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RESIDUALLY REDUCIBLE REPRESENTATIONS AND MODULAR FORMS

by C.M. SKINNER [1] *and* A.J. WILES [2]

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1. Introduction

In this paper we give criteria for the modularity of certain two-dimensional Galois representations. Originally conjectural criteria were formulated for compatible systems of λ-adic representations, but a more suitable formulation for our work was given by Fontaine and Mazur. Throughout this paper p will denote an odd prime.

Conjecture (Fontaine-Mazur [FM]). — *Suppose that* $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\mathrm{E})$ *is a continuous representation, irreducible and unramified outside a finite set of primes, where* E *is a finite extension of* $\mathbf{Q}_p$. *Suppose also that*

(i) $\rho|_{\mathrm{I}_p} \simeq \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$, *where* I_p *is an inertia group at* p

(ii) $\det\rho = \psi\varepsilon^{k-1}$ *for some* $k \geqslant 2$ *and is odd,*

where ε *is the cyclotomic character and* ψ *is of finite order. Then* ρ *comes from a modular form.*

To say that ρ comes from a modular form is to mean that there exists a modular form f with the property that $\mathrm{T}(\ell)\, f = \mathrm{trace}\rho(\mathrm{Frob}_\ell)\, f$ for all ℓ at which ρ is unramified. Here $\mathrm{T}(\ell)$ is the ℓ^{th} Hecke operator, and an arbitrary embedding of E into $\mathbf{C}$ is chosen so that $\mathrm{trace}\rho(\mathrm{Frob}_\ell)$ can be viewed in $\mathbf{C}$.

Fontaine and Mazur actually state a more general conjecture where condition (i) is replaced by a more general, but more technical, hypothesis. The condition which we use, which we refer to as the condition that ρ be *ordinary*, is essential to the methods of this paper.

If we pick a stable lattice in E^2, and reduce ρ modulo a uniformizer λ of $\mathcal{O}_\mathrm{E}$, the ring of integers of E, we get a representation $\overline{\rho}$ of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ into $\mathrm{GL}_2(\mathcal{O}_\mathrm{E}/\lambda)$. If $\overline{\rho}$ is irreducible, then it is uniquely determined by ρ. In general we write $\overline{\rho}^{ss}$ for the semisimplification of $\overline{\rho}$, and this is uniquely determined by ρ in all cases. Previous work on this conjecture has mostly focused on the case where $\overline{\rho}$ is irreducible (cf. [W1], [D1]). In that case the main theorems prove weakened versions of the conjecture under the important additional hypothesis that $\overline{\rho}$ has some lifting which is modular. This hypothesis, which is in fact a conjecture of Serre, is as yet unproved.

In this paper we consider the case where $\overline{\rho}$ is reducible, and we prove the following theorem.

Theorem. — *Suppose that* $\rho : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\mathrm{E})$ *is a continuous representation, irreducible and unramified outside a finite set of primes, where* E *is a finite extension of* $\mathbf{Q}_p$. *Suppose also that* $\overline{\rho}^{ss} \simeq 1 \oplus \chi$ *and that*

(i) $\chi|_{\mathrm{D}_p} \neq 1$, *where* D_p *is a decomposition group at* p,

(ii) $\rho|_{\mathrm{I}_p} \simeq \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$,

(iii) $\det\rho = \psi\varepsilon^{k-1}$ *for some* $k \geqslant 2$ *and is odd,*

where ε *is the cyclotomic character and* ψ *is of finite order. Then* ρ *comes from a modular form.*

We also prove similar but weaker statements when $\mathbf{Q}$ is replaced by a general totally real number field: see Theorems A and B of §4.5.

In the irreducible case the proof consists of identifying certain universal deformation rings associated to $\bar{\rho}$ with certain Hecke rings. However in the reducible case even for a fixed $\bar{\rho}^{ss} \simeq 1 \oplus \chi$ we have to consider all the deformation rings corresponding to the possible extensions of χ by 1. These deformation rings are not nearly as well-behaved as in the irreducible case. They are not in general equidimensional. Indeed there is a part corresponding to the reducible representations whose dimension grows with Σ, the finite set of primes at which we permit ramification in the deformation problem. Just as in the irreducible case, we do not know whether there is an irreducible lifting for each extension of χ by 1, but happily we do not need to assume this.

In a previous paper [SW] we examined some special cases where we could identify the deformation rings with Hecke rings. These cases roughly corresponded to the condition that there is a unique extension of 1 by χ. In this paper we proceed quite differently. In particular we do not identify the deformation rings with Hecke rings. As we mentioned earlier, we consider the problem over a general totally real number field. This is not just to extend the theorem but is, in fact, an essential part of the proof. For it allows us by base change to restrict ourselves to situations where the part of the deformation ring corresponding to reducible representations has large codimension inside the full deformation ring. It should be noted that the base change we choose depends on Σ.

We now give an outline of the paper. In §2 we introduce and give a detailed analysis of certain deformation rings $R_{\mathscr{D}}$. These are associated to an extension c of χ by 1. They are given as the universal deformation ring of the representation

$$\rho_c : \mathrm{Gal}(\mathbf{Q}_\Sigma/\mathbf{Q}) \longrightarrow \begin{pmatrix} 1 & * \\ 0 & \chi \end{pmatrix}$$

where the implied extension is given by c. Here $\mathbf{Q}_\Sigma$ is the maximal extension of $\mathbf{Q}$ unramified outside Σ and ∞ although in the main body of the paper $\mathbf{Q}$ is replaced by a totally real field F. More precise definitions are given in §2.1. In §3 we give a corresponding detailed analysis of certain nearly ordinary Hecke rings introduced by Hida. We say that a prime of $R_{\mathscr{D}}$ is pro-modular if the trace of the corresponding representation occurs in a Hecke ring in a sense that is made precise in §4.1. If all the primes on an irreducible component of $R_{\mathscr{D}}$ are pro-modular then we say that the component is pro-modular. If all the irreducible components of $R_{\mathscr{D}}$ are pro-modular then we say that $R_{\mathscr{D}}$ is pro-modular.

The above theorem is deduced from our main result which establishes the pro-modularity of $R_{\mathscr{D}}$ for suitable $\mathscr{D}$. There are three main steps in the proof of this latter result:

(I) We show that if $\mathfrak{p}$ is a "nice" prime of $R_{\mathscr{D}}$ then every component containing $\mathfrak{p}$ is pro-modular. (The definition of a nice prime is given in §4.2; it includes the requirement that $\mathfrak{p}$ itself is pro-modular).

(II) We show that $R_{\mathscr{D}}$ has a nice prime $\mathfrak{p}$.

(III) We show that $R_{\mathscr{D}}$ is pro-modular.

The proof of step (I) is modelled on that for the residually irreducible case and is given in §5-8. The point is that the representation associated to $R_{\mathscr{D}}/\mathfrak{p}$ is irreducible of dimension one and pro-modular. However the techniques of the irreducible case have to be modified as this representation, which we now view as our residual representation, takes values in an infinite field of characteristic p. We should note also that the analog of the patching argument of [TW] is here performed on the deformation rings rather than on the Hecke rings.

The proof of step (III) is given in Proposition 4.1. Steps (I) and (II) show that some irreducible component at the minimum level is modular. Then we use a connectivity result of M. Raynaud (see §A) to show that there is a nice prime in every component at the minimum level. By step (I) again we deduce pro-modularity at the minimum level. A more straightforward argument then shows that there is a nice prime in every component of $R_{\mathscr{D}}$, so that we can again apply step (I) to deduce pro-modularity.

For step (II) we proceed as follows. First we show, using the main result of §3.4 (which in turn uses techniques for proving the existence of congruences between cusp forms and Eisenstein series), that $R_{\mathscr{D}}$ has a nice prime for *some* extension c_0 of χ by 1. Using commutative algebra we show that there are primes in the subring of traces of $R_{\mathscr{D}}$ which correspond to representations with other reduction types, i.e. corresponding to a different extension c (the pair $1, \chi$ are fixed though). We make a construction to show that we can achieve all extensions in this way, and hence find nice primes for all extensions c. These primes are necessarily primes of the ring of traces which do not extend to $R_{\mathscr{D}}$ itself. The proof of step (II) is given in Proposition 4.2. At the start of the proof of this proposition is a more detailed outline of how we carry out step (II).

We now briefly indicate the extra restriction in the case of a general totally real field F. We need to be able to make large solvable extensions of $F(\chi)$, the splitting field of χ, with prescribed local behavior at a finite number of primes and such that the relative class number is controlled. When $F(\chi)$ is abelian over $\mathbf{Q}$ we can do this using a theorem of Washington about the behavior of the p-part of the class number of $\mathbf{Z}_\ell$-extensions. In the general case such a result is not known.

Finally we note that the ordinary hypothesis which is essential to our method is frequently satisfied in applications. For example, suppose that ρ (with $\bar{\rho}$ reducible) arises as the λ-adic representation associated to an abelian variety A over $\mathbf{Q}$ with a field of endomorphisms $K \hookrightarrow \mathrm{End}_{\mathbf{Q}}(A) \otimes \mathbf{Q}$ such that $\dim A = [K : \mathbf{Q}]$. Then the nearly ordinary hypothesis will hold provided A is semistable at p, or even if A acquires semistability over an extension of $\mathbf{Q}_p$ with ramification degree $< p-1$. This can be verified by considering the Zariski closure of $\ker(\lambda)$ in the Neron model of A.

2. Deformation data and deformation rings

2.1. Generators and relations

Let F be a totally real number field of degree d. For any finite set of finite places Σ, let F_Σ be the maximal extension of F unramified outside of Σ and all $v|\infty$. For each place v, fix once and for all an embedding of $\overline{F}$ into $\overline{F}_v$. Doing so fixes a choice of decomposition group D_v and inertia group I_v for each finite place v and a choice of complex conjugation for each infinite place. Let $z_1, \ldots, z_d$ be the d complex conjugations so chosen, and let $v_1, \ldots, v_t$ be the places dividing p. Write D_i and I_i for the decomposition group and inertia group chosen for the place v_i. Let d_i be the degree of F_{v_i} over $\mathbf{Q}_p$. Normalize the reciprocity maps of Class Field Theory so that uniformizers correspond to arithmetic Frobenii and write Frob_v for a Frobenius at a place v. Suppose that k_0 is a finite field of characteristic p and that $\chi : \mathrm{Gal}(\overline{F}/F) \longrightarrow k_0^\times$ is a character such that

- $\chi|_{D_i} \neq 1$ for $i = 1, \ldots, t$
- $\chi(z_i) = -1$ for $i = 1, \ldots, d$.

A *deformation datum* for F is a 4-tuple $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$ consisting of the ring of integers $\mathscr{O}$ of a local field with finite residue field k containing k_0, a finite set of finite places Σ containing all those at which χ is ramified together with $\mathscr{P} = \{v_1, \ldots, v_t\}$, a non-zero cohomology class

$$0 \neq c \in \ker\left\{ H^1(F_\Sigma/F,\ k(\chi^{-1})) \xrightarrow{\text{res}} \bigoplus_{i=1}^{t} H^1(D_i,\ k(\chi^{-1}))\right\}, \tag{2.1}$$

and a set of places $\mathscr{M} \subseteq \Sigma\backslash\mathscr{P}$ at each of which either c is ramified or $\chi|_{I_v}$ is non-trivial. For future reference write $H_\Sigma(F, k)$ for the kernel of the map in (2.1). A cocycle class $c \in H_\Sigma(F, k)$ is called *admissible.*

Let $F(\chi)$ be the splitting field of χ. There is a canonical isomorphism (via the restriction map)

$$H^1(F_\Sigma/F,\ k(\chi^{-1})) \simeq H^1(F_\Sigma/F(\chi),\ k(\chi^{-1}))^{\mathrm{Gal}(F(\chi)/F)}.$$

Using this identification, one sees that for any cocycle c there is a unique representation

$$\rho_c : \mathrm{Gal}(F_\Sigma/F) \longrightarrow \mathrm{GL}_2(k), \qquad \rho_c = \begin{pmatrix} 1 & * \\ & \chi \end{pmatrix}$$

such that

$$\rho_c(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$
$$\rho_c(\sigma) = \begin{pmatrix} 1 & c(\sigma) \\ & 1 \end{pmatrix} \quad \text{for} \quad \sigma \in \mathrm{Gal}(F_\Sigma/F(\chi)).$$

If c is admissible, then ρ_c also satisfies

$$\rho_c|_{D_i} \simeq \begin{pmatrix} 1 & \\ & \chi \end{pmatrix}, \qquad i = 1, \ldots, t.$$

A *deformation* of ρ_c is a local complete Noetherian ring A with residue field k and maximal ideal $\mathfrak{m}_A$ together with a strict equivalence class of continuous representations $\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(A)$ satisfying $\rho_c = \rho \bmod \mathfrak{m}_A$. Such a deformation is *of type-$\mathscr{D}$* if

- A is an $\mathscr{O}$-algebra,
- ρ is unramified outside of Σ and the places above ∞,
- $\rho|_{D_i} \simeq \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$ with $\chi = \psi_1^{(i)} \bmod \mathfrak{m}_A$ for each i, and
- $\rho|_{I_w} \simeq \begin{pmatrix} 1 & * \\ & \tilde{\chi} \end{pmatrix}$ for each $w \in \mathscr{M}$.

Here $\tilde{\chi}$ denotes the Teichmüller lift of χ to A. We usually denote a deformation by a single member of its equivalence class.

For any deformation datum $\mathscr{D}$, there is a universal deformation of type-$\mathscr{D}$

$$\rho_{\mathscr{D}} : \mathrm{Gal}(F_\Sigma/F) \longrightarrow \mathrm{GL}_2(R_{\mathscr{D}}).$$

We omit the precise formulation of the universal property as well as the proof of existence as these are now standard (see [M], [R], [W1]).

A totally real finite extension F′ of F is *permissible for $\mathscr{D}$* provided

- $\rho_c|_{\mathrm{Gal}(\overline{F}/F')}$ is non-split;
- if $v \in \mathscr{M}$ and $\chi|_{I_v} \neq 1$, then $\chi|_{I_w} \neq 1$ for each place w of F′ dividing v;
- if $v \in \mathscr{M}$ and $\rho_c|_{I_v} \neq 1$ but $\chi|_{I_v} = 1$, then $\rho_c|_{I_w} \neq 1$ for each place w of F′ dividing v;
- if w is a place of F′ dividing p, then $\chi|_{D_w} \neq 1$.

Remark **2.1.** — If F′ is permissible for $\mathscr{D}$, then $\mathscr{D}$ determines a deformation datum $\mathscr{D}' = (\mathscr{O}, \Sigma', c, \mathscr{M}')$ for F′ with Σ' and $\mathscr{M}'$ being the sets of places of F′ dividing those in Σ and $\mathscr{M}$, respectively. Clearly, if $\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(A)$ is a deformation of type-$\mathscr{D}$, then $\rho|_{\mathrm{Gal}(\overline{F}/F')}$ is a deformation of type-$\mathscr{D}'$.

In this subsection we give a preliminary analysis of the structure of $R_{\mathscr{D}}$ as an abstract ring. To start, we analyze the *versal* deformation rings associated to representations $\rho : D_i \longrightarrow GL_2(A)$ satisfying $\rho \bmod \mathfrak{m}_A = \chi \oplus 1$ and $\det\rho = \tilde{\chi}$. Such a deformation is a *local $\mathcal{O}$-deformation* if A is an $\mathcal{O}$-algebra, and it is *nearly ordinary* if in addition $\rho \simeq \begin{pmatrix} \tilde{\chi}\psi & * \\ & \psi^{-1} \end{pmatrix}$ with $1 = \psi \bmod \mathfrak{m}_A$. Applying the criteria of Schlessinger as in [M], one sees that there is a versal local $\mathcal{O}$-deformation and a versal nearly ordinary deformation

$$\rho^{(i)} : D_i \longrightarrow GL_2(R^{(i)}) \quad \text{and} \quad \rho^{(i)}_{\mathrm{ord}} : D_i \longrightarrow GL_2(R^{(i)}_{\mathrm{ord}})$$

respectively. The representative $\rho^{(i)}_{\mathrm{ord}}$ can be chosen so that

$$\rho^{(i)}_{\mathrm{ord}} = \begin{pmatrix} \tilde{\chi}\Psi & * \\ & \Psi^{-1} \end{pmatrix}.$$

The following lemma gives a ring-theoretic description of $R^{(i)}_{\mathrm{ord}}$.

Lemma **2.2.** — *Let* ω *be the character giving the action of* D_i *on the pth roots of unity. There is an isomorphism*

$$R^{(i)}_{\mathrm{ord}} \simeq \begin{cases} \mathcal{O}[[x_1, \ldots, x_{2d_i+2}]]/(f) & \text{if } \chi|_{D_i} = \omega \text{ or if } \omega = 1, \\ \mathcal{O}[[x_1, \ldots, x_{2d_i+1}]] & \text{otherwise.} \end{cases}$$

Proof. — Our proof follows along the lines of that of [M, Proposition 2]. Let V be the representation space of $\rho_0|_{D_i}$ where $\rho_0 = \rho_c$ with $c = 0$. Clearly, $V \simeq k \oplus k(\chi)$. Let $\mathrm{ad}\rho_0 = \mathrm{Hom}_k(V, V)$ be the adjoint representation, and let $\mathrm{ad}^0\rho_0$ be the submodule consisting of homomorphisms whose trace is zero. The reduced tangent space of $R^{(i)}_{\mathrm{ord}}$ has dimension equal to

$$r = \dim_k \ker\{ H^1(D_i, \mathrm{ad}^0\rho_0) \longrightarrow H^1(D_i, \mathrm{Hom}_k(k(\chi), k)) \}.$$

A simple calculation using local class field theory and local Galois duality shows that

$$r = \begin{cases} 2d_i + 2 & \text{if } \chi|_{D_i} = \omega \text{ or if } \omega = 1, \\ 2d_i + 1 & \text{otherwise.} \end{cases}$$

It follows that $R^{(i)}_{\mathrm{ord}}$ is a quotient of the power series ring $P = \mathcal{O}[[x_1, \ldots, x_r]]$ by some ideal I. Consider the exact sequence

$$0 \longrightarrow I/\mathfrak{m}I \longrightarrow P/\mathfrak{m}I \longrightarrow R^{(i)}_{\mathrm{ord}} \longrightarrow 0$$

where $\mathfrak{m}$ is the maximal ideal of P. The universal deformation ring $R^{(i)}_1$ for deformations of the trivial character satisfies

$$R^{(i)}_1 \simeq \mathcal{O}[[y_1, \ldots, y_s]]/(h)$$

where

$$\textbf{(2.2)} \qquad s = \dim_k H^1(D_i, k) \quad \text{and if} \quad \omega \neq 1 \quad \text{then} \quad h = 0.$$

This follows immediately from local class field theory. There is a natural map $R_1^{(i)} \longrightarrow R_{\mathrm{ord}}^{(i)}$ corresponding to the character Ψ. Choose a compatible homomorphism $\mathcal{O}[[y_1, \dots, y_s]] \longrightarrow P/\mathfrak{m}I$. This induces a continuous character $\Phi : D_i \longrightarrow (P/(\mathfrak{m}I, h))^\times$ projecting to Ψ. Choose a (continuous) set-theoretic map $\theta : D_i \longrightarrow GL_2(P/(\mathfrak{m}I, h))$ projecting to $\rho_{\mathrm{ord}}^{(i)}$ such that

$$\theta(\sigma) = \begin{pmatrix} \tilde{\chi}\Phi(\sigma) & * \\ & \Phi^{-1}(\sigma) \end{pmatrix}.$$

Define a 2-cocycle $\gamma : D_i \longrightarrow I/(\mathfrak{m}I, h)(\chi)$ by

$$\theta(\sigma_1\sigma_2)\,\theta(\sigma_2)^{-1}\theta(\sigma_1)^{-1} = \begin{pmatrix} 1 & \gamma(\sigma_1, \sigma_2) \\ & 1 \end{pmatrix}$$

and consider its class $[\gamma]$ in $H^2(D_i, \ (I/(\mathfrak{m}I, h))\,(\chi)) \simeq H^2(D_i, k(\chi)) \otimes I/(\mathfrak{m}I, h)$. The map

$$\textbf{(2.3)} \qquad (I/(\mathfrak{m}I, h))^* \longrightarrow H^2(D_i, k(\chi)), \quad f \longmapsto (1 \otimes f)([\gamma])$$

is injective. Here the superscript '$*$' denotes the k-dual. For if $f \in (I/(\mathfrak{m}I, h))^*$ maps to zero, then $\gamma \bmod (\mathfrak{m}I, h, \ker f)$ equals $d\beta$ for some map $\beta : D_i \longrightarrow (I/(\mathfrak{m}I, h, \ker f))(\chi)$, and $\theta' = \begin{pmatrix} 1 & -\beta \\ & 1 \end{pmatrix}\theta$ is a representation into $GL_2(P/(\mathfrak{m}I, h, \ker f))$ that is clearly a nearly ordinary deformation. By the versality of $R_{\mathrm{ord}}^{(i)}$ there is then a homomorphism $R_{\mathrm{ord}}^{(i)} \longrightarrow P/(\mathfrak{m}I, h, \ker f)$ inducing θ', and its composition with the projection $P/(\mathfrak{m}I, h, \ker f) \twoheadrightarrow R_{\mathrm{ord}}^{(i)}$ is an isomorphism. Comparing maps on reduced tangent spaces shows that $R_{\mathrm{ord}}^{(i)} \simeq P/(\mathfrak{m}I, h, \ker f)$, which is possible only if $f = 0$.

Let $g = \dim_k I/\mathfrak{m}I$. This is the minimal number of generators of the ideal I. By (2.2) and the injectivity of (2.3),

$$\begin{aligned} g &\leqslant \dim_k H^2(D_i, k(\chi)) + \begin{cases} 1 & \text{if } \omega = 1 \\ 0 & \text{otherwise} \end{cases} \\ &\leqslant \dim_k H^0(D_i, k(\chi^{-1}\omega)) + \begin{cases} 1 & \text{if } \omega = 1 \\ 0 & \text{otherwise.} \end{cases} \\ &\leqslant \begin{cases} 1 & \text{if } \omega = 1 \ \text{ or if } \ \chi = \omega \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This proves the lemma. □

Corollary **2.3.** — *The ring* $R^{(i)}_{\mathrm{ord}}$ *is a quotient of* $R^{(i)}$ *by an ideal generated by*

$$d_i + \begin{cases} 2 & \text{if } \chi|_{D_i} = \omega = \chi^{-1}|_{D_i} \\ 1 & \text{if } \chi|_{D_i} = \omega \neq \chi^{-1}|_{D_i},\ \chi|_{D_i} \neq \omega = \chi^{-1}|D_i, \text{ or } \omega = 1 \\ 0 & \text{otherwise} \end{cases}$$

elements.

Proof. — The ring $R^{(i)}$ is a quotient of $\mathcal{O}[[y_1, \ldots, y_{r'}]]$ with

$$\begin{aligned} r' &= \dim_k H^1(D_i, \mathrm{ad}^0\rho_0) \\ &= 3d_i + \begin{cases} 3 & \text{if } \chi = \omega = \chi^{-1} \\ 2 & \text{if } \omega = 1 \text{ or } \chi = \omega \neq \chi^{-1} \text{ or } \chi \neq \omega = \chi^{-1} \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Combining this with the previous lemma and the fact that $R^{(i)}_{\mathrm{ord}}$ is a quotient of $R^{(i)}$ yields the corollary. □

The above lemma and its corollary, together with minor variations of the methods used to prove them, yield the following ring-theoretic description of $R_{\mathcal{D}}$. Let δ_F be the $\mathbf{Z}_p$-rank of the Galois group of the maximal abelian pro-p-extension of F unramified away from primes above p.

Proposition **2.4.** *Suppose that* $\mathcal{D} = (\mathcal{O}, \Sigma, c, \mathcal{M})$ *is a deformation datum. There exist integers* g *and* r*, depending on* $\mathcal{D}$*, such that*

$$R_{\mathcal{D}} \simeq \mathcal{O}[[x_1, \ldots, x_g]]/(f_1, \ldots, f_r)$$

and

$$g - r \geqslant d + \delta_F - 2t - 3 \cdot \#\mathcal{M}.$$

Recall that t is the number of places of F dividing p.

Proof. — First we introduce an auxiliary deformation problem. A deformation $\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(A)$ of ρ_c is of *auxiliary type-*$\mathcal{D}$ if

- A is an $\mathcal{O}$-algebra
- $\det \rho = \tilde{\chi}$
- ρ is unramified outside of Σ and the places above ∞.

There is a universal deformation of auxiliary type-$\mathcal{D}$

$$\rho^{\mathrm{aux}}_{\mathcal{D}} : \mathrm{Gal}(F_\Sigma/F) \longrightarrow \mathrm{GL}_2(R^{\mathrm{aux}}_{\mathcal{D}}).$$

Clearly, there are natural maps $\varphi_i : R^{(i)} \longrightarrow R_{\mathscr{D}}^{aux}$ corresponding to $\rho_{\mathscr{D}}^{aux}|_{D_i}$ for $i = 1, \ldots, t$. Let J_i be the kernel of the projection $R^{(i)} \longrightarrow R^{(i)}_{ord}$, and let J be the ideal generated by $\cup \varphi_i(J_i)$. It follows from Corollary 2.3 that

$$\text{(2.4)} \qquad \text{J is generated by} \quad \sum_{i=1}^{t}(d_i + 2) = d + 2t \text{ elements.}$$

Now $\rho_{\mathscr{D}}^{aux} \bmod J$ is clearly a deformation of type-$\mathscr{D}'$, where $\mathscr{D}' = (\mathscr{O}, \Sigma, c, \phi)$. Using the versality of the various rings one finds that

$$\text{(2.5)} \qquad R_{\mathscr{D}'} \simeq R_{\mathscr{D}}^{aux}/J \otimes_{\mathscr{O}} \mathscr{O}[[\mathrm{Gal}(L(\Sigma)/F)]]$$

where $L(\Sigma)$ is the maximal abelian pro-p-extension of F unramified away from Σ. (One difference between deformations of type-$\mathscr{D}'$ and auxiliary deformations of type-$\mathscr{D}$ is that the former include deformations of the determinant whereas the latter do not.) It is easy to see that there is an isomorphism

$$\text{(2.6)} \qquad \mathscr{O}[[\mathrm{Gal}(L(\Sigma)/F)]] \simeq \mathscr{O}[[x_1, \ldots, x_{\delta_F}, y_1, \ldots, y_s]]/(g_1, \ldots, g_s).$$

Let I be the kernel of $R_{\mathscr{D}'} \twoheadrightarrow R_{\mathscr{D}}$. For each $v \in \mathscr{M}$ let $\tau_v \in I_v$ be a generator of the p-part of tame inertia. Choose for each $v \in \mathscr{M}$ a basis for $\rho_{\mathscr{D}'}$ such that $\rho_{\mathscr{D}'}(\tau_v) \bmod I = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$. Write $\rho_{\mathscr{D}'}(\tau_v) = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}$ with respect to the basis. Clearly, I is generated by the set $\{ a_v - 1,\ d_v - 1,\ c_v : v \in \mathscr{M} \}$. It follows that

$$\text{(2.7)} \qquad \text{I is generated by} \quad 3 \cdot \#\mathscr{M} \text{ elements.}$$

Arguing as in [M, Proposition 2] shows that

$$R_{\mathscr{D}}^{aux} \simeq \mathscr{O}[[x_1, \ldots, x_{g'}]]/(f_1, \ldots, f_{r'})$$

where

$$g' = \dim_k H^1\left(F_\Sigma/F,\ \mathrm{ad}^0\rho_c\right), \quad r' \leqslant \dim_k H^2(F_\Sigma/F,\ \mathrm{ad}^0\rho_c).$$

Combining this with (2.4), (2.5), (2.6), and (2.7) shows that

$$R_{\mathscr{D}} \simeq \mathscr{O}[[x_1, \ldots, x_g]]/(f_1, \ldots, f_r)$$

where

$$g = g' + s + \delta_F \quad \text{and} \quad r = r' + s + d + 2t + 3 \cdot \#\mathscr{M}.$$

The desired bound for $g - r$ is a consequence of the global Euler characteristic formula for $\mathrm{ad}^0\rho_c$. □

We conclude this subsection with two simple facts about deformation rings. Suppose that $\mathscr{D}$ is a deformation datum. Fix a basis for $\rho_{\mathscr{D}}$. With respect to the basis write $\rho_{\mathscr{D}}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$ for each $\sigma \in \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$. Let $\mathrm{R}' \subseteq \mathrm{R}_{\mathscr{D}}$ be the $\mathscr{O}$-subalgebra generated by $\{a_\sigma, b_\sigma, c_\sigma, d_\sigma \mid \sigma \in \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})\}$. Let $\mathfrak{m}' = \mathfrak{m}_{\mathscr{D}} \cap \mathrm{R}'$, where $\mathfrak{m}_{\mathscr{D}}$ is the maximal ideal of $\mathrm{R}_{\mathscr{D}}$. Let $\mathrm{R}_1 = \mathrm{R}'_{\mathfrak{m}'}$ and denote by $\widehat{\mathrm{R}}_1$ the completion of R_1 at its maximal ideal. The inclusion $\mathrm{R}_1 \subseteq \mathrm{R}_{\mathscr{D}}$ induces a map $i : \widehat{\mathrm{R}}_1 \longrightarrow \mathrm{R}_{\mathscr{D}}$.

