$ such that $\overline{D} \circ p = p \circ D$. In fact, it is A-linear since, for $a \in A$ and $b \in B$,

$$\overline{D}(p(\phi(a)b)) = p(D(\phi(a)b)) = p(\phi(a)D(b)) = p(\phi(a))\overline{D}(p(b));$$

on the other hand,

$$\begin{aligned}\overline{D}(p(b)p(b')) = \overline{D}(p(bb')) = p(D(bb')) &= p(bD(b') + b'D(b)) \\ &= p(b)\overline{D}(p(b')) + p(b')\overline{D}(p(b)).\end{aligned}$$

Let us set $P = \mathbf{C}[x_1, \ldots, x_n]$ and $\mathscr{A} = P[\zeta_1, \ldots, \zeta_n]$ where the ζ_i's are indeterminates over P. These are commutative **C**-algebras. Since P is a noetherian ring, so is $\mathscr{A}$.

For any integer $1 \leqslant i \leqslant n$, let us define $\mathscr{L}_i := x_i \partial/\partial x_i$ as a derivation of P. We have $\mathscr{L}_i(c) = 0$ and $\mathscr{L}_i(x_k) = \delta_{i,k} x_i$, where $c \in \mathbf{C}$, $1 \leqslant k \leqslant n$ and $\delta_{i,k}$ is zero if $k \neq i$ and 1 otherwise. We extend this derivation into a derivation of $\mathscr{A}$ by setting

$$\mathscr{L}_i(\zeta_k) = -\delta_{k,i}\zeta_k.$$

Let us show that, for any pair of indices $1 \leqslant i, j \leqslant n$, $\mathscr{L}_i \circ \mathscr{L}_j = \mathscr{L}_j \circ \mathscr{L}_i$.

In fact, we have

$$\begin{aligned}
\mathscr{L}_i \circ \mathscr{L}_j(x^{\mathrm{Q}}\zeta^{\mathrm{P}}) &= \mathscr{L}_i\left(x^{\mathrm{Q}}\mathscr{L}_j(\zeta^{\mathrm{P}}) + \zeta^{\mathrm{P}}\mathscr{L}_j(x^{\mathrm{Q}})\right)\\
&= (q_j - p_j)\mathscr{L}_i(x^{\mathrm{Q}}\zeta^{\mathrm{P}})\\
&= (q_j - p_j)(q_i - p_i)x^{\mathrm{Q}}\zeta^{\mathrm{P}}\\
&= (q_i - p_i)\mathscr{L}_j(x^{\mathrm{Q}}\zeta^{\mathrm{P}})\\
&= \mathscr{L}_j \circ \mathscr{L}_i(x^{\mathrm{Q}}\zeta^{\mathrm{P}}).
\end{aligned}$$

Let $\mathscr{I}$ be the ideal of $\mathscr{A}$ generated by $\zeta_i\zeta_j$ and $x_i\zeta_i$, $1 \leqslant i, j \leqslant n$. Let $\mathscr{B}$ be the $\mathbf{C}$-subspace of $\mathscr{A}$ whose basis is the set of monomials x^{P}, $\mathrm{P} \in \mathbf{N}^n$, and for $1 \leqslant i \leqslant n$, $x^{\mathrm{Q}}\zeta_i$, $\mathrm{Q} \in \mathbf{N}^n$ with $q_i = 0$. Then, we have the decomposition into direct sum $\mathscr{A} = \mathscr{B} \oplus \mathscr{I}$. Let $p : \mathscr{A} \to \mathrm{A} := \mathscr{A}/\mathscr{I}$ be the quotient map onto the quotient ring $\mathscr{A}/\mathscr{I}$. The latter is a noetherian commutative ring as well as a $\mathbf{C}$-algebra. If we set $\mathrm{X}_i = p(x_i)$, $\mathrm{Z}_i = p(\zeta_i)$, $1 \leqslant i \leqslant n$, then the set

$$\{\mathrm{X}^{\mathrm{P}},\ \mathrm{P} \in \mathbf{N}^n\} \cup_{i=1}^{n} \{\mathrm{X}^{\mathrm{Q}}\mathrm{Z}_i,\ \mathrm{Q} \in \mathbf{N}^n \text{ with } q_i = 0\}$$

is a basis of A over $\mathbf{C}$. Each derivation $\mathscr{L}_i$ leaves $\mathscr{I}$ invariant. In fact, if $f \in \mathscr{A}$, then

$$\begin{aligned}
\mathscr{L}_i(f\zeta_k\zeta_l) &= \zeta_k\zeta_l\mathscr{L}_i(f) + f\mathscr{L}_i(\zeta_k\zeta_l)\\
&= \zeta_k\zeta_l\mathscr{L}_i(f) - \zeta_l\zeta_k f(\delta_{k,i} + \delta_{l,i})\\
\mathscr{L}_i(fx_k\zeta_k) &= x_k\zeta_k\mathscr{L}_i(f) + f\mathscr{L}_i(x_k\zeta_k)\\
&= x_k\zeta_k\mathscr{L}_i(f).
\end{aligned}$$

It follows that each derivation $\mathscr{L}_i$ defines a derivation L_i of A which satisfies $\mathrm{L}_i \circ p = p \circ \mathscr{L}_i$ and we have

$$\begin{aligned}
\mathrm{L}_i \circ \mathrm{L}_j \circ p &= \mathrm{L}_i \circ p \circ \mathscr{L}_j = p \circ \mathscr{L}_i \circ \mathscr{L}_j\\
&= p \circ \mathscr{L}_j \circ \mathscr{L}_i = \mathrm{L}_j \circ p \circ \mathscr{L}_i\\
&= \mathrm{L}_j \circ \mathrm{L}_i \circ p.
\end{aligned}$$

Let S be a diagonal Lie morphism from $\mathfrak{g}$, a nilpotent Lie algebra of finite dimension over $\mathbf{C}$, to $\mathscr{P}_n^1$; i.e. for all $g \in \mathfrak{g}$, $\mathrm{S}(g) = \sum_{i=1}^{n} \lambda_i(g) x_i \partial/\partial x_i$. For all $g \in \mathfrak{g}$, $\mathrm{S}(g)$ can be regarded as a derivation of P that can be extended to a derivation of $\mathscr{A}$ by setting $\mathrm{S}(g) = \sum_{i=1}^{n} \lambda_i(g)\mathscr{L}_i$; therefore, it induces a derivation $\tilde{\mathrm{S}}(g) = \sum_{i=1}^{n} \lambda_i(g)\mathrm{L}_i$ of A. We claim that $\tilde{\mathrm{S}}$ is a Lie morphism from $\mathfrak{g}$ to $\mathrm{Der}_{\mathbf{C}}(\mathrm{A})$, the associative algebra of

C-derivations of A. We just have to show that $\tilde{S}([g_1, g_2]) = \tilde{S}(g_1)\circ\tilde{S}(g_2) - \tilde{S}(g_2)\circ\tilde{S}(g_1)$. In fact, on the one hand, we have $\tilde{S}(g_1)\circ\tilde{S}(g_2) = \sum_{i=1}^{n} \lambda_i(g_1)L_i\left(\sum_{j=1}^{n} \lambda_j(g_2)L_j\right) = \tilde{S}(g_2)\circ\tilde{S}(g_1)$; on the other hand, we just have to recall the fact that the λ_i's are linear forms on $\mathfrak{g}$ which vanish on $[\mathfrak{g}, \mathfrak{g}]$ and we are done. Thus, $\tilde{S}$ is a representation of $\mathfrak{g}$ in A. With the notations of lemma 5.2.3, an easy computation shows that, for any $g \in \mathfrak{g}$,

$$\tilde{S}(g)(X^Q) = (Q, \lambda(g))X^Q = \beta_Q(g)X^Q, \quad \text{if } Q \in \mathbf{N}^n;$$
$$\tilde{S}(g)(X^Q Z_i) = ((Q, \lambda(g)) - \lambda_i(g))X^Q Z_i = \alpha_{Q,i}(g)X^Q Z_i, \quad \text{if } Q \in \mathbf{N}^n,\ q_i = 0.$$

Let us define $A_{k,i}$ to be the finite-dimensional **C**-subspace of A whose basis is the set $\{X^Q Z_i, |Q| = k, q_i = 0\}$ and $A_{k,0}$ to be the **C**-subspace of A whose basis is the $\{X^Q, |Q| = k\}$; their elements will be called homogeneous of degree (k, i). An element of $A_k := \oplus_{i=0}^{n} A_{k,i}$ will be called homogeneous of degree k. An easy computation shows that $A_k A_l \subset A_{k+l}$ for any couple of nonnegative integers k, l. This is due to the fact that, for all $(P, R) \in \mathbf{N}^n$, and all $Q \in \mathbf{N}^n$ with $q_i = 0$, $X^P X^R = X^{P+R}$, $(X^P Z_j)(X^Q Z_i) = 0$, $X^P(X^Q Z_i) = 0$ if $p_i \neq 0$ and $X^P(X^Q Z_i) = X^{P+Q} Z_i$ if $p_i = 0$. It follows that the ring A is a graded ring: $A = \bigoplus_{k\geqslant 0} A_k$. Each of the spaces A_k are invariant with respect to $\tilde{S}$.

Let us define $\mathrm{Inv}(\tilde{S}) = \{f \in A \mid \forall g \in \mathfrak{g}, \tilde{S}(g)(f) = 0\}$; this is the 0-th cohomology space of the Chevalley-Koszul complex associated to the representation $\tilde{S}$ in A. Let $\mathcal{J}$ be the ideal of A generated by the nonconstant homogeneous elements of $\mathrm{Inv}(\tilde{S})$. If $\mathrm{Inv}(\tilde{S}) = \mathbf{C}$ then $\mathcal{J} = (0)$. Let us assume this is not the case; then, since A is noetherian, $\mathcal{J}$ is generated by a finite number of elements. We may require these generators to belong to $\mathrm{Inv}(\tilde{S})$ and to be homogeneous. We shall denote them by $U_1, \ldots, U_r$. In fact, let us start with the nonconstant homogeneous elements of $\mathrm{Inv}(\tilde{S})$ of degree 0. These are the Z_i's for which $\lambda_i \equiv 0$. Let $\mathcal{J}_0$ be the ideal of A they generate. Now, let us consider the homogeneous elements of $\mathrm{Inv}(\tilde{S})$ of degree 1. If they don't belong to $\mathcal{J}_0$, then let us define $\mathcal{J}_1$ to be the ideal generated by $\mathcal{J}_0$ and the homogeneous elements of degree 1. In this way, we define an increasing sequence (which may be finite) of ideals $\{\mathcal{J}_k\}_{k\in K}$ and let us set $\mathcal{J} = \bigcup_{k\in K} \mathcal{J}_k$. If K is finite then $\mathcal{J}$ is generated by homogeneous elements of $\mathrm{Inv}(\tilde{S})$. Otherwise, since A is noetherian, the ascending chain of ideals $\mathcal{J}_k$ must be stationary. Therefore, $\mathcal{J}$ is generated by homogeneous elements of $\mathrm{Inv}(\tilde{S})$. By definition of the action of $\tilde{S}$ on A, $\mathrm{Inv}(\tilde{S})$ is the **C**-vector space generated by the monomials X^Q with $(Q, \lambda) \equiv 0$ and $X^Q Z_i$ with $(Q, \lambda) - \lambda_i \equiv 0$ and $q_i = 0$. Therefore, we may replace the generators of $\mathcal{J}$ by the monomials they are sums of. Thus, we may assume that U_i is a monomial of degree $(m_i, 0)$, $m_i > 0$, if $1 \leqslant i \leqslant p$, and U_i is a monomial of degree (m_i, j_i), $m_i \geqslant 0$, if $p + 1 \leqslant i \leqslant r$ with $1 \leqslant j_i \leqslant n$.

We claim that $\mathrm{Inv}(\widetilde{S}) = \mathbf{C}[U_1, \ldots, U_p] \oplus \left(\sum_{i=p+1}^{r} \mathbf{C}[U_1, \ldots, U_p]U_i\right)$. We shall prove it by induction on the degree k of homogeneous elements of $\mathrm{Inv}(\widetilde{S})$. If $f \in \mathrm{Inv}(\widetilde{S})$ is of degree $k = 0$, then $f \in \mathbf{C} \oplus_{i|\lambda_i \equiv 0} \mathbf{C}Z_i$. Thus, by definition of the U_i's, it belongs to the space. Let us assume that the result holds of all homogeneous elements of $\mathrm{Inv}(\widetilde{S})$ of degree $q < k$. Let $f \in \mathrm{Inv}(\widetilde{S})$ be homogeneous of degree k; by definition, it belongs to $\mathscr{J}$. So, there exists $a_1, \ldots, a_p \in A$ such that $f = \sum_{i=1}^{r} a_i U_i$. Let a be the maximum of the degrees of the a_i's. We can decompose each a_i along the weight spaces of $\widetilde{S}$ on the subspace $\oplus_{q=0}^{a} A_q$, so that $a_i = a_i^0 + \sum_{\alpha \neq 0} a_{i,\alpha}$ where a_i^0 belongs to 0-weight space of $\widetilde{S}$ and $a_{i,\alpha}$ belongs to the α-weight space of $\widetilde{S}$. For any weight α and any $g \in \mathfrak{g}$, $\widetilde{S}(g)(a_{i,\alpha}U_i) = U_i\widetilde{S}(g)(a_{i,\alpha}) = \alpha(g)a_{i,\alpha}U_i$; thus, $a_{i,\alpha}U_i$ belongs to the α-weight space of $\widetilde{S}$. It follows that, since $\widetilde{S}(g)(f) = 0$, then $f = \sum_{i=1}^{r} a_i^0 U_i$. On the other hand, f is homogeneous of degree k and U_i of degree m_i; thus $f = \sum_{i=1}^{r} \widetilde{a}_i^0 U_i$ where $\widetilde{a}_i^0$ is the homogeneous part of degree $k - m_i$ of a_i^0 if $k - m_i \geqslant 0$, and 0 otherwise. For each i such that $m_i > 0$, we may apply the induction argument to $\widetilde{a}_i^0$: $\widetilde{a}_i^0 = P_i(U_1, \ldots, U_p) + \sum_{j=p+1}^{r} P_{i,j}(U_1, \ldots, U_p)U_j$. Since, $U_iU_j = 0$ in the ring A, for all $p+1 \leqslant i, j \leqslant r$, we have:

$$f - \sum_{i|m_i=0} \widetilde{a}_i^0 U_i = \sum_{i=1,\, m_i>0}^{p} P_i(U_1, \ldots, U_p)U_i + \sum_{j=p+1}^{r} \left(\sum_{i=1,\, m_i>0}^{p} P_{i,j}(U_1, \ldots, U_p)U_i + P_j(U_1, \ldots, U_p) \right) U_j.$$

Let π be the projection of A onto its subspace $\mathbf{C}[X_1, \ldots, X_n]$. The projection $\pi(f)$, which is homogeneous of degree k, belongs to $\mathbf{C}[U_1, \ldots, U_p]$ since it is equal to $\sum_{i=1,\, m_i>0}^{p} P_i(U_1, \ldots, U_p)U_i$. Thus, any $f \in \mathbf{C}[X_1, \ldots, X_n] \cap \mathrm{Inv}(\widetilde{S})$ of degree $\leqslant k$ belongs to $\mathbf{C}[U_1, \ldots, U_p]$. Moreover, for any i such that $m_i = 0$, the element $\widetilde{a}_i^0$ such that $\widetilde{a}_i^0 U_i \neq 0$, is homogeneous of degree k and belongs to $\mathbf{C}[X_1, \ldots, X_n] \cap \mathrm{Inv}(\widetilde{S})$ ($Z_iZ_j = 0$ in A). Therefore, by the result just proved, $\widetilde{a}_i^0 \in \mathbf{C}[U_1, \ldots, U_p]$. Hence, we have $f \in \mathbf{C}[U_1, \ldots, U_p] \oplus \left(\sum_{i=p+1}^{r} \mathbf{C}[U_1, \ldots, U_p]U_i\right)$ and this concludes the induction.

Let $\phi : A \to \mathbf{C}[x_1, \ldots, x_n] \oplus_{i=1}^{n} \mathbf{C}[x_1, \ldots, x_n]\,\partial/\partial x_i$ be the $\mathbf{C}$-linear map defined by $\phi(X^Q) = x^Q$, $\phi(X^Q Z_i) = x^Q \partial/\partial x_i$; let us set $u_i = \phi(U_i)$ for any $1 \leqslant i \leqslant p$. Let us show that $\widehat{\mathscr{O}}_n^S = \mathbf{C}[[u_1, \ldots, u_p]]$. In fact, let $f = \sum_{k \geqslant 0} f_k \in \widehat{\mathscr{O}}_n^S$ where f_k is the homogeneous component of degree k. We have $\phi^{-1}(f_k) \in \mathrm{Inv}(\widetilde{S})$ so that $\phi^{-1}(f_k) \in \mathbf{C}[U_1, \ldots, U_p]$ and thus $f_k \in \mathbf{C}[u_1, \ldots, u_p]$. Since each u_i is a monomial, there can only be a finite number of f_k's with a nonzero component on every monomial u^Q where $Q \in \mathbf{N}^p$. Thus, $f \in \mathbf{C}[[u_1, ..., u_p]]$.

For any $p+1 \leqslant i \leqslant r$, we have $U_i = X^{Q_i} Z_{j_i}$ where $Q_j \in \mathbf{N}^n$ and $1 \leqslant j_i \leqslant n$. Let us show that $\widehat{\mathscr{X}}_n^S$ is the $\mathbf{C}[[u_1, ..., u_p]]$-module generated by $x_i \partial/\partial x_i$, $1 \leqslant i \leqslant n$, and $x^{Q_i} \partial/\partial x_{j_i}$, $p+1 \leqslant i \leqslant r$. In fact, let $x^Q \partial/\partial x_j \in \widehat{\mathscr{X}}_n^S$, so that the weight $\alpha_{Q,j} \equiv 0$. If $q_j \neq 0$, then $Q' := Q - E_j \in \mathbf{N}^n$ and $\beta_{Q'} \equiv 0$. It follows that $x^{Q'} \in \mathbf{C}[[u_1, ..., u_p]]$ and $x^Q \partial/\partial x_j = x^{Q'} x_j \partial/\partial x_j$. If $q_j = 0$, then $\phi^{-1}(x^Q \partial/\partial x_j) = X^Q Z_j \in \mathrm{Inv}(\tilde{S})$ so that $X^Q Z_j \in \sum_{i=p+1}^r \mathbf{C}[U_1, ..., U_p] U_i$ and $x^Q \partial/\partial x_j \in \sum_{i=p+1}^r \mathbf{C}[u_1, ..., u_p] x^{Q_i} \partial/\partial x_{j_i}$. □

Remark **5.3.3.** — *The previous result may be thought as surprising as far as we have the "naïve geometry" in mind, that is the geometry of the set of eigenvalues of a diagonal linear vector field in the complex plane. For instance, this is the way to define the classical Poincaré domain: the diagonal vector field s belongs to that domain if there is a line* L *passing through the origin such that the set of eigenvalues of s belongs to one of the (strict) half-spaces defined by L. In that case, we can show, with elementary geometry in the complex plane, that there are only a finite number of resonances (see* [Arn80, p. 181]*).*

The previous result shows that, in fact, as soon as s has only trivial polynomial first integrals, that is, constants, then it has only a finite number of resonances.

Corollary **5.3.4.** — *There exist monomials* $u_1, ..., u_p \in \mathbf{C}[x_1, ..., x_n]$ *and homogeneous vector fields* $nf_1, ..., nf_\kappa \in \mathscr{X}_n^1$ *such that* $\widehat{\mathscr{O}}_n^S = \mathbf{C}[[u_1, ..., u_p]]$ *and* $\left(\widehat{\mathscr{X}}_n^1\right)^S = \sum_{j=1}^{\kappa} \mathbf{C}[[u_1, ..., u_p]] nf_j$ *(with the convention that* $\widehat{\mathscr{O}}_n^S = \mathbf{C}$ *whenever* $p = 0$*).*

5.4. *The canonical singular fibration over an algebraic variety*

Together with Walcher and with the notation of the previous proposition, if $p \neq 0$, we may assume that $\{u_1, ..., u_s\}$ $(1 \leqslant s \leqslant p)$ is a maximal algebraically independent (over $\mathbf{C}$) family of $\{u_1, ..., u_p\}$. Thus, the integer s is the degree of transcendence of the field of fractions of $\mathbf{C}[u_1, ..., u_p]$ over $\mathbf{C}$. Although p may be greater than n, s must be smaller than n. Let $\mathscr{T}_\phi$ be the kernel of the ring morphism $\mathbf{C}[X_1, ..., X_p] \to \mathbf{C}[u_1, ..., u_p]$ which maps X_i to u_i. This ideal defines an **affine algebraic variety** $\mathscr{C}_\phi$ of $\mathbf{C}^p$. Therefore, to ϕ is associated, in a canonical way, a **singular fibration** $\pi_\phi : (\mathbf{C}^n, 0) \to \mathscr{C}_\phi \cap (\mathbf{C}^p, 0)$ which maps x to $(u_1(x), ..., u_p(x))$. The algebraic variety $\mathscr{C}_\phi$ has dimension s. Moreover, the vector space $\mathscr{D}$ of diagonal vector fields D of $\mathbf{C}^n$ such that $\mathscr{L}_D(\widehat{\mathscr{O}}_n^S) = 0$ has complex dimension $n-s$. In fact, let $R_i \in \mathbf{N}^n$ be such that $u_i = x^{R_i}$; since $u_1, ..., u_s$ are algebraically independent monomials, $R_1, ..., R_s$ are independent over $\mathbf{Q}$. Hence the matrix B whose rows are $R_1, ..., R_s$ has rank s. Therefore, as a linear mapping from $\mathbf{C}^n$ to $\mathbf{C}^s$, B has rank s, so that its kernel has dimension $n-s$. But, $D = \sum_{i=1}^n \mu_i x_i \partial/\partial x_i$ belongs to $\mathscr{D}$ if and only if $(\mu_1, ..., \mu_n)$ belongs to the kernel of B.

If $p=0$, then $\widehat{\mathcal{O}}^{\phi}=\mathbf{C}$. In this case, we set $\mathcal{C}_{\phi}=\{*\}$, that is the variety is reduced to a point and we set $\pi_{\phi}:\mathbf{C}^n\to\{*\}$.