Lemma **2.5.** — *The map* $i : \widehat{\mathrm{R}}_1 \longrightarrow \mathrm{R}_{\mathscr{D}}$ *is surjective.*

Proof. — Let $\mathfrak{m}_1$ be the maximal ideal of $\widehat{\mathrm{R}}_1$. Let $\rho_1 : \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F}) \longrightarrow \mathrm{GL}_2(\widehat{\mathrm{R}}_1)$ be defined by $\rho_1(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$. Clearly, composing ρ_1 with the homomorphism $\mathrm{GL}_2(\widehat{\mathrm{R}}_1) \longrightarrow \mathrm{GL}_2(\mathrm{R}_{\mathscr{D}})$ induced by i yields $\rho_{\mathscr{D}}$. It follows from the definitions of R_1 and ρ_1 that $\rho_1 \bmod \mathfrak{m}_1 = \rho_c$. Let $\mathfrak{a} = \mathfrak{m}_1 \mathrm{R}_{\mathscr{D}}$. The deformation $\rho_{\mathscr{D}} \bmod \mathfrak{a}$ is the same as the deformation obtained by composing $\rho_1 \bmod \mathfrak{m}_1$ with the homomorphism $\mathrm{GL}_2(k) = \mathrm{GL}_2(\widehat{\mathrm{R}}_1/\mathfrak{m}_1) \longrightarrow \mathrm{GL}_2(\mathrm{R}_{\mathscr{D}}/\mathfrak{a})$ obtained from i. As $\rho_1 \bmod \mathfrak{m}_1 = \rho_c$, it follows from the universality of $\mathrm{R}_{\mathscr{D}}$ that there is a unique map $\mathrm{R}_{\mathscr{D}} \longrightarrow \widehat{\mathrm{R}_1}/\mathfrak{m}_1$ whose kernel is necessarily $\mathfrak{m}_{\mathscr{D}}$. The composition $\mathrm{R}_{\mathscr{D}} \longrightarrow \widehat{\mathrm{R}_1}/\mathfrak{m}_1 \longrightarrow \mathrm{R}_{\mathscr{D}}/\mathfrak{a}$ must be the same as the canonical map $\mathrm{R}_{\mathscr{D}} \longrightarrow \mathrm{R}_{\mathscr{D}}/\mathfrak{a}$. Therefore $\mathfrak{a} = \mathfrak{m}_{\mathscr{D}}$. This proves that $\dim_k(\mathrm{R}_{\mathscr{D}}/\mathfrak{m}_1\mathrm{R}_{\mathscr{D}}) = 1$, from which it follows that $\mathrm{R}_{\mathscr{D}}$ is generated as an $\widehat{\mathrm{R}_1}$-module by one element (cf. [Mat, Theorem 8.4]). □

For future reference we also record the following fact.

Lemma **2.6.** — *If* $\mathfrak{p} \subseteq \mathrm{R}_{\mathscr{D}}$ *is not the maximal ideal, and if* $\mathfrak{p}_1 = \mathrm{R}_1 \cap \mathfrak{p}$, *then* $\dim \mathrm{R}_1/\mathfrak{p}_1 \geqslant 1$.

Proof. — If $\mathfrak{p}_1$ is maximal, then $\mathfrak{p}_1$, and hence also $\mathfrak{p}$, contains a uniformizer of $\mathscr{O}$. In the deformation $\rho_{\mathscr{D}} \bmod \mathfrak{p}_1\mathrm{R}_{\mathscr{D}}$ the matrix entries are in k. Therefore the deformation $\rho_{\mathscr{D}} \bmod \mathfrak{p}$ is obtained by composing $\rho_{\mathscr{D}} \bmod \mathfrak{m}_{\mathscr{D}}$ with the natural inclusion $k \hookrightarrow \mathrm{R}_{\mathscr{D}}/\mathfrak{p}$. From the universality of $\mathrm{R}_{\mathscr{D}}$ it then follows that $\mathfrak{p} = \mathfrak{m}_{\mathscr{D}}$. □

2.2. Reducible deformations

A reducible deformation of ρ_c is a deformation ρ such that $\rho \simeq \begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix}$. In this subsection we analyze the universal reducible deformation of type-$\mathscr{D}$ where $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$. Write

$$\rho_{\mathscr{D}}^{\mathrm{red}} : \mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F}) \longrightarrow \mathrm{GL}_2(\mathrm{R}_{\mathscr{D}}^{\mathrm{red}})$$

for the universal reducible deformation. A consequence of our analysis of $\rho_{\mathscr{D}}^{\mathrm{red}}$ will be an upper bound for the dimension of $\mathrm{R}_{\mathscr{D}}^{\mathrm{red}}$. This bound will be important in our subsequent analysis of $\mathrm{R}_{\mathscr{D}}$.

Choose a basis for $\rho_{\mathscr{D}}$ such that $\rho_{\mathscr{D}}(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. For $\sigma \in \mathrm{Gal}(F_\Sigma/F)$ write $\rho_{\mathscr{D}}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$, and let I be the ideal generated by the c_σ's. Clearly,

$$R_{\mathscr{D}}^{\mathrm{red}} = R_{\mathscr{D}}/I \quad \text{and} \quad \rho_{\mathscr{D}}^{\mathrm{red}} = \rho_{\mathscr{D}} \bmod I.$$

Unfortunately, this description of $R_{\mathscr{D}}^{\mathrm{red}}$ does not easily yield a non-trivial bound for the dimension. Therefore we take a more pedestrian approach.

Let $L(\Sigma)$ be the maximal abelian pro-p-extension of F unramified away from Σ. Write

$$G = \mathrm{Gal}(L(\Sigma)(\chi)/F) \simeq \Delta \times \Gamma \times \mathbf{Z}_p^{\delta_F}$$

where $\Delta \simeq \mathrm{Gal}(F(\chi)/F)$, Γ is a finite p-group, and $L(\Sigma)(\chi) = L(\Sigma) \cdot F(\chi)$. Let M be the maximal abelian pro-p-extension of $L(\Sigma)(\chi)$ unramified away from $\Sigma\backslash\mathscr{P}$ and such that Δ acts on $\mathrm{Gal}(M/L(\Sigma)(\chi))$ via the unique representation over $\mathbf{Z}_p$ associated to χ^{-1}. Any reducible deformation of type-$\mathscr{D}$ factors through Gal(M/F).

Put

$$A = \mathbf{Z}_p[[G]] \simeq \mathbf{Z}_p[[\Delta \times \Gamma]]\,[[T_1, \ldots, T_{\delta_F}]].$$

The group $H = \mathrm{Gal}(M/L(\Sigma)(\chi))$ is a finitely generated A-module generated by m elements where

$$\begin{aligned} m &= \dim_{k'} H/\mathfrak{m}_A \\ &= \dim_k \ker\{H^1(F_\Sigma/F, k(\chi^{-1})) \longrightarrow \bigoplus_{i=1}^{t} H^1(I_i, k(\chi^{-1}))\} \\ &= \dim_k H_\Sigma(F, k). \end{aligned}$$

Note that by our hypothesis that $\chi|_{D_i} \neq 1$, $H^1(D_i, k(\chi^{-1}) \simeq H^1(I_i, k(\chi^{-1}))^{D_i}$. Here $\mathfrak{m}_A$ is the maximal ideal of A corresponding to χ^{-1} and k' is the residue field of $\mathbf{Z}_p[\Delta]$ associated to χ^{-1}. Fix a presentation

$$\mathfrak{a} \longrightarrow \bigoplus_{i=1}^{m} Ae_i \twoheadrightarrow H$$

such that e_m projects to an element h_m of H for which $\rho_c(h_m) = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}$ with $u \neq 0$ and such that if $i \neq m$ then e_i projects to an element h_i of H for which $\rho_c(h_i) = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. Choose an element $u_0 \in \mathscr{O}$ reducing to u. Put

$$A_1 = \mathscr{O}[[\Gamma]]\,[[T_1, \ldots, T_{\delta_F}]], \quad A_2 = \mathscr{O}[[\Gamma]]\,[[S_1, \ldots, S_{\delta_F}]]$$

and fix embeddings

$$\varphi_i : A \hookrightarrow A \otimes_{\mathbf{Z}_p[\Delta]} \mathscr{O} \simeq A_i$$

where for φ_1 the map $\mathbf{Z}_p[\Delta] \longrightarrow \mathcal{O}$ is that induced by $\tilde{\chi}^{-1}$ and for φ_2 it is that induced by $\tilde{\chi}$. Let J be the ideal of $A_1[[x_1, \ldots, x_{m-1}]]$ generated by

$$\{ \varphi_1(a_1)x_1 + \varphi_1(a_2)x_2 + \cdots + \varphi_1(a_{m-1})x_{m-1} + \varphi_1(a_m)u_0 : \Sigma a_i e_i \in \mathfrak{a} \}.$$

Fix a homomorphism of A-modules

$$\tau : H \longrightarrow B = A_1[[x_1, \ldots, x_{m-1}]]/J; \qquad \begin{array}{l} e_i \longmapsto x_i, \quad i = 1, \ldots, m-1 \\ e_m \longmapsto u_0. \end{array}$$

Put $R = B\hat{\otimes}_{\mathcal{O}} A_2$. Observe that $\mathrm{Gal}(M/F) \simeq G \rtimes H$. We may therefore define a reducible deformation of type-$\mathcal{D}'$, $\mathcal{D}' = (\mathcal{O}, \Sigma, c, \emptyset)$

$$\rho : \mathrm{Gal}(M/F) \longrightarrow \mathrm{GL}_2(R)$$

by

$$\rho(g) = \begin{pmatrix} \varphi_1(g) \otimes \varphi_2(g) & \\ & \varphi_2(g) \end{pmatrix}, \qquad g \in G,$$

$$\rho(h) = \begin{pmatrix} 1 & \tau(h) \\ & 1 \end{pmatrix}, \qquad h \in H.$$

The deformation ρ is readily seen to be the universal reducible deformation of type-$\mathcal{D}'$. As easy consequences of this explicit description of $R^{\mathrm{red}}_{\mathcal{D}'}$ we obtain the following estimates.

Lemma **2.7.** — *We have* $\dim R^{\mathrm{red}}_{\mathcal{D}} \leqslant 1 + 2\delta_F + \dim_k H_\Sigma(F, k)$.

Lemma **2.8.** — *If* $\mathfrak{q} \subseteq R_{\mathcal{D}}$ *is a prime containing* p *such that* $\rho_{\mathcal{D}} \bmod \mathfrak{q}$ *is reducible and its determinant has finite order, then* $\dim R_{\mathcal{D}}/\mathfrak{q} \leqslant \delta_F + \dim_k H_\Sigma(F, k)$.

A *diagonal deformation* of the representation $1 \oplus \chi$ is a representation $\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(A)$, A a complete local Noetherian ring with residue field k, such that $\rho = \begin{pmatrix} \phi_1 & \\ & \phi_2 \end{pmatrix}$ with $1 = \phi_1 \bmod \mathfrak{m}_A$ and $\chi = \phi_2 \bmod \mathfrak{m}_A$. Such a representation $\rho : \mathrm{Gal}(F_\Sigma/F) \longrightarrow \mathrm{GL}_2(A)$ is a diagonal deformation *of type-*$(\mathcal{O}, \Sigma)$ if A is a local $\mathcal{O}$-algebra ($\mathcal{O}$ and Σ as in the definition of a deformation datum). There is a universal diagonal deformation of type-$(\mathcal{O}, \Sigma)$: $\rho^{\mathrm{diag}}_{(\mathcal{O},\Sigma)} : \mathrm{Gal}(F_\Sigma/F) \longrightarrow \mathrm{GL}_2(R^{\mathrm{diag}}_{(\mathcal{O},\Sigma)})$. Later we shall need to know an upper bound for the dimension of various primes of $R^{\mathrm{diag}}_{(\mathcal{O},\Sigma)}$. The proof of the following lemma is similar to, but much simpler than, the proofs of Lemmas 2.7 and 2.8 and hence is omitted.

Lemma **2.9.** — (i) $\dim R^{\mathrm{diag}}_{(\mathcal{O},\Sigma)} \leqslant 1 + 2\delta_F$. (ii) *If* $\mathfrak{q} \subseteq R^{\mathrm{diag}}_{(\mathcal{O},\Sigma)}$ *is a prime containing* p *such that* $\det \rho^{\mathrm{diag}}_{(\mathcal{O},\Sigma)} \bmod \mathfrak{q}$ *has finite order, then* $\dim R^{\mathrm{diag}}_{(\mathcal{O},\Sigma)}/\mathfrak{q} \leqslant \delta_F$.

2.3. Some special deformations

In the course of our analysis of the rings $R_{\mathscr{D}}$ we shall sometimes have to consider some augmented deformation problems as well as deformations of various restricted type. Here we introduce these deformations and, when applicable, their universal deformation rings.

Let $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$ be a deformation datum and Q a finite set of finite places, $Q = \{w_1, \ldots, w_r\}$, disjoint from Σ. A deformation ρ of type-$(\mathscr{O}, \Sigma \cup Q, c, \mathscr{M})$ is *of type-*$\mathscr{D}_Q$ if

- $\det\rho$ is unramified at each $w_i \in Q$.

There exists a universal deformation of type-$\mathscr{D}_Q$:

$$\rho_{\mathscr{D}_Q} : \mathrm{Gal}\,(F_{\Sigma\cup Q}/F) \longrightarrow GL_2(R_{\mathscr{D}_Q}).$$

For a deformation datum $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$, $\Sigma_c \subseteq \Sigma$ is the subset of places at which ρ_c is ramified together with the set $\mathscr{P}$. Similarly, $\mathscr{M}_c = \Sigma_c \backslash \mathscr{P}$. Also, we write Σ_0 for the set of finite places at which χ is ramified together with $\mathscr{P}$. Given $\mathscr{D}$ we write $\mathscr{D}_c$ for the deformation datum $\mathscr{D}_c = (\mathscr{O}, \Sigma_c, c, \mathscr{M}_c)$.

A deformation $\rho : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(A)$ of type-$\mathscr{D}$ is *nice* if

- A is a one-dimensional domain of characteristic p,
- ρ is a deformation of type-$\mathscr{D}_c$,
- $\rho|_{D_i} \simeq \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$ with $\psi_1^{(i)}/\psi_2^{(i)}$ having infinite order for $i = 1, \ldots, t$.

For a deformation datum $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$ let L_Σ/F be the maximal abelian pro-p-extension of F unramified away from Σ, and let N_Σ be the torsion subgroup of $\mathrm{Gal}(L_\Sigma/F)$. A deformation $\rho : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(A)$ of type-$\mathscr{D}_Q$ is $\mathscr{D}_Q$*-minimal* ($\mathscr{D}$*-minimal* if $Q = \emptyset$) if $\det\rho$ is trivial on N_Σ. Let

$$\rho^{\min}_{\mathscr{D}_Q} : \mathrm{Gal}\,(F_{\Sigma\cup Q}/F) \longrightarrow GL_2(R^{\min}_{\mathscr{D}_Q})$$

be the universal $\mathscr{D}_Q$-minimal deformation. If $Q = \emptyset$, then we just write $\rho^{\min}_{\mathscr{D}}$ and $R^{\min}_{\mathscr{D}}$. There is a simple relation between $R_{\mathscr{D}_Q}$ and $R^{\min}_{\mathscr{D}_Q}$. We fix for each Σ a free $\mathbf{Z}_p$-summand $H_\Sigma \subseteq \mathrm{Gal}(L_\Sigma/F)$ such that $\mathrm{Gal}(L_\Sigma/F) \simeq H_\Sigma \oplus N_\Sigma$. We choose the H_Σ's to be compatible with varying Σ. Let $\Psi_\Sigma : \mathrm{Gal}(L_\Sigma/F) \twoheadrightarrow N_\Sigma$ denote the character obtained by projecting modulo H_Σ. The representation $\rho^{\min}_{\mathscr{D}_Q} \otimes \Psi_\Sigma : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(R^{\min}_{\mathscr{D}_Q} \otimes_{\mathscr{O}} \mathscr{O}[N_\Sigma])$ is easily seen to be a deformation of type-$\mathscr{D}_Q$. It follows from the universal properties of $R_{\mathscr{D}_Q}$ and $R^{\min}_{\mathscr{D}_Q}$ that

$$R_{\mathscr{D}_Q} \simeq R^{\min}_{\mathscr{D}_Q} \otimes_{\mathscr{O}} \mathscr{O}[N_\Sigma] \quad \text{and} \quad \rho_{\mathscr{D}_Q} \simeq \rho^{\min}_{\mathscr{D}_Q} \otimes \Psi_\Sigma.$$

Suppose that $\mathscr{K}$ is a field and that $\rho : \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F}) \longrightarrow \mathrm{GL}_2(\overline{\mathscr{K}})$ is a representation. For each place $w \nmid p$ at which ρ is ramified we distinguish for future reference four possibilities for $\rho|_{\mathrm{I}_w}$:

Type A $\rho|_{\mathrm{I}_w} \simeq \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$, $* \not\equiv 0$.

Type B $\rho|_{\mathrm{I}_w} \simeq \begin{pmatrix} \phi & \\ & 1 \end{pmatrix}$, ϕ a finite character.

Type B′ $\rho|_{\mathrm{I}_w} \simeq \begin{pmatrix} \phi & \\ & \phi^{-1} \end{pmatrix}$, ϕ a finite character.

Type C $\rho|_{\mathrm{D}_w} = \mathrm{Ind}_{\mathrm{F}_{w^2}}^{\mathrm{F}_w} \psi$ where F_{w^2} is the unique unramified quadratic extension of F_w and ψ is a character of $\mathrm{Gal}(\overline{\mathrm{F}}_w/\mathrm{F}_{w^2})$ such that $\psi|_{\mathrm{I}_w} \not\equiv \psi^{\mathrm{Frob}_w}|_{\mathrm{I}_w}$, $\psi|_{\mathrm{I}_w}$ has p-power order, and $\det \rho|_{\mathrm{D}_w}$ has order prime to p.

Note that Type C can only occur if the characteristic of $\mathscr{K}$ is zero.

2.4. Pseudo-deformations

For our purposes, following [W2] a (2-dimensional) *pseudo-representation* of $\mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$ into a topological ring A is a set $\rho = \{a, d, x\}$ of continuous functions $a, d : \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F}) \longrightarrow \mathrm{A}$ and $x : \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})^2 \longrightarrow \mathrm{A}$ such that

- $a(\sigma\tau) = a(\sigma)a(\tau) + x(\sigma, \tau)$,
- $d(\sigma\tau) = d(\sigma)d(\tau) + x(\tau, \sigma)$,
- $x(\sigma, \tau)x(\alpha, \beta) = x(\sigma, \beta)x(\alpha, \tau)$,
- $x(\sigma\tau, \alpha\beta) = a(\sigma)a(\beta)x(\tau, \alpha) + a(\beta)d(\tau)x(\sigma, \alpha) + a(\sigma)d(\alpha)x(\tau, \beta) + d(\tau)d(\alpha)x(\sigma, \beta)$,
- $a(1) = 1 = d(1)$,
- $a(z_1) = 1 = -d(z_1)$, and
- $x(\sigma, g) = 0 = x(g, \sigma)$ if $g = 1$ or z_1.

The *trace* and *determinant* of ρ are

$$\mathrm{trace}\,\rho(\sigma) = a(\sigma) + d(\sigma) \quad \text{and} \quad \det \rho(\sigma) = a(\sigma)d(\sigma) - x(\sigma, \sigma).$$

Suppose that $\rho : \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F}) \longrightarrow \mathrm{GL}_2(\mathrm{A})$ is a continuous representation such that $\rho(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Write $\rho(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$. The functions $a(\sigma) = a_\sigma$, $d(\sigma) = d_\sigma$, and $x(\sigma, \tau) = b_\sigma c_\tau$ form a pseudo-representation, and the trace and determinant of this pseudo-representation are merely the trace and determinant of the representation ρ.

Let ρ_0 be the pseudo-representation associated to the representation $1 \oplus \chi$ (i.e., $a = 1$, $d = \chi$, and $x = 0$). A *pseudo-deformation* of ρ_0 is a pair (A, ρ) consisting of a local complete Noetherian ring A with residue field k (which we assumed finite) and maximal ideal $\mathfrak{m}_\mathrm{A}$ and a pseudo-representation ρ of $\mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$ into A such that $\rho \bmod \mathfrak{m}_\mathrm{A} = \rho_0$. We often just write ρ to mean such a pair (A, ρ). A *pseudo-datum* for F is a pair $\mathscr{D}^{\mathrm{ps}} = (\mathscr{O}, \Sigma)$ consisting of the ring of integers $\mathscr{O}$ of some local field having residue field k and a finite set of finite places Σ of F containing $\mathscr{P}$ and those places at which χ is ramified. A pseudo-deformation ρ of ρ_0 is *of type-*$\mathscr{D}^{\mathrm{ps}}$ if

- A is an $\mathscr{O}$-algebra and
- ρ is unramified outside of Σ (i.e., a, d, and x factor through $\mathrm{Gal}(F_\Sigma/F)$).

It is relatively straightforward to verify that the functor $F_{\mathscr{D}^{\mathrm{ps}}}$ from the category of local complete Noetherian $\mathscr{O}$-algebras with residue field k to the category of sets given by

$$F_{\mathscr{D}^{\mathrm{ps}}}(A) = \{\text{pseudo-deformations into A of type-}\mathscr{D}^{\mathrm{ps}}\}$$

satisfies the criteria of Schlessinger [Sch]. The only non-trivial point is the finiteness of the tangent space, and this is provided by the following lemma.

Lemma **2.10.** — *Let* $k[\varepsilon]$ *be the "dual numbers". Then* $\#F_{\mathscr{D}^{\mathrm{ps}}}(k[\varepsilon]) = (\#k)^r$, *where*

$$r \leqslant 4(\#\mathrm{Gal}(F(\chi)/F)) + 2(\dim_k H^1(F_\Sigma/F(\chi), k))^2 + 4.$$

Proof. — If $\rho = \{a, d, x\} \in F_{\mathscr{D}^{\mathrm{ps}}}(k[\varepsilon])$, then

$$a = 1 + \varepsilon a_1, \quad d = \chi + \varepsilon d_1, \quad \text{and } x = \varepsilon x_1.$$

If $\rho' = \{a', d', x'\}$ is another such pseudo-deformation, and if $\alpha \in k$, then $\{1 + \varepsilon\alpha(a_1 + a_1'),\ \chi + \varepsilon\alpha(d_1 + d_1'),\ \varepsilon\alpha(x_1 + x_1')\}$ is in $F_{\mathscr{D}^{\mathrm{ps}}}(k[\varepsilon])$. In particular $F_{\mathscr{D}^{\mathrm{ps}}}(k[\varepsilon])$ is a k-space.

Let $G_\chi = \mathrm{Gal}(F_\Sigma/F(\chi))$. From the relations defining pseudo-representations it follows that $x_1|_{G_\chi\times G_\chi}$ determines an element of $\mathrm{Hom}(G_\chi, \mathrm{Hom}(G_\chi, k))$ via $x_1 \longmapsto \{g \longmapsto x_1(\cdot, g)\}$. If ρ is in the kernel of the k-linear map $F_{\mathscr{D}^{\mathrm{ps}}}(k[\varepsilon]) \longrightarrow \mathrm{Hom}(G_\chi, \mathrm{Hom}(G_\chi, k))$ given by $\rho \longmapsto x_1|_{G_\chi\times G_\chi}$, then $a_1|_{G_\chi}, d_1|_{G_\chi} \in \mathrm{Hom}(G_\chi, k)$. Thus

$$\#F_{\mathscr{D}^{\mathrm{ps}}}(k[\varepsilon]) \leqslant (\#k)^{s^2+2s} \cdot \#\{\rho : a_1|_{G_\chi} = d_1|_{G_\chi} = x_1|_{G_\chi^2} = 0\},$$

where

$$s = \dim_k H^1(F_\Sigma/F(\chi), k).$$

Now let $G = \mathrm{Gal}(F_\Sigma/F)$ and suppose that $\rho = \{a, d, x\} \in F_{\mathscr{D}^{\mathrm{ps}}}(k[\varepsilon])$ satisfies $a_1|_{G_\chi} = d_1|_{G_\chi} = x_1|_{G_\chi^2} = 0$. Then $x_1|_{G_\chi\times G}$ determines a 1-cocycle $G \longrightarrow \mathrm{Hom}(G_\chi, k(\chi))$ via $g \longmapsto x_1(\cdot, g)$. Moreover, this cocycle vanishes upon restricting to G_χ. Thus the number of possibilities for $x_1|_{G_\chi\times G}$ is at most $\#\mathrm{Hom}(G_\chi, k)$. A similar argument shows that the number of possibilities for $x_1|_{G\times G_\chi}$ is also bounded by the same quantity. Thus

$$\begin{aligned}\#F_{\mathscr{D}^{\mathrm{ps}}}(k[\varepsilon]) &\leqslant (\#k)^{s^2+4s} \cdot \#\{\rho : a_1|_{G_\chi} = d_1|_{G_\chi} = x_1|_{G_\chi\times G} = x_1|_{G\times G_\chi} = 0\} \\ &\leqslant (\#k)^{s^2+4s+4\cdot\#\mathrm{Gal}(F(\chi)/F)}.\end{aligned}$$

Here we have used that for any pseudo-deformation $\rho = \{a, d, x\}$ satisfying $a_1|_{G_\chi} = d_1|_{G_\chi} = x_1|_{G_\chi \times G} = x_1|_{G\times G_\chi} = 0$ the functions a_1, d_1 and x_1 are constant on cosets of G_χ in G. □

There is therefore a universal pseudo-deformation $(R_{\mathscr{D}^{ps}}, \rho_{\mathscr{D}^{ps}})$ of type-$\mathscr{D}^{ps}$.

Clearly, any deformation $\rho : \mathrm{Gal}(F_\Sigma/F) \longrightarrow \mathrm{GL}_2(A)$ of some ρ_c with A an $\mathscr{O}$-algebra gives rise to a pseudo-deformation of type-$(\mathscr{O}, \Sigma)$. Choose a basis for ρ such that $\rho(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Write $\rho(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$. As we have previously noted, $\{a_\sigma, d_\sigma, x_{\sigma,\tau} = b_\sigma c_\tau\}$ is a pseudo-representation, and its reduction modulo $\mathfrak{m}_A$ is ρ_0, so it is a pseudo-deformation. One easily checks that it is also of type-$(\mathscr{O}, \Sigma)$. The entries of $\rho(\sigma)$ with respect to any other basis for which $\rho(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ are obtained by conjugating the chosen basis by a diagonal matrix. Such a conjugation does not change a_σ, d_σ or $b_\sigma c_\tau$. We call $\{a_\sigma, d_\sigma, x_{\sigma,\tau} = b_\sigma c_\tau\}$ the *pseudo-deformation associated to* ρ and sometimes denote it by ρ as well. There is a unique map $R_{\mathscr{D}^{ps}} \longrightarrow A$ $(\mathscr{D}^{ps} = (\mathscr{O}, \Sigma))$ inducing the pseudo-deformation associated to ρ. This argument shows that to any deformation ρ of ρ_c, where c is some cocycle in $H^1(F_\Sigma/F, k(\chi^{-1}))$, one can associate a well-defined pseudo-deformation. In particular, if $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$ is a deformation datum, and if $\mathscr{D}^{ps} = (\mathscr{O}, \Sigma)$, then we obtain a unique map $r_{\mathscr{D}} : R_{\mathscr{D}^{ps}} \longrightarrow R_{\mathscr{D}}$ inducing the pseudo-deformation associated to $\rho_{\mathscr{D}}$. We write $r_{\mathscr{D}}^{\min} : R_{\mathscr{D}^{ps}} \longrightarrow R_{\mathscr{D}}^{\min}$ for the composition of $r_{\mathscr{D}}$ with the canonical map $R_{\mathscr{D}} \longrightarrow R_{\mathscr{D}}^{\min}$.

Let $\mathscr{D}^{ps} = (\mathscr{O}, \Sigma)$ be a pseudo-datum and let Q be a finite set of finite places disjoint from Σ. A pseudo-deformation ρ of type-$(\mathscr{O}, \Sigma \cup Q)$ is of type-$\mathscr{D}_Q^{ps}$ if

- det ρ is unramified at each $w \in Q$.

There exists a universal pseudo-deformation of type-$\mathscr{D}_Q$: $(R_{\mathscr{D}_Q^{ps}}, \rho_{\mathscr{D}_Q^{ps}})$. If $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$ is a deformation datum, then as in the preceding paragraph there is a unique map $r_{\mathscr{D}_Q} : R_{\mathscr{D}_Q^{ps}} \longrightarrow R_{\mathscr{D}_Q}$ inducing the pseudo-deformation associated to $\rho_{\mathscr{D}_Q}$. Of course, if $Q = \emptyset$, then $R_{\mathscr{D}_Q^{ps}} = R_{\mathscr{D}^{ps}}$, $\rho_{\mathscr{D}_Q^{ps}} = \rho_{\mathscr{D}^{ps}}$, and $r_{\mathscr{D}_Q} = r_{\mathscr{D}}$.

Suppose $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$ is a deformation datum and $\mathscr{D}^{ps} = (\mathscr{O}, \Sigma)$. The following proposition reflects the relation of $R_{\mathscr{D}_Q^{ps}}$ to $R_{\mathscr{D}_Q}$.