6. Formal complete integrability

6.1. *Extensions of linear morphisms*

Let $\phi' : \mathfrak{g}\to\mathcal{P}_n^1$ and $\phi'' : \mathfrak{g}\to\mathcal{P}_m^1$ be linear morphisms. We shall use the following canonical injections and projections:

$$\begin{array}{ll} i' : \mathbf{C}^n \to \mathbf{C}^{n+m} & i'' : \mathbf{C}^m \to \mathbf{C}^{n+m} \\ \quad x\mapsto (x,0) & y\mapsto (0,y) \\ p' : \mathbf{C}^{n+m}\to\mathbf{C}^n & p'' : \mathbf{C}^{n+m}\to\mathbf{C}^m \\ \quad (x,y)\mapsto x & (x,y)\mapsto y \end{array}$$

We will call the linear morphism $\phi:=i'_*(\phi')+i''_*(\phi''):\mathfrak{g}\to\mathcal{P}_{n+m}^1$ the **extension of** ϕ' **by** ϕ''. It is clear that $S:=i'_*S'+i''_*S''$ is a diagonal morphism, $N:=i'_*N'+i''_*N''$ is strictly upper triangular and, for any $g\in\mathfrak{g}$,

$$\begin{aligned}[S(g),N(g)] = {} & i'_*[S'(g),N'(g)]+i''_*[S''(g),N''(g)] \\ & +[i'_*S'(g),i''_*N''(g)]+[i''_*S''(g),i'_*N'(g)]=0.\end{aligned}$$

Let us write the weights of the extension as a function of the weights of ϕ' and ϕ''. Let $Q\in\mathbf{N}^{n+m}$ and $1\leqslant j\leqslant n+m$. By an easy computation, we have:

$$\begin{aligned}\beta_Q(\phi)&=\beta_{p'(Q)}(\phi')+\beta_{p''(Q)}(\phi''),\\ \alpha_{Q,j}(\phi)&=\alpha_{p'(Q),j}(\phi')+\beta_{p''(Q)}(\phi'')\ \text{if } j\leqslant n,\\ \alpha_{Q,j}(\phi)&=\beta_{p'(Q)}(\phi')+\alpha_{p''(Q),j}(\phi'')\ \text{if } j>n.\end{aligned}$$

Definition **6.1.1.** — *Let* $\mathfrak{g}$ *be a finite dimensional nilpotent Lie algebra over* $\mathbf{C}$. *Let* $\phi:\mathfrak{g}\to\mathcal{P}_n^1$ *be a linear morphism. As usual, we shall write* $\phi=S+N$ *where* S *is a diagonal morphism,* N *a strictly upper triangular morphism such that, for all* $g\in\mathfrak{g}$, $[S(g),N(g)]=0$. *We assume that* $S\not\equiv 0$.

1. *The linear morphism* ϕ *will be called* **flat** *if* $\widehat{\mathcal{O}}_n^{S}=\mathbf{C}$;
2. *an extension of a linear morphism* ϕ *by* ϕ'' *will be called a* **flat extension** *of* ϕ *when* ϕ'' *is flat;*
3. *a nonflat linear morphism* ϕ *will be called* **complete** *if for any linear injection* $i:\mathbf{C}^k\to\mathbf{C}^n$, $0<k<n$, *whose image is left invariant by* S, *then* $i_*\left(\widehat{\mathcal{O}}_k^{i^*S}\right)\not\equiv\widehat{\mathcal{O}}_n^{S}$. *In other words,* S *is complete if its ring of formal first integral depends on all the coordinates.*

Definition **6.1.2.** — *Let $\phi : \mathfrak{g} \to \mathscr{P}_n^1$ be a nonnilpotent linear morphism. Let $0 \leqslant n' \leqslant n$ be an integer. A pair of linear morphisms (ϕ', ϕ'') from $\phi' : \mathfrak{g} \to \mathscr{P}_{n'}^1$ (resp. $\phi'' : \mathfrak{g} \to \mathscr{P}_{n-n'}^1$) where ϕ'' is flat, will be called a* **splitting** *of ϕ if ϕ is the flat extension of ϕ' by ϕ''.*

Lemma **6.1.3.** — *Let $\mathfrak{g}$ be a finite dimensional nilpotent complex Lie algebra. Let $\phi : \mathfrak{g} \to \mathscr{P}_n^1$ be a nonnilpotent linear morphism. If ϕ is not flat nor complete, then there exists $1 \leqslant m < n$ such that, after a renumbering of the coordinates, ϕ can be regarded as a flat extension of a complete linear morphism $\tilde{\phi}' : \mathfrak{g} \to \mathscr{P}_m^1$ by a flat linear morphism $\tilde{\phi}'' : \mathfrak{g} \to \mathscr{P}_{n-m}^1$. This will be called the* **natural algebraic splitting** *of ϕ.*

Proof. — Since ϕ is not flat, S has a nontrivial formal first integral; according to proposition 5.3.2, $\widehat{\mathscr{O}}_n^{\,S} = \mathbf{C}[[u_1, \ldots, u_p]]$ for some integer $p \geqslant 1$ and some monomials $u_1, \ldots, u_p$. Moreover, we may reorder the coordinates in such a way that there exists an integer $1 \leqslant m < n$ such that for all $1 \leqslant i \leqslant m$, there is $1 \leqslant j \leqslant p$ with $\partial u_j / \partial x_i \not\equiv 0$ and, for all $1 \leqslant j \leqslant p$, $\partial u_j / \partial x_i \equiv 0$ for all $i > m$; this means that the set of monomials $\{u_1, \ldots, u_p\}$ depends only on the first m coordinates and it depends on all these coordinates. Note that $m < n$ since ϕ is not complete. Thus, there is a monomial u wich belongs to $\widehat{\mathscr{O}}_n^{\,S}$ and which depends on all the m first coordinates; let us write $u = x^R$ with $R \in \mathbf{N}^n$. Let us define

$$\begin{array}{ll} \tilde{\imath}' : \mathbf{C}^m \to \mathbf{C}^n & \tilde{\imath}'' : \mathbf{C}^{n-m} \to \mathbf{C}^n \\ \quad x \mapsto (x, 0) & \quad y \mapsto (0, y) \\ \tilde{p}' : \mathbf{C}^n \to \mathbf{C}^m & \tilde{p}'' : \mathbf{C}^n \to \mathbf{C}^{n-m} \\ \quad (x, y) \mapsto x & \quad (x, y) \mapsto y \end{array}$$

and let us set $\tilde{\phi}' = (\tilde{\imath}')^* \phi$ and $\tilde{\phi}'' = (\tilde{\imath}'')^* \phi$; then $\tilde{\phi}'$ is complete, $\tilde{\phi}''$ is flat and we have $\widehat{\mathscr{O}}_n^{\,S} = \widehat{\mathscr{O}}_m^{\,\tilde{S}'}$ with $\tilde{S}' = (\tilde{\imath}')^* S$. □

6.2. *Diophantine and Poincaré linear morphisms*

Let ϕ be a linear morphism from a nilpotent complex l dimensional Lie algebra $\mathfrak{g}$ to $\mathscr{P}_n^1$. The weights of ϕ in $\widehat{\mathscr{X}}_n^1$ are linear forms on $\mathfrak{g}$ which vanish on $[\mathfrak{g}, \mathfrak{g}]$. Let $Ab(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ be the abelianization of $\mathfrak{g}$ and l' its dimension. Thus, the weights belong to $\mathrm{Hom}_{\mathbf{C}}(Ab(\mathfrak{g}), \mathbf{C})$.

The $\mathbf{C}$-vector space $\mathrm{Hom}_{\mathbf{C}}(Ab(\mathfrak{g}), \mathbf{C})$ has complex dimension l'; it can be turned into an $\mathbf{R}$-vector space $V(Ab(\mathfrak{g}))$ of dimension $2l'$ in a canonical way. On this space, we use the following norm : if $G = \{g_1, \ldots, g_{l'}\}$ denotes a basis of $Ab(\mathfrak{g})$ and if $\alpha \in V(Ab(\mathfrak{g}))$, then we set $\|\alpha\|_G = \max_{1 \leqslant i \leqslant l'} |\alpha(g_i)|$. Let $\mathscr{H}(\phi)$ be the closed convex hull of the linear forms $\lambda_1, \ldots, \lambda_n$ in $V(Ab(\mathfrak{g}))$. They will be called the eigenvalues of ϕ.

Definition **6.2.1.** — *A linear morphism* ϕ *whose convex hull* $\mathscr{H}(\phi)$ *does not contain* 0 *will be called a* **Poincaré morphism.**

Lemma **6.2.2.** — *Let* ϕ *be a Poincaré morphism. Then there exists a real hyperplane* H *of* $V(Ab(\mathfrak{g}))$, *a real line* D *such that if* p *denotes the projection onto* D *relative to* H, *we have* $0 < p(\lambda_1) \leqslant \ldots \leqslant p(\lambda_n)$ *up to a reordering of the coordinates. There are positive constants* c, c' *such that, for any nonzero weight* $\alpha_{Q,i}$ *of* ϕ *in* $\widehat{\mathscr{X}}_n^1$, *we have* $c' \leqslant \|\alpha_{Q,i}\|$ *and* $|Q| \leqslant c\|\alpha_{Q,i}\|$. *Moreover, if* $p(\alpha_{Q,i}) \neq 0$ *then* $c' \leqslant p(\alpha_{Q,i})$ *and* $|Q| \leqslant cp(\alpha_{Q,i})$. *Therefore, a Poincaré morphism is flat.*

Proof. — For any nonzero $Q \in \mathbf{N}^n$ and $1 \leqslant i \leqslant n$, $|Q|p(\lambda_1) - p(\lambda_i) \leqslant p((Q, \lambda) - \lambda_i)$. If $|Q| \geqslant (1 + p(\lambda_i))/p(\lambda_1)$ then, $1 \leqslant p(\alpha_{Q,i}(\phi))$. Since p is linear, there exists $\tilde{c} > 0$ such that $|p(\alpha_{Q,i})(\phi)| < \tilde{c}\|\alpha_{Q,i}(\phi)\|$. On the contrary, if $|Q| < (1 + p(\lambda_i))/p(\lambda_1)$, the $\|\alpha_{Q,i}(\phi)\|$'s and the $|p(\alpha_{Q,i}(\phi))|$'s assume only a finite number of nonnegative real values. Hence, there exists a positive constant c' such that $\|\alpha_{Q,i}\| > c'$ for all nonzero weights of ϕ in $\widehat{\mathscr{X}}_n^1$ and $p(\alpha_{Q,i}) > c'$ if $p(\alpha) \neq 0$.

On the other hand, we have

$$(|Q| - 1)p(\lambda_1) - p(\lambda_n) \leqslant |Q|p(\lambda_1) - p(\lambda_i) \leqslant p((Q, \lambda) - \lambda_i);$$

hence, if $1/2|Q| \geqslant 1 + p(\lambda_n)/p(\lambda_1)$, then $|Q| \leqslant 2p(\alpha_{Q,i}(\phi)\|) \leqslant 2\tilde{c}\|\alpha_{Q,i}(\phi)\|$. If $|Q| < 1 + p(\lambda_n)/p(\lambda_1)$, then, by the first part of the lemma, we have $c' \leqslant \|\alpha_{Q,i}\|$ if $\alpha_{Q,i} \not\equiv 0$ and $c' \leqslant p(\alpha)$ if $p(\alpha) \neq 0$; hence $|Q| \leqslant \dfrac{1 + p(\lambda_n)/p(\lambda_1)}{c'}\|\alpha_{Q,i}\|$ and the same holds for $p(\alpha)$. □

Corollary **6.2.3.** — *Let* ϕ *be a Poincaré morphism and let H be the real hyperplane as above. Let* α *be a weight of* ϕ *which belongs to* H. *Then, the associated weight space is a complex finite-dimensional vector space.*

Definition **6.2.4.** — *Let* $\phi' : \mathfrak{g} \to \mathscr{P}_{n'}^1$ *and* $\phi'' : \mathfrak{g} \to \mathscr{P}_{n-n'}^1$ *be linear morphisms such that* ϕ'' *is a Poincaré morphism. The extension of* ϕ' *by* ϕ'' *will be called a* **Poincaré extension** *if there is a real hyperplane* H *of* $V(Ab(\mathfrak{g}))$ *which contains the eigenvalues of* ϕ' *whereas the eigenvalues of* ϕ'' *remain on the same side of* H. *A Poincaré extension will be called* **proper** *if* H *can be chosen such that* ϕ'' *has no nonzero weight in* H.

Lemma **6.2.5.** — *Let* ϕ *be a linear morphism from* $\mathfrak{g}$ *to* $\mathscr{P}_n^1$. *Then* ϕ *is the Poincaré extension of* $\phi' : \mathfrak{g} \to \mathscr{P}_{n'}^1$ *which is a flat extension of* $\tilde{\phi}'$ *(the natural algebraic splitting of* ϕ*) by a Poincaré morphism* $\phi'' : \mathfrak{g} \to \mathscr{P}_{n-n'}^1$. *The integer* n' *satisfies* $0 \leqslant n' \leqslant n$. *The splitting* (ϕ', ϕ'') *will be called the* **analytic splitting** *of* ϕ *whenever* n' *is minimal. If* $n' \neq n$, *then there exists* *a real hyperplane* H *of* V($Ab(\mathfrak{g})$) *such that* H *contains the* λ'*'s while the* λ''*'s all lie on one side of* H *(*H *not included).*

Remark **6.2.6.** — *The algebraic splitting* $(\widetilde{\phi}', \widetilde{\phi}'')$ *of* ϕ *is a priori different from its analytic splitting* (ϕ', ϕ'') *in the sense that there is no reason why* $\widetilde{\phi}''$ *should be a Poincaré morphism. We only know that a Poincaré morphism is flat but not the converse.*

Proof. — There are three cases to be considered:

- $0 \notin \mathcal{H}(\mathrm{S})$,
- $0 \in Int(\mathcal{H}(\mathrm{S}))$,
- $0 \in \partial(\mathcal{H}(\mathrm{S}))$.

In the first case, we may set $n' = 0$. In the second case, we may set $n' = n$. In the last case, up to a reordering of the coordinates, there is an hyperplane in V($Ab(\mathfrak{g})$) which contains $\lambda_1, \ldots, \lambda_{n'}$, whereas $\lambda_{n'+1}, \ldots, \lambda_n$ all belong to the same side of the hyperplane. Let us set

$$\begin{array}{ll} i' : \mathbf{C}^{n'} \to \mathbf{C}^n & i'' : \mathbf{C}^{n-n'} \to \mathbf{C}^n \\ \quad x \mapsto (x, 0) & \quad y \mapsto (0, y) \\ p' : \mathbf{C}^n \to \mathbf{C}^{n'} & p'' : \mathbf{C}^n \to \mathbf{C}^{n-n'} \\ \quad (x, y) \mapsto x & \quad (x, y) \mapsto y \end{array}$$

as well as $\phi' = (i')^*\phi$ and $\phi'' = (i'')^*\phi$. Then ϕ' is a flat extension of $\widetilde{\phi}'$ and ϕ'' is a Poincaré morphism. Of course, we have $n' \geqslant m$. □

Lemma **6.2.7.** — *Let* (ϕ', ϕ'') *with* $\phi' : \mathfrak{g} \to \mathcal{P}^1_{n'}$ *and* $\phi'' : \mathfrak{g} \to \mathcal{P}^1_{n-n'}$ *be the analytic splitting of* ϕ. *Let us assume that* $n' < n$ *and let* H *be the real hyperplane of* V($Ab(\mathfrak{g})$) *with properties of the previous lemma. Then there exists a constant* $Sep(\phi) > 0$ *such that if* α *is a weight of* ϕ *in* $\widehat{\mathcal{X}}^1_n$ *such that* $\|\alpha\| < Sep(\phi)$ *then it belongs to* H. *The constant* $Sep(\phi)$ *will be called a* **separating constant**. *Moreover, there exists* $d > 0$ *such that if* $\alpha = \alpha_{\mathrm{Q},i}$ *and* $\|\alpha\| \geqslant Sep(\phi)$ *then, in fact we have* $|p''(\mathrm{Q})| \leqslant d\|\alpha\|$.

Remark **6.2.8. If** $n' = n$, **we shall set** $Sep(\phi) = +\infty$ **and** $\mathrm{H} = \mathrm{V}(Ab(\mathfrak{g}))$ *(which is not an hyperplane!). This will enable us to do only one proof.*

Proof. — According to our definitions and our assumptions, there exists a real hyperplane H of V($Ab(\mathfrak{g})$) such that the linear forms $\lambda_1, \ldots, \lambda_{n'}$ belong to H whereas $\lambda_{n'+1}, \ldots, \lambda_n$ all belong to the same side of H. As in lemma 6.2.2, let D be a transversal real 1-dimensional space transverse to H and let p be the projection onto D relative to H. We may assume that $0 < p(\lambda_{n'+1}) \leqslant \ldots \leqslant p(\lambda_n)$.

Since p is continuous, for any weight α of ϕ we have $|p(\alpha)| \leqslant c\|\alpha\|$ for some positive constant c. Moreover, ϕ'' is a Poincaré morphism and any weight α of ϕ is the sum of a weight α' of ϕ' and a weight α'' of ϕ''. In fact, by the formulas of section 6.1, α is the sum of α' and β'' or β' and α''. But, by definition of the β's and α's, we have $\beta_Q = \alpha_{Q+E_i}$. Thus, the result holds. According to lemma 6.2.2, if $p(\alpha'') \neq 0$ then $c'/c \leqslant \|\alpha\|$ since $c' \leqslant |p(\alpha)| = |p(\alpha'')| \leqslant c\|\alpha\|$. Thus if $\|\alpha\| < c'/c$ then $p(\alpha) = 0$. According to lemma 6.2.2, if $p(\alpha) \neq 0$, there is $a > 0$ such that $|Q''| \leqslant a|p(\alpha'')| = a|p(\alpha)| \leqslant ac\|\alpha\|$. □

We recall that $\|\alpha\|_G = \max_{1\leqslant i\leqslant \ell} |\alpha(g_i)|$

Next, we define for $k \geqslant 2$,

$$\omega_{k,G}(\phi) = \inf\{\|\alpha\|_G, \alpha \in \mathscr{W}^{2,2^k}_{v,n}(\phi) \setminus \{0\}\}.$$

Definition **6.2.9.** — *We shall say that the linear morphism ϕ is* **diophantine** *if*

$$-\sum_{k\geqslant 0} \frac{\ln \omega_{k,G}(\phi)}{2^k} < +\infty. \tag{6.2.1}$$

Remark **6.2.10.** — *This condition does neither depend on the choice of the basis nor on the chosen norm due to the fact that, in finite dimensional vector spaces, all norms are equivalent.*

Remark **6.2.11.** — *We recall that if* $S = \sum_{i=1}^n \lambda_i x_i \partial/\partial x_i$ *belongs to* $\phi(\mathfrak{g})$, *then Bruno's condition* (ω) *associated to it is defined by:*

$$(\omega) \quad -\sum_{k\geqslant 0} \frac{\ln \omega_k}{2^k} < +\infty$$

where $\omega_k = \inf\{|(Q, \lambda) - \lambda_i| \neq 0,\ 1 \leqslant i \leqslant n,\ Q \in \mathbf{N}^n, 2 \leqslant |Q| \leqslant 2^k\}$. *It should be noticed that such an* S *may* **not** *satisfy Bruno's condition while* ϕ *is diophantine.*

Definition **6.2.12.** — *We will say that the morphism* **doesn't have small denominators** *if the sequence* $\{\omega_{k,G}(\phi)\}$ *is bounded from below by a positive constant.*

6.3. *Formal complete integrability of a nonlinear deformation*

Let $X \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \widehat{\mathscr{X}}^1_n\right)$ be a morphism from $\mathfrak{g}$ to the Lie algebra of formal vector fields vanishing at the origin. This morphism defines a representation $g \mapsto [X(g), .]$ of $\mathfrak{g}$ in $\widehat{\mathscr{X}}^1_n$. This allows us to define the associated Chevalley-Koszul complex as well as the cohomology spaces. The proof of the following proposition is an adaptation of a result of Walcher ([Wal91] [prop. 1.8]):

Proposition **6.3.1.** — *Let* $\mathfrak{g}$ *be a nilpotent finite dimensional Lie algebra over* $\mathbf{C}$. *Let* $\phi : \mathfrak{g} \to \mathscr{P}^1_n$ *be a nonnilpotent Lie morphism. Let* $\phi + \varepsilon \in \mathrm{Compat}\left(\mathfrak{g}, \phi, \widehat{\mathscr{X}}^2_n\right)$ *be a compatible* *deformation of* ϕ *of order* $m \geqslant 2$ *and* $\hat{\Phi}$ *its formal normalizing transformation. Then, the map* $\hat{\Phi}^* : \widehat{\mathcal{O}}_n \to \widehat{\mathcal{O}}_n$ *which maps* f *to* $f \circ \hat{\Phi}$ *defines an injection* $H^0_{\phi+\varepsilon}\left(\mathfrak{g}, \widehat{\mathcal{O}}_n\right) \hookrightarrow H^0_S\left(\mathfrak{g}, \widehat{\mathcal{O}}_n\right)$ *as well as an injection* $H^0_{\phi+\varepsilon}\left(\mathfrak{g}, \widehat{\mathcal{X}}^1_n\right) \hookrightarrow H^0_S\left(\mathfrak{g}, \widehat{\mathcal{X}}^1_n\right)$. *In other words, each first integral of* $\hat{\Phi}^*(\phi + \varepsilon)$ *is a first integral of* S; *each formal vector field commuting with* $\hat{\Phi}^*(\phi + \varepsilon)$ *commutes with* S.

Proof. — Let us write $\hat{\Phi}^*(\phi+\varepsilon) =: \phi+\eta = \phi+\eta_2+\eta_3+\cdots$ where $\eta_i \in \mathrm{Hom}_{\mathbb{C}}\left(\mathfrak{g}, \mathcal{P}^i_n\right)$. Let f be a first integral of $\hat{\Phi}^*(\phi + \varepsilon)$, that is, for all $g \in \mathfrak{g}$, $\mathcal{L}_{\phi+\eta(g)}(f) = 0$. We may assume that f is not constant and $f(0) = 0$. Let us write $f = f_r + f_{r+1} + \cdots$ where $f_i \in p^i_n$ and let us decompose the above equation along the space of homogeneous polynomials of degree $r+j \geqslant r$. This leads to $\mathcal{L}_{\phi(g)}(f_r) = 0$ and, for $j \geqslant 1$, to

$$\mathcal{L}_{\phi(g)}(f_{r+j}) + \mathcal{L}_{\eta_2(g)}(f_{r+j-1}) + \cdots + \mathcal{L}_{\eta_{j+1}(g)}(f_r) = 0.$$

Let us show by induction on j that $\mathcal{L}_{S(g)}(f_{r+j}) = 0$ for all $g \in \mathfrak{g}$. This is true for $j = 0$ since $f_r \in \left(p^r_n\right)^{\phi} \subset \left(p^r_n\right)^0(\phi) = \left(p^r_n\right)^S$. So, we may assume that it is so for $k < j$. Let us apply the operator $\mathcal{L}_{S(g)}$ to the above equation:

$$\mathcal{L}_{S(g)}\mathcal{L}_{\phi(g)}(f_{r+j}) + \mathcal{L}_{S(g)}\mathcal{L}_{\eta_2(g)}(f_{r+j-1}) + \cdots + \mathcal{L}_{S(g)}\mathcal{L}_{\eta_{j+1}(g)}(f_r) = 0.$$

But, according to remark (5.2.2), for all $(g_1, g_2) \in \mathfrak{g}^2$, $[S(g_1), \eta(g_2)] = 0$, so that $\mathcal{L}_{S(g)}\mathcal{L}_{\eta_2(g)} = \mathcal{L}_{\eta_2(g)}\mathcal{L}_{S(g)}$. By induction, we have $\mathcal{L}_{S(g)}(f_{r+k}) = 0$ if $k < j$. Thus, we obtain $\mathcal{L}_{S(g)}\mathcal{L}_{\phi(g)}(f_{r+j}) = 0$. As we have seen, $S(g)$ commutes with $\phi(g)$, so that $\mathcal{L}_{\phi(g)}\mathcal{L}_{S(g)}(f_{r+j}) = 0$. But, $\mathcal{L}_{\phi(g)}$ is invertible on the image of $\mathcal{L}_{S(g)}$ (as endomorphism of p^{r+j}_n). If follows that $\mathcal{L}_{S(g)}(f_{r+j}) = 0$ and we are done. The same proof holds in the case of vector fields. □

This motivates the following definition:

Definition **6.3.2.** — *A compatible nonlinear deformation* $\phi + \varepsilon \in \mathrm{Compat}(\mathfrak{g}, \phi, \widehat{\mathcal{X}}^2_n)$ *of a nonnilpotent morphism* ϕ *will be called* **formally integrable** *if the injection* $H^0_{\phi+\varepsilon}(\mathfrak{g}, \widehat{\mathcal{O}}_n) \hookrightarrow H^0_S(\mathfrak{g}, \widehat{\mathcal{O}}_n)$ *is an isomorphism.*

In other words, a compatible nonlinear deformation of ϕ is formally integrable if its normal form has the same formal first integrals as S.