Proposition **2.11.** — *If* $\mathfrak{p} \subseteq R_{\mathscr{D}_Q}$ *is a one-dimensional prime such that* $\rho_{\mathscr{D}_Q} \bmod \mathfrak{p}$ *is irreducible, and if* $\mathfrak{p}^{ps} = r_{\mathscr{D}_Q}^{-1}(\mathfrak{p}) \subseteq R_{\mathscr{D}_Q^{ps}}$, *then the canonical map* $\widehat{R}_{\mathscr{D}_Q^{ps}, \mathfrak{p}^{ps}} \longrightarrow \widehat{R}_{\mathscr{D}_Q, \mathfrak{p}}$ *is surjective.*

Proof. — As $\rho_{\mathscr{D}} \bmod \mathfrak{p}$ is not reducible, $\mathfrak{p}^{ps}$ is not the maximal ideal $\mathfrak{m}^{ps}$. Therefore $\mathfrak{m}^{ps} R_{\mathscr{D}_Q} \not\subseteq \mathfrak{p}$. Let $A = R_{\mathscr{D}_Q}/\mathfrak{p}$ and $A^{ps} = R_{\mathscr{D}_Q^{ps}}/\mathfrak{p}^{ps}$. Let K^{ps} be the field of fractions of A^{ps}. The map $r_{\mathscr{D}} : R_{\mathscr{D}_Q^{ps}} \longrightarrow R_{\mathscr{D}_Q}$ induces an inclusion $A^{ps} \hookrightarrow A$.

As we have observed, $\mathfrak{m}^{ps}A \neq 0$, whence $A/\mathfrak{m}^{ps}A$ is a zero-dimensional Noetherian local ring with residue field k. It follows that $\#(A/\mathfrak{m}^{ps}A) < \infty$ and hence that A is a finite A^{ps}-module (cf. [Mat, Theorem 8.4]). Thus A is an integral extension of A^{ps} and $\dim A^{ps} = \dim A = 1$.

Fix a basis of $\rho_{\mathscr{D}_Q}$ such that $\rho_{\mathscr{D}_Q}(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $\rho_{\mathscr{D}_Q}(\sigma_0) = \begin{pmatrix} * & u \\ * & * \end{pmatrix}$, $u \in \mathscr{O}^\times$, for some $\sigma_0 \in \mathrm{Gal}(\overline{F}/F)$. With respect to this basis write $\rho_{\mathscr{D}_Q}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$. It is easily checked that $\{ a_\sigma, d_\sigma, c_\sigma, b_\sigma c_\tau \mid \sigma, \tau \in \mathrm{Gal}(\overline{F}/F) \}$ is contained in $\mathrm{im}(r_{\mathscr{D}})$. Let $R' \subseteq R_{\mathscr{D}_Q}$ be the subring generated by $\mathrm{im}(r_{\mathscr{D}})$ and the set $\{ b_\sigma \mid \sigma \in \mathrm{Gal}(\overline{F}/F) \}$. Let $\mathfrak{m}' = R' \cap \mathfrak{m}_{\mathscr{D}_Q}$, where $\mathfrak{m}_{\mathscr{D}_Q}$ is the maximal ideal of $R_{\mathscr{D}_Q}$. Put $R_0 = R'_{\mathfrak{m}'}$. Let $\mathfrak{p}' = R' \cap \mathfrak{p}$ and $\mathfrak{p}_0 = R_0 \cap \mathfrak{p}$. It is a standard fact about localizations that $\mathfrak{p}_0 = \mathfrak{p}'R_0$. Let $A' = R'/\mathfrak{p}'$ and $A_0 = R_0/\mathfrak{p}_0$.

Our first claim is that $A_0 = A$ and $\mathfrak{p}_0 R_{\mathscr{D}_Q} = \mathfrak{p}$. To see this, first note that there are inclusions $A^{ps} \subseteq A_0 \subseteq A$. Since A is a finite A^{ps}-module and A^{ps} is Noetherian, A_0 is also a finite A^{ps}-module as is any ideal of A_0. It follows that A_0 is a Noetherian ring and that it is complete as an A^{ps}-module. Since A_0 is local and the radical of $\mathfrak{m}^{ps}A_0$ is the maximal ideal of A_0, it follows that A_0 is a complete local Noetherian domain. It now follows from Lemma 2.5 that the map from A_0 to $R_{\mathscr{D}_Q}/\mathfrak{p}_0 R_{\mathscr{D}_Q}$ is surjective, so we have $A_0 \twoheadrightarrow R_{\mathscr{D}_Q}/\mathfrak{p}_0 R_{\mathscr{D}_Q} \twoheadrightarrow R_{\mathscr{D}_Q}/\mathfrak{p} R_{\mathscr{D}_Q} = A$. The claim follows.

We next claim that the canonical map $R_{\mathscr{D}^{ps}_Q, \mathfrak{p}^{ps}} \longrightarrow R'_{\mathfrak{p}'}$ is surjective. As $\rho_{\mathscr{D}} \bmod \mathfrak{p}$ is irreducible there exists some τ_0 such that $c_{\tau_0} \notin r_{\mathscr{D}}(\mathfrak{p}^{ps})$. It follows easily that $\{ b_\sigma \mid \sigma \in \mathrm{Gal}(\overline{F}/F) \} \subseteq R^* = \mathrm{im}(R_{\mathscr{D}^{ps}_Q, \mathfrak{p}^{ps}} \longrightarrow R'_{\mathfrak{p}'})$. Therefore the image of the canonical map $R' \longrightarrow R'_{\mathfrak{p}'}$ is contained in R^*. The inverse image in R' of the prime $\mathfrak{p}^{ps}R^*$ is just $\mathfrak{p}'$, whence localization induces a map $R'_{\mathfrak{p}'} \longrightarrow R^*$ whose composition with the inclusion $R^* \hookrightarrow R'_{\mathfrak{p}'}$ is the identity map. It follows that $R^* = R'_{\mathfrak{p}'}$. As a consequence we have $\mathfrak{p}^{ps}R'_{\mathfrak{p}} = \mathfrak{p}'R'_{\mathfrak{p}'}$.

Combining the results of the preceding two paragraphs yields

$$\mathfrak{p}^{ps}R_{\mathscr{D}_Q, \mathfrak{p}} = \mathfrak{p}'R_{\mathscr{D}_Q, \mathfrak{p}} = \mathfrak{p}_0 R_{\mathscr{D}_Q, \mathfrak{p}} = \mathfrak{p}R_{\mathscr{D}_Q, \mathfrak{p}}.$$

We also find that A^{ps}, A', A_0 and A all have the same field of fractions, namely K^{ps}. It follows that $\dim_{K^{ps}}(\widehat{R}_{\mathscr{D}_Q, \mathfrak{p}}/\mathfrak{p}^{ps}\widehat{R}_{\mathscr{D}_Q, \mathfrak{p}}) = 1$. Therefore the canonical map $\widehat{R}_{\mathscr{D}^{ps}_Q, \mathfrak{p}^{ps}} \longrightarrow \widehat{R}_{\mathscr{D}_Q, \mathfrak{p}}$ is surjective (see [Mat, Theorem 8.4]). □

As a corollary of this we have the following important result.

Corollary **2.12.** — *If* $Q \subseteq R_{\mathscr{D}}$ *is any prime such that* $\rho_{\mathscr{D}} \bmod Q$ *is irreducible, and if* $Q^{ps} = r_{\mathscr{D}}^{-1}(Q) \subseteq R_{\mathscr{D}^{ps}}$, *then*

$$\dim R_{\mathscr{D}^{ps}}/Q^{ps} \geqslant \dim R_{\mathscr{D}}/Q$$

with equality holding if Q *is a dimension one prime.*

Proof. — Equality of $\dim R_{\mathscr{D}^{ps}}/Q^{ps}$ and $\dim R_{\mathscr{D}}/Q$ when Q is a dimension one prime was shown in the first paragraph of the proof of Proposition 2.11. We may therefore assume that Q is a prime of dimension at least two. Choose a prime $\mathfrak{p} \supseteq Q$ of $R_{\mathscr{D}}$ having dimension one and such that $\rho_{\mathscr{D}} \bmod \mathfrak{p}$ is also irreducible. Let $\mathfrak{p}^{ps} = r_{\mathscr{D}}^{-1}(\mathfrak{p})$. We then have

$$\begin{aligned}\dim R_{\mathscr{D}}/Q = 1 + \dim \widehat{R}_{\mathscr{D},\mathfrak{p}}/Q &\leqslant 1 + \dim \widehat{R}_{\mathscr{D}^{ps},\mathfrak{p}^{ps}}/Q^{ps}\\ &\leqslant \dim R_{\mathscr{D}^{ps}}/Q^{ps}\end{aligned}$$

with the first inequality following from Proposition 2.11. □

We now collect a few results connecting deformations and pseudo-deformations. Suppose that A is a complete DVR with residue field k. Let K be the field of fractions of A, and let λ be a uniformizer of A. Suppose that $\rho : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(K)$ is a continuous representation. As $\mathrm{Gal}(F_\Sigma/F)$ is compact, there exists a $\mathrm{Gal}(F_\Sigma/F)$-stable A lattice L in the representation space of ρ. Such a lattice, being a free A-module of rank 2, gives rise to a representation $\rho_L : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(A)$ such that $\rho_L \otimes_A K \simeq \rho$. It is well-known that whereas the reduction $\bar\rho_L = \rho_L \bmod \lambda$ is not necessarily independent of L, its semisimplification $\bar\rho_L^{ss}$ is. We call $\bar\rho_L^{ss}$ the *reduction* of ρ.

Lemma **2.13.** — *Suppose that the reduction of* ρ *is* $1 \oplus \chi$ *and that* ρ *is irreducible.*

(i) *There exists a* $\mathrm{Gal}(F_\Sigma/F)$*-stable lattice* L *in the representation space of* ρ *such that* $\bar\rho_L(\sigma) = \begin{pmatrix} 1 & b_\sigma \\ & \chi(\sigma)\end{pmatrix}$ *for all* σ, *and* $\bar\rho_L$ *has scalar centralizer.*

(ii) *For two lattices* L_1 *and* L_2 *as in* (i), *the classes in* $H^1(F_\Sigma/F, k(\chi^{-1}))$ *of the cocycles* $\sigma \longmapsto \chi(\sigma)^{-1} b_i(\sigma)$, $i = 1, 2$, *are non-zero scalar multiples of one another.*

Proof. — Choose any $\mathrm{Gal}(F_\Sigma/F)$-stable lattice L and pick a basis for ρ_L such that $\rho_L(z_1) = \begin{pmatrix} 1 & \\ & -1\end{pmatrix}$. Write $\rho_L(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma\end{pmatrix}$, and let $n = \min_\sigma \mathrm{ord}_\lambda(b_\sigma)$. As ρ is irreducible, $n < \infty$. Let L′ be the lattice obtained by scaling L by $\begin{pmatrix} \lambda^{-n} & \\ & 1\end{pmatrix}$. The representation $\rho_{L'}$ is just

$$\rho_{L'} = \begin{pmatrix} \lambda^{-n} & \\ & 1\end{pmatrix} \rho_L \begin{pmatrix} \lambda^{n} & \\ & 1\end{pmatrix}.$$

The representation clearly has the properties desired for part (i). To prove part (ii) it suffices to show that the representations $\bar\rho_{L_1}$ and $\bar\rho_{L_2}$ are equivalent. Choose bases for the ρ_{L_i}'s such that $\rho_{L_i}(z_1) = \begin{pmatrix} 1 & \\ & -1\end{pmatrix}$. As $\rho_{L_i} \otimes K \simeq \rho$, there exists $g \in GL_2(K)$ such that $g^{-1}\rho_{L_1} g = \rho_{L_2}$. Since g must commute with $\begin{pmatrix} 1 & \\ & -1\end{pmatrix}$, one may assume that $g = \begin{pmatrix} 1 & \\ & a\end{pmatrix}$. Write $\rho_{L_1}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma\end{pmatrix}$. By hypothesis, there exists some σ_0 such that b_{σ_0} is a unit. As $ab_{\sigma_0} \in A$, it must be that $\mathrm{ord}_\lambda(a) \geqslant 0$. As the reduction of ab_σ is not always zero, it must be that $\mathrm{ord}_\lambda(a) \leqslant 0$. Thus a is a unit and $\bar\rho_{L_1}$ and $\bar\rho_{L_2}$ are equivalent. □

Corollary **2.14.** — *If* A *is a complete DVR with residue field k as above, and if* (A, φ) *is a pseudo-deformation of* ρ_0 *unramified away from* Σ *for which* $x(\sigma, \tau)$ *is not identically zero, then there exists a non-zero cocycle* $c \in H^1(F_\Sigma/F, k(\chi^{-1}))$ *and a deformation* $\rho_\varphi : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(A)$ *of* ρ_c *whose associated pseudo-deformation is* φ. *Moreover, c is unique up to multiplication by a non-zero scalar.*

Proof. — We need only observe that there is some irreducible representation $\rho : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(A)$ whose associated pseudo-representation is φ, for then the claim follows from the lemma. Fix $\sigma_0, \tau_0 \in \mathrm{Gal}(F_\Sigma/F)$ such that $\mathrm{ord}_\lambda x(\sigma_0, \tau_0)$ is minimal. Define ρ by

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & x(\sigma, \tau_0)/x(\sigma_0, \tau_0) \\ x(\sigma_0, \sigma) & d(\sigma) \end{pmatrix}.$$

□

Our next result also associates deformations to pseudo-deformations. Suppose that R is a local complete Noetherian domain with residue field k and maximal ideal $\mathfrak{m}$. Suppose that $\rho = \{a, d, x\}$ is a pseudo-representation of $\mathrm{Gal}(F_\Sigma/F)$ into R such that $\rho_0 = \rho \bmod \mathfrak{m}$ (i.e., (R, ρ) is a pseudo-deformation). Let $\mathfrak{p}$ be a prime of R such that the dimension of $R/\mathfrak{p}$ is one. Let A be the integral closure of $R/\mathfrak{p}$ in its field of fractions K. This is a complete DVR with residue field a finite extension of k, say k'. Suppose that $x \bmod \mathfrak{p}$ is not identically zero. By Corollary 2.14 there exists a cocycle $0 \neq c$ in $H^1(F_\Sigma/F, k(\chi^{-1}))$ and a deformation $\rho_\mathfrak{p} : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(A)$ of ρ_c such that the pseudo-deformation associated to $\rho_\mathfrak{p}$ is $\rho \bmod \mathfrak{p}$. We will construct a local complete Noetherian domain R^+ having the same dimension as R, an injective local homomorphism $R \hookrightarrow R^+$, and a deformation $\rho^+ : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(R^+)$ of ρ_c whose associated pseudo-deformation is ρ. Moreover, R^+ will have a prime $\mathfrak{p}^+$ of dimension one such that $\mathfrak{p} = R \cap \mathfrak{p}^+$ and $\rho_\mathfrak{p} = \rho^+ \bmod \mathfrak{p}^+$.

Let L be the field of fractions of R. Pick $\alpha, \beta \in \mathfrak{m}$, $\beta \notin \mathfrak{p}$, such that α/β is a uniformizer of A. Put $R' = R[\alpha/\beta] \subseteq L$. This is a Noetherian domain with maximal ideal $\mathfrak{m}' = (\mathfrak{m}, \alpha/\beta)$. To see that $\mathfrak{m}'$ is in fact a maximal ideal, let $\varphi' : R' \twoheadrightarrow A$ be given by $\varphi'(f(\alpha/\beta)) = \bar{f}(\bar{\alpha}/\bar{\beta})$ for any polynomial f with coefficients in R. Here the "bar" denotes reduction modulo $\mathfrak{p}$. This is well-defined, for if $f(\alpha/\beta) = 0$, then $\bar{f}(\bar{\alpha}/\bar{\beta}) = 0$ as can be seen by first clearing denominators and then reducing. Let $\mathfrak{p}'$ be the kernel of φ', and let I be the ideal of R' generated by the set $\{x(\sigma, \tau)\}$. Let $\{i_1, \ldots, i_r\}$ be a set of generators of I taken from among the $x(\sigma, \tau)$'s. As $\rho_\mathfrak{p}$ is irreducible, the image of I under φ' is non-trivial. Pick an $i \in \{i_1, \ldots, i_r\}$ whose image has minimal valuation in A. Define R^* by

$$R^* = R'[i_1/i, \ldots, i_r/i] \subseteq L.$$

This is a Noetherian integral domain with maximal ideal $\mathfrak{m}^*$ defined as the inverse image of the maximal ideal of A under the homomorphism $\varphi^* : R^* \twoheadrightarrow A$ given by

$f(i_1/i, \ldots, i_r/i) \longmapsto \bar{f}(\bar{i}_1/\bar{i}, \ldots, \bar{i}_r/\bar{i})$ for any polynomial f with coefficients in R'. (Now the "bar" denotes reduction modulo $\mathfrak{p}'$.) Let $\mathfrak{p}^*$ be the kernel of the map φ^*. Let $\mathrm{R}^+ = \widehat{\mathrm{R}}^*_{\mathfrak{m}^*}$, and let $\mathfrak{m}^+$ be its maximal ideal and $\mathfrak{p}^+$ the kernel of the induced map $\mathrm{R}^+ \twoheadrightarrow \mathrm{A}$. The ring R^+ is clearly a local complete Noetherian ring with residue field k. Moreover, the inclusion $\mathrm{R} \hookrightarrow \mathrm{R}^+$ is a local homomorphism. To see that the dimension of R^+ is the same as that of R, observe that it follows from the construction of R^* that $\mathrm{R}^*_{\mathfrak{p}^*} = \mathrm{R}_{\mathfrak{p}}$, whence

$$\dim \mathrm{R}^+ = \dim \mathrm{R}^*_{\mathfrak{m}^*} = \dim \mathrm{R}^*_{\mathfrak{p}^*} + 1 = \dim \mathrm{R}_{\mathfrak{p}} + 1 = \dim \mathrm{R}.$$

Unfortunately, R^+ need not be a domain. However, since the "going-down" property holds for the pair $\mathrm{R}^*_{\mathfrak{m}^*}$ and R^+ (see [Mat, Theorem 9.5]), there is a minimal prime $\mathfrak{q}^+$ of R^+ contained in $\mathfrak{p}^+$ and such that $\mathfrak{q}^+ \cap \mathrm{R}^*_{\mathfrak{m}^*} = (0)$ and $\dim \mathrm{R}^+/\mathfrak{q}^+ = \dim \mathrm{R}^+$. We replace R^+ by the quotient $\mathrm{R}^+/\mathfrak{q}^+$. This ring has all of the desired properties.

In the ring R^+ the ideal $\mathrm{I}\mathrm{R}^+ = \{i\}$ is principal. As $i = x(\sigma_0, \tau_0)$ for some $\sigma_0, \tau_0 \in \mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F})$, one can define a representation $\rho^+ : \mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F}) \longrightarrow \mathrm{GL}_2(\mathrm{R}^+)$ by

$$\rho^+(\sigma) = \begin{pmatrix} a(\sigma) & x(\sigma, \tau_0)/i \\ x(\sigma_0, \sigma) & d(\sigma) \end{pmatrix}.$$

The reduction of $\rho^+ \bmod \mathfrak{m}^+$ is non-semisimple as $\rho^+(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $\rho^+(\sigma_0) = \begin{pmatrix} * & 1 \\ & * \end{pmatrix}$. Thus ρ^+ is a deformation of $\rho_{c'}$ for some $0 \neq c' \in \mathrm{H}^1(\mathrm{F}_\Sigma/\mathrm{F}, k'(\chi^{-1}))$. Reducing ρ^+ modulo $\mathfrak{p}^+$ gives a deformation of $\rho_{c'}$ into $\mathrm{GL}_2(\mathrm{A})$ whose associated pseudo-deformation is $\rho \bmod \mathfrak{p}$. It follows from Corollary 2.14 that c' is a non-zero scalar multiple of c. Thus, after possibly replacing ρ^+ by a conjugate, we may assume that $c = c'$ and $\rho^+ \bmod \mathfrak{p}^+ = \rho_{\mathfrak{p}}$.

Finally, suppose that A is an $\mathcal{O}$-algebra with $\mathcal{O}$ the ring of integers of some local field having residue field k' and that $\mathcal{M}, \mathrm{Q} \subseteq \Sigma$ are sets of finite places (possibly empty) such that

(2.8) (i) $\mathcal{M} \subseteq \Sigma \backslash \mathcal{P}$ consists of places w such that ρ_c is ramified at w ;
(ii) $\mathrm{Q} \subseteq \Sigma \backslash \mathcal{P} \cup \mathcal{M}$ consists of places w at which ρ_c is unramified.

Let $\mathcal{L}$ be the field of fractions of R^+. It is easily checked that if

(2.9) (i) $\rho^+ \otimes \mathcal{L}|_{\mathrm{D}_i} \simeq \begin{pmatrix} \tilde{\chi}\psi_{1,i} & * \\ & \psi_{2,i} \end{pmatrix}$, $\psi_{j,i} \bmod \mathfrak{m}^+ = 1$, $i = 1, \ldots, t$,
(ii) $\rho^+ \otimes \mathcal{L}|_{\mathrm{I}_w} \simeq \begin{pmatrix} 1 & * \\ & \tilde{\chi} \end{pmatrix}$ for all $w \in \mathcal{M}$, and
(iii) $\det \rho^+|_{\mathrm{I}_w} = 1$ for all $w \in \mathrm{Q}$,

then c is admissible and ρ^+ is of type-$\mathcal{D}_\mathrm{Q}$ where $\mathcal{D} = (\mathcal{O}, \Sigma, c, \mathcal{M})$. For ease of reference we summarize these results in a proposition.

Proposition **2.15.** — *Suppose* (R, ρ) *is a pseudo-deformation of type-*$(\mathcal{O}, \Sigma)$. *Suppose also that* $\mathfrak{p} \subseteq \mathrm{R}$ *is a prime of dimension one such that* $x(\sigma, \tau) \bmod \mathfrak{p}$ *is not identically zero* $(\rho = \{a, d, x\})$. *Let* k' *be the residue field of the integral closure of* $\mathrm{R}/\mathfrak{p}$ *in its field of fractions. There exists a cocycle* $0 \neq c \in \mathrm{H}^1(\mathrm{F}_\Sigma/\mathrm{F}, k'(\chi^{-1}))$ *unique up to multiplication by a scalar, a local complete Noetherian* $\mathcal{O}$*-domain* R^+ *with residue field* k' *and having the same dimension as* R, *a local homomorphism* $\mathrm{R} \hookrightarrow \mathrm{R}^+$ *of* $\mathcal{O}$*-algebras, a dimension one prime* $\mathfrak{p}^+ \subseteq \mathrm{R}^+$ *extending* $\mathfrak{p}$, *and a deformation* $\rho^+ : \mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F}) \longrightarrow \mathrm{GL}_2(\mathrm{R}^+)$ *of* ρ_c *whose associated pseudo-deformation is that induced from* ρ. *Moreover, if* $\mathrm{Q}, \mathcal{M} \subseteq \Sigma$ *are sets of places satisfying* (2.8), *and if* $\rho^+ \otimes \mathcal{L}$, $\mathcal{L}$ *the field of fractions of* R^+, *satisfies* (2.9), *then* ρ^+ *is a deformation of type-*$\mathcal{D}_\mathrm{Q}$ *with* $\mathcal{D} = (\mathcal{O}, \Sigma\backslash \mathrm{Q}, c, \mathcal{M})$, *with* $\mathcal{O}' = \mathcal{O} \otimes_{\mathrm{W}(k)} \mathrm{W}(k')$.

For a finite field $\mathbf{F}$, $\mathrm{W}(\mathbf{F})$ denotes the ring of Witt vectors of $\mathbf{F}$.

2.5. The Iwasawa algebra

In this subsection we describe how each of the deformation rings $\mathrm{R}_\mathcal{D}$ and $\mathrm{R}_{\mathcal{D}_\mathrm{Q}}$ is an algebra over a certain multivariate "Iwasawa algebra". Let L_0 be the maximal abelian pro-p-extension of F unramified away from $\mathcal{P}$. Let $\mathrm{I} \subseteq \mathrm{Gal}(\mathrm{L}_0/\mathrm{F})$ be the subgroup generated by the images of the inertia groups I_i, $i = 1, \dots, t$. We fix once and for all a maximal free $\mathbf{Z}_p$-summand I_0 of I (necessarily of rank δ_F). Fix also a free $\mathbf{Z}_p$-summand G_0 of $\mathrm{Gal}(\mathrm{L}_0/\mathrm{F})$ containing I_0 (this also has rank δ_F). Finally, fix elements $\gamma_1, \dots, \gamma_{\delta_\mathrm{F}} \in \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$ whose images in $\mathrm{Gal}(\mathrm{L}_0/\mathrm{F})$ generate G_0 and for which there exist integers $r_1, \dots, r_{\delta_\mathrm{F}}$ such that $\gamma_1^{p^{r_1}}, \dots, \gamma_{\delta_\mathrm{F}}^{p^{r_{\delta_\mathrm{F}}}}$ generate I_0. For each $0 \leqslant i \leqslant t$ fix once and for all $y_1^{(i)}, \dots, y_{d_i}^{(i)} \in \mathrm{U}_i$ (the units of F_{v_i}) generating a free $\mathbf{Z}_p$-summand of rank d_i. Put

$$\Lambda_\mathcal{O} = \mathcal{O}[[\mathrm{T}_1, \dots, \mathrm{T}_{\delta_\mathrm{F}}, \mathrm{Y}_1^{(1)}, \dots, \mathrm{Y}_{d_t}^{(t)}]].$$

The rings $\mathrm{R}_\mathcal{D}$ (and hence the $\mathrm{R}_{\mathcal{D}_\mathrm{Q}}$) are algebras over $\Lambda_\mathcal{O}$ via

- $\mathrm{T}_i \longmapsto \det \rho_\mathcal{D}(\gamma_i) - 1, \quad i = 1, \dots, \delta_\mathrm{F};$
- $\mathrm{Y}_j^{(i)} \longmapsto \psi_2^{(i)}(y_j^{(i)}) - 1$, where $\rho_\mathcal{D}|_{\mathrm{D}_i} \simeq \begin{pmatrix} \psi_1^{(i)}\widetilde{\chi} & * \\ & \psi_2^{(i)} \end{pmatrix}$ and U_i is identified with

the inertia subgroup of $\mathrm{D}_i^{\mathrm{ab}}$ via local reciprocity.

Suppose that $\mathcal{D} = (\mathcal{O}, \Sigma, c, \mathcal{M})$ and $\mathcal{D}' = (\mathcal{O}, \Sigma', c, \mathcal{M}')$ are deformation data with $\Sigma \subseteq \Sigma'$ and $\mathcal{M}' \subseteq \mathcal{M}$. The natural map $\mathrm{R}_{\mathcal{D}'} \twoheadrightarrow \mathrm{R}_\mathcal{D}$ is a map of $\Lambda_\mathcal{O}$-algebras.

Each universal pseudo-deformation ring $\mathrm{R}_{\mathcal{D}^{\mathrm{ps}}}$ and $\mathrm{R}_{\mathcal{D}_\mathrm{Q}^{\mathrm{ps}}}$ is a $\Lambda_\mathcal{O}$-algebra in a manner compatible with the canonical maps $\mathrm{R}_{\mathcal{D}^{\mathrm{ps}}} \longrightarrow \mathrm{R}_\mathcal{D}$ and $\mathrm{R}_{\mathcal{D}_\mathrm{Q}^{\mathrm{ps}}} \longrightarrow \mathrm{R}_{\mathcal{D}_\mathrm{Q}}$. To see this, for each $i = 1, \dots, t$ fix $g_i \in \mathrm{D}_i$ such that $\chi(g_i) \neq 1$ and for each $j = 1, \dots, d_i$ let $\sigma_j^{(i)} \in \mathrm{D}_i$ be a lift of $y_j^{(i)}$. By the choice of g_i the polynomial

$X^2 - \operatorname{trace}\rho_{\mathscr{D}^{\mathrm{ps}}}(g_i)X + \det\rho_{\mathscr{D}^{\mathrm{ps}}}(g_i)$ has distinct roots in $R_{\mathscr{D}^{\mathrm{ps}}}$, say α_i and β_i with α_i reducing to $\chi(g_i)$ modulo the maximal ideal of $R_{\mathscr{D}^{\mathrm{ps}}}$. (The images of α_i and β_i in $R_{\mathscr{D}}$ are just the eigenvalues of $\rho_{\mathscr{D}}(g_i)$). We define a map $\Lambda_{\mathscr{O}} \longrightarrow R_{\mathscr{D}^{\mathrm{ps}}}$ by

- $T_i \longmapsto \det\rho_{\mathscr{D}^{\mathrm{ps}}}(\gamma_i) - 1, \quad i = 1, \ldots, \delta_F,$
- $Y_j^{(i)} \longmapsto (\operatorname{trace}\rho_{\mathscr{D}^{\mathrm{ps}}}(g_i\sigma_j^{(i)}) - \alpha_i \operatorname{trace}\rho_{\mathscr{D}^{\mathrm{ps}}}(\sigma_j^{(i)}))/(\beta_i - \alpha_i) - 1.$

The compatibility with the $\Lambda_{\mathscr{O}}$-algebra structure of $R_{\mathscr{D}}$ is clear. Also, if $\mathscr{D}_1^{\mathrm{ps}} = (\mathscr{O}, \Sigma_1)$ and $\mathscr{D}_2^{\mathrm{ps}} = (\mathscr{O}, \Sigma_2)$ are two pseudo-data with $\Sigma_2 \supseteq \Sigma_1$, then the natural map $R_{\mathscr{D}_2^{\mathrm{ps}}} \longrightarrow R_{\mathscr{D}_1^{\mathrm{ps}}}$ is a map of $\Lambda_{\mathscr{O}}$-algebras.