As we have seen, we have $\hat{\Phi}^*(\phi + \varepsilon) \in \mathrm{Hom}_{\mathbb{C}}\left(\mathfrak{g}, \left(\widehat{\mathcal{X}}^1_n\right)^S\right)$. This leads us to define the following space:

$$\widehat{\mathcal{F}}_n = \left\{X \in \left(\widehat{\mathcal{X}}^1_n\right)^S \mid \mathcal{L}_X\left(\widehat{\mathcal{O}}^S_n\right) = 0\right\}.$$

Lemma **6.3.3.** — *$\left(\mathcal{X}_n^1\right)^{\mathrm{S}}$ is a Lie algebra over* $\mathbf{C}$ *and the $\widehat{\mathcal{O}}_n^{\mathrm{S}}$-submodule $\widehat{\mathcal{F}}_n$ of $\left(\mathcal{X}_n^1\right)^{\mathrm{S}}$ has a Lie algebra structure over $\widehat{\mathcal{O}}_n^{\mathrm{S}}$ (induced by its natural Lie algebra structure) and it is a maximal Lie subalgebra over $\widehat{\mathcal{O}}_n^{\mathrm{S}}$ of $\left(\mathcal{X}_n^1\right)^{\mathrm{S}}$.*

Proof. — By Jacobi identity, we have, for all $g \in \mathfrak{g}$,

$$[\mathrm{S}(g), [\mathrm{X}, \mathrm{Y}]] = -[\mathrm{X}, [\mathrm{Y}, \mathrm{S}(g)]] - [\mathrm{Y}, [\mathrm{S}(g), \mathrm{X}]].$$

Thus, if $\mathrm{X}, \mathrm{Y} \in \left(\mathcal{X}_n^1\right)^{\mathrm{S}}$, then $[\mathrm{S}(g), \mathrm{X}] = [\mathrm{S}(g), \mathrm{Y}] = 0$ so that $[\mathrm{S}(g), [\mathrm{X}, \mathrm{Y}]] = 0$; that is $[\mathrm{X}, \mathrm{Y}] \in \left(\mathcal{X}_n^1\right)^{\mathrm{S}}$.

The $\mathbf{C}$-space $\widehat{\mathcal{F}}_n$ is clearly an $\widehat{\mathcal{O}}_n^{\mathrm{S}}$-submodule of the $\widehat{\mathcal{O}}_n^{\mathrm{S}}$-module $\left(\mathcal{X}_n^1\right)^{\mathrm{S}}$. Moreover, if $(\mathrm{X}, \mathrm{Y}) \in \widehat{\mathcal{F}}_n$ and $f \in \widehat{\mathcal{O}}_n^{\mathrm{S}}$, then

$$[f\mathrm{X}, \mathrm{Y}] = f[\mathrm{X}, \mathrm{Y}] + \mathcal{L}_{\mathrm{Y}}(f)\mathrm{X} = f[\mathrm{X}, \mathrm{Y}] = f[\mathrm{X}, \mathrm{Y}] - \mathcal{L}_{\mathrm{X}}(f)\mathrm{Y} = [\mathrm{X}, f\mathrm{Y}]$$

and $[\mathrm{X}, \mathrm{Y}](f) = \mathrm{X}(\mathrm{Y}(f)) - \mathrm{Y}(\mathrm{X}(f)) = 0$. The Jacobi identity follows from the Lie algebra structure of $\left(\mathcal{X}_n^1\right)^{\mathrm{S}}$; thus $\widehat{\mathcal{F}}_n$ has a Lie algebra structure over $\widehat{\mathcal{O}}_n^{\mathrm{S}}$.

Let us assume that there exists an $\widehat{\mathcal{O}}_n^{\mathrm{S}}$-submodule $\widehat{\mathrm{V}}$ of $\left(\mathcal{X}_n^1\right)^{\mathrm{S}}$, which is a Lie algebra over $\widehat{\mathcal{O}}_n^{\mathrm{S}}$ and which contains strictly $\widehat{\mathcal{F}}_n$. Then there exists $\mathrm{Z} \in \left(\mathcal{X}_n^1\right)^{\mathrm{S}}$ such that $\mathrm{Z} \notin \widehat{\mathcal{F}}_n$. For all $\mathrm{X} \in \widehat{\mathcal{F}}_n$ and any $f \in \widehat{\mathcal{O}}_n^{\mathrm{S}}$, we have, in the one hand,

$$[f\mathrm{Z}, \mathrm{X}] = f[\mathrm{Z}, \mathrm{X}] + \mathcal{L}_{\mathrm{X}}(f)\mathrm{Z} = f[\mathrm{Z}, \mathrm{X}];$$

on the other hand, we have

$$[f\mathrm{Z}, \mathrm{X}] = [\mathrm{Z}, f\mathrm{X}] = f[\mathrm{Z}, \mathrm{X}] - \mathcal{L}_{\mathrm{Z}}(f)\mathrm{X};$$

thus, we have $\mathcal{L}_{\mathrm{Z}}(f) = 0$ that is, $\mathrm{Z} \in \widehat{\mathcal{F}}_n$. This is a contradiction. □

Let ϕ be a nonnilpotent linear morphism. Let (ϕ', ϕ'') be its analytic splitting; $\phi' : \mathfrak{g} \to \mathcal{P}_{n'}^1$, $\phi'' : \mathfrak{g} \to \mathcal{P}_{n-n'}^1$. If X is a formal vector field on $\mathbf{C}^n$, we shall write X′ (resp. X″) for the projection of X onto $\{\partial/\partial x_1, \dots, \partial/\partial x_{n'}\}$ (resp. $\{\partial/\partial x_{n'+1}, \dots, \partial/\partial x_n\}$). The morphism S′ is assumed to be **injective**. As a consequence, $\mathfrak{g}$ is **commutative** and its dimension l is lower than or equal to $n' - s$.

Definition **6.3.4.** — *Let* S *be an injective diagonal linear morphism from a commutative Lie algebra* $\mathfrak{g}$ *to* $\mathcal{P}_n^1$. *The* **isoresonant hull** *IsoRes*(S) *of* S *is the largest Lie subalgebra of the Lie algebra of diagonal linear vector fields of* $\mathbf{C}^n$ *which has the same invariants as* S. *More precisely,* *let $\tilde{g}$ be the largest commutative Lie algebra for which there is an injective diagonal Lie morphism $\tilde{S} : \tilde{\mathfrak{g}} \to \mathscr{P}_n^1$ as well as an injection i: $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ such that*

- $\tilde{S} \circ i = S$,
- $\widehat{\mathscr{O}}_n^{\tilde{S}} = \widehat{\mathscr{O}}_n^{S}$,
- $\left(\widehat{\mathscr{X}}_n^1\right)^{\tilde{S}} = \left(\widehat{\mathscr{X}}_n^1\right)^{S}$.

Then, we set $IsoRes(S) = \tilde{S}(\tilde{\mathfrak{g}})$.

Definition **6.3.5.** — *Let S be an injective diagonal linear morphism from a commutative Lie algebra $\mathfrak{g}$ to $\mathscr{P}_n^1$. A* **diophantine hull** *of S is a Lie subalgebra of the isoresonant hull $IsoRes(S)$ of S defined to be the image of an injective diagonal Lie morphism $\overline{S} : \overline{\mathfrak{g}} \to \mathscr{P}_n^1$ from a (commutative) Lie algebra $\bar{g}$ as well as injections k: $\mathfrak{g} \hookrightarrow \overline{\mathfrak{g}}$, j: $\overline{\mathfrak{g}} \hookrightarrow \tilde{\mathfrak{g}}$ such that*

- $i = j \circ k$
- $\overline{S} \circ k = S$
- *there is a constant $c > 0$ such that for any weight α of $\overline{S}$ in $\widehat{\mathscr{X}}_n^1$, we have $\|\alpha\| \leqslant c\|\alpha \circ k\|$.*

As an exemple, let us assume that $S_i = \sum_{j=1}^n \lambda_{i,j} x_j \partial/\partial x_j$, $1 \leqslant i \leqslant l$, is a basis of $S(\mathfrak{g})$. Let us set $T_i = \sum_{j=1}^l \bar{\lambda}_{i,j} x_j \partial/\partial x_j$, $1 \leqslant i \leqslant l$ where $\bar{\lambda}_{i,j}$ denotes the complex conjugate of $\lambda_{i,j}$. Let $0 \leqslant k \leqslant l$ such that $\{S_1, ..., S_l, T_1, ..., T_k\}$ is a family of linearly independent vector fields. If $k > 0$, then the vector space generated by $S_1, ..., S_l, T_1, ..., T_k$ is a diophantine hull of S. In fact, let us define $\mathfrak{h} = \mathfrak{g} \oplus \mathbf{C}^k$ and let us set $\{e_1, ..., e_k\}$ the canonical basis of $\mathbf{C}^k$; let $i : \mathfrak{g} \to \mathfrak{h}$ be the injection. Now, let $\mathscr{S} : \mathfrak{h} \to \mathscr{P}_n^1$ be the Lie morphism defined by $\mathscr{S}(g_i) = S_i$ and $\mathscr{S}(e_k) = T_k$. An easy computation shows that if α is a weight of $\mathscr{S}$ then $\alpha(e_k) = \alpha(\bar{g}_k)$. Thus α is zero if and only if the weight $\alpha \circ i$ of S is zero. As a consequence, $\mathscr{S}(\mathfrak{h})$ is a Lie subalgebra of $Isores(S)$ and contains $S(\mathfrak{g})$. As the computation on the weight α shows, we have $\|\alpha\| = \|\alpha \circ i\|$. Therefore, $\mathscr{S}(\mathfrak{h})$ is a diophantine hull of S.

Remark **6.3.6.** — *Actually we don't know whether the diophantine hulls of S contain other vector fields than those which are linear combinations of $S_1, ..., S_l, T_1, ..., T_l$.*

Remark **6.3.7.** — *As we shall see in the definition of the complete integrability, the notion of a diophantine hull for a Poincaré morphism is irrelevant. Therefore, we shall define a diophantine hull of S as S itself. This will enable us to define the notion of complete integrability without considering too many cases.*

Definition **6.3.8.** — *Let S be a Poincaré extension of S′ by S″ and let H be a real hyperplane of $\mathfrak{g}^*$ which contains the "eigenvalues" of S′ whereas the eigenvalues of S″ remain on one* *side of* H. *Let* h *denotes the set of weights of* S″ *which belong to* H. *It is a finite set and*

$$\left(\widehat{\mathscr{X}}^{1}_{n-n'}\right)_{h}(\mathrm{S}''):=\oplus_{\alpha''\in\mathrm{H}}\left(\widehat{\mathscr{X}}^{1}_{n-n'}\right)_{\alpha''}(\mathrm{S}'')$$

is a finite dimensional $\mathbf{C}$*-vector space. Let* $\mathscr{L}_h(\mathrm{S}'')$ *be the* $\mathbf{C}$*-subspace of the space of diagonal linear vector fields of* $\mathbf{C}^{n-n'}$ *whose elements commute with* $\left(\widehat{\mathscr{X}}^{1}_{n-n'}\right)_{h}(\mathrm{S}'')$.

Remark **6.3.9.** — *Since the elements of* $\mathscr{L}_h(\mathrm{S}'')$ *are linear diagonal vector fields on* $\mathbf{C}^{n-n'}$, $\mathscr{L}_h(\mathrm{S}'')$ *commutes with* S″, *so that it belongs to* $\left(\widehat{\mathscr{X}}^{1}_{n-n'}\right)_{h}(\mathrm{S}'')$.

Definition **6.3.10.** — *A* **good deformation** *of* $0\in\mathbf{C}^{n-n'}$ *relative to the analytic splitting of* $\mathrm{S}=\mathrm{S}'\oplus\mathrm{S}''$ *is a morphism*

$$\mathrm{D}''+\mathcal{N}il''+\mathrm{R}'':\mathfrak{g}\to\left(\widehat{\mathscr{O}}_{n'}\otimes_{\mathbf{C}} i''_{*}\left(\widehat{\mathscr{X}}^{1}_{n-n'}\right)_{h}(\mathrm{S}'')\right)\cap\left(\widehat{\mathscr{X}}^{1}_{n}\right)^{\mathrm{S}},$$

such that

$$\mathrm{D}''\in\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g},\left(\widehat{\mathscr{O}}^{\mathrm{S}'}_{n'}\otimes_{\mathbf{C}} i''_{*}\mathscr{L}_h(\mathrm{S}'')\right)\cap\left(\widehat{\mathscr{X}}^{1}_{n}\right)^{\mathrm{S}}\right)$$

is a diagonal deformation of 0,

$$\mathcal{N}il''\in\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g},\widehat{\mathscr{O}}^{\mathrm{S}'}_{n'}\otimes_{\mathbf{C}} i''_{*}\left(\widehat{\mathscr{P}}^{1}_{n-n'}\right)^{\mathrm{S}''}\right)$$

is a nilpotent deformation of 0 *and*

$$\mathrm{R}''\in\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g},\left(\widehat{\mathscr{O}}_{n'}\otimes_{\mathbf{C}} i''_{*}\left(\widehat{\mathscr{X}}^{2}_{n-n'}\right)\right)\cap\left(\widehat{\mathscr{X}}^{1}_{n}\right)^{\mathrm{S}}\right).$$

In these conditions, we have $[\mathrm{D}'',\mathcal{N}il''+\mathrm{R}'']=0$.

Remark **6.3.11.** — *If* S *is a* **proper Poincaré extension** *then* $\mathscr{L}_h(\mathrm{S}'')$ *is reduced to the linear vector fields of* $\mathbf{C}^{n-n'}$ *which commute with the centralizer* $\left(\widehat{\mathscr{X}}^{1}_{n-n'}\right)^{\mathrm{S}''}$ *of* S″. *This is due to the fact that the only weight of* S″ *which belongs to the hyperplane* H *is the zero weight. Therefore, if* $\alpha=\alpha'+\alpha''$ *is a zero weight of* S *then both* α' *and* α'' *must be zero so that*

$$\left(\widehat{\mathscr{X}}^{1}_{n}\right)^{\mathrm{S}}=\left(\widehat{\mathscr{X}}^{1}_{n'}\right)^{\mathrm{S}'}\oplus\widehat{\mathscr{O}}^{\mathrm{S}'}_{n'}\otimes_{\mathbf{C}} i''_{*}\left(\widehat{\mathscr{X}}^{1}_{n-n'}\right)^{\mathrm{S}''}.$$

It follows that, in this case, a good deformation is a **trivial deformation** *over* $\widehat{\mathscr{O}}^{\mathrm{S}'}_{n'}$ *that is,* $\mathrm{D}''+\mathcal{N}il''+\mathrm{R}'':\mathfrak{g}\to\widehat{\mathscr{O}}^{\mathrm{S}'}_{n'}\otimes_{\mathbf{C}} i''_{*}\left(\widehat{\mathscr{X}}^{1}_{n-n'}\right)^{\mathrm{S}''}$ *where* $\mathrm{D}''\in\mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g},\widehat{\mathscr{O}}^{\mathrm{S}'}_{n'}\otimes_{\mathbf{C}} i''_{*}\mathit{IsoRes}(\mathrm{S}'')\right)$.

Since we have $[D'', Nil'' + R''] = 0$, *we can view* $D'' + Nil'' + R''$ *as a normal form (relative to* D''*) with coefficients in* $\widehat{\mathcal{O}}_{n'}^{S'}$.

Definition **6.3.12.** — *Let* $\mathfrak{g}$ *be a* **commutative** *Lie algebra over* $\mathbf{C}$ *of dimension* l. *Let* S *be a semi-simple linear morphism from* $\mathfrak{g}$ *to* $\mathcal{P}_n^1$. *Let* $(S' : \mathfrak{g} \to \mathcal{P}_{n'}^1, S'' : \mathfrak{g} \to \mathcal{P}_{n-n'}^1)$ *be its analytic splitting.*

1. *When* $n' = 0$ *or when* $n' > 0$ *and* S *doesn't have small divisors, then a compatible nonlinear deformation* $S + \varepsilon$ *of* S *is said to be* **formally completely integrable** *if its formal normal form is the sum of a nonlinear deformation of* S *in* $\widehat{\mathcal{O}}_{n'}^{S'} \otimes_{\mathbf{C}} Dioph(S)$, *where* $Dioph(S)$ *denotes a diophantine hull of* S, *and a nonlinear deformation of* $0 \in \mathbf{C}^{n-n'}$ *relative to the analytic splitting of* S.
2. *When* $n' > 0$ *and* S *is diophantine, then a compatible nonlinear deformation* $S + \varepsilon$ *of* S *is said to be* **formally completely integrable** *if its normal form is the sum of a nonlinear deformation of* S *in* $\widehat{\mathcal{O}}_{n'}^{S'} \otimes_{\mathbf{C}} Dioph(S)$, *where* $Dioph(S)$ *denotes a diophantine hull of* S, *and a good deformation of* $0 \in \mathbf{C}^{n-n'}$ *relative to the analytic splitting of* S.

Remark **6.3.13.**

- *If* $n' = 0$, *then* $\mathcal{O}_{n'}^{S'} = \mathbf{C}$. *Therefore, a nonlinear deformation of* S *in* $\widehat{\mathcal{O}}_{n'}^{S'} \otimes_{\mathbf{C}} Dioph(S)$ *is reduced to* S *itself.*
- *When* $n' \neq 0$, *the projection on* $\mathbf{C}^{n'}$ *of a diophantine hull of* S *is a diophantine hull of* S'. *In fact, let* $\overline{S} : \overline{\mathfrak{g}} \to \mathcal{P}_n^1$ *be a diophantine hull of* S. *Let* $X \in \left(\widehat{\mathcal{X}}_{n'}^1\right)^{S'}$, *then we have, for all* $g \in \mathfrak{g}$, $[S(g), X] = [S'(g), X] = 0$. *Therefore, for all* $\bar{g} \in \overline{\mathfrak{g}}$, $0 = [\overline{S}(\bar{g}), X] = [(\overline{S}(\bar{g}))', X]$. *It follows that* $(\overline{S}(\overline{\mathfrak{g}}))'$ *belongs to the isoresonant hull of* S'. *Moreover, let* $X \in \widehat{\mathcal{X}}_{n'}^1$ *belong to the* α*-weight space of* $\overline{S}$. *Then, for all* $\bar{g} \in \overline{\mathfrak{g}}$, $[\overline{S}(\bar{g}), X] = \alpha(\bar{g})X = [\overline{S}(\bar{g})', X]$. *Thus* α *is a weight of* $(\overline{S})'$. *The converse is also true. It follows that for any weight* α *of* $(\overline{S})'$, *we have* $\|\alpha\| \leqslant c\|\alpha \circ i\|$. *Thus there is a subalgebra* $\tilde{\mathfrak{g}}$ *of* $\overline{\mathfrak{g}}$ *such that the restriction of* $(\overline{S})'$ *to* $\tilde{\mathfrak{g}}$ *is a diophantine hull of* S'.

7. Newton cohomology with bounds

Let $\mathfrak{g}$ be a **nilpotent** complex Lie algebra of dimension l. Let ϕ be a nonnilpotent morphism from $\mathfrak{g}$ to $\mathcal{P}_n^1$. After a linear change of coordinates, we may assume that $\phi = S + N$ where, for any $g \in \mathfrak{g}$, $S(g)$ is a diagonal vector field and $N(g)$ is a nilpotent one commuting with $S(g)$.

Let $m \geqslant 1$ be an integer and let NF^m be the m-jet of the formal normal form NF of a compatible non-linear deformation $\phi + \varepsilon$ of ϕ.

In order to handle the Newtonian normalization process, we have to use a natural representation of $\mathfrak{g}$ into $\mathcal{P}_n^{m+1, 2m}$ associated to the m-jet of the normal form. We shall show, in this section, that the linear map $g \mapsto J^{2m}([NF^m(g), .])$ defines a representation of $\mathfrak{g}$ into $\mathscr{P}_n^{m+1,2m}$ which leaves invariant the weight subspaces of ϕ. We shall call it a *Newtonian representation* of order m. Let α be a nonzero weight of ϕ into $\mathscr{P}_n^{m+1,2m}$, then the 0-th and the 1-st cohomology spaces of the associated the Chevalley-Koszul complex of the $\mathfrak{g}$-module $\mathscr{P}_{n,\alpha}^{m+1,2m}$ vanish. We shall provide the spaces of this complex with norms which make it a continuous complex of normed spaces.

If $\alpha \neq 0$, then for any 1-cocycle z_α, there exists a unique u_α such that $d^0 u_\alpha = z_\alpha$. This defines a map s_α inverse of d^0. We shall show that s_α is continuous, and we shall provide a bound for its norm.

7.1. *The Newton complex*

Let us make a few remarks which will be of constant use:

1. For all $(g_1, g_2) \in \mathfrak{g}^2$, we have $[S(g_1), NF^m(g_2)] = 0$; this follows from (5.2.2).
2. Since NF is a Lie morphism, we have $[NF(g_1), NF(g_2)] = NF([g_1, g_2])$. Taking its m-jet at 0 leads to $J^m([NF^m(g_1), NF^m(g_2)]) = NF^m([g_1, g_2])$.