3. Nearly ordinary Hecke algebras and Galois representations

3.1. Modular forms and Hecke operators

We keep our previous conventions for the field F. We write $\mathbf{A}$ and $\mathbf{A}_f$ for the adeles and the finite adeles of F, respectively. If G is any algebraic group over F, then we identify $G(\mathbf{A})$ with the restricted direct product of the groups $G(F_w)$ with respect to the subgroups $G(\mathscr{O}_{F,w})$ (for finite w), writing x_w for the w-component of $x \in G(\mathbf{A})$, and similarly for $G(\mathbf{A}_f)$. For a finite place w, we sometimes write $x_{\mathfrak{p}}$ for x_w with $\mathfrak{p}$ the prime ideal of F corresponding to w. Let I be the set of infinite places of F (equivalently, the set of embeddings $\tau : F \hookrightarrow \mathbf{R}$). This description of $G(\mathbf{A})$ identifies $G(F \otimes \mathbf{R})$ with $G(\mathbf{R})^I$. We also fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ and an embedding of $\overline{\mathbf{Q}} = \overline{F}$ into $\overline{\mathbf{Q}}_p$.

For an ideal $\mathfrak{n}$ of $\mathscr{O}_F$ we define various standard open compact subgroups of $GL_2(\mathbf{A}_f)$ as follows:

$$U_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathscr{O}_F \otimes \widehat{\mathbf{Z}}) : c \equiv 0 \mod \mathfrak{n} \right\},$$

$$U_1(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\mathfrak{n}) : a \equiv 1 \mod \mathfrak{n} \right\}, \quad \text{and}$$

$$U(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_1(\mathfrak{n}) : d \equiv 1 \mod \mathfrak{n} \right\}.$$

For $k = \Sigma k_\tau \tau \in \mathbf{Z}[I]$ and $x \in \mathbf{C}^I$ write x^k for the product $\Pi x_\tau^{k_\tau}$. Let $t = \Sigma\tau$. To each $k = \Sigma k_\tau \tau$, with each $k_\tau \geqslant 2$ and having the same parity as the others, we associate quantities $m, v \in \mathbf{Z}[I]$ and $\mu \in \mathbf{Z}$ as follows:

$$m = k - 2t$$

and

$$v = \Sigma v_\tau \tau, \ \ v_\tau \geqslant 0, \ \ \text{some } v_\tau = 0; \ m + 2v = \mu \cdot t.$$

Let $\mathbf{H}$ denote the complex upper-half plane. Define $j : \mathrm{GL}_2(\mathrm{F} \otimes \mathbf{R}) \times \mathbf{H}^{\mathrm{I}} \longrightarrow \mathbf{C}^{\mathrm{I}}$ by

$$j(u_\infty, z) = (c_\tau z_\tau + d_\tau), \quad u_\infty = \begin{pmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{pmatrix} \in \mathrm{GL}_2(\mathbf{R})^{\mathrm{I}} = \mathrm{GL}_2(\mathrm{F} \otimes \mathbf{R}).$$

Define also an action of $\mathrm{GL}_2^+(\mathrm{F} \otimes \mathbf{R}) = \mathrm{GL}_2^+(\mathbf{R})^{\mathrm{I}}$ on $\mathbf{H}^{\mathrm{I}}$ by

$$u_\infty(z) = \left(\frac{a_\tau z_\tau + b_\tau}{c_\tau z_\tau + d_\tau}\right).$$

Denote by z_0 the point $(i, \ldots, i) \in \mathbf{H}^{\mathrm{I}}$.

We now recall the notion of a (holomorphic) modular form on GL_2. First, for any congruence subgroup $\Gamma \subseteq \mathrm{GL}_2(\mathrm{F})$, denote by $\mathrm{M}_k(\Gamma)$ and $\mathrm{S}_k(\Gamma)$ the spaces of (classical, Hilbert) modular forms and cusp forms on $\mathbf{H}^{\mathrm{I}}$, respectively, of weight k (cf. [Sh]). For a function $f : \mathrm{GL}_2(\mathbf{A}) \longrightarrow \mathbf{C}$ and $u = u_f \cdot u_\infty \in \mathrm{GL}_2(\mathbf{A}) = \mathrm{GL}_2(\mathbf{A}_f) \cdot \mathrm{GL}_2(\mathrm{F} \otimes \mathbf{R})$ we define $f|_k u$ by

$$(f|_k u)(g) = j(u_\infty, z_0)^{-k} \det(u_\infty)^{\mathrm{v}+k-t} f(gu^{-1}).$$

Write C_∞ for the subgroup $(\mathbf{R}^\times \cdot \mathrm{SO}_2(\mathbf{R}))^{\mathrm{I}} \subseteq \mathrm{GL}_2^+(\mathrm{F} \otimes \mathbf{R})$. A function $f : \mathrm{GL}_2(\mathbf{A}) \longrightarrow \mathbf{C}$ satisfying $f|_k u = f$ for all $u \in \mathrm{C}_\infty$ gives rise to a function $f_x : \mathbf{H}^{\mathrm{I}} \longrightarrow \mathbf{C}$ for each $x \in \mathrm{GL}_2(\mathbf{A}_f)$:

$$f_x(z) = j(u_\infty, z_0)^k \det(u_\infty)^{t-k-\mathrm{v}} f(xu_\infty), \qquad u_\infty(z_0) = z.$$

Let $\mathrm{U} \subseteq \mathrm{GL}_2(\mathbf{A}_f)$ be an open compact subgroup. A function $f : \mathrm{GL}_2(\mathbf{A}) \longrightarrow \mathbf{C}$ is a *modular form of weight k and level* U if

- $f(ax) = f(x) \quad \forall a \in \mathrm{GL}_2(\mathrm{F})$,
- $f|_k u = f \quad \forall u \in \mathrm{U} \cdot \mathrm{C}_\infty$,
- $f_x(z) \in \mathrm{M}_k(\Gamma_x), \quad \Gamma_x = \mathrm{GL}_2(\mathrm{F}) \cap x\mathrm{U} \cdot \mathrm{GL}_2^+(\mathrm{F} \otimes \mathbf{R})x^{-1}, \quad \forall x \in \mathrm{GL}_2(\mathbf{A}_f)$.

Such a function is a *cusp form* if $f_x(z) \in \mathrm{S}_k(\Gamma_x)$ for all $x \in \mathrm{GL}_2(\mathbf{A}_f)$. Denote by $\mathrm{M}_k(\mathrm{U})$ and $\mathrm{S}_k(\mathrm{U})$ the spaces of modular forms and cusp forms of weight k and level U, respectively. For more on such forms see [Sh] and [H1].

If $\mathrm{U} = \mathrm{U}_1(\mathfrak{n})$, then $\mathrm{M}_k(\mathrm{U})$ and $\mathrm{S}_k(\mathrm{U})$ are just the spaces $\mathrm{M}_k(\mathfrak{n})$ and $\mathrm{S}_k(\mathfrak{n})$ defined in [Sh]. For each $\mathfrak{n}$ choose once and for all representatives $t^{(i)} \in \mathbf{A}^\times$ of the ideal classes of F ($i = 1, \ldots, h$) such that $t_w^{(i)} = 1$ for each place $w \mid \mathrm{Nm}(\mathfrak{n}p) \cdot \infty$. Put $x_i = \begin{pmatrix} t^{(i)} & \\ & 1 \end{pmatrix}$ and write Γ_i for the subgroup Γ_{x_i} and for each $f \in \mathrm{M}_k(\mathfrak{n})$ write f_i for f_{x_i}. There are isomorphisms $\mathrm{M}_k(\mathfrak{n}) \simeq \prod_{i=1}^h \mathrm{M}_k(\Gamma_i)$ and $\mathrm{S}_k(\mathfrak{n}) \simeq \prod_{i=1}^h \mathrm{S}_k(\Gamma_i)$ given by $f \longmapsto (f_i)$. Each $f_i(z)$ has a Fourier expansion of the form $f_i(z) = a_i(0) + \sum^+_{\mu \in (t^{(i)})} a_i(\mu)\, e(\mu \cdot z)$ where $(t^{(i)})$ is the ideal of F associated to the idele $t^{(i)}$, the sum is over totally positive elements of $(t^{(i)})$, and

$\mu \cdot z = \Sigma\tau(\mu)\cdot z_\tau$. For a ring $A \subseteq \mathbf{C}$, let $M_k(\mathfrak{n}, A)$ be the space of modular forms $f \in M_k(\mathfrak{n})$ such that each f_i has Fourier coefficients in A. Define $S_k(\mathfrak{n}, A)$ similarly. Shimura has shown that $M_k(\mathfrak{n}, A) = M_k(\mathfrak{n}, \mathbf{Z}) \otimes A$ and $S_k(\mathfrak{n}, A) = S_k(\mathfrak{n}, \mathbf{Z}) \otimes A$. For a ring $R \subseteq \overline{\mathbf{Q}}_p$ define $M_k(\mathfrak{n}, R)$ and $S_k(\mathfrak{n}, R)$ by $M_k(\mathfrak{n}, R) = M_k(\mathfrak{n}, \mathbf{Z})\otimes R$ and $S_k(\mathfrak{n}, R) = S_k(\mathfrak{n}, \mathbf{Z})\otimes R$. If $R \subseteq \overline{\mathbf{Q}}$ as well, then this agrees with the earlier definitions, as Shimura's result shows.

From now on we require each U to satisfy

$$U = \prod_{w \nmid \infty} U_w, \qquad U_w \subseteq GL_2(\mathcal{O}_{F, w}).$$

We also require that $U(\mathfrak{n}) \subseteq U \subseteq U_0(\mathfrak{n})$ for some $\mathfrak{n}$.

Next we recall the connection between modular forms on GL_2 and automorphic representations of GL_2. For details and definitions the reader should consult [De], [Ge], and [J-L]. Let $\mathcal{A}_k^0$ be the space of all cusp forms on GL_2 (over F, of course) of weight k. The group $GL_2(\mathbf{A}_f)$ acts on $\mathcal{A}_k^0$ via $({}^g f)(x) = f(xg)$. Under this action $\mathcal{A}_k^0$ is an admissible representation of $GL_2(\mathbf{A}_f)$. Moreover, $\mathcal{A}_k^0$ decomposes into a direct sum $\mathcal{A}_k^0 = \bigoplus_\pi V_\pi$ where, for each π, V_π is an irreducible admissible representation of $GL_2(\mathbf{A}_f)$ (which we often denote just by π), and the V_π are all non-isomorphic. For an open subgroup $U \subseteq GL_2(\mathcal{O}_F \otimes \hat{\mathbf{Z}})$ let $\Pi_k(U) = \{\pi \mid V_\pi^U \neq 0\}$. Clearly the space $\bigoplus_{\pi\in\Pi_k(U)} V_\pi^U$ is just $S_k(U)$. We recall that each $\pi \in \Pi_k(U)$ can be written as a restricted tensor product $\pi = \bigotimes_v \pi_v$ where v runs over the finite places of F and each π_v is an irreducible admissible representation of $GL_2(F_v)$. Let $V_\pi = \otimes V_{\pi, v}$ be the corresponding tensor product decomposition of V_π. Clearly $V_\pi^U = \bigotimes_v V_{\pi, v}^{U_v}$. It follows from the theory of newforms that $\dim V_{\pi, v}^{U_v} = 1$ for each place v for which $U_v = GL_2(\mathcal{O}_{F, v})$.

For each $g \in GL_2(\mathbf{A}_f)$ define a Hecke operator $[UgU'] : M_k(U) \longrightarrow M_k(U')$ by

(3.1)
$$[UgU']f(x) = \sum_{g_i} f(xg_i^{-1}), \quad UgU' = \sqcup Ug_i.$$

Of course, $[UgU']$ maps $S_k(U)$ to $S_k(U')$. For each prime ideal ℓ of F choose an element $\lambda^{(\ell)} \in \mathcal{O}_F \otimes \hat{\mathbf{Z}}$ such that $\lambda_\ell^{(\ell)}$ is a uniformizer of $\mathcal{O}_{F,\ell}$ and $\lambda_\mathfrak{p}^{(\ell)} = 1$ for $\mathfrak{p} \neq \ell$. If $\mathfrak{p}|p$ then we require that $\lambda_\mathfrak{p}^{(\mathfrak{p})}$ also be an element of $\mathcal{O}_F$ such that $\chi(\lambda_\mathfrak{p}^{(\mathfrak{p})}) \neq 1$ and that $\lambda_\mathfrak{p}^{(\mathfrak{p})} \in \mathcal{O}_{F,\mathfrak{p}'}^\times$ for each $\mathfrak{p}'|p$ but $\mathfrak{p}' \neq \mathfrak{p}$. (For $y \in F_\mathfrak{p}^\times$ we define $\chi(y)$ to be the value obtained from composing χ with the local reciprocity map $F_\mathfrak{p}^\times \longrightarrow Gal(F_\mathfrak{p}^{ab}/F_\mathfrak{p})$.) Denote by $T(\ell)$ and $S(\ell)$ the operators $[U\begin{pmatrix} 1 & \\ & \lambda^{(\ell)} \end{pmatrix}U]$ and $[U\begin{pmatrix} \lambda^{(\ell)} & \\ & \lambda^{(\ell)} \end{pmatrix}U]$, respectively. These operators commute one with another. Moreover, it is easy to see that $T(\ell)$ and $S(\ell)$ are independent of the choice of $\lambda^{(\ell)}$ if $U_\ell = GL_2(\mathcal{O}_{F,\ell})$. Also, if $V \subseteq U$ and if

$GL_2(\mathcal{O}_{F,\ell}) = V_\ell = U_\ell$, then the inclusion $M_k(U) \subseteq M_k(V)$ respects the actions of $T(\ell)$ and $S(\ell)$.

If $U = U_1(\mathfrak{n})$, then $T(\ell)$ and $S(\ell)$ (for $\ell \nmid \mathfrak{n}$) are just the Hecke operators defined and so denoted in [Sh]. These operators stabilize each $M_k(\mathfrak{n}, R)$ and $S_k(\mathfrak{n}, R)$.

As $S_k(U) = \bigoplus_{\pi\in\Pi_k(U)} V_\pi^U$ we find that the Hecke operator $[UgU]$ stabilizes each V_π^U, the action being given by

$$\textbf{(3.1}'\textbf{)} \qquad [UgU]x = \sum_{g_i} \pi(g_i^{-1})x, \quad x \in V_\pi^U, \quad UgU = \sqcup U g_i.$$

For each place v let g_v be the v-component of g. Under the tensor product decomposition $V_\pi^U = \bigotimes_v V_{\pi,v}^{U_v}$, $[UgU]$ decomposes as $[UgU] = \bigotimes_v [U_v g_v U_v]$ with $[U_v g_v U_v] \in \mathrm{End}(V_{\pi,v}^{U_v})$ being given by

$$[U_v g_v U_v]x = \sum_{h_i} \pi_v(h_i^{-1})x, \quad x \in V_{\pi,v}^{U_v}, \quad U_v g_v U_v = \sqcup U_v h_i.$$

3.2. Nearly ordinary Hecke algebras

Keeping the conventions for U introduced in the preceding subsection, for each positive integer a define U_a^0, U_a^1, and U_a by

$$U_a^0 = U \cap U_0(p^a), \quad U_a^1 = U \cap U_1(p^a), \text{ and } U_a = U \cap U(p^a),$$

respectively.

Suppose that $U_v = GL_2(\mathcal{O}_{F,v})$ for each $v|p$. There is an action of the group $\mathbf{G}(U_a) = U_a^0 \cdot \mathcal{O}_F^\times / U_a \cdot \mathcal{O}_F^\times$ on $M_k(U_a)$ with $x \cdot y$ acting via the operator $(U_a x U_a) = (\prod_{v|p} \omega(a_v)^{-\mu})\,[U_a x U_a]$ where $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_a^0$ and $y \in \mathcal{O}_F^\times$. Here $\omega : \mathrm{Gal}(F^{ab}/F) \longrightarrow \overline{F}^\times$ is the Teichmüller character, and $\omega(a_v)$ is defined by composing ω with the global reciprocity map taking $\mathcal{O}_{F,v}^\times$ to the inertia group of v in $\mathrm{Gal}(F^{ab}/F)$. Recall that we have fixed an embedding of $\overline{F}$ into $\overline{\mathbf{Q}}_p$, so ω, which *a priori* takes values in $\mathbf{Q}_p^\times$, can be considered as taking values in $\overline{F}^\times$.

Let $(p) = \mathfrak{p}_i^{e_i}, \ldots, \mathfrak{p}_t^{e_t}$ be the prime factorization of (p) in F with $\mathfrak{p}_i$ the prime corresponding to the place v_i. For each $i = 1, \ldots, t$ define an operator $T_0(\mathfrak{p}_i)$ on $M_k(U_a)$ by $T_0(\mathfrak{p}_i) = (\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)})^{-\nu} T(\mathfrak{p}_i)$. Define an operator $T_0(p)$ similarly by $T_0(p) = p^{-\nu}[U_a \begin{pmatrix} 1 & \\ & \tilde{p} \end{pmatrix} U_a]$ where $\tilde{p}_v = p$ if $v|p$ and $\tilde{p}_v = 1$ otherwise. Note that $T_0(p)$ differs from $\prod_{i=1}^t T_0(\mathfrak{p}_i)^{e_i}$ by multiplication by some $\lambda \cdot [U_a \begin{pmatrix} 1 & \\ & e \end{pmatrix} U_a]$ where $\mathrm{ord}_{\mathfrak{p}_i}(\lambda) = 0$ and $e_{\mathfrak{p}_i} \in \mathcal{O}_{F,\mathfrak{p}_i}^\times$ for each $\mathfrak{p}_i$.

As discussed in the previous subsection, these operators act on each $V_\pi^{U_a}$, $\pi \in \Pi_k(U_a)$. If $V \subseteq U$ and if $V_v = GL_2(\mathcal{O}_{F,v})$ for each $v|p$, then the inclusion $M_k(U_a) \subseteq M_k(V_b)$ ($b \geqslant a$) is compatible with the natural homomorphism $\mathbf{G}(V_b) \longrightarrow \mathbf{G}(U_a)$ and with the actions of $T_0(p)$ and the $T_0(\mathfrak{p}_i)$'s.

Let $k = \Sigma k_\tau \tau$ with each $k_\tau \geqslant 2$. Let $\pi \in \Pi_k(U_a)$. Suppose that $v|p$ and that $\mathfrak{p}_v$ is the corresponding prime ideal. It is an easy consequence of the classification of local automorphic representations that if there exists a line in $V_{\pi,v}^{U_{a,v}}$ on which $T_0(\mathfrak{p}_v)$ acts via an element of $\overline{F}$ that is a unit in the ring of integers of $\overline{\mathbf{Q}}_p$ via the fixed embedding $\overline{F} \hookrightarrow \overline{\mathbf{Q}}_p$, then the line is unique. Call such a line *v-good.* A *v*-good line exists if and only if π_v is either a principal series representation $\pi(\eta_v|\cdot|_v^{-\frac{1}{2}}, \xi_v|\cdot|_v^{-\frac{1}{2}})$ or a special representation $\pi(\xi_v|\cdot|_v^{-\frac{1}{2}}, \xi_v|\cdot|_v^{\frac{1}{2}})$ such that in either case $\lambda_v^{-v}\xi_v(\lambda_v^{-1})$ is a unit in the ring of integers of $\overline{\mathbf{Q}}_p$ (cf. [H3, Corollary 2.2]). Here λ_v is the uniformizer of $\mathcal{O}_{F,v}$ chosen in the definition of $T_0(\mathfrak{p}_v)$. The representation π is said to be *nearly ordinary* if a *v*-good line exists in $V_{\pi,v}$ for each v dividing p. Similarly, a newform $f \in S_k(U_a)$ is called nearly ordinary if the corresponding automorphic representation π_f is nearly ordinary. Let $\Pi_k^{\mathrm{ord}}(U_a) \subseteq \Pi_k(U_a)$ be the subset of nearly ordinary representations. A representation $\pi \in \Pi_k^{\mathrm{ord}}(U_a)$ is said to be *ordinary* if ξ_v is unramified at v for each $v|p$. Similarly, a newform $f \in S_k(U_a)$ is ordinary if the corresponding automorphic representation is ordinary.

Fix an identification of $\mathbf{C}$ with $\mathbf{Q}_p$ extending the fixed embedding of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_p$. For each $\pi \in \Pi_k^{\mathrm{ord}}(U_a)$ let $w(\pi) = \otimes w(\pi, v) \in \bigotimes_v V_{\pi,v}^{U_{a,v}}$ be a vector such that $w(\pi, v)$ spans a *v*-good line for each $v|p$. Each $w(\pi)$ is an eigenvector for the Hecke operators $T_0(\mathfrak{p}_i)$ for $i = 1, \ldots, t$, $T_0(p)$, $T(\ell)$ and $S(\ell)$ for each prime ideal $\ell \nmid p$ for which $U_\ell = GL_2(\mathcal{O}_{F,\ell})$, and for each element of $\mathbf{G}(U_a)$, and the corresponding eigenvalues are integers in $\overline{\mathbf{Q}}_p$. Let $S_k^*(U_a) \subseteq S_k(U_a)$ be the subspace spanned by the $w(\pi)$'s (recall that $S_k(U_a) = \bigoplus_{\pi \in \Pi_k(U_a)} V_\pi^{U_a}$). Let $\mathbf{T}_k(U_a) \subseteq \mathrm{End}_{\mathbf{C}}(S_k^*(U_a))$ be the subalgebra generated over $\mathbf{Z}_p$ by the aforementioned Hecke operators. The ring $\mathbf{T}_k(U_a)$ is a finite, flat, commutative, reduced $\mathbf{Z}_p$-algebra. In fact, we have an injection

$$\mathbf{T}_k(U_a) \hookrightarrow \prod_{\pi \in \Pi_k^{\mathrm{ord}}(U_a)} \overline{\mathbf{Q}}_p.$$

Note that the definition of $\mathbf{T}_k(U_a)$ is independent of the choice of the $w(\pi)$'s. Also, if $V \supseteq U$ is another open compact subgroup, then there is a canonical homomorphism $\mathbf{T}_k(V_b) \longrightarrow \mathbf{T}_k(U_a)$ ($b \geqslant a$).

For $\mathcal{O}$ the ring of integers of some finite extension K of $\mathbf{Q}_p$, put $\mathbf{T}_k(\mathrm{U}_a, \mathcal{O}) = \mathbf{T}_k(\mathrm{U}_a) \otimes_{\mathbf{Z}_p} \mathcal{O}$. Put also

$$\mathbf{T}_\infty(\mathrm{U}, \mathcal{O}) = \varprojlim_a \mathbf{T}_2(\mathrm{U}_a, \mathcal{O}).$$

Suppose k is parallel (i.e., $\nu = 0$). We now give an alternate (but equivalent) definition of $\mathbf{T}_k(\mathrm{U}_a, \mathcal{O})$. As we shall see, both definitions will have their uses. Let $\widetilde{\mathbf{T}}_k(\mathrm{U}_a)$ be the subring of $\mathrm{End}_{\mathbf{C}}(\mathrm{M}_k(\mathrm{U}_a))$ generated over $\mathbf{Z}$ by $\mathrm{T}_0(\mathfrak{p}_i)$ for $i = 1, \dots, t$, $\mathrm{T}_0(p)$, the action of $\mathbf{G}(\mathrm{U}_a)$, and by the operators $\mathrm{T}(\ell)$ and $\mathrm{S}(\ell)$ for each prime ideal $\ell \nmid p$ for which $\mathrm{U}_\ell = \mathrm{GL}_2(\mathcal{O}_{\mathrm{F},\ell})$. Let $\mathbf{T}_k^*(\mathrm{U}_a)$ be the quotient ring obtained by restricting the action of the Hecke operators to the space $\mathrm{S}_k(\mathrm{U}_a)$ of cusp forms. These rings are finite, flat, commutative $\mathbf{Z}$-algebras. Put $\widetilde{\mathbf{T}}_k(\mathrm{U}_a, \mathcal{O}) = \widetilde{\mathbf{T}}_k(\mathrm{U}_a) \otimes \mathcal{O}$ and $\mathbf{T}_k^*(\mathrm{U}_a, \mathcal{O}) = \mathbf{T}_k^*(\mathrm{U}_a) \otimes \mathcal{O}$ with $\mathcal{O}$ as in the preceding paragraph. For all sufficiently divisible integers m, the operator $e = \lim_{n\to\infty} \mathrm{T}_0(p)^{p^n(p^m-1)}$ exists in $\widetilde{\mathbf{T}}_k(\mathrm{U}_a, \mathcal{O})$ and is independent of m. Moreover, e is an idempotent. Put $\mathbf{T}_k(\mathrm{U}_a, \mathcal{O}) = e\mathbf{T}_k^*(\mathrm{U}_a, \mathcal{O})$.

That this definition of $\mathbf{T}_k(\mathrm{U}_a, \mathcal{O})$ yields the same ring as did the previous one can be seen as follows. The ring $\mathbf{T}_k^*(\mathrm{U}_a, \mathcal{O})$ is the subring of $\mathrm{End}_{\mathbf{C}}(\mathrm{S}_k(\mathrm{U}_a))$ generated over $\mathcal{O}$ by $\mathrm{T}_0(\mathfrak{p}_i)$ for $i = 1, \dots, t$, $\mathrm{T}_0(p)$, the action of $\mathbf{G}(\mathrm{U}_a)$, and by $\mathrm{T}(\ell)$ and $\mathrm{S}(\ell)$ for each $\ell \nmid p$ such that $\mathrm{U}_\ell = \mathrm{GL}_2(\mathcal{O}_{\mathrm{F},\ell})$. In particular, e is identified with an idempotent in $\mathrm{End}_{\mathbf{C}}(\mathrm{S}_k(\mathrm{U}_a))$ and $e\mathbf{T}_k^*(\mathrm{U}_a, \mathcal{O})$ is just the image of $\mathbf{T}_k^*(\mathrm{U}_a, \mathcal{O})$ in $\mathrm{End}_{\mathbf{C}}(e\mathrm{S}_k(\mathrm{U}_a))$. Write $e\mathrm{S}_k(\mathrm{U}_a) = \bigoplus_{\pi\in\Pi_k(\mathrm{U}_a)} e\mathrm{V}_\pi^{\mathrm{U}_a}$. From the definition of e we have that $e\mathrm{V}_\pi^{\mathrm{U}_a} = 0$ if $\pi \notin \Pi_k^{\mathrm{ord}}(\mathrm{U}_a)$ and that if $\pi \in \Pi_k^{\mathrm{ord}}(\mathrm{U}_a)$ then $e\mathrm{V}_\pi^{\mathrm{U}_a} = \{ x = \otimes x_v : x_v \text{ spans a } v\text{-good line } \forall v|p \}$. It is now immediate that $e\mathbf{T}_k^*(\mathrm{U}_a, \mathcal{O})$ agrees with the first definition of $\mathbf{T}_k(\mathrm{U}_a, \mathcal{O})$.

Let $\mathbf{G}(\mathrm{U}) = \varprojlim_a \mathbf{G}(\mathrm{U}_a)$, where the transition maps are the maps induced from the inclusions $\mathrm{U}_a^0 \subseteq \mathrm{U}_b^0$, $a \geqslant b$. There is a homomorphism $\mathcal{O}[[\mathbf{G}(\mathrm{U})]] \longrightarrow \mathbf{T}_\infty(\mathrm{U}, \mathcal{O})$. Put

$$\mathrm{U}^f = \mathrm{U} \cap (\mathbf{A}_f)^\times, \quad \mathrm{U}_a^f = \mathrm{U}_a \cap (\mathbf{A}_f)^\times, \quad \mathrm{Z}(\mathrm{U}_a) = \mathrm{U}^f \cdot \mathcal{O}_{\mathrm{F}}^\times / \mathrm{U}_a^f \cdot \mathcal{O}_{\mathrm{F}}^\times,$$

and

$$\mathrm{Z}(\mathrm{U}) = \varprojlim_a \mathrm{Z}(\mathrm{U}_a).$$

The map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto (\prod_{v|p} (a^{-1}d)_v, a)$ induces isomorphisms

$$\mathbf{G}(\mathrm{U}_a) \simeq (\mathcal{O}_{\mathrm{F}}/p^a)^\times \times \mathrm{Z}(\mathrm{U}_a) \quad \text{and} \quad \mathbf{G}(\mathrm{U}) \simeq (\mathcal{O}_{\mathrm{F}} \otimes \mathbf{Z}_p)^\times \times \mathrm{Z}(\mathrm{U}).$$

For $(y, 1) \in \mathbf{G}(\mathrm{U})$ we write T_y for the corresponding Hecke operator. Similarly, we write S_x for the operator corresponding to $(1, x) \in \mathbf{G}(\mathrm{U})$. Let

$$\mathscr{U} = \prod_{v_i} \mathscr{U}_{v_i} \subseteq (\mathscr{O}_F \otimes \mathbf{Z}_p)^\times = \prod_{v_i} \mathscr{O}^\times_{\mathrm{F}, v_i}$$

where $\mathscr{U}_{v_i} \subseteq \mathscr{O}^\times_{\mathrm{F}, v_i}$ is the subgroup of units congruent to one modulo v_i. Let $y_j^{(i)} \in \mathscr{U}$ be as in §2.5. Let $x_1, \ldots, x_{\delta_\mathrm{F}} \in \mathrm{Z}(\mathrm{U})$ be the images of $\gamma_1^{p^{r_1}}, \ldots, \gamma_{\delta_\mathrm{F}}^{p^{r_{\delta_\mathrm{F}}}}$, respectively, via the global reciprocity map (for the definition of γ_i and r_i see §2.5). The x_i's generate a maximal $\mathbf{Z}_p$-free direct summand of $\mathrm{Z}(\mathrm{U})$. The ring $\mathbf{T}_\infty(\mathrm{U}, \mathscr{O})$ is an algebra over the ring $\Lambda'_\mathscr{O} = \mathscr{O}[[\mathrm{X}_1, \ldots, \mathrm{X}_{\delta_\mathrm{F}}, \mathrm{Y}_1^{(1)}, \ldots, \mathrm{Y}_{d_t}^{(t)}]]$ via $\mathrm{X}_i \longmapsto \mathrm{S}_{x_i} - 1$ and $\mathrm{Y}_j^{(i)} \longmapsto \mathrm{T}_{y_j^{(i)}} - 1$.