Let us define the $\mathbf{C}$-linear map $\rho_{N,m} : \mathfrak{g} \to End_{\mathbf{C}}\left(\mathscr{P}_n^{m+1,2m}\right)$ by

$$\rho_{N,m}(g)(X) = J^{2m}([NF^m(g), X])$$

for all $g \in \mathfrak{g}$ and $X \in \mathscr{P}_n^{m+1,2m}$. This map is a representation : for all $(g_1, g_2) \in \mathfrak{g}^2$, $\rho_{N,m}([g_1, g_2]) = \rho_{N,m}(g_1)\rho_{N,m}(g_2) - \rho_{N,m}(g_2)\rho_{N,m}(g_1)$. Indeed, for all $X \in \mathscr{P}_n^{m+1,2m}$, we have

$$\begin{aligned}\rho_{N,m}(g_1)\rho_{N,m}(g_2)(X) &= J^{2m}([NF^m(g_1), J^{2m}([NF^m(g_2), X])]) \\ &= J^{2m}\left([NF^m(g_1), [NF^m(g_2), X]]\right) \quad \text{(Jacobi identity)}, \\ &= J^{2m}\left([NF^m(g_2), [NF^m(g_1), X]] - [X, [NF^m(g_1), NF^m(g_2)]]\right).\end{aligned}$$

According to the remark above, we have

$$J^{2m}([X, [NF^m(g_1), NF^m(g_2)]]) = J^{2m}([X, NF^m([g_1, g_2])]),$$

thus

$$\rho_{N,m}(g_1)\rho_{N,m}(g_2)(X) = \rho_{N,m}(g_2)\rho_{N,m}(g_1)(X) + \rho_{N,m}([g_1, g_2])(X).$$

It follows that $\rho_{N,m}$ provides $\mathscr{P}_n^{m+1,2m}$ with a structure of $\mathfrak{g}$-module. We shall say that $\rho_{N,m}$ is a *Newtonian representation of order m.*

To this representation is associated the Chevalley-Koszul complex (see (4.1.1)):

$$\textbf{(7.1.1)} \quad 0 \to \mathscr{P}_n^{m+1,2m} \xrightarrow{d_0} \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{P}_n^{m+1,2m}\right) \xrightarrow{d_1} \cdots \xrightarrow{d_{l-1}} \mathrm{Hom}_{\mathbf{C}}\left(\wedge^l \mathfrak{g}, \mathscr{P}_n^{m+1,2m}\right) \to 0.$$

We shall call this complex the **Newton complex** of ϕ of order m. We shall denote by $H^i_{N,m}(\mathfrak{g}, \mathscr{P}^{m+1,2m}_n)$ the i-th cohomology space of this complex. We recall that $H^i_\phi(\mathfrak{g}, \mathscr{P}^{m+1,2m}_n)$ denotes the i-th cohomology space for the representation associated to ϕ, and that $\mathscr{W}^{m+1,2m}_{v,n}$ denotes the set of weights of ϕ into $\mathscr{P}^{m+1,2m}_n$; $\mathscr{P}^{m+1,2m}_{n,\alpha}$ denotes the weight space associated to the weight α. The following proposition is fundamental in our construction. In the first version, its proof depended on a four pages computation. Following an idea of B. Malgrange, this computation can be written easily using a classical spectral sequence setting.

Proposition **7.1.1.** — *Let $\alpha \in \mathscr{W}^{m+1,2m}_{v,n}$ be a weight of ϕ. Then, $\rho_{N,m}$ provides $\mathscr{P}^{m+1,2m}_{n,\alpha}(S)$ with a structure of $\mathfrak{g}$-module. Moreover, if α is a nonzero weight, then $H^i_{N,m}\left(\mathfrak{g}, \mathscr{P}^{m+1,2m}_{n,\alpha}(S)\right) = 0$ for $i=0, 1$.*

Proof.

• For the first point, it is sufficient to show that $\mathscr{P}^{m+1,2m}_{n,\alpha}(S)$ is invariant by the Newtonian representation $\rho_{N,m}$. Let $m+1 \leqslant k \leqslant 2m$ be an integer. For all $(g_1, g_2) \in \mathfrak{g}^2$ and all $X \in \mathscr{P}^{m+1,2m}_{n,\alpha}(S)$, we have:

$$\begin{aligned}[S(g_1), J^k([NF^m(g_2), X])] &= J^k([S(g_1), [NF^m(g_2), X]]) \\ &= J^k\left([NF^m(g_2), [S(g_1), X]]) + J^k([X, [NF^m(g_2), S(g_1)]]\right) \\ &= J^k([NF^m(g_2), [S(g_1), X]]) \text{ (by the remark above)}\end{aligned}$$

Therefore, we have

$$[S(g_1), J^k([NF^m(g_2), X])] = \alpha(g_1) J^k([NF^m(g_2), X]).$$

It follows that, for any weight $\alpha \in \mathscr{W}^{m+1,2m}_{v,n}$ of ϕ, the weight space $\mathscr{P}^{m+1,2m}_{n,\alpha}(S)$ is a $\mathfrak{g}$-submodule of $\mathscr{P}^{m+1,2m}_n$ relative to the Newtonian representation.

• Let α be a nonzero weight of ϕ. Let us show that $H^i_{N,m}\left(\mathfrak{g}, \mathscr{P}^{m+1,2m}_{n,\alpha}(S)\right) = 0$, for $i=0, 1$. Let us set, for $0 \leqslant j \leqslant l$, $K^j = \mathrm{Hom}_{\mathbf{C}}\left(\wedge^j \mathfrak{g}, \mathscr{P}^{m+1,2m}_{n,\alpha}(S)\right)$. We set $K^j = 0$ if $j<0$ or $j>l$. We may consider the Newton complex as a differential graded module $K = \oplus_{j=1}^l K^j$. This module is filtered by $F^i K = \oplus_{j=1}^l \mathrm{Hom}_{\mathbf{C}}\left(\wedge^j \mathfrak{g}, \mathscr{P}^{m+1+i,2m}_{n,\alpha}(S)\right)$. Using the convention that $\mathscr{P}^{k,k'}_n = 0$ if $k > k'$, we have:

$$\{0\} = F^m K^j \subset F^{m-1} K^j \subset \cdots \subset F^0 K^j = K^j.$$

Moreover, we have $d(F^i K) \subset F^i K$ since ϕ is linear. The filtration is *homogeneous* since $F^p K$ is the direct sum of the submodules $K^{p+q} \cap F^p K$. We set

$$F^{p,q}K := K^{p+q} \cap F^p K = \mathrm{Hom}_{\mathbf{C}}\left(\wedge^{p+q}\mathfrak{g}, \mathscr{P}^{m+1+p,2m}_{n,\alpha}(S)\right)$$
$$\text{and } E_0^{p,q}(K) = F^{p,q}K / F^{p+1,q-1}K.$$

Clearly, we have $E_0^{p,q}(K) = \mathrm{Hom}_{\mathbf{C}}\left(\wedge^{p+q}\mathfrak{g}, \mathscr{P}_{n,\alpha}^{m+1+p}(S)\right)$. Then we define (as in [CE56] [p. 323]) the classical spectral sequence $\{E_r^{p,q}(K)\}_{r\in\mathbf{N}}$ together with the differentials $d_r^{p,q}$: $E_r^{p,q} \to E_r^{p+r,q-r+1}$. This spectral sequence is defined by $E_{r+1}^{p,q} = \operatorname{Ker} d_r^{p,q} / \operatorname{Im} d_r^{p-r,q+r-1}$. We are mainly interested in $E_r^{p,q}(K)$ with $p+q=0$ or $p+q=1$. We recall the definition of the 0-th and 1-st differentials of the Newton complex: if $U \in \mathscr{P}_{n,\alpha}^{m+1+p,2m}(S)$ and $f \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{P}_{n,\alpha}^{m+1+p,2m}(S)\right)$, then

$$d_{N,m}^0 U(g) = J^{2m}([NF^m(g), U])$$

$$d_{N,m}^1 f(g_1, g_2) = J^{2m}([NF^m(g_1), f(g_2)] - [NF^m(g_2), f(g_1)]) - f([g_1, g_2]).$$

Therefore, we have

$$d_{N,m}^0 U(g) = [\phi(g), J^{m+1+p}(U)]) \bmod \widehat{\mathscr{X}}_n^{m+2+p}$$

$$d_{N,m}^1 f(g_1, g_2) = [\phi(g_1), J^{m+1+p}(f(g_2))] - [\phi(g_2), J^{m+1+p}(f(g_1))] - J^{m+1+p}(f([g_1, g_2])) \bmod \widehat{\mathscr{X}}_n^{m+2+p}.$$

It follows that the differentials $d_0^{p,q}$ with $p+q=0$ or $p+q=1$ are nothing but the differentials of the Chevalley-Koszul complex associated to the linear representation of ϕ into $\mathscr{P}_{n,\alpha}^{m+1+p}$. Thus, by lemma 5.2.1 (we recall that $\mathscr{P}_{n,\alpha}^{m+1+p}(S) = \left(\mathscr{P}_n^{m+1+p}\right)^{\alpha}(\phi)$), we have

$$E_1^{p,q} = H_\phi^0(\mathfrak{g}, \mathscr{P}_{n,\alpha}^{m+1+p}(S)) = 0, \qquad \text{for } p+q=0$$

$$E_1^{p,q} = H_\phi^1(\mathfrak{g}, \mathscr{P}_{n,\alpha}^{m+1+p}(S)) = 0, \qquad \text{for } p+q=1.$$

As a consequence, we have $E_r^{p,q} = 0$ if $p+q=0$ or $p+q=1$ and $r \in \mathbf{N}$. Let us define $F^{p,q}H(K)$ as the image of $H_{N,m}^{p+q}\left(\mathfrak{g}, \mathscr{P}_{n,\alpha}^{m+1+p,2m}(S)\right)$ in $H_{N,m}^{p+q}\left(\mathfrak{g}, \mathscr{P}_{n,\alpha}^{m+1,2m}(S)\right)$ induced by the injection $\mathrm{Hom}_{\mathbf{C}}\left(\wedge^{p+q}\mathfrak{g}, \mathscr{P}_{n,\alpha}^{m+1+p,2m}(S)\right) \hookrightarrow \mathrm{Hom}_{\mathbf{C}}\left(\wedge^{p+q}\mathfrak{g}, \mathscr{P}_{n,\alpha}^{m+1,2m}(S)\right)$. Moreover, the filtration is *regular*, that is, for each n there exists an integer $u(n)$ such that $H^n(F^pK) = 0$ for $p > u(n)$ (just take $u(n) = m$). Therefore, the spectral sequence (strongly) converges to $E_\infty^{p,q}(K) = F^{p,q}H(K)/F^{p+1,q-1}H(K)$. Using the fact that $\mathscr{P}_n^{k,k'} = 0$ if $k > k'$, we have, for $p+q=0$,

$$H_{N,m}^0\left(\mathfrak{g}, \mathscr{P}_{n,\alpha}^{m+1,2m}(S)\right) = F^{0,0}H(K) = F^{1,-1}H(K) = \cdots = F^{m,-m}H(K) = 0;$$

and for $p+q=1$,

$$H_{N,m}^1\left(\mathfrak{g}, \mathscr{P}_{n,\alpha}^{m+1,2m}(S)\right) = F^{0,1}H(K) = F^{1,-2}H(K) = \cdots = F^{m,-m-1}H(K) = 0. \quad \square$$

7.2. *The topological Newton complex*

From now on, we assume that $\mathfrak{g}$ is **commutative**.

We recall that if $f = \sum_{Q} f_Q x^Q$ (resp. $F = (f_1, \ldots, f_n)$) is any formal power series (resp. formal vector field) then, for any $r > 0$, we set $|f|_r = \sum_Q |f_Q| r^{|Q|}$ (resp. $|F|_r = \max_i |f_i|_r$). Obviously, if F is a polynomial vector field then $|F|_r$ is finite. Let $G = \{g_1, \ldots, g_l\}$ be a basis of $\mathfrak{g}$, $1 \leqslant p \leqslant l$ an integer, and $c \in \mathrm{Hom}_{\mathbf{C}}(\wedge^p \mathfrak{g}, \mathscr{P}_n^{m+1,2m})$. Then we define

$$|c|_r^p = \max_{1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant l} |c(g_{i_1}, \ldots, g_{i_p})|_r .$$

It is clear that $\left(\mathrm{Hom}_{\mathbf{C}}(\wedge^p \mathfrak{g}, \mathscr{P}_n^{m+1,2m}), |.|_r^p\right)$ is a normed space.

Let $r > 0$ be a fixed positive number. In this section, we shall only consider the norms relative to r. We shall consider the Newton complex as a complex of normed spaces.

Lemma **7.2.1.** — *The Newton complex (7.1.1) is a continuous complex of normed spaces.*

Proof. — We have to prove that each differential of the complex is continuous. First of all, if $Y \in \mathscr{P}_n^{m+1,2m}$ and $g \in \mathfrak{g}$, then

$$|[\mathrm{NF}^m(g), Y]|_r = \max_i \left| \sum_{j=1}^n Y_j \frac{\partial(\mathrm{NF}^m(g))_i}{\partial x_j} - (\mathrm{NF}^m(g))_j \frac{\partial Y_i}{\partial x_j} \right|_r$$

$$\leqslant \max_i \sum_{j=1}^n |Y_j|_r \left| \frac{\partial(\mathrm{NF}^m(g))_i}{\partial x_j} \right|_r + |(\mathrm{NF}^m(g))_j|_r \left| \frac{\partial Y_i}{\partial x_j} \right|_r .$$

According to inequality (3.1.3) we have $\left| \frac{\partial Y_i}{\partial x_j} \right|_r \leqslant \frac{2m}{r} |Y_i|_r$, thus,

$$|[\mathrm{NF}^m(g), Y]|_r \leqslant n \left(|D(\mathrm{NF}^m(g))|_r + \frac{2m}{r} |\mathrm{NF}^m(g)|_r \right) |Y|_r.$$

It follows that, if $c \in \mathrm{Hom}_{\mathbf{C}}\left(\wedge^p \mathfrak{g}, \mathscr{P}_n^{m+1,2m}\right)$, then

$$|d_p(c)|_r = \max_{1 \leqslant i_1 < i_2 < \cdots < i_{p+1} \leqslant l} |d_p(c)(g_{i_1}, \ldots, g_{i_{p+1}})|_r$$

$$= \max_{1 \leqslant i_1 < i_2 < \cdots < i_{p+1} \leqslant l} \left| \sum_{i=1}^{p+1} (-1)^{i+1} [\mathrm{NF}^m(g_i), c(g_{i_1}, \ldots, \widehat{g_{i_i}}, \ldots, g_{i_{p+1}})] \right|_r$$

$$\leqslant n(p+1) \left(|D(\mathrm{NF}^m)|_r + \frac{2m}{r} |\mathrm{NF}^m|_r \right) |c|_r . \quad \square$$

7.3. *Cohomology with bounds for the Newton complex*

Let α be a nonzero weight of S into $\mathscr{P}_n^{m+1,2m}$ and let $\mathscr{P}_{n,\alpha}^{m+1,2m}$ be the associated weight space. As we have seen in proposition (7.1.1), for all $Z \in Z^1_{N,m}(\mathfrak{g}, \mathscr{P}_{n,\alpha}^{m+1,2m})$, there exists a unique $U \in \mathscr{P}^{m+1,2m}_{n,\alpha}$ such that $J^{2m}([NF^m_i, U]) = Z_i$ for all integer $1 \leqslant i \leqslant l$. This defines a **C**-linear map s_α from $Z^1_{N,m}(\mathfrak{g}, \mathscr{P}^{m+1,2m}_{n,\alpha})$ to $\mathscr{P}^{m+1,2m}_{n,\alpha}$ such that

$$s_\alpha \circ d^0_{N,m} = Id_{\mathscr{P}^{m+1,2m}_{n,\alpha}} \text{ and } d^0_{N,m} \circ s_\alpha = Id_{Z^1_{N,m}(\mathfrak{g},\, \mathscr{P}^{m+1,2m}_{n,\alpha})}.$$

The remaining of this section is devoted to the proof of the continuity of s_α as well as the determination of a bound of its norm under some assumptions. Moreover, we assume that NF^m **is the** m**-jet of the normal form of a completely integrable deformation of** s. More precisely, we shall prove the

Theorem **7.3.1.** — *Under the above assumptions, there exist constants* $d \geqslant 0$, $\eta_1 > 0$ *and* $c_1(\eta_1) > 0$ *such that, if* $1/2 < r \leqslant 1$, $m = 2^k$ *and* $\max(|NF^m - S|_r, |D(NF^m - S)|_r) < \eta_1$, *then for any nonzero weight* α *of* s *in* $\mathscr{P}^{m+1,2m}_n$, *for any* $Z \in Z^1_{N,m}(\mathfrak{g}, \mathscr{P}^{m+1,2m}_{n,\alpha})$, *the unique* $U \in \mathscr{P}^{m+1,2m}_{n,\alpha}$ *such that* $d^0_{N,m}U = Z$ *satisfies*

$$\textbf{(7.3.1)} \qquad |U|_r \leqslant \frac{c_1(\eta_1)}{\omega^d_{k+1,G}} |Z|_r;$$

and d *depends only on* S.

Proof. — First of all, let $Z \in Z^1_{N,m}(\mathfrak{g}, \mathscr{P}^{m+1,2m}_{n,\alpha})$ and let us write $U \in \mathscr{P}^{m+1,2m}_{n,\alpha}$ the unique solution of $J^{2m}([NF^m_i, U]) = Z_i$ for any integer $1 \leqslant i \leqslant l$. These equations can be written: $[NF^m_i, U] = Z_i + \Big([NF^m_i, U] - J^{2m}([NF^m_i, U])\Big)$. Let us set, for any integer $1 \leqslant i \leqslant l$, $\mathfrak{Z}_i = [NF^m_i, U] - J^{2m}([NF^m_i, U])$.

Let us set, as usual, $S(g_i) = S_i$ $i = 1, \dots, l$ and let S'_i denote the projection of S_i onto $\partial/\partial x_1, \dots, \partial/\partial x_{n'}$. Let $S_1, \dots, S_{l+r}$ be a set a linearly independent diagonal vector fields of a diophantine hull *Dioph*(S) of S. Therefore, if $\bar{\mathfrak{g}}$ denotes a complex commutative Lie algebra of dimension $l + r$, we define the Lie morphism $\overline{S} : \bar{\mathfrak{g}} \to \mathscr{P}^1_n$ by $\overline{S}(g_i) = S_i$ (where $\{g_1, \dots g_{l+r}\}$ denotes a basis of $\bar{\mathfrak{g}}$). The value $r = 0$ is a possible value. We recall that, by definition, $\left(\mathscr{X}^1_n\right)^{\overline{S}} = \left(\mathscr{X}^1_n\right)^{S}$.

By assumption, we have, for any $1 \leqslant i \leqslant l$, $NF^m_i = \sum_{j=1}^{r+l} \tilde{a}_{i,j} S_j + V''_i$ with $\tilde{a}_{i,j} \in \mathscr{O}^S_n$, $\tilde{a}_{i,j}(0) = \delta_{i,j}$ and $V''_i = D''_i + \mathcal{N}il_i'' + R''_i$ is a good deformation of S″ relative to S.

It follows that

$$\sum_{j=1}^{r+l} \Big(\tilde{a}_{i,j}[S_j, U] - U(\tilde{a}_{i,j})S_j\Big) + [V''_i, U] = Z_i + \mathfrak{Z}_i \quad \text{for any } 1 \leqslant i \leqslant l,$$

where $U(\tilde{a}_{i,j})$ denotes the Lie derivative of $\tilde{a}_{i,j}$ along U. We recall that U is a weight vector for S for the nonzero weight α; such a weight is the restriction to $\mathfrak{g}$ of a weight $\bar{\alpha}$ of $\overline{S}$. Let us choose an index $1 \leqslant i \leqslant l$ such that $|\alpha(g_i)| = ||\alpha||$; this value is nonzero.

Since U belongs to the α-weight space of $\mathfrak{g}$ into $\mathscr{P}_n^{m+1,2m}$, then U belongs also to the $\overline{\alpha}$-weight space of $\overline{\mathfrak{g}}$ into $\mathscr{P}_n^{m+1,2m}$. Therefore, for all $1 \leqslant i \leqslant l+r$, $[S_i, U] = \overline{\alpha}(g_i)U$. These equations can be written in the following matrix form

$$A(x)\begin{pmatrix}[S_1, U]\\ \vdots \\ [S_l, U]\end{pmatrix} + \begin{pmatrix}D_1(U)\\ \vdots \\ D_l(U)\end{pmatrix} = \begin{pmatrix}Z_1 + \mathfrak{Z}_1\\ \vdots \\ Z_l + \mathfrak{Z}_l\end{pmatrix}$$

where $A = (a_{p,q})_{1\leqslant p,q\leqslant l}$ is the matrix defined by

$$a_{p,q} = \tilde{a}_{p,q} \text{ if } q \neq i \text{ and } a_{p,i} = \left(\tilde{a}_{p,i} + \sum_{j=l+1}^{r+l} \tilde{a}_{p,j}\frac{\overline{\alpha}(g_j)}{\alpha(g_i)}\right);$$

D_i is the $\widehat{\mathscr{O}}_n^S$-linear map defined by

$$D_i : U \in \widehat{\mathscr{X}}_n^2 \mapsto -\sum_{j=1}^{l+r} U(\tilde{a}_{i,j})S_j + [V_i'', U] \in \widehat{\mathscr{X}}_n^2.$$

The $\widehat{\mathscr{O}}_n^S$-linearity is straightforward for the first term while for the second one, one has to remind that $\widehat{\mathscr{O}}_n^S \hookrightarrow \mathbf{C}[[x_1, \ldots, x_{n'}]]$ whereas $\tilde{V}_i''$ involves only the vector fields $\partial/\partial x_k$ with $k > n'$.

Since $A(0) = \mathrm{Id}$, $A(x)$ is formally invertible: if $\overline{A}^t := (c_{i,j})_{1\leqslant i,j\leqslant l}$ denotes the transpose of the cofactors matrix of A, then $\hat{A}^{-1} := \dfrac{1}{\det A}\overline{A}^t := (b_{i,j})_{1\leqslant i,j\leqslant l}$ belongs to $\mathscr{M}_l(\widehat{\mathscr{O}}_{n'}^{S'})$ and satisfies to $\hat{A}^{-1}A = A\hat{A}^{-1} = \mathrm{Id}$. It follows that

$$\begin{pmatrix}[S_1, U]\\ \vdots \\ [S_l, U]\end{pmatrix} + \begin{pmatrix}\tilde{D}_1(U)\\ \vdots \\ \tilde{D}_l(U)\end{pmatrix} = \hat{A}^{-1}\begin{pmatrix}Z_1 + \mathfrak{Z}_1\\ \vdots \\ Z_l + \mathfrak{Z}_l\end{pmatrix} \text{ where } \tilde{D}_j(U) = \sum_{k=1}^{l} b_{j,k}D_k(U).$$

Thus, the i-th equation of the cohomological equation can be written

(7.3.2) $\qquad U - (P_i + Q_i)(U) = \tilde{Z}_i + \tilde{\mathfrak{Z}}_i.$

Here, we have written

$$\alpha(g_i)\tilde{Z}_i = \sum_{k=1}^{l} b_{i,k}Z_k,$$

$$\alpha(g_i)\tilde{\mathfrak{Z}}_i = \sum_{k=1}^{l} b_{i,k}\mathfrak{Z}_k,$$

$$P_i(U) = \frac{1}{\alpha(g_i)}\sum_{k=1}^{l} b_{i,k}\sum_{j=1}^{l+r} U(\tilde{a}_{k,j})S_j,$$

$$Q_i(U) = \frac{-1}{\alpha(g_i)}\sum_{k=1}^{l} b_{i,k}[V_k'', U].$$

We claim that the operator P_i is nilpotent: it satisfies $P_i \circ P_i = 0$. In fact, we have

$$P_i(P_i(U)) = \frac{1}{\alpha(g_i)} \sum_{k=1}^{l} b_{i,k} \sum_{j=1}^{r+l} P_i(U)(\tilde{a}_{k,j})S_j.$$

But, since for all $1 \leqslant k, p \leqslant l$, $\tilde{a}_{k,p} \in \widehat{\mathcal{O}}_n^{S} = \widehat{\mathcal{O}}_n^{\bar{S}}$ then for all $1 \leqslant q \leqslant l+r$, $S_q(\tilde{a}_{k,p}) = 0$; it follows that $P_i(U)(\tilde{a}_{k,j}) = 0$ and so $P_i \circ P_i = 0$, as claimed.

Let us give bounds for the operators P_i and Q_j. To do so, we shall write $A(x) = Id + R(x)$ where $R(0) = 0$; we shall write $R(x) = (r_{i,j}(x))_{1 \leqslant i,j \leqslant l}$. Let $\mathcal{S}_l$ be the group of permutations of $\{1, \ldots, l\}$. If $\sigma \in \mathcal{S}_l$, then $\varepsilon(\sigma)$ denotes the signature of σ. Recalling the expression of the determinant, we have

$$\begin{aligned} \det(A(x)) &= \sum_{\sigma \in \mathcal{S}_l} \varepsilon(\sigma) \left(\prod_{i=1}^{l} a_{i,\sigma(i)}(x) \right) = \sum_{\sigma \in \mathcal{S}_l} \varepsilon(\sigma) \left(\prod_{i=1}^{l} (a_{i,\sigma(i)}(0) + r_{i,\sigma(i)}(x)) \right) \\ &= 1 + P(r_{i,j}(x)) \end{aligned}$$

where $P(Z) \in \mathbf{C}[Z_1, \ldots, Z_{l^2}]$ is a polynomial functions of l^2 variables without constant term and of degree l; it can be written as $P(Z) = \sum_{\substack{Q \in \mathbf{N}^{l^2} \\ 1 \leqslant |Q| \leqslant l}} d_Q Z^Q$. As a function of $Z_1, \ldots, Z_{l^2}$, it is dominated by $\bar{P}(Z) = \sum_{\substack{Q \in \mathbf{N}^{l^2} \\ 1 \leqslant |Q| \leqslant l}} |d_Q| Z^Q$, thus there exists $\eta > 0$ such that $|P(Z)|_\eta < 1/2$. It follows that, if $|R(x)|_r = |A(x) - A(0)|_r = \max_{i,j} |r_{i,j}|_r < \eta$, then $|P(r_{i,j}(x))|_r < 1/2$. By lemma (3.1.1), if $|A(x) - A(0)|_r < \eta$, then we have

$$\begin{aligned} \left| \frac{1}{\det A(x)} \right|_r &\leqslant \frac{1}{1 - |P(r_{i,j}(x))|_r} \\ &\leqslant 2; \\ |\det A(x)|_r &\leqslant 1 + |P(r_{i,j}(x))|_r \\ &\leqslant \frac{3}{2}. \end{aligned}$$

We recall that $(c_{i,j})_{1 \leqslant i,j \leqslant l} = \bar{A}^t$ is the transpose of the cofactor matrix of A. Thus, there are universal polynomials of degree $\leqslant l-1$, $Q_{i,j}(Z) = \sum_{\substack{S \in \mathbf{N}^{l^2} \\ 1 \leqslant |S| \leqslant l-1}} q_{i,j,S} Z^S \in \mathbf{C}[Z_1, \ldots, Z_{l^2}]$ such that $c_{i,j}(x) = Q_{i,j}(A(x))$. It follows that, for all $1 \leqslant i, j \leqslant l$, $|c_{i,j}(x)|_r \leqslant Q(|A|_r)$ where Q is the universal polynomial of one variable defined by

$$Q(t) = \sum_{\substack{S \in \mathbf{N}^{l^2} \\ 1 \leqslant |S| \leqslant l-1}} \max_{i,j} |q_{i,j,S}| t^{|S|}.$$

As a consequence, if $|A(x) - A(0)|_r < \eta$, we have

$$\text{(7.3.3)} \qquad |b_{i,j}|_r = \left| \frac{c_{i,j}}{\det A} \right|_r \leqslant \left| \frac{1}{\det A} \right|_r |c_{i,j}|_r \leqslant 2Q(|A|_r).$$

By definition, we have, for all integers $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant r+l$

$$\mathrm{S}_j = \sum_{k=1}^{n} \lambda_{j,k} x_k \frac{\partial}{\partial x_k}$$

$$\mathrm{NF}_i^m = \sum_{j=1}^{r+l} \widetilde{a}_{i,j} \mathrm{S}_j := \sum_{k=1}^{n} x_k g_{i,k} \frac{\partial}{\partial x_k} \quad \text{with} \quad g_{i,k} = \left(\sum_{j=1}^{r+l} \lambda_{j,k} \widetilde{a}_{i,j} \right),$$

that is

$$\begin{pmatrix} g_{i,1} \\ \vdots \\ \vdots \\ g_{i,n} \end{pmatrix} = \begin{pmatrix} \lambda_{1,1} & \dots & \lambda_{r+l,1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \lambda_{1,n} & \dots & \lambda_{r+l,n} \end{pmatrix} \begin{pmatrix} \widetilde{a}_{i,1} \\ \vdots \\ \widetilde{a}_{i,r+l} \end{pmatrix},$$

and $r + l \leqslant n$ (in fact, we have $r + l \leqslant n - s$).

By assumption, the S_i's ($1 \leqslant i \leqslant r + l$) are linearly independent over $\mathbf{C}$ ($\overline{\mathrm{S}}$ is injective), the matrix $(\lambda_{j,i})_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant r+l}}$ has rank $r + l$. Without loss of generality, we may assume that the matrix $\mathrm{L} := (\lambda_{j,i})_{1 \leqslant i,j \leqslant r+l}$ is invertible with inverse $\mathrm{L}^{-1} := (\widetilde{\lambda}_{i,j})_{1 \leqslant i,j \leqslant l+r}$.

Thus, we can write, for $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant r + l$,

$$\widetilde{a}_{i,j}(x) - \widetilde{a}_{i,j}(0) = \sum_{k=1}^{r+l} \widetilde{\lambda}_{j,k} (g_{i,k}(x) - g_{i,k}(0)) \prec \sum_{k=1}^{r+l} |\widetilde{\lambda}_{j,k}| \overline{(g_{i,k}(x) - g_{i,k}(0))}.$$

Since $1/2 < r$ then

$$\begin{aligned} |g_{i,k}(x) - g_{i,k}(0)|_r &\leqslant 2r |g_{i,k}(x) - g_{i,k}(0)|_r = 2\, |x_k (g_{i,k}(x) - g_{i,k}(0))|_r \\ &\leqslant 2 |\mathrm{NF}_i^m - \mathrm{S}_i|_r, \end{aligned}$$

so that

$$|\widetilde{a}_{i,j}(x) - \widetilde{a}_{i,j}(0)|_r \leqslant 2(r + l) |\mathrm{L}^{-1}| |\mathrm{NF}_i^m - \mathrm{S}_i|_r.$$

Since, for any $1 \leqslant k \leqslant n$, $\mathrm{NF}_{i,k}^m / x_k = g_{i,k}$ is formal power series, then

$$\frac{\partial x_k \overline{(g_{i,k}(x) - g_{i,k}(0))}}{\partial x_p} = \delta_{k,p} \overline{(g_{i,k}(x) - g_{i,k}(0))} + x_k \frac{\partial \overline{(g_{i,k}(x) - g_{i,k}(0))}}{\partial x_p},$$

that is

$$x_k \frac{\partial \overline{(g_{i,k}(x) - g_{i,k}(0))}}{\partial x_p} = \frac{\partial x_k \overline{(g_{i,k}(x) - g_{i,k}(0))}}{\partial x_p} - \delta_{k,p} \overline{(g_{i,k}(x) - g_{i,k}(0))}.$$

But, since $x_k\frac{\partial\overline{(g_{i,k}(x)-g_{i,k}(0))}}{\partial x_p}$, $\frac{\partial x_k\overline{(g_{i,k}(x)-g_{i,k}(0))}}{\partial x_p}$ and $\overline{(g_{i,k}(x)-g_{i,k}(0))}$ are formal power series with non negative coefficients, we have

$$x_k\frac{\partial\overline{(g_{i,k}(x)-g_{i,k}(0))}}{\partial x_p} \prec \frac{\partial x_k\overline{(g_{i,k}(x)-g_{i,k}(0))}}{\partial x_p}.$$

Since $1/2 < r$, then

$$\begin{aligned}\left|\frac{\partial\overline{(g_{i,k}(x)-g_{i,k}(0))}}{\partial x_p}\right|_r &\leqslant 2r\left|\frac{\partial\overline{(g_{i,k}(x)-g_{i,k}(0))}}{\partial x_p}\right|_r\\ &\leqslant 2\left|x_k\frac{\partial\overline{(g_{i,k}(x)-g_{i,k}(0))}}{\partial x_p}\right|_r\\ &\leqslant 2\left|\frac{\partial x_k\overline{(g_{i,k}(x)-g_{i,k}(0))}}{\partial x_p}\right|_r.\end{aligned}$$

It follows that, for all $1 \leqslant i \leqslant l$, $1 \leqslant j \leqslant l+r$,

$$\begin{aligned}\left|\frac{\partial\overline{(\widetilde{a}_{i,j}(x)-\widetilde{a}_{i,j}(0))}}{\partial x_p}\right|_r &\leqslant \sum_{k=1}^{r+l}|\widetilde{\lambda}_{j,k}|\left|\frac{\partial\overline{(g_{i,k}(x)-g_{i,k}(0))}}{\partial x_p}\right|_r\\ &\leqslant 2\sum_{k=1}^{r+l}|\widetilde{\lambda}_{j,k}|\left|\frac{\partial x_k\overline{(g_{i,k}(x)-g_{i,k}(0))}}{\partial x_p}\right|_r\\ &\leqslant 2(l+r)|L^{-1}||D(NF_i^m-S_i)|_r.\end{aligned}$$

This can be summarized as follows: let us set

$$\widetilde{A}=(\widetilde{a}_{i,j})_{1\leqslant i\leqslant l;1\leqslant j\leqslant l+r},\quad |\widetilde{A}|_r=\max_{i,j}|\widetilde{a}_{i,j}|_r \text{ and } |D(\widetilde{A})|_r=\max_{i,j,k}\left|\frac{\partial\widetilde{a}_{i,j}}{\partial x_k}\right|_r;$$

then we have

$$|\widetilde{A}-\widetilde{A}(0)|_r \leqslant 2(r+l)|L^{-1}||NF^m-S|_r \tag{7.3.4}$$

$$|D(\widetilde{A})|_r \leqslant 2(r+l)|L^{-1}||D(NF^m-S)|_r. \tag{7.3.5}$$

Let us estimate the norm of the matrices $A-A(0)$ and $D(A)$. First of all, we have $a_{p,q}-a_{p,q}(0)=\widetilde{a}_{p,q}-\widetilde{a}_{p,q}(0)$ for all integers $1 \leqslant p, q \leqslant l$ with $q \neq i$. Thus, for these pairs of integers, we have $|a_{p,q}-a_{p,q}(0)|_r \leqslant 2(r+l)|L^{-1}||NF_p^m-S_p|$. On the other hand, for all $1 \leqslant p \leqslant l$, we have

$$a_{p,i} - a_{p,i}(0) = \left((\tilde{a}_{p,i} - \tilde{a}_{p,i}(0)) + \sum_{j=l+1}^{r+l} (\tilde{a}_{p,j} - \tilde{a}_{p,j}(0)) \frac{\overline{\alpha}(g_j)}{\alpha(g_i)} \right).$$

But, according to the fact that $\overline{S}(\overline{\mathfrak{g}})$ is a diophantine hull of S, then there exists a constant $c > 0$ such that, for any weight $\overline{\alpha}$ of $\overline{S}$, $\|\overline{\alpha}\| \leqslant c\|\overline{\alpha} \circ i\|$. As a consequence, we have $\left|\frac{\overline{\alpha}(g_j)}{\alpha(g_i)}\right| \leqslant c$ for all $l+1 \leqslant j \leqslant r+l$. According the above estimates, we obtain

$$|a_{p,i} - a_{p,i}(0)|_r \leqslant 2(r+l)|L^{-1}||NF_p^m - S_p|_r(1 + rc).$$

The same estimates holds for the derivatives since

$$\left| \frac{\partial(\overline{\tilde{a}_{i,j}(x) - \tilde{a}_{i,j}(0)})}{\partial x_p} \right|_r \leqslant 2(r+l)|L^{-1}||D(NF_i^r - S_i^r)|_r.$$

Since $1 + rc \geqslant 1$, the two inequalities found may be written as follows:

(**7.3.6**) $$|A(x) - A(0)|_r \leqslant 2(r+l)(1+rc)|L^{-1}||NF^m - S|_r$$

(**7.3.7**) $$|D(A)|_r \leqslant 2(r+l)(1+rc)|L^{-1}||D(NF^m - S)|_r.$$

Let us set

$$\eta_1 = \frac{\eta}{2(r+l)(1+rc)|L^{-1}|}.$$

If $|NF^m - S|_r < \eta_1$, then by (7.3.6), we have $|A(x) - A(0)|_r < \eta$ so that $|b_{i,j}|_r \leqslant 2Q(|A|_r)$ by (7.3.3). Moreover, we have $|A|_r \leqslant |A(0)| + |A - A(0)|_r \leqslant 1 + \eta$. It follows that $Q(|A|_r) < Q(1 + 2(r+l)(1+rc)|L^{-1}|\eta_1)$.

On the other hand, if $|D(NF^m - S)|_r < \eta_1$ then $|\overline{U}(\tilde{a}_{k,j})|_r \leqslant n|U|_r|D(\tilde{A})|_r \leqslant n\eta|U|_r$. Let us set $\lambda = \max_{1 \leqslant i \leqslant l+r,\, 1 \leqslant j \leqslant n} |\lambda_{i,j}|$, then, since $r \leqslant 1$, for all $1 \leqslant j \leqslant l$, $|S_j|_r \leqslant \lambda$.

We recall (see the section of notation) that, given two vectors of formal power series $Y = (Y_1, \ldots, Y_q)$ and $W = (W_1, \ldots, W_q)$, we say that Y is dominated by W, and we write $Y \prec W$, if $Y_i \prec W_i$ for all $1 \leqslant i \leqslant q$. Moreover, we shall write $\overline{Y} := (\overline{Y}_1, \ldots, \overline{Y}_q)$. Now, we are able to give estimates of P_i and Q_j. We have:

$$P_i(U) = \frac{1}{\alpha(g_i)} \sum_{k=1}^{l} b_{i,k} \sum_{j=1}^{r+l} U(\tilde{a}_{k,j}) S_j.$$

Therefore,

$$P_i(U) \prec \frac{1}{\|\alpha\|} \sum_{k=1}^{l} \overline{b}_{i,k} \sum_{j=1}^{r+l} \overline{U}\left(\overline{\tilde{a}_{k,j}}\right) \overline{S}_j,$$

where $\overline{S}_j$ stands for $\sum_{k=1}^{n} |\lambda_{j,k}| x_k \partial/\partial x_k$. It follows that if

$$1/2 < r \leqslant 1 \text{ and } \max(|NF^m - S|_r, |D(NF^m - S)|_r) < \eta_1,$$

then

$$|P_i(U)|_r \leqslant \frac{c_2(\eta_1)}{\|\alpha\|}|U|_r \tag{7.3.8}$$

with

$$c_2(\eta_1) = 4l(r+l)^2 Q(|A(0)| + 2(r+l)(1+rc)|L^{-1}|\eta_1) n\lambda(1+rc)|L^{-1}|\eta_1.$$

Let U be a polynomial vector field, then let us denote by $d''(U)$ the degree of U in the variable $x_{n'+1}, \ldots, x_n$. Since the vector fields V''_k depend only on the derivatives $\partial/\partial x_{n'+1}, \ldots, \partial/\partial x_n$, then, the estimate of the previous section shows that

$$|[V''_k, U]|_r \leqslant n\left(|D(V''_k)|_r + \frac{d''(U)}{r}|V''_k|_r\right)|U|_r.$$

We recall that $V''_i = (NF^m_i - S_i - \left(\sum_{j=1}^{r+l}(\tilde{a}_{i,j} - \tilde{a}_{i,j}(0))S_j\right)$. Since $r \leqslant 1$, we have

$$|V''_i|_r \leqslant |NF^m - S|_r + 2(l+r)^2\lambda|L^{-1}||NF^m - S|_r \leqslant (1 + 2(l+r)^2\lambda|L^{-1}|)\eta_1.$$

Moreover, we have

$$\frac{\partial V''_i}{\partial x_k} \prec \overline{\frac{\partial(NF^m_i - S_i)}{\partial x_k}} + \left(\sum_{j=1}^{r+l} \overline{\frac{\partial(\tilde{a}_{i,j} - \tilde{a}_{i,j}(0))\,S_j}{\partial x_k}}\right);$$

it follows that

$$\begin{aligned} |D(V''_i)|_r &\leqslant |D(NF^m - S)|_r + 4\lambda(l+r)^2|L^{-1}||NF^m - S|_r \\ &\leqslant (1 + 4(l+r)^2\lambda|L^{-1}|)\eta_1, \end{aligned}$$

hence,

$$|[V''_k, U]|_r \leqslant n\left((1 + 4(l+r)^2\lambda|L^{-1}|) + \frac{d''(U)}{r}(1 + 2(l+r)^2\lambda|L^{-1}|)\right)\eta_1|U|_r.$$

Finally, we obtain

$$|Q_i(U)|_r \leqslant \frac{c_3(\eta_1, d''(U))}{\|\alpha\|}|U|_r, \tag{7.3.9}$$

with

$$\begin{aligned} c_3(\eta_1, d''(U)) = {}& 2lnQ(|A(0)| + 2(r+l)(1+rc)|L^{-1}|\eta_1) \\ & \times \left((1 + 4(l+r)^2\lambda|L^{-1}|) + 2d''(U)(1 + 2(l+r)^2\lambda|L^{-1}|)\right)\eta_1. \end{aligned}$$

We have $\alpha(g_i)\tilde{Z}_i = -\sum_{k=1}^{l} b_{i,k}Z_k$. It follows that

$$|\tilde{Z}|_r \leqslant \frac{2lQ(|A(0)| + 2(r+l)(1+rc)|L^{-1}|\eta_1)}{\|\alpha\|}|Z|_r. \tag{7.3.10}$$

Let $Sep(S) > 0$ be a separating constant for S; according to lemma 6.2.7, if α is a weight of S and $\|\alpha\| < Sep(S)$ then α is a weight of H. We recall that if $n' = n$ then we set $Sep(\phi) = +\infty$ and $H = V(Ab(\mathfrak{g}))$.

In order to give good estimates, we shall consider two different cases:

1. $\|\alpha\| \geqslant Sep(S)$,
2. $\|\alpha\| < Sep(S)$.

For each weight α of each case, we shall solve the cohomological equation with Z in the corresponding weight space and rewrite the solution of equation (7.3.2) in an appropriate way.

1. $\|\alpha\| \geqslant Sep(S)$.

Let us write the equation (7.3.2) as follows:

$$U - P_i(U) = \tilde{Z}_i + \tilde{\mathfrak{Z}}_i + Q_i(U).$$

Since $P_i \circ P_i = 0$ then we have: $U = (Id + P_i)(\tilde{Z}_i + \tilde{\mathfrak{Z}}_i + Q_i(U))$. We can be a bit more precise since U is a polynomial vector field of degree $2m$. In fact, by definition, for any integer $1 \leqslant i \leqslant l$, $\mathfrak{Z}_i$ is a vector field of order greater than or equal to $2m+1$. Since the $b_{i,j}$'s are formal power series, $\alpha(g_i)\tilde{\mathfrak{Z}}_i = -\sum_{k=1}^{l} b_{i,k}\mathfrak{Z}_k$ is a formal vector field of order greater than or equal to $2m+1$. Hence, both $P_i(\tilde{\mathfrak{Z}}_i)$ and $Q_i(\tilde{\mathfrak{Z}}_i)$ are formal vector fields of order greater than or equal to $2m+1$. It follows that $U = J^{2m}\left(\tilde{Z}_i + P_i(\tilde{Z}_i) + Q_i(U) + P_i(Q_i(U))\right)$. Hence, we have

$$U \prec \overline{\tilde{Z}_i} + \overline{P_i(\tilde{Z}_i)} + \overline{Q_i(U)} + \overline{P_i(Q_i(U))}.$$

According to inequalities (7.3.8), (7.3.9), we have

$$|U|_r \leqslant |\tilde{Z}|_r\left(1 + \frac{c_2(\eta_1)}{\|\alpha\|}\right) + \left(\frac{c_3(\eta_1, d''(U))}{\|\alpha\|} + \frac{c_2(\eta_1)c_3(\eta_1, d''(U))}{\|\alpha\|^2}\right)|U|_r.$$

According to lemma 6.2.7, there is a constant $d > 0$ such that, if $\alpha = \alpha_{Q,i}$ then $|p''(Q)| \leqslant d\|\alpha\|$. It follows that, if we set

$$\begin{aligned} c_4(\eta_1) := {} & 2lnQ(1 + 2(r+l)(1+rc)|L^{-1}|\eta_1 \\ & \times\left(\frac{(1 + 4(l+r)^2\lambda|L^{-1}|)}{Sep(S)} + 2d(1 + 2(l+r)^2\lambda|L^{-1}|)\right)\eta_1, \end{aligned}$$

then, $\dfrac{c_3(\eta_1, d''(U))}{\|\alpha\|} \leqslant c_4(\eta_1)$. Now we may choose η_1 small enough so that:

$$\left(c_4(\eta_1) + \frac{c_2(\eta_1)c_4(\eta_1)}{Sep(S)}\right) \leqslant 1/2;$$

hence, by (7.3.10), we obtain

$$\textbf{(7.3.11)} \qquad |U|_r \leqslant |Z|_r \frac{4lQ(1+2(r+l)(1+rc)|L^{-1}|\eta_1)}{Sep(S)}\left(1+\frac{c_2(\eta_1)}{Sep(S)}\right).$$

2. $\|\alpha\| < Sep(S)$.