The principal goal of this subsection is to show that $\mathbf{T}_\infty(\mathrm{U}, \mathscr{O})$ is a finite, torsion-free $\Lambda'_\mathscr{O}$-module. We only prove this for F having even degree, although the result is true in general. Our proof involves analyzing modular forms on a twisted-form of GL_2.

Suppose that F has even degree. Let D be the unique quaternion algebra over F ramified at every infinite place and unramified at all finite places, and let R be a maximal order of D. Let G^D be the unique algebraic group over F such that $\mathrm{G}^\mathrm{D}(\mathrm{F}) = \mathrm{D}^\times$. Let $\nu_\mathrm{D} : \mathrm{G}^\mathrm{D} \longrightarrow \mathbf{G}_m$ be the reduced norm. For each finite place v fix an isomorphism $\mathrm{R} \otimes \mathscr{O}_{\mathrm{F}, v} \simeq \mathrm{M}_2(\mathscr{O}_{\mathrm{F}, v})$. This induces an isomorphism $\mathrm{G}^\mathrm{D}(\mathbf{A}_f) \simeq \mathrm{GL}_2(\mathbf{A}_f)$ which we use to identify these two groups. For each open compact subgroup $\mathrm{U} \subseteq \mathrm{GL}_2(\mathbf{A}_f)$ put

$$\mathscr{F}^\mathrm{D}(\mathrm{U}) = \{ f : \mathrm{D}^\times \backslash \mathrm{G}^\mathrm{D}(\mathbf{A}_f)/\mathrm{U} \longrightarrow \mathbf{C} \}.$$

(Note that $\mathrm{D}^\times \backslash \mathrm{G}^\mathrm{D}(\mathbf{A}_f)/\mathrm{U}$ is a finite set.) We distinguish a subspace

$$\mathrm{I}^\mathrm{D}(\mathrm{U}) = \{ f \in \mathscr{F}^\mathrm{D}(\mathrm{U}) : f \text{ factors through } \mathbf{G}^\mathrm{D}(\mathbf{A}_f)/\mathrm{U} \xrightarrow{\nu_\mathrm{D}} (\mathbf{A}_f)^\times/\nu_\mathrm{D}(\mathrm{U}) \}.$$

For any $g \in \mathrm{G}^\mathrm{D}(\mathbf{A}_f) \simeq \mathrm{GL}_2(\mathbf{A}_f)$ there is a Hecke operator $[\mathrm{U}g\mathrm{U}'] : \mathscr{F}^\mathrm{D}(\mathrm{U}) \longrightarrow \mathscr{F}^\mathrm{D}(\mathrm{U}')$ defined as in (3.1). It is easy to see that $[\mathrm{U}g\mathrm{U}']$ maps $\mathrm{I}^\mathrm{D}(\mathrm{U})$ to $\mathrm{I}^\mathrm{D}(\mathrm{U}')$. A theorem of Jacquet, Langlands, and Shimizu [J-L], [Shi] states that there is a system of isomorphisms

$$\mathrm{S}^\mathrm{D}(\mathrm{U}) = \mathscr{F}^\mathrm{D}(\mathrm{U})/\mathrm{I}^\mathrm{D}(\mathrm{U}) \simeq \mathrm{S}_2(\mathrm{U})$$

compatible with the action of the Hecke operators $[\mathrm{U}g\mathrm{U}']$. Thus $\mathbf{T}_2^*(\mathrm{U}_a)$ can be identified with the subring of $\mathrm{End}_\mathbf{C}(\mathrm{S}^\mathrm{D}(\mathrm{U}_a))$ generated over $\mathbf{Z}$ by $\mathrm{T}_0(p)$, $\mathrm{T}_0(\mathfrak{p}_i)$ for $i = 1, \ldots, t$, $\mathbf{G}(\mathrm{U}_a)$, and $\mathrm{T}(\ell)$ and $\mathrm{S}(\ell)$ for all prime ideals ℓ for which $\mathrm{U}_\ell = \mathrm{GL}_2(\mathscr{O}_{\mathrm{F}, \ell})$.

Put $\mathrm{X}(\mathrm{U}) = \mathrm{D}^\times \backslash \mathrm{G}^\mathrm{D}(\mathbf{A}^f)/\mathrm{U}$, and define

$$\mathrm{H}^0(\mathrm{X}(\mathrm{U}), \mathbf{Z}) = \{ f \in \mathscr{F}^\mathrm{D}(\mathrm{U}) \text{ taking values in } \mathbf{Z} \}.$$

This is a free $\mathbf{Z}$-module of rank equal to $\#X(U)$. For any $\mathbf{Z}$-module R, put $H^0(X(U), R) = H^0(X(U), \mathbf{Z}) \otimes R$. Note that $H^0(X(U), \mathbf{C}) = \mathscr{F}^D(U)$. The action of $[UgU]$ on $H^0(X(U), \mathbf{C})$ stabilizes $H^0(X(U), \mathbf{Z})$ and hence induces an action of $[UgU]$ on $H^0(X(U), R)$ for any $\mathbf{Z}$-module R. If R is an $\mathcal{O}$-module then the operator $e = \lim\limits_{\to n}[U_a\begin{pmatrix}1 & \\ & \tilde{p}\end{pmatrix}U_a]^{p^n(p^m-1)}$ exists in $\mathrm{End}_{\mathcal{O}}(H^0(X(U_a), R))$ for sufficiently divisible m. Moreover, e annihilates $I^D(U_a, \mathbf{Z}) \otimes R$, where $I^D(U_a, \mathbf{Z}) = \{f \in I^D(U_a)$ taking values in $\mathbf{Z}\}$.

Let $T(U_a, \mathcal{O})$ be the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(H^0(X(U_a), \mathcal{O}))$ generated over $\mathcal{O}$ by $T_0(p)$, $T_0(\mathfrak{p}_i)$ for $i = 1, ..., t$, $\mathbf{G}(U_a)$, and $T(\ell)$ and $S(\ell)$ for all prime ideals $\ell \nmid p$ such that $U_\ell = GL_2(\mathcal{O}_{F,\ell})$. It follows that $\mathbf{T}_2(U_a, \mathcal{O})$ can be identified with $eT(U_a, \mathcal{O})$ (equivalently, with the image of $T(U_a, \mathcal{O})$ in $\mathrm{End}_{\mathcal{O}}(eH^0(X(U_a), \mathcal{O})))$. Put

$$H_\infty(U) = \lim_{\to a} eH^0(X(U_a), K/\mathcal{O}).$$

(K is the field of fractions of $\mathcal{O}$.) This is a $\mathbf{T}_\infty(U, \mathcal{O})$-module.

For any open subgroup U put $\overline{U} = U/U\cap F^\times$. For each $x \in G^D(\mathbf{A}_f)$ put $c_U(x) = \#\{u \in \overline{U} \mid xu = x\}$.

Let R be any $\mathcal{O}$-algebra. If each $c_U(x)$ is invertible in R, then define a pairing

$$\langle\ ,\ \rangle_U : H^0(X(U), R) \times H^0(X(U), R) \longrightarrow R$$

by

$$\langle f, g\rangle_U = \sum_{x\in X(U)} \Big| c_U(x)^{-1} f(x) g(x).$$

This is a non-degenerate pairing, and the map $f \longmapsto \langle f, \cdot\rangle_U$ determines an isomorphism $H^0(X(U), R) \simeq \mathrm{Hom}_R(H^0(X(U), R), R)$ that is functorial in R.

The pairing $\langle\ ,\ \rangle_U$ is not Hecke-equivariant, but a straight-forward calculation shows that

$$\langle [UgU]f, h\rangle_U = \langle f, [Ug^{-1}U]h\rangle_U$$

for any $g \in G^D(\mathbf{A}_f)$. It follows that for each $t \in T(U_a, \mathcal{O})$ there exists $t^+ \in \mathrm{End}_{\mathcal{O}}(H^0(X(U_a), \mathcal{O}))$ such that $\langle t\cdot f, h\rangle_{U_a} = \langle f, t^+ \cdot h\rangle_{U_a}$ for all $f, h \in H^0(X(U_a), \mathcal{O})$. Let $T^+(U_a, \mathcal{O}) \subseteq \mathrm{End}(H^0(X(U_a), \mathcal{O}))$ be the $\mathcal{O}$-subalgebra generated by $\{t^+ : t \in T(U_a, \mathcal{O})\}$. Clearly the map $t \longmapsto t^+$ determines an isomorphism of $\mathcal{O}$-algebras $T(U_a, \mathcal{O}) \simeq T^+(U_a, \mathcal{O})$. For any $\mathcal{O}$-module R write $H^0(X(U_a), R)^+$ for the $T(U_a, \mathcal{O})$-module whose underlying $\mathcal{O}$-module is just $H^0(X(U_a), R)$ but the action of $t \in T(U_a, \mathcal{O})$ is via t^+. It follows that the pairing $\langle\ ,\ \rangle_{U_a}$ induces a perfect pairing $\langle\ ,\ \rangle_{U_a}$: $eH^0(X(U_a), R) \times eH^0(X(U_a), R)^+ \longrightarrow R$ of $\mathbf{T}_2(U_a, \mathcal{O})$-modules. Put $H^+_\infty(U_a) = \varinjlim_a eH^0(X(U_a), K/\mathcal{O})^+$.

If $V \subseteq U$ is any subgroup, then we define a trace map $tr(V, U) : H^0(X(V), R) \longrightarrow H^0(X(U), R)$ by

$$tr(V, U)f(x) = \sum_i f(xx_i^{-1}), \quad \overline{U} = \sqcup \overline{V}x_i.$$

If $V = V_b$ and $U = U_a$ ($b \geqslant a$) then it is easy to see that this is independent of the chosen coset representatives and that it is compatible with the actions of t and t^+ for $t \in \mathbf{T}_2(V, \mathcal{O})$. The pairings $\langle\ ,\ \rangle_U$ and $\langle\ ,\ \rangle_V$ satisfy the following compatibility whenever they are both defined: the diagram

$$\begin{array}{ccccc} \langle\ ,\ \rangle_U : H^0(X(U), R) & \times & H^0(X(U), R) & \longrightarrow & R \\ \downarrow & & \uparrow tr(V, U) & & \| \\ \langle\ ,\ \rangle_V : H^0(X(V), R) & \times & H^0(X(V), R) & \longrightarrow & R \end{array}$$

commutes. Since $c_{U_a}(x) = 1$ for all x if a is sufficiently large, it follows that by putting

$$M_\infty(U) = \varprojlim_a eH^0(X(U_a), \mathcal{O})^+,$$

where the transition maps are just the trace maps $tr(U_b, U_a)$ ($b \geqslant a$), we have an identification of $\mathbf{T}_\infty(U, \mathcal{O})$-modules

$$\begin{aligned} M_\infty(U) &\simeq \varprojlim_a \mathrm{Hom}_{\mathcal{O}}(eH^0(X(U_a), \mathcal{O}), \mathcal{O}) \\ &\simeq \varprojlim_a \mathrm{Hom}_{\mathcal{O}}(eH^0(X(U_a), K/\mathcal{O}), K/\mathcal{O}) \\ &\simeq \mathrm{Hom}_{\mathcal{O}}(\varinjlim_a eH^0(X(U_a), K/\mathcal{O}), K/\mathcal{O}) \\ &\simeq \mathrm{Hom}_{\mathcal{O}}(H_\infty(U), K/\mathcal{O}), \end{aligned}$$

the Pontryagin dual of $H_\infty(U)$. Putting

$$M^+_\infty(U) = \varprojlim_a eH^0(X(U_a), \mathcal{O})$$

we obtain a similar identification of $M^+_\infty(U)$ with the Pontryagin dual of $H^+_\infty(U)$.

The following proposition is due to Hida [H2, Theorem 3.8].

Proposition **3.3.** — *If the action of every element of* $U/U \cap F^\times$ *on* $D^\times \backslash G^D(\mathbf{A}^f)$ *is fixed-point free, then* $M_\infty(U)$ *and* $M^+_\infty(U)$ *are free* $\Lambda'_{\mathcal{O}}$ *-modules of rank equal to*

$$\mathrm{rank}_{\mathcal{O}}\ eH^0(X(U_1^0), \mathcal{O}) \times \begin{pmatrix} \text{order of the torsion} \\ \text{subgroup of } \mathbf{G}(U) \end{pmatrix}.$$

Proof. — We first claim that the Pontryagin dual of $e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_a),\mathrm{K}/\mathcal{O})$ is a free $\mathcal{O}[[\mathbf{G}(\mathrm{U}_a)]]$-module of rank equal to the $\mathcal{O}$-rank of the Pontryagin dual of $e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_a^0),\mathrm{K}/\mathcal{O})$. Clearly, it suffices to prove the claim without having applied the operator e, in which case it is a simple consequence of the fact that $\mathrm{H}^0(\mathrm{X}(\mathrm{U}_a^0),\mathrm{K}/\mathcal{O}) = \mathrm{H}^0(\mathrm{X}(\mathrm{U}_a),\mathrm{K}/\mathcal{O})^{\mathbf{G}(\mathrm{U}_a)}$ and that $\#\mathrm{X}(\mathrm{U}_a) = \#\mathrm{X}(\mathrm{U}_a^0)\times\#\mathbf{G}(\mathrm{U}_a)$, the latter equality a consequence of the assumption that $\mathbf{G}(\mathrm{U}_a)$ acts freely on $\mathrm{X}(\mathrm{U}_a)$. The assertion of the proposition for $\mathrm{M}_\infty(\mathrm{U})$ will follow if we can establish that $e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_a^0),\mathrm{K}/\mathcal{O}) = e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_1^0),\mathrm{K}/\mathcal{O})$, for then we will have that the dual of $e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_a),\mathrm{K}/\mathcal{O})$ is a free $\mathcal{O}[[\mathbf{G}(\mathrm{U}_a)]]$-module of rank equal to the $\mathcal{O}$-rank of the dual of $e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_1^0),\mathrm{K}/\mathcal{O})$ which in turn equals $\mathrm{rank}_{\mathcal{O}}\,\mathrm{H}^0(\mathrm{X}(\mathrm{U}_1^0),\mathcal{O})$. Now, if $a\geqslant 2$, then $\mathrm{U}_a^0\begin{pmatrix}1&\\&\tilde{p}\end{pmatrix}\mathrm{U}_a^0 = \mathrm{U}_a^0\begin{pmatrix}1&\\&\tilde{p}\end{pmatrix}\mathrm{U}_{a-1}^0$, so $\mathrm{T}_0(p)\cdot\mathrm{H}^0(\mathrm{X}(\mathrm{U}_a^0),\mathrm{K}/\mathcal{O})\subseteq\mathrm{H}^0(\mathrm{X}(\mathrm{U}_{a-1}^0),\mathrm{K}/\mathcal{O})$, whence $e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_a^0),\mathrm{K}/\mathcal{O}) = e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_1^0),\mathrm{K}/\mathcal{O})$ as desired. The same argument applies to the Pontryagin dual of $e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_a),\mathrm{K}/\mathcal{O})^+$ yielding the assertion of the proposition for $\mathrm{M}_\infty^+(\mathrm{U})$. □

Corollary **3.4.** — *For any* U, $\mathbf{T}_\infty(\mathrm{U},\mathcal{O})$ *is a finite, torsion-free* $\Lambda'_{\mathcal{O}}$*-module. In particular* $\mathbf{T}_\infty(\mathrm{U},\mathcal{O})$ *is a semilocal ring complete with respect to its radical.*

Proof. — Choose a prime ℓ of F for which $\mathrm{V} = \mathrm{U}\cap\mathrm{U}(\ell)$ is such that $\mathrm{V}/\mathrm{V}\cap\mathrm{F}^\times$ acts freely on $\mathrm{D}^\times\backslash\mathrm{G}^\mathrm{D}(\mathbf{A}^f)$. The existence of such an ℓ is proven in Lemma 3.5 below. The induced map $\mathrm{M}_\infty(\mathrm{U})\hookrightarrow\mathrm{M}_\infty(\mathrm{V})$ is compatible with the action of $\mathbf{T}_\infty(\mathrm{V},\mathcal{O})$ and hence is a map of $\Lambda'_{\mathcal{O}}$-modules. As $\mathrm{M}_\infty(\mathrm{V})$ is a free $\Lambda'_{\mathcal{O}}$-module by Proposition 3.3, $\mathrm{M}_\infty(\mathrm{U})$ is a finite, torsion-free $\Lambda'_{\mathcal{O}}$-module. As there is an injection $\mathbf{T}_\infty(\mathrm{U},\mathcal{O})\hookrightarrow\mathrm{End}_{\Lambda'_{\mathcal{O}}}(\mathrm{M}_\infty(\mathrm{U}))$, the same is therefore true of $\mathbf{T}_\infty(\mathrm{U},\mathcal{O})$. □

Lemma **3.5.** — *If* $\ell\nmid 6$ *is an unramified prime ideal of* F, *then* $\mathrm{U}(\ell)/\mathrm{U}(\ell)\cap\mathrm{F}^\times$ *acts freely on* $\mathrm{D}^\times\backslash\mathrm{G}^\mathrm{D}(\mathbf{A}_f)$.

Proof. — If $\delta x = xu$ for some $\delta\in\mathrm{D}^\times$, $x\in\mathrm{G}^\mathrm{D}(\mathbf{A}_f)$, and $u\in\mathrm{U}(\ell)$, then $\delta\in\Gamma_x$ where $\Gamma_x = \mathrm{D}^\times\cap x\mathrm{U}(\ell)x^{-1}$. We claim that $\Gamma_x/\Gamma_x\cap\mathrm{F}^\times$ has finite order. To see this, note that the canonical injection $i:\mathrm{D}^\times/\mathrm{F}^\times\hookrightarrow\mathrm{G}^\mathrm{D}(\mathbf{A})/\mathbf{A}^\times$ identifies $\mathrm{D}^\times/\mathrm{F}^\times$ with a discrete subgroup of $\mathrm{G}^\mathrm{D}(\mathbf{A})/\mathbf{A}^\times$. Now let $\mathrm{V} = \mathrm{G}^\mathrm{D}(\mathbf{R}\otimes\mathrm{F})/(\mathrm{R}\otimes\mathrm{F})^\times\times(x\mathrm{U}(\ell)x^{-1}/\mathrm{U}(\ell)\cap(\mathbf{A}_f)^\times)$. This is a compact open subgroup of $\mathrm{G}^\mathrm{D}(\mathbf{A})/\mathbf{A}^\times$, so $\mathrm{W} = \mathrm{V}\cap\mathrm{im}(i)$ is a finite group, and it is clear from the definitions that $\Gamma_x/\Gamma_x\cap\mathrm{F}^\times\hookrightarrow\mathrm{W}$. This proves the claim. Thus some power of δ lies in $\mathrm{F}^\times$, and the same is true of u. By the choice of ℓ, u must itself be in $\mathrm{F}^\times$. □

Corollary **3.6.** — *Let* M *be the exponent of the torsion subgroup of* $\mathrm{D}^\times/\mathrm{F}^\times$. *If* $\{\ell_1,\ldots,\ell_s\}$ *is a set of unramified primes of* F *such that*

(i) $\ell_i\nmid 6$ *and* $(\mathrm{Nm}(\ell_i)-1,\mathrm{M}) = 2^{r_i}$ *for each* i,

(ii) *for each* $y \in \mathcal{O}_F^\times$ *that is totally positive some* ℓ_i *does not split in* $F(\sqrt{-y})$, *then* $U_1(\ell_1 \dots \ell_s) / U_1(\ell_1 \dots \ell_s) \cap F^\times$ *acts freely on* $D^\times \backslash G^D(\mathbf{A}_f)$.

Proof. — If $\delta x = xu$ for some $\delta \in D^\times$, $x \in G^D(\mathbf{A}_f)$, and $u \in U_1(\ell_1 \dots \ell_s)$, then $\delta^e x = xu^e$ for $e = \prod_{i=1}^{s}(\mathrm{Nm}(\ell_i) - 1)$. Since $u^e \in U(\ell_1 \dots \ell_s)$ it follows from Lemma 3.5 that $u^e \in F^\times$, and therefore $\delta^e \in F^\times$. It then follows from our hypotheses on ℓ_i that $\delta^{2^r} \in F^\times$ for some r. If $r = 0$, then δ, $u \in F^\times$. We may therefore suppose that $r \geqslant 1$ but that $\delta^{2^{r-1}} \notin F^\times$. Put $\gamma = \delta^{2^{r-1}}$ and $\omega = u^{2^{r-1}}$. Then $\gamma, \omega \notin F^\times$, but $\gamma^2, \omega^2 \in F^\times$. Let α and β be the eigenvalues of γ. As $(\alpha/\beta)^2 = 1$ it must be that either $\alpha = \beta$ or $\alpha = -\beta$. If $\alpha = \beta$ then $\gamma \in F^\times$ since $2\alpha = \alpha + \beta \in F$. Therefore $\alpha = -\beta$. Note that $\det(\gamma) = \alpha\beta$ must be totally positive. Since α and β are also the eigenvalues of ω and since $\omega \in U_1(\ell_1 \dots \ell_s)$ we find that α and β are in F_{ℓ_i} for each i and that $\alpha\beta \in \mathcal{O}_F^\times$. Therefore each ℓ_i splits in $F(\sqrt{-\alpha\beta}) = F(\sqrt{\alpha^2})$, contradicting our hypotheses. □

Suppose $\lambda : \mathbf{T}_\infty(U, \mathcal{O}) \longrightarrow \overline{\mathbf{Q}}_p$ is a homomorphism of $\mathcal{O}$-algebras such that $\psi = \lambda|_{Z(U)}$ and $\varphi = \lambda|_{(\mathcal{O}_F \otimes \mathbf{Z}_p)^\times}$ are finite characters. (Recall that we have identified $\mathbf{G}(U)$ with $Z(U) \times (\mathcal{O}_F \otimes \mathbf{Z}_p)^\times$.) It is not difficult to deduce from the definition of $\mathbf{T}_\infty(U, \mathcal{O})$ that λ factors through some $\mathbf{T}_2(U_a, \mathcal{O})$ and hence corresponds to some $\pi \in \Pi_2^{\mathrm{ord}}(U_a)$. That is to say, there exists a unique $\pi \in \Pi_2^{\mathrm{ord}}(U_a)$ and an eigenvector $v \in V_\pi^{U_a}$ for $\mathbf{T}_\infty(U, \mathcal{O})$ such that the eigenvalue of each $t \in \mathbf{T}_\infty(U, \mathcal{O})$ acting on v, viewed as an element of $\overline{\mathbf{Q}}_p$, is just $\lambda(t)$. The existence of such a π follows from the definition of $\mathbf{T}_2(U_a, \mathcal{O})$. It is also easy to see that any $\pi \in \Pi_2^{\mathrm{ord}}(U_a)$ determines such a homomorphism λ. Therefore there is a correspondence between homomorphisms as at the start of this paragraph and nearly ordinary automorphic representations $\pi \in \bigcup_a \Pi_2^{\mathrm{ord}}(U_a)$. This correspondence generalizes to other weights k as summarized in the following remarkable result of Hida [H2, Corollary 2.5].

Proposition **3.7.** — *If* $\lambda : \mathbf{T}_\infty(U, \mathcal{O}) \longrightarrow \overline{\mathbf{Q}}_p$ *is an* $\mathcal{O}$*-algebra homomorphism such that* $\lambda|_{Z(U)} = \psi\varepsilon^\mu$, $\mu \geqslant 0$, *with* ψ *and* $\varphi = \lambda|_{(\mathcal{O}_F \otimes \mathbf{Z}_p)^\times}$ *finite characters, then there exists a nearly ordinary automorphic representation* π *of weight* $k = (\mu + 2) \cdot t$ *for which* $\lambda(T(\ell))$ *and* $\lambda(S(\ell))$ *equal, respectively, the eigenvalues of* $T(\ell)$ *and* $S(\ell)$ *acting on the newform associated to* π *for all prime ideals* $\ell \nmid p$ *for which* $U_\ell = GL_2(\mathcal{O}_{F,\ell})$.

Here, as in the preceding section, ε denotes the cyclotomic character giving the action of $\mathrm{Gal}(F^{ab}/F)$ on the $\mathbf{Z}_p$-module $\varprojlim_n \mu_{p^n}$, μ_{p^n} being the group of p^nth roots of unity. The character ε factors through $\mathrm{Gal}(F^{(p)}/F)$ where $F^{(p)}$ is the maximal abelian extension of F unramified outside of those places dividing p and ∞. Global reciprocity determines a homomorphism $Z(U) \longrightarrow \mathrm{Gal}(F^{(p)}/F)$ via which we view ε as a character on $Z(U)$.

We continue to assume that the degree of F is even. A prime P of $\mathbf{T}_\infty(U, \mathcal{O})$ that is the kernel of a homomorphism as in Proposition 3.7 is called an *algebraic prime*. The associated element $k = m + 2t \in \mathbf{Z}[\Pi]$ is called the *weight of* P. For an algebraic prime P of $\mathbf{T}_\infty(U, \mathcal{O})$ there are finitely many homomorphisms $\lambda : \mathbf{T}_\infty(U, \mathcal{O}) \longrightarrow \overline{\mathbf{Q}}_p$ whose kernel is P since $\mathbf{T}_\infty(U, \mathcal{O})/P$ is a finite extension of $\mathcal{O}$. Let $\mathcal{H}(P)$ denote the set of such homomorphisms. The set of algebraic primes of $\mathbf{T}_\infty(U, \mathcal{O})$ is Zariski dense, as the following lemma shows.

Lemma **3.8.** — *Let* Q *be a minimal prime of* $\mathbf{T}_\infty(U, \mathcal{O})$ *and let* $\mathcal{X}(Q)$ *be the set of algebraic primes of weight* 2 *containing* Q*. The set* $\mathcal{X}(Q)$ *is Zariski dense in* $\mathrm{spec}(\mathbf{T}_\infty(U, \mathcal{O})/Q)$.

Proof. — By Corollary 3.4, $\mathbf{T}_\infty(U, \mathcal{O})/Q$ is an integral extension of $\Lambda'_{\mathcal{O}}$. Call a prime $\mathfrak{p} \subseteq \Lambda'_{\mathcal{O}}$ algebraic (of weight 2) if it is of the form $\mathfrak{p} = \Lambda'_{\mathcal{O}} \cap P$ for some algebraic prime P of $\mathbf{T}_\infty(U, \mathcal{O})/Q$ of weight 2. The algebraic primes of $\Lambda'_{\mathcal{O}}$ are just those corresponding to kernels of homomorphisms $\Lambda'_{\mathcal{O}} \longrightarrow \overline{\mathbf{Q}}_p$ sending $1+Y_j^{(i)} \longmapsto \varphi(y_j^{(i)})$ and $1+X_i \longmapsto \psi(x_i)$ for finite characters φ and ψ of $(\mathcal{O}_F \otimes \mathbf{Z}_p)^\times$ and Z(U), respectively. That such primes are Zariski-dense in $\mathrm{spec}(\Lambda'_{\mathcal{O}})$ is immediate. □

Corollary **3.9.** — *Let* Q *be a minimal prime of* $\mathbf{T}_\infty(U, \mathcal{O})$. *If* $V \supseteq U$ *is such that some* $\lambda \in \mathcal{H}(P)$ *factors through the map* $\mathbf{T}_\infty(U, \mathcal{O}) \longrightarrow \mathbf{T}_\infty(V, \mathcal{O})$ *for all* P *in a subset of* $\mathcal{X}(Q)$ *that is Zariski-dense in* $\mathrm{spec}(\mathbf{T}_\infty(U, \mathcal{O})/Q)$, *then* Q *is the inverse image of a minimal prime of* $\mathbf{T}_\infty(V, \mathcal{O})$.

3.3. Hecke algebras, representations, and pseudo-representations

In this subsection we assume that F has even degree. Let $U \subseteq GL_2(\mathcal{O}_F \otimes \widehat{\mathbf{Z}})$ be as in the preceding subsection. Write $\mathfrak{n}$ for the product of those prime ideals ℓ for which $U_\ell \nsubseteq GL_2(\mathcal{O}_{F,\ell})$. Suppose that $k = \kappa \cdot t$ with $\kappa \geqslant 2$. Let $\pi \in \Pi_k^{\mathrm{ord}}(U_a)$ and let $\lambda : \mathbf{T}_k(U_a, \mathcal{O}) \longrightarrow \overline{\mathbf{Q}}_p$ be the corresponding homomorphism. Suppose that π is ordinary. In [W2] it was shown that there exists a continuous, irreducible representation $\rho_\pi : \mathrm{Gal}(\overline{F}/F) \longrightarrow GL_2(\overline{\mathbf{Q}}_p)$ such that

(3.2)
- $\rho_\pi(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$
- ρ_π is unramified at all primes $\ell \nmid \mathfrak{n}p$
- $\mathrm{trace}\,\rho_\pi(\mathrm{Frob}_\ell) = \lambda(T(\ell))$ for all $\ell \nmid \mathfrak{n}p$
- $\det \rho_\pi(\mathrm{Frob}_\ell) = \lambda(S(\ell))\mathrm{Nm}(\ell)$ for all $\ell \nmid \mathfrak{n}p$
- $\det \rho_\pi(x) = \lambda(S_x)\varepsilon(x)$ for all $x \in Z(U)$
- $\rho_\pi|_{D_i} \simeq \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$ with $\psi_2^{(i)}(y) = \lambda(T_y)$ for all $y \in \mathcal{O}_{F,v_i}^\times$ and $\psi_2^{(i)}(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)}) = \lambda(T_0(\mathfrak{p}_i))$ for all $i = 1, \ldots, t$.

Now suppose that π is any element of $\Pi_k^{\mathrm{ord}}(U_a)$. Given any finite set S of finite places of F distinct from those dividing p, there exists a finite character ψ unramified at S and such that $\pi \otimes \psi$ is ordinary. The representation $\rho_\pi = (\rho_{\pi\otimes\psi}) \otimes \psi$ is independent of ψ, and by varying S one finds that (3.2) also holds for this ρ_π.

Now, for any representation $\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ and for any finite place $v \nmid p$ let $\pi_v(\rho)$ be the automorphic representation of $\mathrm{GL}_2(F_v)$ corresponding to $\rho|_{D_v}$ via the local Langlands' correspondence (see [Ca] and [Ku]) normalized as in [C]. So, in particular, if $\rho|_{D_v} \simeq \begin{pmatrix} \mu_1 & * \\ & \mu_2 \end{pmatrix}$ then $\pi_v(\rho) = \pi(\mu_1^{-1}|\cdot|_v^{-\frac{1}{2}}, \mu_2^{-1}|\cdot|_v^{-\frac{1}{2}})$. For any ordinary representation $\pi \in \Pi_k^{\mathrm{ord}}(U)$ it was shown in [W2, Theorem 2.1.3] that $\pi_v(\rho_\pi) \simeq \pi_v$. Now suppose that π is any element in $\Pi_k^{\mathrm{ord}}(U)$ and that ψ is a finite character unramified at v such that $\pi \otimes \psi$ is ordinary. It follows from the preceding observations that $\pi_v \otimes \psi_v = (\pi \otimes \psi)_v \simeq \pi_v(\rho_{\pi\otimes\psi}) = \pi_v(\rho_\pi \otimes \psi^{-1}) = \pi_v(\rho_\pi) \otimes \psi_v$. We therefore have that

(3.3) $$\pi_v \simeq \pi_v(\rho_\pi), \quad \pi \in \Pi_k^{\mathrm{ord}}(U), \quad v \nmid p\infty.$$

The representation ρ_π can be generalized as follows. Suppose that Q is a prime of $\mathbf{T}_\infty(U, \mathcal{O})$. Let $R = \mathbf{T}_\infty(U, \mathcal{O})/Q$ and let L be the field of fractions of R. Note that R is a complete local domain. Hida has shown that there is a continuous, semi-simple representation $\rho_Q : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(\overline{L})$ such that

(3.4) (i) $\rho_Q(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$

(ii) ρ_Q is unramified at all primes $\ell \nmid \mathfrak{n}p$

(iii) $\mathrm{trace}\,\rho_Q(\mathrm{Frob}_\ell) = T(\ell) \bmod Q$ for all $\ell \nmid \mathfrak{n}p$

(iv) $\det\rho_Q(\mathrm{Frob}_\ell) = S(\ell)\,\mathrm{Nm}(\ell) \bmod Q$ for all $\ell \nmid \mathfrak{n}p$

(v) $\det\rho_Q(x) = S_x\,\varepsilon(x) \bmod Q$ for all $x \in Z(U)$

(vi) $\rho_Q|_{D_i} \cong \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$ with $\psi_2^{(i)}(y) = T_y \bmod Q$ for all $y \in \mathcal{O}_{F, v_i}^\times$ and $\psi_2^{(i)}(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)}) = T_0(\mathfrak{p}_i) \bmod Q$ for all $i = 1, \ldots, t$.

By ρ_Q being continuous we mean that there is a finitely generated $\mathrm{Gal}(\overline{F}/F)$-stable R-module $\mathcal{M}$ in the underlying representation space of ρ_Q such that $\mathrm{Gal}(\overline{F}/F)$ acts continuously on $\mathcal{M}$. We give a proof of the existence of ρ_Q in the next few paragraphs.

If Q is an algebraic prime of weight 2, then the desired representation follows immediately from the existence and properties of the representations ρ_π. For let $\lambda \in \mathcal{H}(Q)$ and let π be the automorphic representation corresponding to λ. The homomorphism $\lambda : \mathbf{T}_\infty(U, \mathcal{O}) \longrightarrow \overline{\mathbf{Q}}_p$ determines an embedding $R \hookrightarrow \overline{\mathbf{Q}}_p$ which extends to an identification of $\overline{L}$ with $\overline{\mathbf{Q}}_p$. Under this identification we may take $\rho_Q = \rho_\pi$. Properties (3.4i-vi) follow from (3.2).

Suppose now that Q is a minimal prime. Using Lemma 3.8 one then deduces the existence of ρ_Q as in the proof of [W2, Theorem 2.2.1]. Of the properties of ρ_Q

listed in (3.4) the only one that is not immediate from the construction is the final one concerning the restrictions $\rho_Q|_{D_i}$. This can be deduced from the corresponding properties of the ρ_P's, P an algebraic prime of weight 2, as follows. First, arguing as in the proof of [W2, Lemma 2.2.4] shows that the semisimplification of $\rho_Q|_{D_i}$ is the sum of two characters $\psi_1^{(i)}$ and $\psi_2^{(i)}$ with $\psi_2^{(i)}(y) = T_y$ for each $y \in \mathcal{O}_{F,v_i}^\times$. Moreover, we may assume that $\psi_1^{(i)}|_{I_{v_i}} \neq \psi_2^{(i)}|_{I_{v_i}}$ for otherwise there would be nothing to prove. Choose $\tau_0 \in I_{v_i}$ such that $\psi_1^{(i)}(\tau_0) \neq \psi_2^{(i)}(\tau_0)$. Let $\mathcal{Y} \subseteq \mathcal{X}(Q)$ be the Zariski-dense subset of primes P for which $\psi_1^{(i)}(\tau_0) \neq \psi_2^{(i)}(\tau_0) \bmod P$. Choose a basis of ρ_Q such that $\rho_Q(\tau_0) = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ and $\rho_Q|_{D_i} = \begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix}$ with either $\chi_1 = \psi_1^{(i)}$ or $\chi_1 = \psi_2^{(i)}$. If $\rho_Q|_{D_i}$ is split, then property (vi) is immediate, so assume otherwise. Let $\sigma_0 \in D_i$ be such that $\rho_Q(\sigma_0) = \begin{pmatrix} * & b_0 \\ & * \end{pmatrix}$ with $b_0 \neq 0$, and let $g_0 \in \mathrm{Gal}(\overline{F}/F)$ be such that $\rho_Q(g_0) = \begin{pmatrix} * & * \\ c_0 & * \end{pmatrix}$ with $c_0 \neq 0$. Let $\tilde{R} \subseteq \tilde{L}$ be a finite integral extension of R containing α and β. Let $\tilde{\mathcal{Y}}$ be the set of primes in $\tilde{R}$ consisting of those primes $\tilde{P}$ such that $\tilde{P} \cap R$ is in $\mathcal{Y}$. Let $\mathcal{Z} \subseteq \tilde{\mathcal{Y}}$ be the subset consisting of primes $\tilde{P}$ such that $b_0c_0 \in \tilde{R}_{\tilde{P}}^\times$. The set $\mathcal{Z}$ is non-empty. For each $\tilde{P} \in \mathcal{Z}$ it is not difficult to see that $\rho'_Q = \begin{pmatrix} 1 & \\ & b_0 \end{pmatrix} \rho_Q \begin{pmatrix} 1 & \\ & b_0^{-1} \end{pmatrix}$ takes values in $GL_2(\tilde{R}_{\tilde{P}})$. As $\rho'_Q|_{D_i} \bmod \tilde{P}$ is nonsplit for each prime $\tilde{P} \in \mathcal{Z}$ it follows that for $\tilde{P} \in \mathcal{Z}$, $\chi_1 \bmod \tilde{P} = \psi_1^{(i)} \bmod \tilde{P}$. Since $\psi_2^{(i)} \neq \psi_1^{(i)} \bmod \tilde{P}$ and either $\chi_1 = \psi_1^{(i)}$ or $\psi_2^{(i)}$ we conclude that $\chi_1 = \psi_1^{(i)}$. Note that arguing again as in [W2, Lemma 2.2.4] shows that either $\psi_2^{(i)}(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)}) = T_0(\mathfrak{p}_i) \bmod Q$ or $\psi_1^{(i)}(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)}) = T_0(\mathfrak{p}_i) \bmod Q$. Arguing as before shows that the former must hold. This proves property (vi).

Note that the representation ρ_Q gives rise to a pseudo-representation into $R = \mathbf{T}_\infty(U, \mathcal{O})/Q$. This is just the pseudo-representation associated to ρ_Q (cf. §2). For a non-minimal $Q' \subseteq Q$ the representation $\rho_{Q'}$ can be constructed in the usual way (cf. end of the proof of [W2, Lemma 2.2.3]) from the pseudo-representation into $\mathbf{T}_\infty(U, \mathcal{O})/Q'$ obtained by reducing modulo Q' the pseudo-representation associated to ρ_Q. The only property that is not immediate is (3.4vi). For this we note that if $\rho_{Q'}$ is reducible then there is nothing more to prove (as one of the characters has the desired property), so assume that ρ'_Q is irreducible. Let $\tilde{R}$ be a finite integral extension of R containing the values of $\psi_1^{(i)}$ and $\psi_2^{(i)}$, and let $\tilde{Q}'$ be an extension of Q' to $\tilde{R}$. It is easy to see that the semisimplication of $\rho_{Q'}|_{D_i}$ is the sum of the characters $\psi_1^{(i)}$ and $\psi_2^{(i)}$ modulo $\tilde{Q}'$. If $\psi_1^{(i)} \equiv \psi_2^{(i)} \bmod \tilde{Q}'$, then there is nothing more to prove. If $\psi_1^{(i)} \not\equiv \psi_2^{(i)} \bmod \tilde{Q}'$, then for a suitable choice of basis ρ_Q takes values in $\tilde{R}_{\tilde{Q}'}$ and satisfies $\rho_Q|_{D_i} = \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$. Reducing modulo $\tilde{Q}'$ yields the representation $\rho_{Q'}$. Property (3.4vi) is now immediate.

Suppose now that $\mathfrak{m}$ is a maximal ideal of $\mathbf{T}_\infty(U, \mathcal{O})$. Patching together the pseudo-representations for the various minimal primes Q contained in $\mathfrak{m}$ yields a pseudo-representation $\rho_{\mathfrak{m}}^{\mathrm{mod}}$ into $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})_{\mathfrak{m}}$ (here we have used the fact that $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})$ is reduced) satisfying

(3.5) (i) $\rho_{\mathfrak{m}}^{\mathrm{mod}}$ is unramified at primes $\ell \nmid \mathfrak{n}p$,
(ii) trace $\rho_{\mathfrak{m}}^{\mathrm{mod}}(\mathrm{Frob}_\ell) = \mathrm{T}(\ell)$ for all $\ell \nmid \mathfrak{n}p$,
(iii) $\det \rho_{\mathfrak{m}}^{\mathrm{mod}}(\mathrm{Frob}_\ell) = \mathrm{S}(\ell)\mathrm{N}m(\ell)$ for all $\ell \nmid \mathfrak{n}p$.

Let χ and k be as in §2. Henceforth $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbf{Q}_p$ having residue field k.

A maximal ideal of $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})$ is *permissible* if $\mathfrak{m} \cap \mathcal{O}[[\mathbf{G}(\mathrm{U})]]$ is the maximal ideal corresponding to the character $\mathbf{G}(\mathrm{U}) \longrightarrow \mathrm{Z}(\mathrm{U}) \xrightarrow{\chi\omega^{-1}} k$, if $\mathfrak{m}$ contains $\mathrm{T}_0(\mathfrak{p}_{v_i}) - 1$ for each $i = 1, \dots, t$, and if $\rho_{\mathfrak{m}} \simeq \chi \oplus 1$. Such a maximal ideal, if it exists, is unique. For this reason we will drop the subscript $\mathfrak{m}$ from the notation for $\rho_{\mathfrak{m}}^{\mathrm{mod}}$ whenever $\mathfrak{m}$ is permissible.

Suppose that $\mathfrak{m}$ is a permissible maximal ideal of $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})$. The ring $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})_{\mathfrak{m}}$ is an algebra over $\Lambda_{\mathcal{O}}$ via $1 + \mathrm{Y}_j^{(i)} \longmapsto \mathrm{T}_{y_j}^{(i)}$ and $1 + \mathrm{T}_j \longmapsto \det \rho^{\mathrm{mod}}(\gamma_j)$. The homomorphism $\Lambda'_{\mathcal{O}} \longrightarrow \Lambda_{\mathcal{O}}$ determined by $1 + \mathrm{X}_j \longmapsto (1 + \mathrm{T}_j)^{p^{r_j}} \varepsilon(\gamma_j^{-p^{r_j}})$ is compatible with the $\Lambda'_{\mathcal{O}}$-algebra structure of $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})_{\mathfrak{m}}$ and makes $\Lambda_{\mathcal{O}}$ a free $\Lambda'_{\mathcal{O}}$-module of rank $r = \Sigma r_j$. Consequently, we obtain the following lemma.

Lemma **3.10.** — *If* $\mathfrak{m}$ *is a permissible maximal ideal of* $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})$, *then*

(i) $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})_{\mathfrak{m}}$ *is a torsion-free, finite* $\Lambda_{\mathcal{O}}$*-algebra,*

(ii) *for* U *satisfying the hypotheses of Proposition* 3.3, $\mathrm{M}_\infty(\mathrm{U})_{\mathfrak{m}}$ *and* $\mathrm{M}_\infty^{+}(\mathrm{U})_{\mathfrak{m}}$ *are free* $\Lambda_{\mathcal{O}}$*-modules of equal rank.*

Let Σ be the places of F for which $\mathrm{U}_v \neq \mathrm{GL}_2(\mathcal{O}_{\mathrm{F},v})$ together with $v_1, \dots, v_t$. If $\mathfrak{m}$ is a permissible maximal ideal of $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})$, then it is easy to see that $\rho_{\mathfrak{m}}^{\mathrm{mod}}$ is a pseudo-deformation of type-$\mathcal{D}^{\mathrm{ps}} = (\mathcal{O}, \Sigma)$. Consequently, there is a map

$$\textbf{(3.6)} \qquad \mathrm{R}_{\mathcal{D}^{\mathrm{ps}}} \longrightarrow \mathbf{T}_\infty(\mathrm{U}, \mathcal{O})_{\mathfrak{m}}$$

inducing ρ^{mod}.

Lemma **3.11.** — *Suppose that* $\mathfrak{m}$ *is a permissible maximal ideal of* $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})$. *If* S *is any finite set of primes of* F *containing all those for which* $\mathrm{U}_\ell \neq \mathrm{GL}_2(\mathcal{O}_{\mathrm{F},\ell})$, *then the ring* $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})_{\mathfrak{m}}$ *is generated over* $\Lambda_{\mathcal{O}}$ *by the operators* $\{\mathrm{T}(\ell), \mathrm{S}(\ell) : \ell \notin \mathrm{S}\}$.

Proof. — Let $\mathbf{T}^{\mathrm{S}} \subseteq \mathbf{T}_\infty(\mathrm{U}, \mathcal{O})_{\mathfrak{m}}$ be the subring generated over $\Lambda_{\mathcal{O}}$ by $\{\mathrm{T}(\ell), \mathrm{S}(\ell) : \ell \notin \mathrm{S}\}$. Note that $\mathbf{T}^{\mathrm{S}}$ is a local, complete $\Lambda_{\mathcal{O}}$-algebra. Let $\rho^{\mathrm{mod}} = \{a(\sigma), d(\sigma), x(\sigma, \tau)\}$ and let $\Sigma = \mathrm{S} \cup \{v_1, \dots, v_t\}$. The pseudo-representation ρ^{mod} factors through $\mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F})$. Since $\mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F})$ is topologically generated by $\{\mathrm{Frob}_\ell : \ell \notin \Sigma\}$ and since trace ρ^{mod} and $\det \rho^{\mathrm{mod}}$ are continuous maps, it follows that $\mathbf{T}^{\mathrm{S}}$ contains trace $\rho^{\mathrm{mod}}(\sigma)$ and det $\rho^{\mathrm{mod}}(\sigma)$ for every $\sigma \in \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F})$. It remains to show that $\mathbf{T}^{\mathrm{S}}$ contains T_y for each $y \in (\mathcal{O}_{\mathrm{F}} \otimes \mathbf{Z}_p)^{\times}$ as well as $\mathrm{T}_0(\mathfrak{p}_i)$ for each $i = 1, \ldots, t$.

Let $g_i \in \mathrm{D}_i$ be such that $\chi(g_i) \not\equiv 1$. Let α_i and $\beta_i \in \mathbf{T}^{\mathrm{S}}$ be the roots of the polynomial $\mathrm{X}^2 - \text{trace } \rho^{\mathrm{mod}}(g_i)\mathrm{X} + \det \rho^{\mathrm{mod}}(g_i)$ with α_i reducing to 1 modulo $\mathfrak{m}$. One then has

$$\mathrm{T}_y = (\beta_i - \alpha_i)\,(\beta_i \,\text{trace } \rho^{\mathrm{mod}}(\sigma_y) - \text{trace } \rho^{\mathrm{mod}}(g_i\sigma_y)) \in \mathbf{T}^{\mathrm{S}}$$

for $y \in \mathcal{O}_{\mathrm{F}, v_i}^{\times}$, where $\sigma_y \in \mathrm{D}_i$ is any lift of the element of $\mathrm{Gal}(\mathrm{F}_{v_i}^{\mathrm{ab}}/\mathrm{F}_{v_i})$ corresponding to y via local reciprocity. Similarly, one also has

$$\mathrm{T}_0(\mathfrak{p}_i) = (\beta_i - \alpha_i)\,(\beta_i \,\text{trace } \rho^{\mathrm{mod}}(\tau_i) - \text{trace } \rho^{\mathrm{mod}}(g_i\tau_i)) \in \mathbf{T}^{\mathrm{S}}$$

where $\tau_i \in \mathrm{D}_i$ is any lift of an element of $\mathrm{Gal}(\mathrm{F}_{v_i}^{\mathrm{ab}}/\mathrm{F}_{v_i})$ corresponding to $(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)})$. These expressions for T_y and $\mathrm{T}_0(\mathfrak{p}_i)$ can be checked for each ρ_{Q}, Q a minimal prime of $\mathbf{T}_{\infty}(\mathrm{U}, \mathcal{O})_{\mathfrak{m}}$, using (3.4). □

Corollary **3.12.** — *If* $\mathrm{V} \subseteq \mathrm{U}$, *then the natural map* $\mathbf{T}_{\infty}(\mathrm{V}, \mathcal{O})_{\mathfrak{m}} \longrightarrow \mathbf{T}_{\infty}(\mathrm{U}, \mathcal{O})_{\mathfrak{m}}$ *is surjective.*

Corollary **3.13.** — *The map* (3.6) *is surjective.*

We conclude this subsection with a few results about the "level" of a prime of $\mathbf{T}_{\infty}(\mathrm{U}, \mathcal{O})$. The first of these is a generalization of Carayol's $\pi_v \simeq \pi(\sigma_v)$ result (see [C]). Indeed, its proof boils down to Carayol's result as generalized in [W2]. For w a finite place of F write ℓ_w for the prime ideal of $\mathcal{O}_{\mathrm{F}}$ associated to w and write Δ_w for the Sylow p-subgroup of $(\mathcal{O}_{\mathrm{F}}/\ell_w)^{\times}$.

Proposition **3.14.** — *Let* $w \nmid p$ *be a finite place of* F. *Suppose that* $\mathrm{U} \subseteq \mathrm{GL}_2(\mathcal{O}_{\mathrm{F}} \otimes \widehat{\mathbf{Z}})$ *is such that* $\mathrm{U}_w \supseteq \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_{\mathrm{F}, w}) : c, a - 1 \in \ell_w^s\}$ *for some s. Given a minimal prime* $\mathrm{Q} \subseteq \mathbf{T}_{\infty}(\mathrm{U}, \mathcal{O})$ *there exists a subgroup* $\mathrm{V} \supseteq \mathrm{U}$ *such that* Q *is the inverse image of a prime of* $\mathbf{T}_{\infty}(\mathrm{V}, \mathcal{O})$ *and* V *satisfies*

(i) *if* ρ_{Q} *is unramified at w, then* $\mathrm{V} \supseteq \mathrm{GL}_2(\mathcal{O}_{\mathrm{F}, w})$;

(ii) *if* ρ_{Q} *is type* A *at w, then* $\mathrm{V} \supseteq \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_{\mathrm{F}, w}) : \begin{array}{c} c \in \ell_w \\ a \bmod \ell_w \in \Delta_w \end{array} \right\}$;

(iii) *if* ρ_{Q} *is type* B *at w, then* $\mathrm{V} \supseteq \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_{\mathrm{F}, w}) : a - 1, c \in \ell_w^r \right\}$, *where* ℓ_w^r *is the conductor of* $\phi = \det \rho_{\mathrm{Q}}|_{\mathrm{I}_w}$;

(iv) *if* ρ_{Q} *is type* C *at w, then* $\mathrm{V} \supseteq \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_{\mathrm{F}, w}) : a - 1, c \in \ell_w^2 \right\}$.

Proof. — Recall that types A, B, and C were defined in §2.3. We first claim that for $P \in \mathscr{X}(Q)$ the representation ρ_P is of the same type at w as ρ_Q and that if $\rho_Q|_{I_w} = \begin{pmatrix} 1 & \\ & \phi \end{pmatrix}$ with $\mathrm{cond}_w(\phi) = \ell_w^r$ then $\rho_P|_{I_w} \simeq \begin{pmatrix} 1 & \\ & \phi' \end{pmatrix}$ with $\mathrm{cond}_w(\phi') = \ell_w^r$ as well. In light of Corollary 3.9 it then suffices to show that some $\lambda \in \mathscr{H}(P)$ factors through $\mathbf{T}_\infty(V, \mathscr{O})$ for some V as in the statement of the proposition.

To prove the claim, first assume that ρ_Q is unramified at w. In this case it is obvious that each ρ_P, $P \in \mathscr{X}(Q)$, is also unramified at w. This can be seen, for instance, by observing that the pseudo-representation associated to ρ_Q is trivial on I_w and hence the same is true of the pseudo-representation associated to ρ_P. As ρ_P is irreducible, this forces ρ_P to be trivial on I_w. Next assume that ρ_Q is type A at w. If $P \in \mathscr{X}(Q)$, then it is easily deduced that the semisimplification of $\rho_P|_{D_w}$ is just $\psi \oplus \varepsilon\psi$ for some character ψ unramified at w. If $\rho_P|_{D_w}$ were unramified then this would contradict (3.3). Therefore, it must be that $\rho_P|_{D_w}$ is ramified, and it follows from the description of its semisimplification that it must be of type A. Now suppose that ρ_Q is type B at w. Write $\rho_Q|_{I_w} = \psi_1\psi_2 \oplus 1$ with ψ_1 of order prime to p and ψ_2 of p-power order. Note that $\mathrm{cond}_w(\psi_1\psi_2) = \max(\mathrm{cond}_w(\psi_1), \mathrm{cond}_w(\psi_2))$ and that both ψ_1 and ψ_2 take values in $\mathbf{T}_\infty(U, \mathscr{O})/Q$. It follows that $\rho_P|_{I_w} \simeq (\psi_1\psi_2 \bmod P) \oplus 1$. As $p \notin P$ one sees that $\mathrm{cond}_w(\psi_1\psi_2 \bmod P) = \max(\mathrm{cond}_w(\psi_1), \mathrm{cond}_w(\psi_2))$, proving the claim in this case. The remaining case (i.e., ρ_Q being of type C at w) is proved similarly.

Now let $P \in \mathscr{X}(Q)$ and choose $\lambda \in \mathscr{H}(P)$. Let π be the automorphic representation corresponding to λ. To prove that λ factors through $\mathbf{T}_\infty(V, \mathscr{O})$ for some V as in the statement of the proposition we need only show that $\pi \in \Pi_2^{\mathrm{ord}}(V)$. In other words, we need to show that if W_π is the underlying representation space for π, then $W_\pi^V \neq 0$. Let $x = \otimes x_v \in W_\pi^U$ and let $x'_w \in W_{\pi, w}$ be the new vector at w. It follows from our hypotheses on U_w and the theory of newforms that $x'_w \in W_{\pi, w}^{U_w}$. Put $y = \bigotimes_{v \neq w} x_v \otimes x'_w$. We claim that y is fixed by a subgroup of the desired type. For this we note that it follows easily from (3.3) that x'_w is fixed by a subgroup of $GL_2(\mathscr{O}_{F, w})$ of the desired type necessarily containing U_w. □

As a variant of the above we have the following result. For a place w of F, let $(\mathscr{O}_F/\ell_w)^\times = \Delta_w \times \Delta'_w$. (Recall that Δ_w is the p-Sylow subgroup of $(\mathscr{O}_F/\ell_w)^\times$.)

Proposition **3.15.** — *Let $w \nmid p$ be a finite place of* F *and let* $U'_w = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathscr{O}_{F, w}) :$ $c \in \ell_w$, $ad^{-1} \bmod \ell_w \in \Delta'_w\}$. *Suppose that* $Q \subseteq \mathbf{T}_\infty(U, \mathscr{O})$ *is a minimal prime such that* $\rho_Q|_{I_w} \simeq \begin{pmatrix} \phi & \\ & \phi^{-1} \end{pmatrix}$ *with ϕ of p-power order. Put* $U' = \prod_{v \neq w} U_v \times U'_w$. *There exists a minimal prime* $Q' \subseteq \mathbf{T}_\infty(U', \mathscr{O})$ *such that* $\rho_Q \simeq \rho_{Q'}$ *and such that* Q *and* Q′ *have the same inverse image in* $\mathbf{T}_\infty(U \cap U', \mathscr{O})$.

Proof. — We prove the existence of a minimal prime $Q' \subseteq \mathbf{T}_\infty(U', \mathscr{O})$ such that Q′ and Q have the same inverse image in $\mathbf{T}_\infty(U \cap U', \mathscr{O})$. Clearly the assertion that

$\rho_Q \simeq \rho_{Q'}$ will follow from this. Upon replacing U by $U \cap U'$ and Q by its inverse image in $\mathbf{T}_\infty(U \cap U', \mathcal{O})$ we need only show that Q is the inverse image of some minimal prime in $\mathbf{T}_\infty(U', \mathcal{O})$. By Corollary 3.9 it then suffices to show for each $P \in \mathscr{X}(Q)$ and $\lambda \in \mathscr{H}(P)$ that λ factors through $\mathbf{T}_\infty(U', \mathcal{O})$. Fix such a P and λ. Let $\pi \in \Pi_2^{\mathrm{ord}}(U_a)$ be the automorphic representation associated to λ. To know that λ factors through $\mathbf{T}_\infty(U', \mathcal{O})$ it suffices to know that $\pi \in \Pi_2^{\mathrm{ord}}(U'_a)$. Now suppose that $x = \otimes x_v \in V_\pi^{U_a}$. We will establish the existence of a non-zero vector $x'_w \in V_{\pi, w}^{U'_w}$. The non-zero vector $y = \bigotimes_{v \neq w} x_v \otimes x'_w$ will then lie in $V_\pi^{U'_a}$ showing that $V_\pi^{U'_a} \neq 0$, from which it follows that $\pi \in \Pi_2^{\mathrm{ord}}(U'_a)$.

We now establish the existence of x'_w. First we note that $\rho_P|_{I_w} \simeq \begin{pmatrix} \phi' & \\ & \phi'^{-1} \end{pmatrix}$ with ϕ' a character of p-power order. To see this observe that by hypothesis $\rho_Q|_{I_w}$ factors through a quotient of I_w of p-power order and $\det \rho_Q|_{I_w} = 1$. Hence the same is true of the pseudo-representation associated to ρ_Q. As the pseudo-representation associated to ρ_P is obtained by reducing modulo P the one associated to ρ_Q, it follows easily that $\rho_P|_{I_w}$ factors through a finite quotient of p-power order and that $\det \rho_P|_{I_w} = 1$. That $\rho_P|_{I_w}$ has the form asserted is now immediate. It follows from (3.3) that π_w is a principal series representation $\pi(\mu_1, \mu_2)$ with $\mu_1\mu_2$ trivial on $\mathcal{O}_{F,w}^\times$ and each μ_i trivial on a subgroup of $\mathcal{O}_{F,w}^\times$ of index a power of p. Let $v_0 \in V_{\pi, w}$ be the vector corresponding to a new vector of $\pi_w \otimes \phi$ where ϕ is a finite character such that $\phi|_{\mathcal{O}_{F,w}^\times} \simeq \mu_1^{-1}|_{\mathcal{O}_{F,w}^\times}$. It follows that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} v_0 = \mu_1(a)\mu_2(d)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U'_w$. Thus $x'_w = v_0$ is the desired vector in $V_{\pi, w}^{U'_w}$. □

Next we record for later reference the following relations between $\rho_Q|_{I_w}$ and the subgroups U_w.