We recall that if $X \in \widehat{\mathscr{X}}_n^1$, then X' (resp. X'') denotes its projection onto the space generated by $\partial/\partial x_1, \ldots, \partial/\partial x_{n'}$ (resp. $\partial/\partial x_{n'+1}, \ldots, \partial/\partial x_n$).

Let us project the equation (7.3.2) onto these two natural subspaces; we obtain:

$$\textbf{(7.3.12)} \qquad U' - P_i'(U') = \tilde{Z}_i' + \mathfrak{Z}_i' + Q_i'(U') + Q_i'(U''),$$

$$\textbf{(7.3.13)} \qquad U'' - Q_i''(U'') = \tilde{Z}_i'' + \mathfrak{Z}_i'' + P_i''(U') + Q_i''(U').$$

It should be noticed that, since $\widehat{\mathscr{O}}_n^S \hookrightarrow \widehat{\mathscr{O}}_{n'}$, we have $P_i(U) = P_i(U')$. Since $\|\alpha\| < Sep(S)$, then α belongs to H (unless $n' = n$, in which case equation (7.3.13) doesn't make sense). Since U belongs to the α-weight space, then both U' and U'' belong to the α-weight space.

First of all, let us consider equation (7.3.12). We have $P_i' \circ P_i' = 0$ by the same argument which shows that $P_i \circ P_i = 0$; moreover, $Q_i'(U') = 0$. In fact, we have $\alpha = \alpha_{Q,j}(\phi) = \alpha_{p'(Q),j}(\phi') + \beta_{p''(Q)}(\phi'')$ for some $Q \in \mathbf{N}^n$ and $1 \leqslant j \leqslant n'$. But, $\beta_{p''(Q)}(\phi'') = 0$ (this is due to the fact that $\alpha \in H$ and $\beta_{p''(Q)}(\phi'')$ lies on one side of H) and since ϕ'' is flat then $p''(Q) = 0$. It follows that U' doesn't depend on $x_{n'+1}, \ldots, x_n$; hence $Q_i(U')$ doesn't have any component along $\partial/\partial x_1, \ldots, \partial/\partial x_{n'}$. Moreover, we have $Q_i'(U'') = 0$ since both U'' and V'' don't have any nonzero component along $\partial/\partial x_1, \ldots, \partial/\partial x_{n'}$. Therefore, equation (7.3.12) may be written $U' - P_i'(U') = \tilde{Z}_i' + \mathfrak{Z}_i'$; it follows that

$$U' = J^{2m}\left((Id + P_i')\tilde{Z}_i'\right).$$

Therefore, we have

$$U' \prec \overline{\tilde{Z}_i'} + \overline{P_i'(\tilde{Z}_i')}.$$

Hence, we have the following estimates

$$\begin{aligned}
|U'|_r &\leqslant |\tilde{Z}_i'|_r + |P_i'(\tilde{Z}_i')|_r \\
&\leqslant \left(1+\frac{c_2(\eta_1)}{\|\alpha\|}\right)|\tilde{Z}_i'|_r \\
&\leqslant \left(1+\frac{c_2(\eta_1)}{\|\alpha\|}\right)\frac{2lQ(1+2(l+r)(1+rc)|L^{-1}|\eta_1)}{\|\alpha\|}|Z|_r.
\end{aligned}$$

By definition, if $m = 2^k$ then $\omega_{k+1,G} \leqslant \|\alpha\|$. Let us set

$$c_1(\eta_1) = (Sep(S) + c_2(\eta_1))2lQ(|A(0)| + 2(l+r)(1+rc)|L^{-1}|\eta_1).$$

It follows that, if $1/2 < r \leqslant 1$, $m = 2^k$ and $\max(|\mathrm{NF}^m - \mathrm{S}|_r, |\mathrm{D}(\mathrm{NF}^m - \mathrm{S})|_r) < \eta_1$, then

$$\textbf{(7.3.14)} \qquad |\mathrm{U}'|_r \leqslant \frac{c_1(\eta_1)}{\omega^2_{k+1,\,\mathrm{G}}}|\mathrm{Z}|_r.$$

Let us consider equation (7.3.13) and let us set $\tilde{\mathfrak{w}}_i = \tilde{\mathrm{Z}}''_i + \mathfrak{Z}''_i + \mathrm{P}''_i(\mathrm{U}') + \mathrm{Q}''_i(\mathrm{U}')$. Then, at least formally, the solution of equation (7.3.13) may be written as $\mathrm{U}'' = \sum_{k\geqslant 0} \mathrm{Q}^k_i(\tilde{\mathfrak{w}}_i)$. But since U'' is a polynomial of degree $\leqslant 2m$ then

$$\textbf{(7.3.15)} \qquad \mathrm{U}'' = \sum_{k\geqslant 0} \mathrm{J}^{2m}(\mathrm{Q}^k_i(\mathfrak{w}_i)) \quad \text{with } \mathfrak{w}_i = \tilde{\mathrm{Z}}''_i + \mathrm{P}''_i(\mathrm{U}') + \mathrm{Q}''_i(\mathrm{U}').$$

We shall show that there exists an integer k_0, **independent of** m, such that $\mathrm{U}'' = \sum_{k=0}^{k_0} \mathrm{J}^{2m}(\mathrm{Q}^k_i(\mathfrak{w}_i))$. We recall that both $\mathfrak{w}_i$ and U'' are finite sums of the weight spaces associated to the weight $\alpha_{\mathrm{Q},i}$ such that $\mathrm{Q} \in \mathbf{N}^n$ with $1 \leqslant |\mathrm{Q}| \leqslant 2m$, $n'+1 \leqslant i \leqslant n$ and $\alpha_{\mathrm{Q},i} = \alpha$. Since $\|\alpha\| < \mathit{Sep}(\mathrm{S})$ then $\alpha_{p''(\mathrm{Q}),i}(\mathrm{S}'')$ belongs to the hyperplane H. On the other hand, we have $\mathrm{V}''_i = \mathrm{D}''_i + \mathcal{N}il_i'' + \mathrm{R}''_i$ where D''_i (resp. $\mathcal{N}il''_i$) is a linear diagonal (resp. nilpotent) deformation of 0 (resp. 0) in $\widehat{\mathcal{O}}^{\mathrm{S}'}_{n'} \otimes_{\mathbf{C}} i''_* \mathcal{L}_h(\mathrm{S}'')$ (resp. in $\widehat{\mathcal{O}}^{\mathrm{S}'}_{n'} \otimes_{\mathbf{C}} i''_* \left(\mathcal{P}^1_{n-n'}\right)^{\mathrm{S}''}$), R''_i is a nonlinear deformation of 0 in $\widehat{\mathcal{O}}_{n'} \otimes_{\mathbf{C}} i''_* \left(\widehat{\mathcal{X}}^2_{n-n'}\right) \cap \left(\widehat{\mathcal{X}}^1_n\right)^{\mathrm{S}}$ (it is not, *a priori*, a morphism) and $[\mathrm{D}''_i, \mathcal{N}il''_i + \mathrm{R}''_i] = 0$. By lemma (6.2.2), $\left(\widehat{\mathcal{X}}^1_{n-n'}\right)_h$ (that is the weight spaces relative to the weights of S'' which belong to H) is a finite dimensional $\mathbf{C}$-vector space. Hence, there is a smallest positive integer $d(\mathrm{S}'')$ such that $\left(\widehat{\mathcal{X}}^1_{n-n'}\right)_h = \left(\mathcal{P}^{1,d(\mathrm{S}'')}_{n-n'}\right)_h$.

Let us recall that $\mathrm{Q}_i(\mathrm{U}) = \dfrac{-1}{\alpha(g_i)} \sum_{k=1}^{l} b_{i,k}[\mathrm{V}''_k, \mathrm{U}]$. It should be noticed that $[\mathrm{V}''_k, \mathfrak{w}_i] = [\mathcal{N}il''_k, \mathfrak{w}_i] + [\mathrm{R}''_k, \mathfrak{w}_i] := \mathcal{B}_{\mathcal{N}il''_k}(\mathfrak{w}_i) + \mathcal{B}_{\mathrm{R}''_k}(\mathfrak{w}_i)$; this is due to the fact that $\mathfrak{w}_i$ belongs to $\mathcal{O}_{n'} \otimes_{\mathbf{C}} \left(i''_* \left(\mathcal{P}^{1,d(\mathrm{S}'')}_{n-n'}\right)_h\right)$ and that D'' commutes with every element of this space. Moreover, both $\mathcal{B}_{\mathrm{N}''_k}$ and $\mathcal{B}_{\mathrm{R}''_k}$ are $\mathcal{O}_{n'}$-linear maps on $\mathcal{O}_{n'} \otimes_{\mathbf{C}} \left(i''_* \left(\mathcal{P}^{1,d(\mathrm{S}'')}_{n-n'}\right)_h\right)$.

In fact, first of all, let α and β denotes weights of S''; let $\mathrm{X} \in \left(\widehat{\mathcal{X}}^1_n\right)_\alpha(\mathrm{S}'')$ and $\mathrm{Y} \in \left(\widehat{\mathcal{X}}^1_n\right)_\beta(\mathrm{S}'')$ denotes elements of the weight spaces. From Jacobi Identity, we show that $[\mathrm{X}, \mathrm{Y}] \in \left(\widehat{\mathcal{X}}^1_n\right)_{\alpha+\beta}(\mathrm{S}'')$. Indeed, we have

$$[\mathrm{S}'', [\mathrm{X}, \mathrm{Y}]] = -[\mathrm{X}, [\mathrm{Y}, \mathrm{S}'']] - [\mathrm{Y}, [\mathrm{S}'', \mathrm{X}]] = (\alpha + \beta)[\mathrm{X}, \mathrm{Y}].$$

Therefore, if α and β belong to H then so does $\alpha + \beta$. By assumption, both $\mathcal{N}il_i''$ and R''_i belong to $\mathcal{O}_{n'} \otimes_{\mathbf{C}} \left(i''_* \left(\mathcal{P}^{1,d(\mathrm{S}'')}_{n-n'}\right)_h\right)$. Thus, if X belongs to that space so do

$[\mathcal{N}il_i'', X]$ and $[R_i'', X]$. The $\mathcal{O}_{n'}$-linearity comes from the fact that the vector fields depend only on the derivations $\partial/\partial x_{n'+1}, \ldots, \partial/\partial x_n$.

Let k be a positive integer; we have for any $X \in \mathcal{O}_{n'} \otimes_{\mathbf{C}} \left(i_*'' \left(\mathcal{P}_{n-n'}^{1,d(S'')}\right)_h\right)$,

$$\begin{aligned} Q_i^k(X) &= \frac{(-1)^k}{\alpha(g_i)^k} \sum_{i_1, \ldots, i_k = 1}^{l} \left(\prod_{j=1}^{k} b_{i,i_j}\right) ad_{V''_{i_1}} \circ \cdots \circ ad_{V''_{i_k}}(X) \\ &= \frac{(-1)^k}{\alpha(g_i)^k} \sum_{i_1, \ldots, i_k = 1}^{l} \left(\prod_{j=1}^{k} b_{i,i_j}\right) (\mathcal{B}_{\mathcal{N}il''_{i_1}} + \mathcal{B}_{R''_{i_1}}) \circ \cdots \circ (\mathcal{B}_{\mathcal{N}il''_{i_k}} + \mathcal{B}_{R''_{i_k}})(X). \end{aligned}$$

Let us first notice that $ord''(\mathcal{B}_{\mathcal{N}il''_i}(X)) \geqslant ord''X$ and $ord''(\mathcal{B}_{R''_i}(X)) \geqslant ord''X + 1$. Let us set for $\mathbf{k} = (k_1, \ldots, k_l) \in \mathbf{N}^l$

$$\mathcal{B}_N^{\mathbf{k}} := \mathcal{B}_{N''_1}^{k_1} \circ \cdots \circ \mathcal{B}_{\mathcal{N}''_l}^{k_l}.$$

Let us set, for any nonnegative integer p, $\mathbf{i} = (i_1, \ldots, i_p) \in \{1, \ldots, l\}^p$ and $\mathbf{K} = (\mathbf{k}_0, \ldots, \mathbf{k}_p) \in (\mathbf{N}^l)^{p+1}$:

$$\mathcal{B}_{p,\mathbf{i},\mathbf{K}} := \mathcal{B}_N^{\mathbf{k}_0} \circ \mathcal{B}_{R''_{i_1}} \circ \mathcal{B}_N^{\mathbf{k}_1} \circ \cdots \circ \mathcal{B}_{R''_{i_p}} \circ \mathcal{B}_N^{\mathbf{k}_p}.$$

It follows that $(\mathcal{B}_{N''_{i_1}} + \mathcal{B}_{R''_{i_1}}) \circ \cdots \circ (\mathcal{B}_{N''_{i_k}} + \mathcal{B}_{R''_{i_k}})(X)$ is a sum of the $\mathcal{B}_{p,\mathbf{i},\mathbf{K}}(X)$'s with $|\mathbf{K}| + p := |\mathbf{k}_0| + \cdots + |\mathbf{k}_p| + p = k$; furthermore, we have $ord''(\mathcal{B}_{p,\mathbf{i},\mathbf{K}}(X)) \geqslant ord''X + p$. As a consequence, since $\mathcal{O}_{n'} \otimes_{\mathbf{C}} \left(i_*'' \left(\mathcal{P}_{n-n'}^{1,d(S'')}\right)_h\right)$ is left invariant by both $\mathcal{B}_{N''_i}$ and $\mathcal{B}_{R''_i}$, then for any positive integer p such that $ord''X + p > d(S'')$, for any $\mathbf{i} = (i_1, \ldots, i_p) \in \{1, \ldots, l\}^p$ and for any $\mathbf{K} = (\mathbf{k}_0, \ldots, \mathbf{k}_p) \in (\mathbf{N}^l)^{p+1}$, we have $\mathcal{B}_{p,\mathbf{i},\mathbf{K}}(X) = 0$. Thus, $Q_i^k(X)$ is a linear combination (with coefficients in $\mathcal{O}_{n'}^{S'}$) of the $\mathcal{B}_{p,\mathbf{i},\mathbf{K}}(X)$'s for which $p \leqslant d(S'') - ord''X \leqslant d(S'')$, $\mathbf{i} = (i_1, \ldots, i_p) \in \{1, \ldots, l\}^p$, $\mathbf{K} = (\mathbf{k}_0, \ldots, \mathbf{k}_p) \in (\mathbf{N}^l)^{p+1}$ and $|\mathbf{K}| + p = k$.

Let us show that $ord(\mathcal{B}_N^{\mathbf{k}}(X)) \geqslant m + ord(X)$ if $|\mathbf{k}|$ is large enough. In fact, by assumption, each $\mathcal{N}il_i''$ is the m-jet of a nilpotent vector field $\tilde{N}_i''$ with coefficients in $\mathcal{O}_{n'}^{S'}$; thus $ad_{\tilde{N}_i''}$ is nilpotent too. Let us notice that $ad_{\tilde{N}_1''}^{k_1} \circ \cdots \circ ad_{\tilde{N}_l''}^{k_l}(X) - \mathcal{B}_N^{\mathbf{k}}(X)$ is a sum of compositions of adjoint operators in which appears at least one of the $ad_{\tilde{N}_i'' - \mathcal{N}il_i''}$'s for some i. Since $\tilde{N}_i'' - \mathcal{N}il_i''$ is of order $\geqslant m + 1$, it follows that $ord(ad_{\tilde{N}_1''}^{k_1} \circ \cdots \circ ad_{\tilde{N}_l''}^{k_l}(X) - \mathcal{B}_N^{\mathbf{k}}(X)) \geqslant ord\, X + m$. Now, let nil be the maximum of the orders of nilpotency of the $ad_{\tilde{N}''_i}$'s; if $|\mathbf{k}| \geqslant nil \times l$, then there is at least one $k_i \geqslant nil$, hence $ad^{k_1}_{\tilde{N}''_1} \circ \cdots \circ ad^{k_l}_{\tilde{N}''_l}(X) = 0$. It follows that if $|\mathbf{k}| \geqslant nil \times l$, then $ord(\mathscr{B}^{\mathbf{k}}_{\mathrm{N}}(X)) \geqslant m + ord(X)$.

Let us go back to the study of $\mathscr{B}_{p,\mathbf{i},\mathbf{K}}(X)$ with $p \leqslant d(S'')$. If $|\mathbf{K}| + p = k > d(S'')(1 + nil \times l) + nil \times l$ then $|\mathbf{k}_0| + \cdots + |\mathbf{k}_p| > (d(S'') + 1)(nil \times l) \geqslant (p+1)(nil \times l)$. Hence, there is at least one of the $\mathbf{k}_i$'s which is of norm greater than or equal to $nil \times l$; thus, $ord(\mathscr{B}_{p,\mathbf{i},\mathbf{K}}(X)) > m + ord(X) \geqslant 2m + 1$ if $ord(X) \geqslant m + 1$.

As a conclusion, we can say that for all $X \in \mathscr{O}_{n'} \otimes_{\mathbf{C}} \left(i''_* \left(\mathscr{P}^{1,d(S'')}_{n-n'}\right)_h\right)$ of order greater than m, then $J^{2m}(Q^k_i(X)) = 0$ for all $k > d(S'')(1 + nil \times l) + nil \times l$.

Let us go back to our first problem, that is, the solution U″. According to what has just been said, equation (7.3.15) can be written as

$$\textbf{(7.3.16)} \qquad U'' = \sum_{k=0}^{d(S'')(1+nil\times l)+nil\times l} J^{2m}(Q^k_i(\mathfrak{w}_i)),$$

where $\mathfrak{w}_i = \tilde{Z}''_i + P''_i(U') + Q''_i(U')$. As a consequence,

$$U'' \prec \sum_{k=0}^{d(S'')(1+nil\times l)+nil\times l} \overline{Q^k_i(\mathfrak{w}_i)},$$

$$|U''|_r \leqslant \sum_{k=0}^{d(S'')(1+nil\times l)+nil\times l} |Q^k_i(\overline{\mathfrak{w}_i})|_r.$$

We recall that $\mathfrak{w}_i$ belongs to $\left(\mathscr{O}_{n'} \otimes_{\mathbf{C}} \left(i''_* \left(\mathscr{P}^{1,d(S'')}_{n-n'}\right)_h\right)\right) \cap \left(\widehat{\mathscr{X}}^1_n\right)^S$; thus, according to estimate (7.3.9), we have $|Q_i(\mathfrak{w}_i)|_r \leqslant \dfrac{c_4(\eta_1)}{\|\alpha\|}|\mathfrak{w}_i|_r$.

Therefore, since $\omega_{k+1,G} \leqslant \|\alpha\| < Sep(S)$, we have

$$|U''|_r \leqslant |\mathfrak{w}_i|_r \sum_{k=0}^{d(S'')(1+nil\times l)+nil\times l} \left(\frac{c_4(\eta_1)}{\|\alpha\|}\right)^k,$$

$$\leqslant \frac{|\mathfrak{w}_i|_r}{\omega_{k+1,G}^{d(S'')(1+nil\times l)+nil\times l}} \sum_{k=0}^{d(S'')(1+nil\times l)+nil\times l} (c_4(\eta_1)Sep(S))^k.$$

Let us set

$$c_5(\eta_1) = \sum_{k=0}^{d(S'')(1+nil\times l)+nil\times l} (c_4(\eta_1)Sep(S))^k.$$

Using first estimates (7.3.8) and (7.3.9) and then, estimate (7.3.10), we obtain

$$\begin{aligned}|\mathfrak{w}_i|_r &\leqslant |\tilde{Z}_i''|_r + |P_i''(U')|_r + |Q_i''(U')|_r \\ &\leqslant \frac{2lQ(|A(0)| + 2(l+r)(1+rc)|L^{-1}|\eta_1)}{\|\alpha\| |\det A(0)|}|Z|_r + \frac{c_2(\eta_1)}{\|\alpha\|}|U'|_r + |Q_i''(U')|_r.\end{aligned}$$

According to estimate (7.3.9), we have $|Q_i''(U')|_r \leqslant \frac{c_3(\eta_1, d''(U'))}{\|\alpha\|}|U'|_r$; but U', as an element of the weight space does not depend on $x_{n'+1}, \ldots, x_n$; therefore, in this case, we have

$$c_3(\eta_1, d''(U')) \leqslant c_6(\eta_1) := \frac{2lnQ(|A(0)| + 2(l+r)(1+rc)|L^{-1}|\eta_1)}{|\det A(0)|}\left((1 + 4(l+r)^2\lambda|L^{-1}|)\right)\eta_1.$$

Hence, we have

$$\begin{aligned}|\mathfrak{w}_i|_r &\leqslant \frac{2lQ(|A(0)| + 2(l+r)(1+rc)|L^{-1}|\eta_1)}{\|\alpha\| |\det A(0)|}|Z|_r + \frac{c_2(\eta_1) + c_6(\eta_1)}{\|\alpha\|}|U'|_r, \\ &\leqslant \left(\frac{(c_2(\eta_1) + c_6(\eta_1))c_1(\eta_1)}{\omega_{k+1,G}^3} + \frac{2lQ(|A(0)| + 2(l+r)(1+rc)|L^{-1}|\eta_1)}{\omega_{k+1,G}|\det A(0)|}\right)|Z|_r.\end{aligned}$$

Finally, we obtain

$$\text{(7.3.17)} \qquad |U''|_r \leqslant \frac{c_7(\eta_1)}{\omega_{k+1,G}^{d(S'')(1+nil\times l)+nil\times l+3}}|Z|_r,$$

where we have set

$$c_7(\eta_1) = c_5(\eta_1)\left((c_2(\eta_1) + c_6(\eta_1))c_1(\eta_1) + \frac{\omega_1(G)^2 2lQ(|A(0)| + 2(l+r)|L^{-1}|\eta_1)}{|\det A(0)|}\right).$$

This ends the proof of the theorem. □

Furthermore, we may assume that $c_1 \geqslant \omega_{k+1,G}^d$ (if not, we can take a greater value for c_1), so we can write $c_1 = \omega_{k+1,G}^d \gamma_k^{-m}$ with $\gamma_k \leqslant 1$. Then, the previous estimate becomes

$$\text{(7.3.18)} \qquad |U|_r \leqslant \gamma_k^{-m}|Z|_r.$$

8. The induction argument

We assume that $\mathfrak{g}$ is a **commutative Lie algebra** of dimension l and that the morphism S is **diagonal**.

Let $1/2 < r \leqslant 1$ be a real number and let $\eta_1 > 0$ be the positive number defined in theorem 7.3.1. For any integer $m \geqslant [8n/\eta_1] + 1$, let us set

$$\mathcal{NF}_m(r) = \left\{ X \in \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathcal{P}_n^{1,m}) \mid \max(|X - S|_r, |D(X - S)|_r) < \eta_1 - \frac{8n}{m} \right\}$$

$$\mathcal{B}_{m+1}(r) = \left\{ X \in \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathcal{X}_n^{m+1}) \mid |X|_r < 1 \right\}.$$

If $m = 2^k$ for some integer $k \geqslant 1$, let us define

$$\rho = m^{-1/m} r \quad \text{and} \quad R = \gamma_k m^{-2/m} r \text{ where } \gamma_k = \left(\frac{c_1}{\omega_{k+1,G}^d} \right)^{-1/m} \text{ is defined in (7.3.18).}$$

It is clear that $m^{1/m} \geqslant 1$. According to the fact that all the numbers $\omega_{k,G}$ are assumed to be smaller than 1, we have $\ln\omega_{k,G} < 1/m \ln m$ so that $\ln\omega_{k,G} - 2/m \ln m < -1/m \ln m < 0$, that is, $R < \rho < r \leqslant 1$.

Let $S + \varepsilon$ be a holomorphic deformation of S, which is formally completely integrable. Let us assume that $S + \varepsilon$ is normalized up to order m. Thus, we may write

$$S + \varepsilon = NF^m + R_{m+1}$$

where NF^m is the m-jet of the normal form of $S + \varepsilon$, $R_{m+1} \in \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathcal{X}_n^{m+1})$.

The core of this section is the following proposition:

Proposition **8.0.2.** — *With the above notations, let us assume that* $(NF^m, R_{m+1}) \in \mathcal{NF}_m(r) \times \mathcal{B}_{m+1}(r)$. *If m is sufficiently large (say $m > m_0$ independent of r), then there exists a unique* $U \in \bigoplus_{\alpha \in \mathcal{W}_{v,n}^{m+1,2m} \setminus \{0\}} \mathcal{P}_{n,\alpha}^{m+1,2m}$ *such that*

1. $\Phi := (Id + U)^{-1} \in \mathit{Diff}_1(\mathbf{C}^n, 0)$ *is a diffeomorphism such that* $D_R \subset \Phi(D_\rho)$,
2. $\Phi^*(S + \varepsilon) = NF^{2m} + R_{2m+1}$ *is normalized up to order* $2m$,
3. $(NF^{2m}, R_{2m+1}) \in \mathcal{NF}_{2m}(R) \times \mathcal{B}_{2m+1}(R)$.

The proof of this proposition will require several steps.

8.1. *The normalizing diffeomorphism*

Let us write

$$S + \varepsilon = NF^m + B + C$$

where $NF^m \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{P}_n^{1,m}\right)$, $B \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{P}_n^{m+1,2m}\right)$ and $C \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{X}_n^{2m+1}\right)$. We may decompose B along the weight spaces of S: $B = B_0 + B^*$ where

$$B_0 \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{P}_{n,0}^{m+1,2m}\right) \text{ and } B^* \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \bigoplus_{\alpha\in\mathscr{W}_{v,n}^{m+1,2m}\setminus\{0\}} \mathscr{P}_{n,\alpha}^{m+1,2m}\right).$$

We claim that B^* is a 1-cocycle for the Newtonian complex of order m associated to NF^m. In fact, since $S+\varepsilon$ is a Lie morphism, we have, for all $g_1, g_2 \in \mathfrak{g}$,

$$\begin{aligned}&[NF^m(g_1) + B(g_1) + C(g_1), NF^m(g_2) + B(g_2) + C(g_2)]\\ &= (NF^m + B + C)([g_1, g_2]);\end{aligned}$$

that is

$$\begin{aligned}&[NF^m(g_1), B(g_2)] - [NF^m(g_2), B(g_1)] - B([g_1, g_2])\\ &\quad + [NF^m(g_1), NF^m(g_2)] - NF^m([g_1, g_2]) = -[B(g_1) + C(g_1), B(g_2) + C(g_2)]\\ &\qquad - [NF^m(g_1), C(g_2)]\\ &\qquad + [NF^m(g_2), C(g_1)] + C([g_1, g_2]).\end{aligned}$$

Since the right hand side of this equation has an order greater than or equal to $2m+1$, we conclude that

$$\begin{aligned}&J^{2m}([NF^m(g_1), B(g_2)] - [NF^m(g_2), B(g_1)] - B([g_1, g_2])\\ &\quad + [NF^m(g_1), NF^m(g_2)] - NF^m([g_1, g_2])) = 0.\end{aligned}$$

Since we have the decomposition

$$\mathscr{P}_n^{m+1,2m} = \mathscr{P}_{n,0}^{m+1,2m} \bigoplus_{\alpha\in\mathscr{W}_{v,n}^{m+1,2m}\setminus\{0\}} \mathscr{P}_{n,\alpha}^{m+1,2m}$$

into $\mathfrak{g}$-modules for the Newtonian representation, we obtain

$$\begin{aligned}&J^{2m}([NF^m(g_1), B^*(g_2)] - [NF^m(g_2), B^*(g_1)]) = B^*([g_1, g_2])\\ &J^{2m}([NF^m(g_1), B_0(g_2)] - [NF^m(g_2), B_0(g_1)] + [NF^m(g_1), NF^m(g_2)])\\ &= NF^m([g_1, g_2]) + B_0([g_1, g_2]),\end{aligned}$$

that is $B^* \in Z^1_{N,m}\left(\mathfrak{g}, \bigoplus_{\alpha\in\mathscr{W}_{v,n}^{m+1,2m}\setminus\{0\}} \mathscr{P}_{n,\alpha}^{m+1,2m}\right)$.

According to proposition (7.1.1), there exists a unique U belonging to the weight spaces associated to a non zero weight of S into $\mathscr{P}^{m+1,2m}$ such that $J^{2m}([NF^m, U]) = B^*$.

Let us set $\Phi^{-1} = \mathrm{Id} + \mathrm{U} \in \mathrm{Diff}_1(\mathbf{C}^n, 0)$. Since, $\Phi^*(\mathrm{S}+\varepsilon)(\Phi(x)) = \mathrm{D}(\Phi)(x)(\mathrm{S}+\varepsilon)(x)$ then, by setting $x = \Phi^{-1}(y)$, we obtain

$$\begin{aligned}\Phi^*(\mathrm{S}+\varepsilon)(y) &= \mathrm{D}(\Phi)(\Phi^{-1}(y))(\mathrm{S}+\varepsilon)(\Phi^{-1}(y))\\ &= \mathrm{D}(\Phi^{-1})^{-1}(y)(\mathrm{S}+\varepsilon)(\Phi^{-1}(y))\\ &\quad (\mathrm{D}(\Phi)(\Phi^{-1}(y))\mathrm{D}(\Phi^{-1})(y) = Id).\end{aligned}$$

It follows that

$$(\mathbf{8.1.1})\qquad \mathrm{D}(\Phi^{-1})(y)\Phi^*(\mathrm{S}+\varepsilon)(y) = (\mathrm{S}+\varepsilon)(\Phi^{-1}(y)).$$

Let us write $\Phi^*(\Phi+\varepsilon)(y) = \mathrm{NF}^m(y) + \mathrm{B}'(y) + \mathrm{C}'(y)$, with $\mathrm{B}' \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{P}_n^{m+1,\,2m+1}\right)$ and $\mathrm{C}' \in \mathrm{Hom}_{\mathbf{C}}\left(\mathfrak{g}, \mathscr{X}_n^{2m+1}\right)$. Thus, the conjugacy equation (8.1.1) can be written in the following form:

$$\begin{aligned}(\mathbf{8.1.2})\qquad (\mathrm{Id}+\mathrm{D}(\mathrm{U})(y))(\mathrm{NF}^m + \mathrm{B}' + \mathrm{C}')(y) &= (\mathrm{NF}^m + \mathrm{B} + \mathrm{C})(\Phi^{-1}(y))\\ &= \mathrm{NF}^m(y) + \mathrm{D}(\mathrm{NF}^m)(y)\mathrm{U}(y) + \mathrm{B}(y)\\ &\quad + \left(\mathrm{B}(\Phi^{-1}(y)) - \mathrm{B}(y)\right) + \mathrm{C}(\Phi^{-1}(y))\\ &\quad + \mathrm{NF}^m(\Phi^{-1}(y)) - \left(\mathrm{NF}^m(y) + \mathrm{D}(\mathrm{NF}^m)(y)\mathrm{U}(y)\right).\end{aligned}$$

Here, $\mathrm{D}(\mathrm{NF}^m)$ denotes the application which maps $g \in \mathfrak{g}$ to the differential $\mathrm{D}(\mathrm{NF}^m(g))$. This can be written

$$\begin{aligned}\mathrm{C}'(y) + \left(\mathrm{B}'(y) - \mathrm{B}(y) + [\mathrm{NF}^m, \mathrm{U}](y)\right) &= \left(\mathrm{B}(\Phi^{-1}(y)) - \mathrm{B}(y)\right) + \mathrm{C}(\Phi^{-1}(y))\\ &\quad + \mathrm{NF}^m(\Phi^{-1}(y)) - \left(\mathrm{NF}^m(y) + \mathrm{D}(\mathrm{NF}^m)(y)\mathrm{U}(y)\right)\\ &\quad - \mathrm{D}(\mathrm{U})(y)(\mathrm{B}' + \mathrm{C}')(y)\\ &=: \mathrm{D}(y).\end{aligned}$$

Since the order of $(\mathrm{B}(\Phi^{-1}(y)) - \mathrm{B}(y))$ is greater than or equal to the order of $\mathrm{D}(\mathrm{B})(y)\mathrm{U}(y)$ and the order of $\mathrm{NF}^m(\Phi^{-1}(y)) - \left(\mathrm{NF}^m(y) + \mathrm{D}(\mathrm{NF}^m)(y)\mathrm{U}(y)\right)$ is greater than or equal to the order of $\mathrm{D}^2(\mathrm{NF}^m)(y)(\mathrm{U}(y), \mathrm{U}(y))$, then the order of D is greater than or equal to $2m+1$. It follows that $\mathrm{J}^{2m}(\mathrm{B}'(y) - \mathrm{B}(y) + [\mathrm{NF}^m, \mathrm{U}]) = 0$. According to the definition of U, we obtain that $\mathrm{B}' - \mathrm{B}_0 + \mathrm{J}^{2m}\left(-[\mathrm{NF}^m, \mathrm{U}] + [\mathrm{NF}^m, \mathrm{U}]\right) = 0$. Hence $\mathrm{B}' = \mathrm{B}_0 \in \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathscr{P}_{n,0}^{m+1,\,2m})$ and $\Phi^*(\mathrm{S}+\varepsilon)$ is normalized up to order $2m$.

8.2. *Computation of the remainder*

In order to obtain an estimate of C', we will write the equation (8.1.2) in the following way

$$\begin{aligned}\mathrm{C}'(y) = &\left(\mathrm{NF}^m(\Phi^{-1}(y)) - \mathrm{NF}^m(y)\right) + (\mathrm{B}+\mathrm{C})(\Phi^{-1}(y))\\ &- \mathrm{B}'(y) - \mathrm{D}(\mathrm{U})(y)(\mathrm{NF}^m + \mathrm{B}' + \mathrm{C}')(y).\end{aligned}$$

Since $\mathrm{NF}^m(\Phi^{-1}(y)) - \mathrm{NF}^m(y) = \int_0^1 D(\mathrm{NF}^m)(y + t\mathrm{U}(y))\mathrm{U}(y)dt$, we shall use the following definition in order to estimate C′,

$$\text{(8.2.1)} \qquad C'(y) = \int_0^1 D(\mathrm{NF}^m)(y + t\mathrm{U}(y))\mathrm{U}(y)dt + (\mathrm{B}+\mathrm{C})(\Phi^{-1}(y)) - \mathrm{B}'(y) - D(\mathrm{U})(y)(\mathrm{NF}^m + \mathrm{B}' + \mathrm{C}')(y).$$

8.3. *Estimate for the diffeomorphism*

Let $\Phi = (Id+\mathrm{U})^{-1}$ be the normalizing diffemorphism. By assumption, NF^m belongs to $\mathcal{NF}_m(r)$; thus, we can apply proposition (7.3.1) so that

$$|\mathrm{U}|_r \leqslant \frac{c_1}{\omega_{k+1,\mathrm{G}}^d}|\mathrm{B}^*|_r.$$

Since $\mathrm{B}^* \prec \overline{\mathrm{B}}^* + \overline{\mathrm{B}}_0 \prec \overline{\mathrm{R}}_{m+1}$, we have $|\mathrm{B}^*|_r < 1$. It follows that $|\mathrm{U}|_r \leqslant \gamma_k^{-m}$.

Lemma **8.3.1.** — *Under the above assumption and if m is large enough (say $m > m_0$), then for all $0 < \theta \leqslant 1$ and any integer $1 \leqslant i \leqslant n$, we have $|y_i + \theta\mathrm{U}_i(y)|_\mathrm{R} < \rho$. As a consequence, $\Phi(\mathrm{D}_\rho) \supset \mathrm{D}_\mathrm{R}$.*

Proof. — We borrow the proof of Bruno [Bru72] [p. 203]. It is sufficient to show that $\mathrm{R} + |\mathrm{U}|_\mathrm{R} < \rho$. Since the order of U is greater than or equal to $m+1$ then, by (3.1.2) and the inequality above, we have

$$\text{(8.3.1)} \qquad \begin{aligned} |\mathrm{U}|_\mathrm{R} &\leqslant \left(\frac{\mathrm{R}}{r}\right)^{m+1}|\mathrm{U}|_r \\ &\leqslant \left(\gamma_k m^{-2/m}\right)^{m+1}\gamma_k^{-m} \\ &\leqslant \gamma_k m^{-2-2/m} \\ &\leqslant m^{-2-2/m}. \end{aligned}$$

Since $\mathrm{R} = \gamma_k m^{-2/m} r \leqslant m^{-2/m} r$, it is sufficient to show that $m^{-2/m}(r + m^{-2}) < \rho = m^{-1/m} r$, that is $\frac{m^{-2}}{m^{1/m} - 1} < r$. But,

$$\frac{m^{-2}}{m^{1/m}-1} = \frac{m^{-2}}{\exp^{1/m\ln m} - 1} \leqslant \frac{m^{-2}}{1/m\ln m} \leqslant \frac{1}{m\ln m}$$

since $1 + x \leqslant \exp x$ for all $x \in \mathbf{R}^+$. But, for $0 < x$ sufficiently large, we have $2 < x\ln x$. Thus, since $1/2 < r$, we obtain the result: $\frac{m^{-2}}{m^{1/m}-1} < \frac{1}{2} < r$. □

8.4. *Estimates for the remainder*

We have $(R/r)^{m+1} = \gamma_k^{m+1} m^{-2-2/m}$ as well as $\frac{\rho}{r} = m^{-1/m}$. Thus,

$$|U|_R \leqslant \gamma_k m^{-2-2/m} \quad \text{(by (8.3.1))},$$

(8.4.1) $$\leqslant \frac{R}{r} m^{-2}$$

(8.4.2) $$< m^{-2}.$$

As we have seen, $\Phi^*(S + \varepsilon)$ is normalized up to order $2m$ and we have $NF^{2m} = NF^m + B_0$. Since the polynomial vector field of degree $2m$, B_0, is dominated by $\overline{B}$, which is dominated by $\overline{R}_{m+1}$, we have $|B_0|_r \leqslant |R_{m+1}|_r < 1$. It follows that

$$\begin{aligned} |B_0|_R &\leqslant (\gamma_k m^{-2/m})^{m+1} |B_0|_r \\ &\leqslant m^{-2}; \\ |D(B_0)|_R &\leqslant \frac{2m}{R} |B_0|_R \quad \text{(by (3.1.3))} \\ &\leqslant \frac{2m(\gamma_k m^{-2/m})^{m+1}}{R} |B_0|_r \quad \text{(by (3.1.2))} \\ &\leqslant \frac{2m(\gamma_k m^{-2/m})^{m}}{r} |B_0|_r \\ &\leqslant \frac{4}{m} \quad (1/2 < r). \end{aligned}$$

It follows that

$$\begin{aligned} |NF^{2m} - S|_R &= |(NF^m - S) + B_0|_R \leqslant |NF^m - S|_R + |B_0|_R, \\ &< \eta_1 - \frac{8n}{m} + \frac{1}{m^2} \\ &< \eta_1 - \frac{8n}{2m} \quad \text{if} \quad 1 < 4nm; \\ |D(NF^{2m} - S)|_R &= |D(NF^m - S) + D(B_0)|_R \leqslant |D(NF^m - S)|_R + |D(B_0)|_R, \\ &< \eta_1 - \frac{8n}{m} + \frac{4}{m} \leqslant \eta_1 - \frac{8n}{2m}. \end{aligned}$$

That is, $NF^{2m} \in NF_{2m}(R)$. It remains to show that $R_{2m+1} \in \mathscr{B}_{2m+1}(R)$.

We have the following estimates:

$$
\begin{aligned}
|(B+C)\circ\Phi^{-1}|_R &\leqslant |B+C|_\rho \quad \text{by (8.3.1)}\\
&\leqslant \left(m^{-1/m}\right)^{m+1}|B+C|_r\\
&\quad (B+C \text{ is of order } \geqslant m+1)\\
&\leqslant m^{-1}\\
|D(U)(NF^m+B_0)|_R &\leqslant n|D(U)|_R(|NF^m|_R+|B_0|_R),\\
&\leqslant \frac{2nm}{R}|U|_R(|NF^m|_R+|B_0|_R)\\
&\quad (U \text{ is a polynomial of degree } 2m)\\
&\leqslant \frac{2nm}{r}m^{-2}(|NF^m|_R+|B_0|_R) \quad \text{(by (8.4.1))}\\
&\leqslant 4nm^{-1}(|NF^m|_r+m^{-2}) \quad (r\geqslant 1/2);\\
|D(U)(y)C'|_R &\leqslant 4nm^{-1}|C'|_R \quad \text{by the same argument.}
\end{aligned}
$$

Furthermore, for all $0\leqslant\theta\leqslant 1$,

$$
D(NF^m)(y+\theta U(y))U(y) \prec D(\overline{NF^m})(y+\overline{U}(y))\overline{U}(y).
$$

It follows that

$$
\begin{aligned}
\left|\int_0^1 D(NF^m)(y+\theta U(y))U(y)d\theta\right|_R &\leqslant |D(\overline{NF^m})(y+\overline{U}(y))\overline{U}(y)|_R,\\
&\leqslant n|D(NF^m)|_\rho|U|_R\\
&\leqslant \frac{nm}{\rho}|NF^m|_\rho|U|_R\\
&\quad (NF^m \text{ is a polynomial of degree } m),\\
&\leqslant \frac{nm}{r}|NF^m|_r|U|_R \quad (\text{since } NF^m \text{ vanishes at } 0)\\
&\leqslant \frac{nm}{r}|NF^m|_r\frac{R}{r}m^{-2} \quad \text{(by (8.4.1))}\\
&\leqslant 2n|NF^m|_r m^{-1}.
\end{aligned}
$$

According to (8.2.1), we have

$$
\begin{aligned}
|C'|_R &\leqslant \left| \int_0^1 D(NF^m)(y + tU(y))U(y)dt \right|_R + |(B + C)(\Phi^{-1}(y))|_R \\
&\quad + |B'|_R + |D(U)(y)(NF^m + B_0)|_R + |D(U)(y)C'|_R \\
&\leqslant 2n|NF^m|_r m^{-1} + m^{-1} + 4nm^{-1}(|NF^m|_r + m^{-2}) + 4nm^{-1}|C'|_R \\
&\leqslant 6nm^{-1}|NF^m|_r + 4nm^{-3} + 4nm^{-1}|C'|_R.
\end{aligned}
$$

It follows that, if $m > 4n$ then

$$
\begin{aligned}
|C'|_R &\leqslant \frac{2n}{m - 4n}\left(3|NF^m|_r + \frac{2}{m^2}\right) \\
&\leqslant \frac{2n}{m - 4n}(3(|S|_r + \eta_1) + 2).
\end{aligned}
$$

Thus, if $m > 2n(3(|S|_1 + \eta_1) + 2) + 4n$ then $|C'|_R < 1$, that is $C' = R_{2m+1} \in \mathscr{B}_{2m+1}(R)$. This ends the proof of the proposition.

9. Proof of the theorem

In this section, we shall prove our main result:

Theorem **9.0.1.** — *Let* S *be an injective diagonal Lie morphism from a commutative complex Lie algebra* $\mathfrak{g}$ *of dimension* l *to the Lie algebra of linear vector fields* $\mathscr{P}_n^1$ *in* $\mathbf{C}^n$. *Assume S is diophantine. Let* (S', S'') *be its analytic splitting. Then, any holomorphic nonlinear deformation of* S *which is formally completely integrable is holomorphically normalizable. As a consequence it is holomorphically integrable.*

Remark **9.0.2.** — *The theorem does not depend on any choice of a basis of* $\mathfrak{g}$. *Consequently, if* $S + \varepsilon$ *is holomorphically normalizable, so is* $S \circ i + \varepsilon \circ i$ *where* i *is a Lie automorphism of* $\mathfrak{g}$ *(the latter being commutative,* i *is just an automorphism of linear spaces). As already noticed, some of the linear part* $S(i(g_j))$ *may be very wild and may not satisfy Bruno's condition.*

Let $1/2 < r \leqslant 1$ be a positive number and let us consider the sequence $\{R_k\}_{k\geqslant 0}$ of positive numbers defined by induction as follows:

$$
\begin{aligned}
R_0 &= r \\
R_{k+1} &= \gamma_k m^{-2/m} R_k \quad \text{where} \quad m = 2^k.
\end{aligned}
$$

Lemma **9.0.3.** — *The sequence* $\{R_k\}_{k\geqslant 0}$ *converges and there exists an integer* m_1 *such that for all integer* $k > m_1$, $R_k > R_{m_1}/2$.

Proof. — We have $R_{k+1} = r\prod_{i=1}^{k}\gamma_i(2^i)^{-2^{1-i}}$, with $\gamma_i = \left(\frac{c_1}{\omega_{i+1,G}^d}\right)^{-1/2^i}$. Hence, we obtain

$$\ln R_{k+1} = +d\sum_{i=1}^{k}\frac{\ln\omega_{i+1,G}}{2^i} - \ln c_1\sum_{i=1}^{k}\frac{1}{2^i} - 2\ln 2\sum_{i=1}^{k}\frac{i}{2^i} + \ln r.$$

The last two sums are convergent and the first is also convergent by assumption. It follows that there exists an integer m_1, such that

$$\prod_{i=m_1+1}^{+\infty}\gamma_i(2^i)^{-2^{1-i}} > 1/2.$$

Thus, if $k > m_1$ then $R_k = R_{m_1}\prod_{i=m_1+1}^{k}\gamma_i(2^i)^{-2^{1-i}} > \frac{R_{m_1}}{2}$. □

Let $S+\varepsilon$ be a holomorphic deformation of S in a neighbourhood of the origin in $\mathbf{C}^n$. We may assume that it is holomorphic in a neighbourhood of the closed polydisc D_1. Let $m_2 = 2^{k_0}$ be the smallest power of 2 which is greater than $\max(m_0, 2^{m_1})$ where m_0 is the integer defined in proposition (8.0.2). By a polynomial change of coordinates, we can normalize $S+\varepsilon$ up to order m_2: in these coordinates, $S+\varepsilon$ can be written as $NF^{m_2} + R_{m_2+1}$. If necessary, we may apply a diffeomorphism $a\mathrm{Id}$ with $a\in\mathbf{C}^*$ sufficiently small so that $(NF^{m_2}, R_{m_2+1}) \in \mathcal{NF}_{m_2}(1)\times\mathcal{B}_{m_2+1}(1)$. We may define as above the sequence $\{R_k\}_{k\geqslant k_0}$, with $R_{k_0} = 1$. Thus, for all integers $k > k_0$, we have $1/2 < R_k \leqslant 1$.