Lemma **3.16.** — *Suppose that $w \nmid p$ is a finite place of* F. *Suppose also that* $Q \subseteq \mathbf{T}_\infty(U, \mathcal{O})$ *is a minimal prime and that* $U_w \supseteq \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \in \ell_w^r,\ a \bmod \ell_w^r \in \Delta_w\}$ *for some* $r \geqslant 1$.

(i) *If* $\rho_Q|_{I_w} \simeq \begin{pmatrix} \phi_1 & \\ & \phi\phi_2 \end{pmatrix}$ *with* ϕ_1 *and* ϕ_2 *of p-power order and* ϕ *non-trivial of order prime to p, and if* $\mathrm{cond}_w(\phi) = \ell_w^r$, *then* ϕ_1 *and* ϕ_2 *are trivial.*

(ii) *If* $\rho_Q|_{D_w} \simeq \begin{pmatrix} \varepsilon\phi & * \\ & \phi \end{pmatrix}$ *and if* $(\mathrm{Nm}(w) - 1, p) = 1$, *then* $\phi|_{I_w}$ *is a finite character of order prime to p, and if* $r = 1$, *then* $\phi|_{I_w} = 1$.

Proof. — Let $R = \mathbf{T}_\infty(U, \mathcal{O})/Q$. We first prove (i). The characters ϕ_1, ϕ_2 and ϕ take values in R, so they may be reduced modulo P for any $P \in \mathscr{X}(Q)$. We denote these reductions by $\phi_{1,P}, \phi_{2,P}$, and ϕ_P, respectively. The reduced characters have the same orders as the corresponding non-reduced characters. Now fix a choice of $P \in \mathscr{X}(Q)$ and $\lambda \in \mathscr{H}(P)$. Let π be the nearly ordinary automorphic representation corresponding to λ. It is easy to see that $\rho_P|_{I_w} \simeq \begin{pmatrix} \phi_{1,P} & \\ & \phi_P\phi_{2,P} \end{pmatrix}$. It follows from this description of $\rho_P|_{I_w}$ and from (3.3) that $\mathrm{cond}(\pi_w) = \mathrm{cond}_w(\phi_P\phi_{2,P})\mathrm{cond}_w(\phi_{1,P}) = \ell_w^r \cdot \mathrm{cond}(\phi_{1,P})$. However, since by hypothesis π_w has a vector fixed by $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : c \in \ell_w^r,\ a \bmod \ell_w^r \in \Delta_w\}$ it follows that $\mathrm{cond}(\pi_w) \mid \ell_w^r$ and that the restriction to I_w of the central character of π_w has order prime to p. From this we deduce that $\phi_{1,P}$ and $\phi_{2,P}$ are both trivial. As $\phi_{1,P}$ and $\phi_{2,P}$ have the same order as ϕ_1 and ϕ_2, respectively, the latter are trivial as well.

We now prove (ii). Our hypothesis on $\mathrm{Nm}(w)$ ensures that ϕ takes values in R. As in the proof of (i) we write ϕ_P for the reduction of ϕ modulo P. The character $\phi_P|_{I_w}$ has the same order as does $\phi|_{I_w}$. Again, fix a $P \in \mathscr{X}(Q)$ and a $\lambda \in \mathscr{H}(P)$. Let π be the automorphic representation corresponding to λ. From the hypotheses on U we find that $\mathrm{cond}(\pi_w) \mid \ell_w^r$ and that the restriction to I_w of the central character of π_w has order prime to p. It is easy to see that $\rho_P|_{D_w} \simeq \begin{pmatrix} \varepsilon\phi_P & * \\ & \phi_P \end{pmatrix}$. It follows from (3.3) that $\mathrm{cond}(\pi_w) = \max(\ell_w, \mathrm{cond}(\phi_P)^2)$ and that the restriction to I_w of the central character of π_w is just $\phi_P^2|_{I_w}$. From this, one deduces that $\phi_P|_{I_w}$, and hence $\phi|_{I_w}$, has order prime to p. And moreover, if $r = 1$, then $\mathrm{cond}(\phi_P)^2 \mid \ell_w$, hence $\phi_P|_{I_w}$ (and so $\phi|_{I_w}$) is trivial. □

We conclude this subsection with a simple observation about twists of the representations ρ_Q.

Lemma **3.17.** — *Suppose that* $Q \subseteq \mathfrak{p} \subseteq \mathbf{T}_\infty(U, \mathscr{O})$ *are primes with* Q *minimal. Let* L *be the field of fractions of* $\mathbf{T}_\infty(U, \mathscr{O})/Q$ *and suppose that* $R \subseteq \bar{L}$ *is a finite integral extension of* $\mathbf{T}_\infty(U, \mathscr{O})/Q$*. If* $\Psi : \mathrm{Gal}(\bar{F}/F) \longrightarrow R^\times$ *is a character of finite order, then there exists primes* $Q' \subseteq \mathfrak{p}' \subseteq \mathbf{T}_\infty(U \cap U_1(\mathrm{cond}^{(p)}(\Psi)^2), \mathscr{O})$ *with* Q' *minimal and such that* $\rho_{Q'} \simeq \rho_Q \otimes \Psi$ *and* $\rho_{\mathfrak{p}'} \simeq \rho_{\mathfrak{p}} \otimes \Psi$.

Here $\mathrm{cond}^{(p)}(\Psi)$ denotes the prime-to-p part of the conductor of Ψ.

Proof. — Let $V = U \cap U_1(\mathrm{cond}(\Psi)^2)$ and let $R_0 = \mathbf{T}_\infty(U, \mathscr{O})/Q$. Let $\mathscr{X}'(Q)$ be the set of primes of R extending those in $\mathscr{X}(Q)$. For each $P \in \mathscr{X}'(Q)$ we write $\mathscr{H}(P)$ for $\mathscr{H}(P \cap R_0)$. Now, for each $P \in \mathscr{X}'(Q)$, let $\Psi_P = \Psi \bmod P$. Let $\mathfrak{n}$ be the product of those primes ℓ such that either $\ell | \mathrm{cond}^{(p)}(\Psi)$ or $U_\ell \neq \mathrm{GL}_2(\mathscr{O}_{F,\ell})$. We claim that for each $P \in \mathscr{X}'(Q)$ there exists a homomorphism $\tau_P : \mathbf{T}_\infty(V, \mathscr{O}) \longrightarrow R/P$ such that

(3.7)
- $\tau_P(T(\ell)) = (T(\ell) \bmod P) \cdot \Psi_P(\mathrm{Frob}_\ell)$ for all $\ell \nmid \mathfrak{n}p$;
- $\tau_P(S(\ell)) = (S(\ell) \bmod P) \cdot \Psi_P^2(\mathrm{Frob}_\ell)$ for all $\ell \nmid \mathfrak{n}p$;
- $\tau_P(S_x) = (S_x \bmod P) \cdot \Psi_P^2(x)$ for all $x \in Z(U)$;
- $\tau_P(T_0(\mathfrak{p}_i)) = (T_0(\mathfrak{p}_i) \bmod P) \cdot \Psi_P(\lambda_{\mathfrak{p}_i}^{(\mathfrak{p}_i)})^{-1}$ for $i = 1, ..., t$;
- $\tau_P(T_y) = (T_y \bmod P) \cdot \Psi_P(y)$ for all $y \in \mathscr{O}_{F,v_i}^\times$ and each $i = 1, ..., t$.

We construct τ_P as follows. Let $\lambda \in \mathcal{H}(P)$ and let $\pi \in \Pi_2^{\mathrm{ord}}(U_a)$ be the corresponding automorphic representation. We fix an embedding $R/P \hookrightarrow \overline{\mathbf{Q}}_p$ extending the embedding $R_0/\ker(\lambda) \hookrightarrow \overline{\mathbf{Q}}_p$ coming from λ. We thus view Ψ_P as taking values in $\overline{\mathbf{Q}}_p^\times$ and hence in $\overline{F}^\times$ (via the fixed embedding $\overline{F} \hookrightarrow \overline{\mathbf{Q}}_p$). Clearly $\pi \otimes \Psi_P \in \Pi_2^{\mathrm{ord}}(V_b)$ for some $b \geqslant a$. Therefore there exists an algebraic prime P_Ψ of $\mathbf{T}_\infty(V, \mathcal{O})$ whose corresponding representation is just $\rho_{P_\Psi} \simeq \rho_P \otimes \Psi_P$. Let $\tau_P \in \mathcal{H}(P_\Psi)$ be the homomorphism corresponding to $\pi \otimes \Psi_P$. Viewing R/P as an $\mathcal{O}$-subalgebra of $\overline{\mathbf{Q}}_p$ as above, we see from the fact that $\rho_{P_\Psi} \simeq \rho_P \otimes \Psi_P$ that τ_P takes values in R/P and satisfies (3.7).

Now consider the map $\tau : \mathbf{T}_\infty(V, \mathcal{O}) \longrightarrow \prod_{P \in \mathcal{X}'(Q)} R/P$ given by $\tau(t) = \Pi\tau_P(t)$. It is easily deduced from (3.7) that the image of τ is contained in the image of the diagonal embedding $R \hookrightarrow \prod_{P \in \mathcal{X}'(Q)} R/P$. In particular, τ determines a homomorphism $\tau : \mathbf{T}_\infty(V, \mathcal{O}) \longrightarrow R$ such that

$$\textbf{(3.8)} \qquad \begin{aligned} \tau(T(\ell)) &= (T(\ell) \bmod Q) \cdot \Psi(\mathrm{Frob}_\ell) \\ \tau(S(\ell)) &= (S(\ell) \bmod Q) \cdot \Psi^2(\mathrm{Frob}_\ell). \end{aligned} \qquad \text{for all } \ell \nmid \mathfrak{n}p$$

Let $Q' = \ker(\tau)$. By (3.8) we have $\rho_{Q'} \simeq \rho_Q \otimes \Psi$. Moreover, by comparing dimensions one sees that Q' is minimal. Let $\mathfrak{p}_1$ be any prime of R extending $\mathfrak{p}$. Let $\mathfrak{p}'$ be the kernel of the composition $\mathbf{T}_\infty(V, \mathcal{O}) \xrightarrow{\tau} R \longrightarrow R/\mathfrak{p}_1$. Obviously $\mathfrak{p}' \supseteq Q'$. Also, it follows from (3.8) that $\rho_{\mathfrak{p}'} \simeq \rho_{\mathfrak{p}} \otimes \Psi$ as well. □

3.4. Eisenstein maximal ideals (existence)

In this subsection we establish sufficient conditions for the existence of a permissible ideal of $\mathbf{T}_\infty(U, \mathcal{O})$. We continue to assume that the degree of F is even. An equivalent definition of permissible maximal ideal is a maximal ideal $\mathfrak{m}$ of $\mathbf{T}_\infty(U, \mathcal{O})$ such that

(3.9)
- $\mathfrak{m} \cap \mathcal{O}[[\mathbf{G}(U)]]$ is the maximal ideal corresponding to the character $\mathbf{G}(U) \longrightarrow Z(U) \xrightarrow{\chi\omega^{-1}} k$,
- $\mathfrak{m}$ contains $T_0(\mathfrak{p}_i) - 1$ for $i = 1, \ldots, t$, and
- $\mathfrak{m}$ contains $T(\ell) - 1 - \tilde{\chi}(\mathrm{Frob}_\ell)$ for each $\ell \nmid p$ for which $U_\ell = GL_2(\mathcal{O}_{F,\ell})$.

We will also call a maximal ideal $\mathfrak{m}$ of any $\mathbf{T}_2(U_a, \mathcal{O})$ satisfying (3.9) a permissible maximal ideal since any such ideal determines a permissible maximal ideal of $\mathbf{T}_\infty(U, \mathcal{O})$. Clearly, to conclude that $\mathbf{T}_\infty(U, \mathcal{O})$ has a permissible maximal ideal it suffices to show that $\mathbf{T}_2(U_a, \mathcal{O})$ does for some a (and hence for all sufficiently large a). This we do, provided a certain p-adic L-function is not a unit.

Let $\mathfrak{n}$ be the prime-to-p part of the conductor of $\chi\omega^{-1}$. For each prime $\ell | \mathfrak{n}$ let $\ell^{r(\ell)} \| \mathfrak{n}$ and write Δ_ℓ for the Sylow p-subgroup of $(\mathcal{O}_F/\ell)^\times$, which we think of as a subgroup of $(\mathcal{O}_F/\ell^{r(\ell)})^\times$. Define an open compact subgroup $U^\chi = \prod_\ell U^\chi_\ell \subseteq GL_2(\mathcal{O}_F \otimes \widehat{\mathbf{Z}})$ as follows:

$$U^\chi_\ell = \begin{cases} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_{F,\ell}) : c \in \ell^{r(\ell)},\ a \bmod \ell^{r(\ell)} \in \Delta_\ell \right\} & \text{if } \ell \mid \mathfrak{n}, \\ GL_2(\mathcal{O}_{F,\ell}) & \text{otherwise.} \end{cases}$$

Let $L_p(F, s, \chi\omega)$ be the p-adic L-function associated to $\chi\omega$ (cf. [Co], [D-R]). Let π denote a uniformizer of $\mathcal{O}$.

Proposition **3.18.** — *If* $\mathrm{ord}_\pi(L_p(F, -1, \chi\omega)) > 0$, *then some* $\mathbf{T}_2(U^\chi_a, \mathcal{O})$ *(and hence* $\mathbf{T}_\infty(U^\chi, \mathcal{O})$*) has a permissible maximal ideal.*

Proof. — Let $\psi = \chi\omega^{-1}$. For each integer $n \geqslant 2$ let $M_n(p\mathfrak{n}, \psi) \subseteq M_n(p\mathfrak{n})$ be the subspace of modular forms having nebentypus character ψ. Define $S_n(p\mathfrak{n}, \psi)$ similarly. For a pair of characters ϕ_1 and ϕ_2 for which $\phi_1\phi_2 = \psi$ and $\mathrm{cond}(\phi_1)\mathrm{cond}(\phi_2) | \mathfrak{n}p$ let $E_n(\phi_1, \phi_2) \in M_n(p\mathfrak{n}, \psi)$ be the Eisenstein series whose associated Dirichlet series is $L(F, s, \phi_1)L(F, s-n+1, \phi_2)$ (cf. [Sh]). A complement for the space $S_n(\mathfrak{n}, \psi)$ in $M_n(\mathfrak{n}, \psi)$ is spanned by the set $\{E_n(\phi_1, \phi_2)\}$. It is well-known that

$$\textbf{(3.10)} \qquad a_i(E_n(1, \psi), 0) = 2^{-d}L(F, 1-n, \psi)\,\mathrm{Nm}(t_i)^{\frac{n}{2}}$$

where the $a_i(E_n(1, \psi), 0)$ are the constant terms of the Fourier expansions of $E_n(1, \psi)$ described in §3.1.

Let γ be a generator of the Galois group of the cyclotomic $\mathbf{Z}_p$-extension of F, and let $f = \mathrm{ord}_p(\varepsilon(\gamma) - 1)$. Now let $n = 2 + p^f(p-1)m \in \mathbf{Z}$ be so big that there exists a modular form $g \in M_n(\mathfrak{n}p, \psi) \cap M_n(\mathfrak{n}p, \mathcal{O})$ such that

$$\textbf{(3.11)} \qquad a_i(g, 0) = 2^{-d}\mathrm{Nm}(t_i)^{\frac{n}{2}}, \quad i = 1, \dots, h.$$

The existence of such a g for large n is proven in [Ch, §4.5]. Let $E_0 = E_n(1, \psi)$. It follows from our choice of n and standard facts about p-adic L-functions that $\mathrm{ord}_\pi(L(F, 1-n, \psi)) > 0$ as well. Thus $E_0 \in M_n(\mathfrak{n}p, \mathcal{O})$.

Consider the form $f = L(F, 1-n, \psi)g - E_0$. By (3.10) and (3.11) this form satisfies

$$a_i(f, 0) = 0, \quad i = 1, \dots, h.$$

We also have $f \in M_n(\mathfrak{n}p, \mathcal{O})$. Let $e \in \widetilde{\mathbf{T}}_n(U, (\mathfrak{n}) \cap U(p), \mathcal{O})$ be the operator defined in §3.2. It is easily checked that $E_1 = eE_0$ is the modular form whose associated Dirichlet series is just $\zeta_F(s)L^{(p)}(s-n+1, \psi)$, where for an ideal $\mathfrak{a}$ we write $L^{\mathfrak{a}}(\,\cdot\,)$ to mean that the Euler factors at places dividing $\mathfrak{a}$ have been removed. In particular, E_1 is an eigenform for the ring $\widetilde{\mathbf{T}}_n(U_1(\mathfrak{n}) \cap U(p), \mathcal{O})$. Let $\varepsilon_0 \in \widetilde{\mathbf{T}}_n(U_1(\mathfrak{n}) \cap U(p), \mathcal{O})$ be the idempotent associated to the corresponding maximal ideal. The form f can be expressed as $f = \mathrm{F} + \mathrm{G}$ with $\mathrm{F} \in \mathrm{S}_n(\mathfrak{n}p, \mathscr{O})$ and G a linear combination of Eisenstein series, say $\mathrm{G} = \Sigma c(\phi)\mathrm{E}_n(\phi, \phi^{-1}\psi)$, where the sum is over those ϕ such that $\mathrm{cond}(\phi) \cdot \mathrm{cond}(\phi^{-1}\psi) | \mathfrak{n}p$. It follows that $\varepsilon_0 \mathrm{G} = \Sigma d(\lambda)\mathrm{E}_n(\lambda, \lambda^{-1}\psi)$ where the sum is now over the unramified characters of p-power order. Since the constant terms of f are zero, the same is true of those of $\varepsilon_0 f$. Thus $\varepsilon_0\mathrm{G}$ must have all constant terms zero (i.e. $a_i(\varepsilon_0\mathrm{G}, 0) = 0$ for $i = 1, \ldots, h$). Arguing as in the proof of [W3, Proposition 1.6] shows that $\varepsilon_0\mathrm{G} = 0$. It follows that $\mathrm{F}_1 = \varepsilon_0 e f$ is a cusp form. Moreover, since $\pi | \mathrm{L}(\mathrm{F}, 1-n, \psi)$ by hypothesis, we have

$$\mathrm{F}_1 \equiv \mathrm{E}_1 \mod \pi.$$

As $\mathrm{E}_1 \not\equiv 0 \mod \pi$, $\mathrm{F}_1 \neq 0$.

As E_1 is an ordinary eigenform (in the sense of [W2]) and $\mathrm{F}_1 \equiv \mathrm{E}_1 \not\equiv 0$, it is easily seen that there must be an ordinary newform f_1 such that $\varepsilon_0 f_1 \neq 0$. The form $e f_1$ is a p-stabilized newform in the sense of [W2]. Let $\mathfrak{M}$ be the maximal ideal of the ring of integers of $\overline{\mathbf{Q}}_p$. The non-vanishing of $\varepsilon_0 f_1$ means that the coefficients of the Dirichlet series $\mathrm{L}^{\mathfrak{n}}(e f_1, s)$ associated to $e f_1$ are congruent modulo $\mathfrak{M}$ to those of $\zeta_{\mathrm{F}}^{\mathfrak{n}}(s)\mathrm{L}^{\mathfrak{n}p}(\mathrm{F}, s-n+1, \psi)$. By the theory of "$\Lambda$-adic forms" developed in [W2] (see especially [W2, Theorem 1.4.1]) there is some p-stabilized newform $f_2 \in \mathrm{S}_2(\mathfrak{n}p^a, \psi)$, for some large a, such that the coefficients of $\mathrm{L}^{\mathfrak{n}}(f_2, s)$ are congruent to those of $\mathrm{L}^{\mathfrak{n}}(e f_1, s)$ modulo $\mathfrak{M}$, and hence are congruent to the coefficients of $\zeta_{\mathrm{F}}^{\mathfrak{n}}(s)\mathrm{L}^{\mathfrak{n}p}(\mathrm{F}, s-n+1, \psi)$. Being a p-stabilized newform, f_2 spans a v-good line in the associated local automorphic representation for each place $v|p$. Thus f_2 is an eigenform for the ring of operators $\mathbf{T}_2(\mathrm{U}_a, \mathscr{O})$ $(\mathrm{U} = \mathrm{U}_1(\mathfrak{n}))$. Let $\mathfrak{m}$ be the corresponding maximal ideal. We claim that $\mathfrak{m}$ satisfies (3.9). The second and third properties listed in (3.9) are consequences of the connection between the eigenvalues of the Hecke operators and the coefficients of the Dirichlet series $\mathrm{L}^{\mathfrak{n}}(f_2, s)$. The first property listed in (3.9) follows from the fact that f_2 is ordinary (so the operators T_y, $y \in (\mathscr{O}_{\mathrm{F}} \otimes \mathbf{Z}_p)^\times$, act trivially) and that $f_2 \in \mathrm{S}_2(\mathfrak{n}p^a, \psi)$ (so S_x, $x \in \mathrm{Z(U)}$, acts via $\chi\omega^{-1}(x)$).

It remains to show that this maximal ideal occurs in $\mathbf{T}_2(\mathrm{U}_a^\chi, \mathscr{O})$. This follows from the fact that π_{f_2}, the automorphic representation associated to the p-stabilized newform f_2, is in $\Pi_2^{\mathrm{ord}}(\mathrm{U}_a^\chi)$. This last fact can be seen by considering the possibilities for $\rho_{\mathrm{P}_2}|_{\mathrm{D}_\ell}$ at primes $\ell \mid \mathfrak{n}$ (P_2 being the algebraic prime of $\mathbf{T}_\infty(\mathrm{U}_1(\mathfrak{n}), \mathscr{O})$ corresponding to f_2) and invoking [W2, Theorem 2.1.3] or (3.3). □

3.5. Some miscellaneous results

We keep the conventions of the previous sections. Suppose that $\mathrm{U} = \Pi \mathrm{U}_v \subseteq \mathrm{GL}_2(\mathscr{O}_{\mathrm{F}} \otimes \widehat{\mathbf{Z}})$ is a compact open subgroup as usual. In this subsection we consider the effect of altering U at one selected place w.

Let $w \nmid p$ be a place of F such that $U_w = GL_2(\mathcal{O}_{F,w})$. Let Δ_w be the Sylow p-subgroup of $(\mathcal{O}_F/\ell_w)^\times$ and let Δ'_w be a complementary subgroup (so $(\mathcal{O}_F/\ell_w)^\times \simeq \Delta_w \times \Delta'_w$). Put

$$U'_w = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_{F,w}) : c \in \ell_w\right\}$$

and

$$U''_w = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U'_w : ad^{-1} \bmod \ell_w \in \Delta'_w\right\}.$$

Put also

$$U' = U'_w \cdot \prod_{v \neq w} U_v \quad \text{and} \quad U'' = U''_w \cdot \prod_{v \neq w} U_v.$$

There is a natural isomorphism

$$\textbf{(3.12)} \qquad U'_a/U''_a \xrightarrow{\sim} \Delta_w$$

given by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$ (image of $(ad^{-1})_w$ in Δ_w). Recall that $U'_a = U' \cap U(p^a)$ and similarly for U''_a. The group U'_a acts on $S_k(U''_a)$ with $g \in U'_a$ acting via the Hecke operator $[U''_a g^{-1} U''_a]$. This action clearly factors through the quotient U'_a/U''_a and hence determines via the isomorphism (3.12) an action of Δ_w on $S_k(U''_a)$. Under the Jacquet-Langlands correspondence (see §3.2) this action is compatible with usual action of U'_a/U''_a on $\{f : D^\times\backslash G^D(\mathbf{A}_f)/U''_a \longrightarrow \mathbf{C}\} = H^0(X(U''_a), \mathbf{C})$ given by $(gf)(x) = f(xg)$ for $g \in U'_a$. This action clearly stabilizes $H^0(X(U''_a), \mathbf{Z})$ and hence we obtain an action of Δ_w on $H^0(X(U''_a), R)$ for any $\mathbf{Z}$-module R. It is straight-forward to check that the action of Δ_w commutes with that of $\mathbf{G}(U''_a)$ and the Hecke operators $T_0(\mathfrak{p}_i)$, $i = 1, \ldots, t$, and $T(\ell)$ and $S(\ell)$ for primes $\ell \nmid pw$ for which $U_\ell = GL_2(\mathcal{O}_{F,\ell})$. Moreover, the action of Δ_w is compatible with varying a. We also have that

$$\textbf{(3.13)} \qquad H^0(X(U''_a), R)^{\Delta_w} = H^0(X(U'_a), R).$$

If every element of $U'/F^\times \cap U'$ acts without fixed points on $D^\times\backslash G^D(\mathbf{A}_f)$ then much more is true, as the following lemma shows.

Lemma **3.19.** — *If each element of* $U'/F^\times \cap U'$ *acts without fixed points on* $D^\times\backslash G^D(\mathbf{A}_f)$ *then*

(i) $M_\infty(U'')$ *and* $M^+_\infty(U'')$ *are free* $\Lambda'_{\mathcal{O}}[[\Delta_w]]$*-modules,*
(ii) $M_\infty(U'')_{\Delta_w} = M_\infty(U')$ *and* $M^+_\infty(U'')_{\Delta_w} = M^+_\infty(U')$.

Proof. — By (3.13) we have

$$\#H^0(X(U''_a), \mathcal{O}/\pi^n)_{\Delta_w} = \#H^0(X(U''_a), \mathcal{O}/\pi^n)^{\Delta_w} = \#H^0(X(U'_a), \mathcal{O}/\pi^n).$$

On the other hand, it follows from the hypothesis on $U'/F^\times \cap U'$ that $\#X(U''_a) = \#\Delta_w \#X(U'_a)$. Combining these observations we find that $H^0(X(U''_a), \mathcal{O}/\pi^n)$ is a free $\mathcal{O}/\pi^n[[\Delta_w]]$-module. The lemma follows from this, the definitions of $M_\infty(U'')$ and $M^+_\infty(U'')$, and Proposition 3.3. □

It is a consequence of the lemma that many characteristic p primes of $\mathbf{T}_\infty(U'', \mathcal{O})$ (i.e., primes containing p) come from primes of $\mathbf{T}_\infty(U', \mathcal{O})$. We state this more precisely in the next proposition. Note in particular that we are *not* assuming anything about $U'/U' \cap F^\times$.

Suppose that $\mathfrak{p} \subseteq \mathbf{T}_\infty(U'', \mathcal{O})$ is a prime such that

- $p \in \mathfrak{p}$,
- $\det \rho_\mathfrak{p} = \chi$,
- $\rho_\mathfrak{p}|_{D_v} \simeq \begin{pmatrix} \psi_1 & * \\ & \psi_2 \end{pmatrix}$ with $(\psi_1/\psi_2)|_{I_v}$ having infinite order for some $v|p$,
- $\rho_\mathfrak{p}$ is irreducible but not dihedral (i.e., not induced from a one-dimensional representation over a quadratic extension).

Proposition **3.20.** — *The prime* $\mathfrak{p}$ *is the inverse image of a prime of* $\mathbf{T}_\infty(U', \mathcal{O})$.

Proof. — Choose $\sigma \in I_w$ such that $\omega(\sigma) = 1$, $\det \rho_\mathfrak{p}(\sigma) = 1$, and $\rho_\mathfrak{p}(\sigma)$ has infinite order. Such a σ exists by the hypothesis on $\rho_\mathfrak{p}|_{D_v}$. We claim that there exists $\tau \in \mathrm{Gal}(\overline{F}/F)$ for which $\omega(\tau) \neq 1$ and an n such that $\rho_\mathfrak{p}(\sigma^n\tau)$ has infinite order. To see this, choose a basis of $\rho_\mathfrak{p}$ for which $\rho_\mathfrak{p}(\sigma) = \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix}$. If $\rho_\mathfrak{p}(\tau) \in \left\{ \begin{pmatrix} & * \\ * & \end{pmatrix} \right\}$ for each τ such that $\omega(\tau) \neq 1$, then it would follow that $\rho_\mathfrak{p}$ is dihedral. Therefore there exists some τ_0, $\omega(\tau_0) \neq 1$, for which $\rho_\mathfrak{p}(\tau_0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with either $a \neq 0$ or $d \neq 0$. Suppose now that $\rho_\mathfrak{p}(\sigma^n\tau_0)$ always has finite order. In this case, the roots of $X^2 - (\alpha^n a + \alpha^{-n} d)X + \chi(\tau_0)$ are roots of unity lying in some quadratic extension of the field of fractions of $\mathbf{T}_\infty(U'', \mathcal{O})/\mathfrak{p}$. As there are only finitely many such roots of unity, there are only finitely many possible values for $\alpha^n a + \alpha^{-n} d$, which is easily seen to be absurd.