Let us prove by induction on $k\geqslant k_0$, that there exists a diffeomorphism Ψ_k of $(\mathbf{C}^n, 0)$ such that $\Psi_k^*(NF^{m_2} + R_{m_2+1}) := NF^{2^{k+1}} + R_{2^{k+1}+1}$ is normalized up to order 2^{k+1}, $(NF^{2^{k+1}}, R_{2^{k+1}+1}) \in \mathcal{NF}_{2^{k+1}}(R_{k+1})\times\mathcal{B}_{2^{k+1}+1}(R_{k+1})$ and $|\mathrm{Id} - \Psi_k^{-1}|_{R_{k+1}} \leqslant \sum_{p=k_0}^{k}\frac{1}{2^{2p}}$.

- For $k = k_0$. According to proposition (8.0.2), there exists a diffeomorphism Φ_{k_0} such that $\Phi_{k_0}^*(NF^{2^{k_0}} + R_{2^{k_0}+1}) = NF^{2^{k_0+1}} + R_{2^{k_0+1}+1}$ is normalized up to order 2^{k_0+1}, $(NF^{2^{k_0+1}}, R_{2^{k_0+1}+1}) \in \mathcal{NF}_{2^{k_0+1}}(R_{k_0+1})\times\mathcal{B}_{2^{k_0+1}+1}(R_{k_0+1})$ and $|\mathrm{Id} - \Phi_{k_0}^{-1}|_{R_{k_0+1}} < 1/2^{2k_0}$.
- Let us assume that the result holds for all integers $i\leqslant k-1$. By assumption, $\Psi_{k-1}^*(NF^{m_2} + R_{m_2+1}) = NF^{2^k} + R_{2^k+1}$ is normalized up to order 2^k and $(NF^{2^k}, R_{2^k+1}) \in \mathcal{NF}_{2^k}(R_k)\times\mathcal{B}_{2^k+1}(R_k)$. Since $1/2 < R_k \leqslant 1$, we may apply proposition (8.0.2) : there exists a diffeomorphism Φ_k such that $(\Phi_k\circ\Psi_{k-1})^*(NF^{2^{k_0}} + R_{2^{k_0}+1}) = NF^{2^{k+1}} + R_{2^{k+1}+1}$ is normalized up to order 2^{k+1} and $(NF^{2^{k+1}}, R_{2^{k+1}+1}) \in \mathcal{NF}_{2^{k+1}}(R_{k+1})\times\mathcal{B}_{2^{k+1}+1}(R_{k+1})$.

Let us set $\Psi_k = \Phi_k \circ \Psi_{k-1}$. According to proposition (8.0.2) (or lemma (8.3.1)), we have $|\mathrm{Id} - \Phi_k^{-1}|_{R_{k+1}} < 1/2^{2k}$. It follows that

$$\begin{aligned}|\mathrm{Id} - \Psi_k^{-1}|_{R_{k+1}} &\leqslant \left|(\mathrm{Id} - \Psi_{k-1}^{-1}) \circ \Phi_k^{-1} + (\mathrm{Id} - \Phi_k^{-1})\right|_{R_{k+1}}, \\ &\leqslant \left|(\mathrm{Id} - \Psi_{k-1}^{-1}) \circ \Phi_k^{-1}\right|_{R_{k+1}} + \left|(\mathrm{Id} - \Phi_k^{-1})\right|_{R_{k+1}}.\end{aligned}$$

According to proposition (8.0.2), we have $\Phi_k^{-1}(D_{R_{k+1}}) \subset D_{R_k}$. It follows that

$$\begin{aligned}|\mathrm{Id} - \Psi_k^{-1}|_{R_{k+1}} &\leqslant \left|(\mathrm{Id} - \Psi_{k-1}^{-1})\right|_{R_k} + \left|(\mathrm{Id} - \Phi_k^{-1})\right|_{R_{k+1}} \\ &\leqslant \sum_{p=k_0}^{k-1} \frac{1}{2^{2p}} + \frac{1}{2^{2k}}.\end{aligned}$$

This ends the proof of the induction.

Since $D(1/2) \subset D_{R_k}$ for all integers $k \geqslant k_0$, then the sequence $\{|\Psi_k^{-1}|_{1/2}\}_{k \geqslant k_0}$ is uniformally bounded. Moreover, the sequence $\{\Psi_k^{-1}\}_{k \geqslant k_0}$ converges coefficientwise to a formal diffeomorphism $\hat{\Psi}^{-1}$ (the inverse of the formal normalizing diffeomorphism). Therefore, this sequence converges in $\mathscr{H}_n^n(r)$ (for all $r < 1/2$) to $\hat{\Psi}^{-1}$ (see [GR71]). This means that the normalizing transformation is holomorphic in a neighbourhood of $0 \in \mathbf{C}^n$.

10. Consequences

In this section we shall show how we can obtain the results of Bruno and Vey from our main result. Let us begin with a corollary of our result in the most simple case.

10.1. *Linearization*

Corollary **10.1.1.** — *Let* S *be a diophantine injective diagonal linear morphism from a complex commutative Lie algebra* $\mathfrak{g}$ *to* $\mathscr{P}_n^1$. *Let us assume that* $\widehat{\mathscr{O}}_n^S = \mathbf{C}$. *Let* ($S' : \mathfrak{g} \to \mathscr{P}_{n'}^1$, $S'' : \mathfrak{g} \to \mathscr{P}_{n-n'}^1$) *be the analytic splitting of* S. *If the normal form of a holomorphic nonlinear deformation* $S + \varepsilon$ *of* S *is the direct sum of* S′ *and a formal normal form with respect to* S″, *then* $S + \varepsilon$ *is holomorphically normalizable.*

Corollary **10.1.2.** — *Let* S *be a diophantine injective diagonal linear morphism from a complex commutative Lie algebra* $\mathfrak{g}$ *to* $\mathscr{P}_n^1$. *Let* $S + \varepsilon$ *be a holomorphic nonlinear deformation of* S. *If* S *is formally linearizable, then it is holomorphically linearizable.*

These are direct corollaries of our main result since, in both cases, the nonlinear deformations are formally completely integrable. It should be noticed that the last corollary is stronger than the corresponding results known for commutative local holomorphic diffeomorphisms of $(\mathbf{C}, 0)$ having a common fixed point (resp. C^∞-diffeomorphisms of the circle which are close to rotations) due to Delatte [DeL97] (resp. Moser [Mos90]). The first of these results states that, under a diophantine type condition, which is sligthly weaker than ours, if Φ_1, Φ_2 are commuting local biholomorphisms which map $0 \in \mathbf{C}$ to itself and if one of their linear parts doesn't satisfy any resonance relation and if these linear parts satisfy some diophantine condition, then they are simultaneously formally linearizable and they are simultaneously and holomorphically linearizable. The result of Delatte has been improved to the case of biholomorphisms of $(\mathbf{C}^n, 0)$ by Gramchev and Yoshino in an article [GY99] communicated to the author. Our corresponding result would only require that they are formally linearizable and that their linear parts define a diophantine morphism.

10.2. *Therorems of J. Vey*

Theorem **10.2.1.** — [Vey79] *Let* $X_1, \ldots, X_{n-1}$ *be* $n-1$ *holomorphic vector fields in a neighbourhood of* $0 \in \mathbf{C}^n$*, vanishing at this point. We assume that:*

- *each* X_i *is a volume preserving vector field* ($\mathcal{L}_{X_i}\omega = 0$ *with* ω *a non singular holomorphic n-differential form),*
- *the* 1*-jets* $J^1(X_1), \ldots, J^1(X_{n-1})$ *are diagonal and independent,*
- $[X_i, X_j] = 0$ *for all indices* i, j.

Then, $X_1, \ldots, X_{n-1}$ *are holomorphically and simultaneously normalizable.*

Proof. — Let $\mathfrak{g}$ be a $n-1$-dimensional commutative Lie algebra with a basis $G = \{g_1, \ldots, g_{n-1}\}$. Let ψ be the linear semi-simple and injective morphism defined by $\psi(g_i) = x_i\,\partial/\partial x_i - x_{i+1}\,\partial/\partial x_{i+1}$ for $1 \leqslant i \leqslant n-1$. The weight associated to $Q = (q_1, \ldots, q_n) \in \mathbf{N}^n$, $|Q| \geqslant 2$ and $1 \leqslant j \leqslant n$ is $\alpha_{Q,j}(g_i) = q_i - q_{i+1} + (\delta_{i,j} + \delta_{i+1,j})(-1)^{\delta_{i,j}}$ (the last expression in the sum is 0 if $j \neq i, i+1$, 1 if $j = i+1$ and -1 if $j = i$) and $\beta_Q(g_i) = q_i - q_{i+1}$. First of all, the values of the nonzero weights of ψ on the g_i's are integers; thus, they cannot accumulate at the origin, so that ψ is diophantine. Moreover, if we set $u = x_1 \cdots x_n$, then $\widehat{\mathcal{O}}_n^{\psi} = \mathbf{C}[[u]]$ and $\left(\widehat{\mathcal{X}}_n^1\right)^{\psi}$ is the $\mathbf{C}[[u]]$-module generated by $x_i\,\partial/\partial x_i$, $1 \leqslant i \leqslant n$. An easy computation shows that $X \in \left(\widehat{\mathcal{X}}_n^1\right)^{\psi}$ satisfies $\mathcal{L}_X(u) = 0$ if and only if X belongs to the $\mathbf{C}[[u]]$-module generated by the $\psi(g_i)$'s.

Let us write $J^1(X_i) = \sum_{j=1}^n \mu_{i,j} x_i\,\partial/\partial x_j$. Since X_i is volume preserving then, $\mu_{i,1} + \cdots + \mu_{i,n} = 0$; it follows that $J^1(X_i) = \sum_{j=1}^{n-1} a_{i,j}\psi(g_j)$. By the independence of the

1-jets, the $(n-1)\times(n-1)$ matrix $A_0=(a_{i,j})$ is invertible. Let us set

$$\begin{pmatrix} Y_1 \\ \vdots \\ Y_{n-1} \end{pmatrix} = A_0^{-1} \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \end{pmatrix};$$

they satisfy the same condition as the X_i's and define a nonlinear deformation $\psi+\varepsilon$ of ψ by $\psi+\varepsilon(g_i)=Y_i$. Thus there exists a formal diffeomorphism $\hat{\Phi}$ such that $\hat{\Phi}^*(\psi+\varepsilon)(g_i)=\sum_{j=1}^n \hat{F}_{i,j}(u)x_i\frac{\partial}{\partial x_i}$ for some $\hat{F}_{i,j}\in \mathbf{C}[[u]]$. Since they are volume preserving, $\operatorname{div}\hat{\Phi}^*(\psi+\varepsilon)(g_i)=0$, that is:

$$\sum_{j=1}^n \frac{\partial x_j \hat{F}_{i,j}(u)}{\partial x_j}=0=(\sum_{j=1}^n \hat{F}_{i,j})(u)+u\frac{d(\sum_{j=1}^n \hat{F}_{i,j})}{du}(u).$$

An easy computation shows that $\sum_{j=1}^n \hat{F}_{i,j}=0$. Thus, $\hat{\Phi}^*(\psi+\varepsilon)(g_i)$ admits u as a first integral and $\psi+\varepsilon$ is formally completely integrable. According to our main result, the diffeomorphism $\hat{\Phi}$ is holomorphic in a neighbourhood of the origin. □

Theorem **10.2.2** [Vey78]. — *Let* $X_1,\dots,X_n$ *be* n *holomorphic vector fields in a neighbourhood of* $0\in\mathbf{C}^{2n}$*, vanishing at this point. We assume that:*

- *each* X_i *is a hamiltonian vector field,*
- *the* 1*-jets* $J^1(X_1),\dots,J^1(X_n)$ *are diagonal and independent,*
- $[X_i,X_j]=0$ *for all indices* i,j.

Then $X_1,\dots,X_n$ *are holomorphically and simultaneously normalizable.*

Proof. — This works exactly as before with the following ingredients : let $\mathfrak{g}$ be an n-dimensional commutative Lie algebra. Let ψ be the injective semi-simple linear morphism defined by : $\psi(g_i)=x_i\partial/\partial x_i - y_i\partial/\partial y_i$. As before, the values of its non zero weights on the g_i's are integers. Hence ψ is diophantine. We have $\widehat{\mathcal{O}}_{2n}^{\psi}=\mathbf{C}[[u_1,\dots,u_n]]$ with $u_i=x_iy_i$ and $\left(\widehat{\mathscr{X}}_{2n}^1\right)^{\psi}$ is the $\mathbf{C}[[u_1,\dots,u_n]]$-module generated by $x_i\partial/\partial x_i$ and $y_i\partial/\partial y_i$ with $1\leqslant i\leqslant n$. An easy computation shows that $X\in\left(\widehat{\mathscr{X}}_{2n}^1\right)^{\psi}$ satisfies $\mathscr{L}_X(u_i)=0$, $i=1,\dots,n$, if and only if X belongs to the $\mathbf{C}[[u_1,\dots,u_n]]$-module generated by the $\psi(g_i)$'s.

We define the Y_i's as above; let $\hat{\Phi}$ be a normalizing diffeomorphism. Since the Y_i's are hamiltonian, then $\hat{\Phi}^*Y_i$ admits $u_1,\dots,u_n$ as first integrals. Therefore, $\hat{\Phi}^*Y_i$ belongs to the $\mathbf{C}[[u_1,\dots,u_n]]$-module generated by the $\psi(g_i)$'s. Thus the deformation is formally completely integrable and, by our theorem, the normalizing diffeomorphism is holomorphic in a neighbourhood of $0 \in \mathbf{C}^n$. □

In fact, in his article [Vey78], J. Vey worked with holomorphic functions. Nevertheless, we have a one-to-one correspondence beetwen $\mathcal{M}_{2n}^2$, the square of the maximal ideal of $\mathcal{O}_{2n}$, and the germs of holomorphic hamiltonian vector fields vanishing at 0. This correspondence is defined by

$$\mathrm{H} \mapsto \sum_{i=1}^{n} \left(\frac{\partial \mathrm{H}}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial \mathrm{H}}{\partial x_i} \frac{\partial}{\partial y_i} \right).$$

He worked with functions $\mathrm{H}_1, ..., \mathrm{H}_n$ which are Poisson commuting, whose homogeneous polynomials of smallest degree are independent linear combinations of the $x_i y_i$'s. These functions are "transformed" into a so-called Birkhoff normal form which is nothing but our normal form by the correspondence. We mention here the work of H. Ito [Ito89, Ito92] who generalized the work of Vey in the hamiltonian case (1). His main result shows that the Vey's result holds when the second assumption is changed to: one of the vector fields has a well chosen linear part but nothing is assumed on the others. The linear part of the first one has all the cohomological informations whereas the other may not have any linear part.

The proofs of Vey are as follows: first, there are holomorphic 1-differential forms (1 in the volume-preserving case, n in the hamiltonian case) which have formal first integrals. Since their singular locus are of codimension $\geqslant 2$, then, by Malgrange theorems [Mal76, Mal77, Ram79], these first integrals are holomorphic in a neighbourhood of the origin. Using a theorem of Artin [Tou72] [p. 58], we can show that, in good holomorphic coordinate systems, these first integrals are the generators of the $\mathbf{C}$-algebra of common first integrals of the linear parts. Therefore, in this new coordinate system, the vector fields are linear combinations of the linear parts with holomorphic functions as coefficients. Now, we can use a procedure which has been generalized by the author [Sto97] to show that, by a holomorphic change of coordinates, the vector fields are in normal form.

10.3. *Theorems of A. Bruno*

Let $\mathrm{X} = \mathrm{S} + \mathrm{R}$ be a holomorphic vector field in a neighbourhood of its singular point $0 \in \mathbf{C}^n$ with $\mathrm{S} = \sum_{i=1}^{n} \lambda_i x_i \partial/\partial x_i$ and R a nonlinear vector field. We assume that the following condition is satisfied:

$$(\omega) \qquad -\sum_{k \geqslant 0} \frac{\ln \omega_k}{2^k} < +\infty$$

(1) Added in proof: the author has generalized the work of Ito [Ito89] to the nonhamiltonian case [Sto00a, Sto00b]

where $\omega_k = \inf\{|(Q, \lambda) - \lambda_i| \neq 0,\ 1 \leqslant i \leqslant n,\ Q \in \mathbf{N}^n, 2 \leqslant |Q| \leqslant 2^k, \}$. Let $\dot{x}_i = \psi_i$, $i = 1, \ldots, n$ be its formal normal form.

Let us recall, almost verbatim, the results of A. Bruno [Bru72] [p. 141]. Let us look at complex numbers $\lambda_1, \ldots, \lambda_n$ in the complex plane. Two cases can occur:

• there exists a real line passing through the origin such that all the λ_i's lie on the same side of it;

• for each line passing through the origin, there are λ_i's which lie on each side of this line.

Condition (A_2): if the λ_i's are in the second case, then there exists power series a, b such that $\psi_i = \lambda_i a + \bar{\lambda}_i b$, $i = 1, \ldots, n$.

Let us consider the first case; then we may assume that $\lambda_1, \ldots, \lambda_l$ belong to the real line d passing through the origin of $\mathbf{C}$ whereas $\lambda_{l+1}, \ldots, \lambda_n$ lie on the same side of d. In this case the formal normal form of X is given by

$$\begin{aligned} \dot{y}_i &= \psi_i & i &= 1, \ldots, l \\ \dot{y}_i &= \sum_{j=l+1}^{n} b_{i,j} y_j + \eta_i(Y) & i &= l+1, \ldots, n \end{aligned}$$

where the ψ_i's and $b_{i,j}$'s are formal power series in $y_1, \ldots, y_l$ and the η_i's contain neither linear terms in $y_1, \ldots, y_l$ nor terms independent of these variables.

Condition (A_1'): if the λ_i's are in the first case, then there exists a formal power series a such that $\psi_i = \lambda_i a$, $i = 1, \ldots, l$.

The first case splits into two different cases:

1*: $\lambda_1, \ldots, \lambda_l$ are pairwise commensurable.

1**: there is an uncommensurable pair in $\lambda_1, \ldots, \lambda_l$.

Let $M = (\mu_1, \ldots, \mu_n)$ be the vector whose coordinates are the distances from the $\lambda_i' s$ to the real line d.

Condition (A_1''): if the λ_i's are case 1*, then all the formal power series $b_{i,j}$ are arbitrary. If the λ_i's are in the case 1**, then there exists formal power series $a_{l+1}, \ldots, a_n \in \mathbf{C}[[y_1, \ldots, y_l]]$ such that

1. if $Q \in N := \{P \in \mathbf{Z}^n | \exists i$ such that $p_i \geqslant -1, p_k \geqslant 0$ if $k \neq i, p_1 + \ldots + p_n \geqslant 0\}$ and $(Q, M) = 0$ then $\sum_{i=l+1}^{n} q_i a_i \equiv 0$;

2. the matrix $(b_{i,j} - \delta_{i,j}(\lambda_i a + a_i))_{l+1 \leqslant i,j \leqslant n}$ is nilpotent.

We shall say that the normal form satisfies condition A if it satisfies condition A_2, or A_1' or A_1'' in the respective three cases concerning the linear part S.

Theorem **10.3.1.** — [Bru72] *Let* $X = S + R$ *be a holomorphic vector field as above. We assume that* S *satisfies the Bruno condition* (ω). *If its formal normal form satisfies condition* A, *then* X *is holomorphically normalizable.*

The proof is a straightforward application of our result: let us set $\mathfrak{g} = \mathbf{C}$ and $S(1) = S$; we have $\mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathbf{C}) = \mathbf{C}$. Let $(S' : \mathfrak{g} \to \mathscr{P}^1_{n'}, S'' : \mathfrak{g} \to \mathscr{P}^1_{n-n'})$ be the analytic splitting of S. In the first case, we have $n' = l$. In the second case, we have $n' = n$. Let us first consider this last case and let us define the Lie morphism $\overline{S} : \mathbf{C}^2 \to \mathscr{P}^1_n$ by $\overline{S}(e_1) = S$, $S(e_2) = \overline{S}$ where $\{e_1, e_2\}$ denotes the canonical basis of the commutative Lie algebra $\mathbf{C}^2$ and $\overline{S} := \sum_{i=1}^n \overline{\lambda}_i x_i \partial/\partial x_i$. By assumption, $\overline{S}$ is injective and has its range in a diophantine hull of S. In fact, the weight of S in the space of vector fields are defined by the numbers $(Q, \lambda) - \lambda_i$. An easy computation shows that the weights of $\overline{S}$ are the linear forms $\tilde{\alpha}$ defined by $\tilde{\alpha}(e_1) = (Q, \lambda) - \lambda_i$, $\tilde{\alpha}(e_2) = \overline{(Q, \lambda) - \lambda_i}$; therefore, $\tilde{\alpha} \equiv 0$ if and only if $\alpha \equiv 0$ and $\|\tilde{\alpha}\| = |(Q, \lambda) - \lambda_i|$. Condition A_2 means that the normal form of X belongs to the $\mathscr{O}_n^S$-module generated by $\overline{S}(\mathbf{C}^2)$. Hence X is formally completely integrable.

Let us consider the first case. If we are case 1^*, then condition A'_1 means that X is formally completely integrable. If we are in the case 1^{**}, condition A''_1 means the following: the vector field $\sum_{i=l+1}^n \left(\sum_{j=l+1}^n b_{i,j} y_j + \eta_i(Y)\right) \partial/\partial x_i$ can be written as a sum of vector fields of $\mathbf{C}^{n-n'}$, $aS'' + D + N + R$ where D is the diagonal vector field defined by

$$D = \sum_{k=1}^{n-n'} a_{k+n'} x_{k+n'} \frac{\partial}{\partial x_{k+n'}};$$

N is the **nilpotent** vector field of $\mathbf{C}^{n-n'}$ with coefficients in $\mathscr{O}_n^S$ defined by

$$N = \sum_{i=1}^{n-n'} \left(\sum_{j=1}^{n-n'} \left(b_{i+n',j+n'} - \delta_{i,j}(\lambda_{i+n'} a + a_{i+n'})\right) x_{j+n'} \right) \frac{\partial}{\partial x_{i+n'}};$$

and R is a nonlinear vector field in $\mathbf{C}^{n-n'}$ and $[D, N + R] = 0$. Hence $D + N + R$ is a good deformation of 0 relative to the analytic splitting of S. Therefore, X is formally completely integrable.

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