Let Σ be the set of finite places v such that $U''_v \neq GL_2(\mathcal{O}_{F,v})$. Fix now an n_0 for which $\rho_\mathfrak{p}(\sigma^{n_0}\tau_0)$ has infinite order. Let $\mathrm{Frob}_\ell \in \mathrm{Gal}(F_\Sigma/F)$ be a Frobenius element such that $\ell \nmid 6$ is unramified in F, $\det \rho_\mathfrak{p}(\mathrm{Frob}_\ell) = \chi(\tau_0)$, $\omega(\mathrm{Frob}_\ell) = \omega(\tau_0) \neq 1$, and $\rho_\mathfrak{p}(\mathrm{Frob}_\ell)$ has infinite order. Such a prime ℓ can be found by choosing a Frob_ℓ sufficiently close to $\sigma^{n_0}\tau_0$ in $\mathrm{Gal}(F_\Sigma/F)$. Put

$$V' = U' \cap U(\ell) \quad \text{and} \quad V'' = U'' \cap U(\ell).$$

The prime $\mathfrak{p}$ of $\mathbf{T}_\infty(U'', \mathcal{O})$ determines a prime of $\mathbf{T}_\infty(V'', \mathcal{O})$ (the inverse image of $\mathfrak{p}$) which we also denote by $\mathfrak{p}$. We now claim that $\mathfrak{p}$ comes from $\mathbf{T}_\infty(V', \mathcal{O})$. To see this, note that by Lemma 3.5 and Lemma 3.19, $M_\infty(V'')$ is a free $\Lambda'_\mathcal{O}[[\Delta_w]]$-module. Let $\mathbf{T} \subseteq \mathrm{End}_{\Lambda'_\mathcal{O}}(M_\infty(V''))$ be the ring generated by Δ_v and $\mathbf{T}_\infty(V'', \mathcal{O})$. This is a finite integral extension of $\mathbf{T}_\infty(V'', \mathcal{O})$. Let $\mathfrak{p}_1$ be an extension of $\mathfrak{p}$ to $\mathbf{T}$. As $\mathfrak{p} \ni p$, it is clear that $\{\delta - 1 : \delta \in \Delta_w\} \subseteq \mathfrak{p}_1$. As $M_\infty(V'', \mathcal{O})$ is a faithful $\mathbf{T}$-module, we have that

$$\mathrm{Fitt}_{\mathbf{T}/\mathfrak{p}_1}(M_\infty(V'', \mathcal{O})/\mathfrak{p}_1) = 0.$$

Since $\mathbf{T}/\mathfrak{p}_1$ is a domain, it follows that it acts faithfully on $M_\infty(V'', \mathcal{O})/\mathfrak{p}_1$. Put $B = \mathbf{T}_\infty(V'', \mathcal{O})/\mathfrak{p} \subseteq \mathbf{T}/\mathfrak{p}_1$. It follows that B acts faithfully on $M_\infty(V'', \mathcal{O})/\mathfrak{p}_1$. On the other hand, $M_\infty(V'', \mathcal{O})/\mathfrak{p}_1$ is a quotient of $M_\infty(V'', \mathcal{O})_{\Delta_w} = M_\infty(V', \mathcal{O})$ by Lemma 3.19(ii). As the action of $\mathbf{T}_\infty(V'', \mathcal{O})$ on $M_\infty(V', \mathcal{O})$ is via the natural map $\mathbf{T}_\infty(V'', \mathcal{O}) \twoheadrightarrow \mathbf{T}_\infty(V', \mathcal{O})$ we have $\mathbf{T}_\infty(V'', \mathcal{O})/\mathfrak{p} = \mathbf{T}_\infty(V', \mathcal{O})/\mathrm{im}(\mathfrak{p})$ which proves the claim. Write $\mathfrak{p}_2$ for the corresponding prime of $\mathbf{T}_\infty(V', \mathcal{O})$ (so $\mathfrak{p}_2 = \mathrm{im}(\mathfrak{p})$).

Our final claim, which proves the proposition, is that $\mathfrak{p}_2$ is the inverse image of a prime of $\mathbf{T}_\infty(U', \mathcal{O})$. Let $Q \subseteq \mathfrak{p}_2$ be a minimal prime of $\mathbf{T}_\infty(V', \mathcal{O})$. It suffices to prove that Q comes from a prime of $\mathbf{T}_\infty(U', \mathcal{O})$. Consider $\rho_Q|_{D_\ell}$. As ℓ does not divide p, $p \nmid (\mathrm{Nm}(\ell) - 1)$, and $\rho_{\mathfrak{p}_2} \simeq \rho_\mathfrak{p}$ is unramified at ℓ, there are three possibilities for $\rho_Q|_{D_\ell}$:

(i) $\rho_Q|_{D_\ell}$ is unramified at ℓ,
(ii) $\rho_Q|_{D_\ell}$ is of type A,
(iii) $\rho_Q|_{D_\ell}$ is of type C.

If the first possibility holds, then the desired claim is a consequence of Proposition 3.14. We will now show that the second and third possibilities cannot occur. If $\rho_Q|_{D_\ell}$ were of type A, then the eigenvalues of $\rho_Q(\sigma_\ell)$ (σ_ℓ a lift of Frob_ℓ), say α and β, would satisfy $\frac{\alpha}{\beta} = \varepsilon(\ell)$ or $\frac{\alpha}{\beta} = \varepsilon(\ell)^{-1}$. The same would then be true of $\rho_{\mathfrak{p}_2}(\sigma_\ell)$. However, since $\mathfrak{p}_2$ contains p and $\det \rho_{\mathfrak{p}_2} = \chi$ it would follow that the eigenvalues of $\rho_{\mathfrak{p}_2}(\sigma_\ell)$ would have finite order, contradicting our choice of ℓ. Similarly, if ρ_Q were of type C at ℓ then $\mathrm{trace}\, \rho_Q(\sigma_\ell) = 0$, but we have chosen ℓ so that $\mathrm{trace}\, \rho_\mathfrak{p}(\sigma_\ell) \not\equiv 0$. This final contradiction completes the proof of the proposition. □

We now assume that $\mathbf{T}_\infty(U, \mathcal{O})$ has a permissible maximal ideal. The same is then true of $\mathbf{T}_\infty(U', \mathcal{O})$ and $\mathbf{T}_\infty(U'', \mathcal{O})$. We also assume that the place w satisfies $\chi(\mathrm{Frob}_w) = 1$ as well as $\omega(\mathrm{Frob}_w) = 1$.

Let $\sigma_w \in I_w$ be a generator of the p-part of tame inertia. We identify σ_w with an element of $\mathcal{O}_{F,w}^\times$ via local reciprocity. The element $\delta_w = \begin{pmatrix} \sigma_w & \\ & 1 \end{pmatrix}$ generates Δ_w via (3.12). Recall that both Δ_w and $\mathbf{T}_\infty(U'', \mathcal{O})_\mathfrak{m}$ act on the module $M_\infty(U'')_\mathfrak{m}$.

Lemma **3.21.** — *The element* $\mathrm{trace}\, \rho^{\mathrm{mod}}(\sigma_w) \in \mathbf{T}_\infty(U'', \mathcal{O})_\mathfrak{m}$ *acts on* $M_\infty(U'')_\mathfrak{m}$ *via* $\delta_w + \delta_w^{-1}$.

Remark **3.22.** — Since both $\mathbf{T}_\infty(U'', \mathcal{O})_\mathfrak{m}$ and Δ_w are contained in $\mathrm{End}_\mathcal{O}(M_\infty(U'')_\mathfrak{m})$, the lemma identifies $\delta_w + \delta_w^{-1}$ with an element of $\mathbf{T}_\infty(U'', \mathcal{O})_\mathfrak{m}$. Moreover, this identification behaves well with respect to varying U.

Proof. — Let $V \subseteq eH^0(X(U''_a), \mathcal{O})_{\mathfrak{m}} \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p$ be an eigenspace for the action of $\mathbf{T}_\infty(U'', \mathcal{O})$. The space V is stable under Δ_w. Under the Jacquet-Langlands correspondence V is identified with a subspace of $S_2(U''_a)$. Let $\lambda : \mathbf{T}_\infty(U'', \mathcal{O}) \longrightarrow \overline{\mathbf{Q}}_p$ be the homomorphism giving the action of $\mathbf{T}_\infty(U'', \mathcal{O})$ on V. Clearly λ factors through $\mathbf{T}_2(U''_a, \mathcal{O})$ and hence is an algebraic homomorphism of weight 2. Let $\pi \in \Pi_2^{\mathrm{ord}}(U''_a)$ be the corresponding automorphic representation. The space V is identified with a subspace of $V_\pi^{U''_a}$. We now determine the action of δ_w on V, which is via $\pi\begin{pmatrix} \sigma_w & \\ & 1 \end{pmatrix}$. First we note that π_w cannot be supercuspidal. To see this, let $P = \ker(\lambda)$. The prime P is clearly contained in $\mathfrak{m}$. It then follows from (3.3) that if π_w is supercuspidal then $\rho_P|_{D_w}$ is type C, but clearly this can only occur if $\chi(\mathrm{Frob}_w) = -1$ contradicting our assumptions on w. Now suppose that $\pi_w = \rho(\mu_1, \mu_1|\ |_w^{-1})$ is a special representation. It follows from the definition of U''_w that μ_1 is unramified. From this we find that $\pi_w\begin{pmatrix} \sigma_w & \\ & 1 \end{pmatrix} = 1$. If $\pi_w \simeq \pi(\mu_1, \mu_2)$ is a principal series representation then it must be that μ_1 and μ_2 are tamely ramified and $\mu_1\mu_2$ is unramified. Moreover, the action of $\begin{pmatrix} \sigma_w & \\ & 1 \end{pmatrix}$ on $V_{\pi, w}^{U''_w}$ is by either $\mu_1(\sigma_w)$ or $\mu_2(\sigma_w) = \mu_1^{-1}(\sigma_w)$.

Now it follows from (3.3) that if π is either a principle series representation or a special representation then $\rho_P(\sigma_w) = \begin{pmatrix} \mu_1(\sigma_w) & * \\ & \mu_1^{-1}(\sigma_w) \end{pmatrix}$. Thus we find that $\mathrm{trace}\, \rho_P(\sigma_w) = \mu_1(\sigma_w) + \mu_1^{-1}(\sigma_w) = \delta_w + \delta_w^{-1}$. □

3.6. The rings $\mathbf{T}_{\mathcal{D}}$ and $\mathbf{T}_{\mathcal{D}}^{\min}$

In this subsection we associate Hecke rings to various deformation data. Essentially this is done by first defining a suitable open compact subgroup of $GL_2(\mathcal{O}_F \otimes \widehat{\mathbf{Z}})$ and then localizing the corresponding Hecke ring at a permissible maximal ideal. To ensure the existence of such a maximal ideal we henceforth assume that $L_p(F, -1, \chi\omega)$ is integral but not a unit (see Proposition 3.18). We are, of course, also assuming that the degree of F is even.

Suppose that $\mathcal{D}_Q = (\mathcal{O}, \Sigma, c, \mathcal{M})_Q$ is an (augmented) deformation datum. As in the previous subsections, for each finite place w we write ℓ_w for the prime ideal of F corresponding to w and we write Δ_w for the Sylow p-subgroup of $(\mathcal{O}_F/\ell_w)^\times$ which we identify with a subgroup of $(\mathcal{O}_F/\ell_w^r)^\times$ for any $r \geqslant 1$. We also write Δ'_w for a complementary subgroup of $(\mathcal{O}_F/\ell_w)^\times$ (so $(\mathcal{O}_F/\ell_w)^\times \simeq \Delta_w \times \Delta'_w$). We define $r(w)$ by

$\ell_w^{r(w)} \| \mathrm{cond}(\chi_w^{-1})$. We define a subgroup $U_{\mathscr{D}_Q} = \prod_{w \nmid \infty} U_{\mathscr{D}_Q, w} \subseteq GL_2(\mathscr{O}_F \otimes \hat{\mathbf{Z}})$ by putting

$$U_{\mathscr{D}_Q, w} = \begin{cases} GL_2(\mathscr{O}_{F,w}) & \text{if } w \notin (\Sigma \backslash \mathscr{P}) \cup Q \\ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathscr{O}_{F,w}) : c, a-1 \in \ell_w^{\max(2, r(w)+1)} \right\} & \text{if } w \in \Sigma \backslash (\mathscr{P} \cup \mathscr{M}) \\ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathscr{O}_{F,w}) : c \in \ell_w^{\max(1, r(w))}, \right. & \\ \qquad \left. a \bmod \ell_w^{\max(1, r(w))} \in \Delta_w \right\} & \text{if } w \in \mathscr{M}. \\ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathscr{O}_{F,w}) : c \in \ell_w, \ ad^{-1} \bmod \ell_w \in \Delta'_w \right\} & \text{if } w \in Q. \end{cases}$$

Let $\mathfrak{m}$ be a permissible maximal ideal of $\mathbf{T}_\infty(U_{\mathscr{D}_Q}, \mathscr{O})$. Put

$$\mathbf{T}_{\mathscr{D}_Q} = \mathbf{T}_\infty(U_{\mathscr{D}_Q}, \mathscr{O})_{\mathfrak{m}}.$$

We define $\mathbf{T}_{\mathscr{D}}$ to be $\mathbf{T}_{\mathscr{D}_\emptyset}$. We write $\rho_{\mathscr{D}_Q}^{\mathrm{mod}}$ for the pseudo-representation into $\mathbf{T}_{\mathscr{D}_Q}$ described in (3.5). This is in fact a pseudo-deformation of type-$\mathscr{D}_Q^{\mathrm{ps}}$, $\mathscr{D}_Q^{\mathrm{ps}} = (\mathscr{O}, \Sigma)_Q$, and we write $\pi_{\mathscr{D}} : R_{\mathscr{D}^{\mathrm{ps}}} \twoheadrightarrow \mathbf{T}_{\mathscr{D}}$ and $\pi_{\mathscr{D}_Q} : R_{\mathscr{D}_Q^{\mathrm{ps}}} \twoheadrightarrow \mathbf{T}_{\mathscr{D}_Q}$ for the corresponding maps ($\mathscr{D}^{\mathrm{ps}} = \mathscr{D}_\emptyset^{\mathrm{ps}}$).

Suppose now that $\mathscr{D}_Q = (\mathscr{O}, \Sigma, c, \mathscr{M})_Q$ is an augmented deformation datum. At times it will be necessary to work with a quotient $\mathbf{T}_{\mathscr{D}_Q}^{\min}$ of $\mathbf{T}_{\mathscr{D}_Q}$. This quotient is defined as follows. As in §2.3 let $L_{\mathscr{D}}/F$ be the maximal abelian p-extension of F unramified away from the places in $\Sigma \backslash M$. Let $\mathrm{Gal}(L_{\mathscr{D}}/F) \simeq H_{\mathscr{D}} \oplus N_{\mathscr{D}}$ be the decomposition fixed in §2.3 ($N_{\mathscr{D}}$ is the torsion subgroup). It is a consequence of our definition of $U_{\mathscr{D}_Q}$ that if $\mathfrak{q} \subseteq \mathbf{T}_{\mathscr{D}_Q}$ is a minimal prime, then $\tilde{\chi}^{-1} \det \rho_{\mathfrak{q}}$ factors through $\mathrm{Gal}(L_{\mathscr{D}}/F)$. Let $\mathscr{M}(\mathscr{D}_Q)$ be the set of minimal primes $\mathfrak{q}$ of $\mathbf{T}_{\mathscr{D}_Q}$ and let $\mathscr{M}^{\min}(\mathscr{D}_Q)$ be the subset of those $\mathfrak{q}$ for which $(\tilde{\chi}^{-1} \det \rho_{\mathfrak{q}})|_{N_\Sigma}$ is trivial. Define $\mathbf{T}_{\mathscr{D}_Q}^{\min}$ by

$$\mathbf{T}_{\mathscr{D}_Q}^{\min} = \mathbf{T}_{\mathscr{D}_Q} / \bigcap_{\mathfrak{q} \in \mathscr{M}^{\min}(\mathscr{D}_Q)} \mathfrak{q}.$$

For this definition to make sense we must show that $\mathscr{M}^{\min}(\mathscr{D}_Q) \neq \emptyset$. To this end, fix another decomposition $\mathrm{Gal}(L_{\mathscr{D}}/F) \simeq M_{\mathscr{D}} \times N_{\mathscr{D}}$ with $M_{\mathscr{D}}$ the free $\mathbf{Z}_p$-summand generated by $\gamma_1, \ldots, \gamma_{\delta_F}$. (For the definition of the γ_i's see §2.5.) Write $\det \rho_{\mathscr{D}_Q}^{\mathrm{mod}} = \Theta \cdot \Psi \cdot \tilde{\chi}$ with Ψ trivial on $N_{\mathscr{D}}$ and Θ trivial on $M_{\mathscr{D}}$. Let Φ be a square root of Θ (i.e., $\Phi^2 = \Theta$). It follows from Lemma 3.17 and from the definition of $U_{\mathscr{D}_Q}$ that given any $\mathfrak{q} \in \mathscr{M}(\mathscr{D}_Q)$ there exists some $\mathfrak{q}_\Phi \in \mathscr{M}(\mathscr{D}_Q)$ such that $\rho_{\mathfrak{q}_\Phi} \simeq \rho_{\mathfrak{q}} \otimes \Phi^{-1}$. Clearly $\tilde{\chi}^{-1} \cdot \det \rho_{\mathfrak{q}_\Phi}$ is trivial on N_Σ, and so $\mathfrak{q}_\Phi \in \mathscr{M}^{\min}(\mathscr{D}_Q)$. This proves that $\mathscr{M}^{\min}(\mathscr{D}_Q)$ is non-empty.

We now relate $\mathbf{T}^{\min}_{\mathscr{D}_Q}$ to $\mathbf{T}_{\mathscr{D}_Q}$ more directly. Let L_Θ/F be the splitting field of Θ. A priori, $\mathrm{Gal}(L_\Theta/F)$ is a quotient of $N_{\mathscr{D}}$. We claim that $\mathrm{Gal}(L_\Theta/F) \simeq N_{\mathscr{D}}$. To see this, let $\xi : N_\Sigma \longrightarrow \overline{\mathbf{Q}}_p^\times$ be any character. Extend ξ to a character of $\mathrm{Gal}(\overline{F}/F)$ by first setting it to be trivial on $M_{\mathscr{D}}$ and then composing with the projection of $\mathrm{Gal}(\overline{F}/F)$ onto $\mathrm{Gal}(L_{\mathscr{D}}/F)$. Choose $\mathfrak{q} \in \mathscr{M}^{\min}(\mathscr{D}_Q)$ and $P \in \mathscr{X}(\mathfrak{q})$. By Lemma 3.17 there is an algebraic prime P_ξ of $\mathbf{T}_{\mathscr{D}_Q}$ such that $\rho_{P_\xi} \simeq \rho_P \otimes \xi$. By the choice of $\mathfrak{q}$, $\widetilde{\chi}^{-1} \det \rho_P$ is trivial on $N_{\mathscr{D}_Q}$, whence $\widetilde{\chi}^{-1} \cdot \det \rho_{P_\xi} \simeq \xi^2$. It follows that $\xi^2 = \Theta \bmod P_\xi$. As ξ^2 can be any character of $N_{\mathscr{D}}$, $\Theta|_{N_{\mathscr{D}}}$ has trivial kernel. This proves the claim.

Now let $X(\mathscr{D})$ be the group of $\overline{\mathbf{Q}}_p^\times$-valued characters of $N_{\mathscr{D}}$, which we view as characters of $\mathrm{Gal}(\overline{F}/F)$ that factor through $\mathrm{Gal}(L_\Theta/F)$. For each $\mathfrak{q} \in \mathscr{M}(\mathscr{D})$ let $R(\mathfrak{q}) = \mathbf{T}_{\mathscr{D}_Q}/\mathfrak{q}$ and let $L(\mathfrak{q})$ be the field of fractions of $R(\mathfrak{q})$. We identify $\overline{\mathbf{Q}}_p$ with an $\mathscr{O}$-subalgebra of $\overline{L(\mathfrak{q})}$. In this way we may view each $\xi \in X(\mathscr{D}_Q)$ as taking values in $\overline{L(\mathfrak{q})}$. For each $\mathfrak{q} \in \mathscr{M}(\mathscr{D}_Q)$ and $\xi \in X(\mathscr{D})$ let $R(\mathfrak{q}, \xi)$ be the subring of $\overline{L(\mathfrak{q})}$ generated by $R(\mathfrak{q})$ and the values of ξ. This is again a complete local Noetherian domain. By Lemma 3.17 there is a prime $\mathfrak{q}_\xi \in \mathscr{M}(\mathscr{D}_Q)$ such that $\rho_{\mathfrak{q}_\xi} \simeq \rho_\mathfrak{q} \otimes \xi$. We next claim that the set $\mathscr{M}' = \{ \mathfrak{q}_\xi : \mathfrak{q} \in \mathscr{M}^{\min}(\mathscr{D}_Q),\ \xi \in X(\mathscr{D}) \}$ is just $\mathscr{M}(\mathscr{D}_Q)$. For let $\mathfrak{q} \in \mathscr{M}(\mathscr{D}_Q)$, and let $\xi \in X(\mathscr{D})$ be the unique character such that $\Phi = \xi \bmod \mathfrak{q}$. Let $\mathfrak{q}' = \mathfrak{q}_{\xi^{-1}}$. Clearly $\mathfrak{q}' \in \mathscr{M}^{\min}(D_Q)$. Also, $\mathfrak{q}'_\xi = \mathfrak{q}$ since $\rho_{\mathfrak{q}'_\xi} \cong \rho_{\mathfrak{q}'} \otimes \xi \simeq \rho_\mathfrak{q} \otimes \xi^{-1} \otimes \xi = \rho_\mathfrak{q}$. This proves the claim.

Given a prime $\mathfrak{q} \in \mathscr{M}^{\min}(\mathscr{D}_Q)$ and a character $\xi \in X(\mathscr{D})$ we have used that Lemma 3.17 ensures that there is a prime $\mathfrak{q}_\xi \in \mathscr{M}(\mathscr{D}_Q)$ such that $\rho_{\mathfrak{q}_\xi} \simeq \rho_\mathfrak{q} \otimes \xi$. However, more is true. It was shown in the proof of Lemma 3.17 that there is a homomorphism $\tau(\mathfrak{q}, \xi) : \mathbf{T}_{\mathscr{D}_Q} \longrightarrow R(\mathfrak{q}, \xi)$ whose kernel is $\mathfrak{q}_\xi$ and such that

$$\textbf{(3.14)} \qquad \begin{aligned} \tau(\mathfrak{q}, \xi)\,(T(\ell)) &= (T(\ell) \bmod \mathfrak{q}) \cdot \xi(\mathrm{Frob}_\ell) \\ \tau(\mathfrak{q}, \xi)\,(S(\ell)) &= (S(\ell) \bmod \mathfrak{q}) \cdot \xi^2(\mathrm{Frob}_\ell) \end{aligned}$$

for all primes $\ell \notin \Sigma \cup Q$. There is also a homomorphism $\phi(\mathfrak{q}, \xi) : \mathbf{T}^{\min}_{\mathscr{D}_Q} \otimes_\mathscr{O} \mathscr{O}[N_{\mathscr{D}}] \longrightarrow R(\mathfrak{q}, \xi)$ such that

$$\textbf{(3.15)} \qquad \begin{aligned} \phi(\mathfrak{q}, \xi)\,(T(\ell) \otimes \mathrm{Frob}_r) &= (T(\ell) \bmod \mathfrak{q}) \cdot \xi(\mathrm{Frob}_r) \\ \phi(\mathfrak{q}, \xi)\,(S(\ell) \otimes \mathrm{Frob}_r) &= (S(\ell) \bmod \mathfrak{q}) \cdot \xi^2(\mathrm{Frob}_r). \end{aligned}$$

Now define

$$\tau : \mathbf{T}_{\mathscr{D}_Q} \longrightarrow \prod_{\substack{\mathfrak{q} \in \mathscr{M}^{\min}(\mathscr{D}_Q) \\ \xi \in X(\mathscr{D})}} R(\mathfrak{q}, \xi)$$

and

$$\phi : \mathbf{T}^{\min}_{\mathscr{D}_Q} \otimes_{\mathscr{O}} \mathscr{O}[N_{\mathscr{D}}] \longrightarrow \prod_{\substack{\mathfrak{q} \in \mathscr{M}^{\min}(\mathscr{D}_Q) \\ \xi \in X(\mathscr{D})}} R(\mathfrak{q}, \xi)$$

by

$$\tau = \prod_{\mathfrak{q}, \xi} \tau(\mathfrak{q}, \xi) \qquad \text{and} \qquad \phi = \prod_{\mathfrak{q}, \xi} \phi(\mathfrak{q}, \xi),$$

respectively. It follows from (3.14) and (3.15) that $\mathrm{im}(\tau) = \mathrm{im}(\phi)$, from which one deduces the following proposition. (Note that $\Theta(\mathrm{Frob}_r)$ and $\Phi(\mathrm{Frob}_r)$ are mapped by each $\tau(\mathfrak{q}, \xi)$ to $\xi^2(\mathrm{Frob}_r)$ and $\xi(\mathrm{Frob}_r)$, respectively.)

Proposition **3.23.** — *There is an isomorphism of* $\Lambda_{\mathscr{O}}$*-algebras* $\mathbf{T}^{\min}_{\mathscr{D}_Q} \otimes_{\mathscr{O}} \mathscr{O}[N_{\mathscr{D}}] \xrightarrow{\sim} \mathbf{T}_{\mathscr{D}_Q}$ *such that* $T(\ell) \otimes n \longmapsto T(\ell) \cdot \Phi(\mathrm{Frob}_\ell^{-1} \cdot n)$ *and* $S(\ell) \otimes n \longmapsto S(\ell) \cdot \Theta(\mathrm{Frob}_\ell^{-1} \cdot n)$.

Corollary **3.24.** — $\mathbf{T}^{\min}_{\mathscr{D}_Q}$ *is a finite, torsion-free* $\Lambda_{\mathscr{O}}$*-algebra.*

Lemma **3.25.** — *Under the isomorphism in Proposition* 3.23 *the element* $\delta_w + \delta_w^{-1} \in \mathbf{T}^{\min}_{\mathscr{D}_Q}$ *maps to* $\delta_w + \delta_w^{-1} \in \mathbf{T}_{\mathscr{D}_Q}$.

We now define a $\mathbf{T}_{\mathscr{D}_Q}$-module $M_{\mathscr{D}_Q}$ for each deformation datum $\mathscr{D}_Q$. The obvious choice for $M_{\mathscr{D}_Q}$ is $M_\infty(U_{\mathscr{D}_Q})_{\mathfrak{m}}$, where $\mathfrak{m}$ is the permissible maximal ideal of $\mathbf{T}_\infty(U_{\mathscr{D}_Q})$. However, for technical reasons we find it better to define $M_{\mathscr{D}_Q}$ to be

$$M_{\mathscr{D}_Q} = M_\infty(U^{\min}_{\mathscr{D}_Q})_{\mathfrak{m}},$$

where $U^{\min}_{\mathscr{D}_Q} = \prod_{w \nmid \infty} U^{\min}_{\mathscr{D}_Q, w} \subseteq \mathrm{GL}_2(\mathscr{O}_F \otimes \widehat{\mathbf{Z}})$ is such that

$$U^{\min}_{\mathscr{D}_Q, w} = \begin{cases} U_{\mathscr{D}_Q, w} & \text{if } w \in \mathscr{P} \cup \mathscr{M} \cup Q \quad \text{or if } w \notin \Sigma \\ U_{\mathscr{D}_Q, w} \cdot \left\{ \begin{pmatrix} a & \\ & 1 \end{pmatrix} : a \bmod \ell_w^{\max(1, r(w))} \in \Delta_w \right\} & \text{if } w \in \mathscr{M}_c \backslash \mathscr{M} \\ \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathscr{O}_{F, w}) : c \equiv 0 \bmod \ell_w^2 \right\} & \text{otherwise.} \end{cases}$$

The module $M_{\mathscr{D}_Q}$ is a $\mathbf{T}_{\mathscr{D}_Q}$-module (and hence a $\mathbf{T}^{\min}_{\mathscr{D}_Q}$-module by Proposition 3.23) via the natural map $\mathbf{T}_{\mathscr{D}_Q} \twoheadrightarrow \mathbf{T}_\infty(U^{\min}_{\mathscr{D}_Q})_{\mathfrak{m}}$.

We write $\pi^{\min}_{\mathscr{D}} : R_{\mathscr{D}^{ps}} \longrightarrow \mathbf{T}^{\min}_{\mathscr{D}}$ for the composition of $\pi_{\mathscr{D}}$ with the canonical surjection $\mathbf{T}_{\mathscr{D}} \twoheadrightarrow \mathbf{T}^{\min}_{\mathscr{D}}$.

3.7. Duality again

We now make some important observations concerning the modules introduced in §3.2. Fix an open subgroup $U \subseteq GL_2(\mathcal{O}_F \otimes \widehat{\mathbf{Z}})$ as in the preceding subsections. We assume that $U/U \cap F^\times$ acts freely on $D^\times \backslash G^D(\mathbf{A}_f)$. Thus by Proposition 3.3 $M_\infty(U)$ and $M_\infty^+(U)$ are free $\mathcal{O}[[\mathbf{G}(U)]]$-modules of (the same) finite rank.

Let $\mathrm{tr}(a) : \mathcal{O}[[\mathbf{G}(U_a)]] \longrightarrow \mathcal{O}$ be the "trace map" given by $\Sigma x_g g \longmapsto x_{\mathrm{id}}$ (where "id" is the identity element in $\mathbf{G}(U_a)$). We have an identification of $\mathbf{T}_\infty(U, \mathcal{O})$-modules

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}[[\mathbf{G}(U)]]}(M_\infty^+(U), \mathcal{O}[[\mathbf{G}(U_a)]]) &= \mathrm{Hom}_{\mathcal{O}[[\mathbf{G}(U_a)]]}(eH^0(X(U_a), \mathcal{O}), \ \mathcal{O}[[\mathbf{G}(U_a)]]) \\ &\overset{\mathrm{tr}(a)\circ(\cdot)}{=} \mathrm{Hom}_{\mathcal{O}}(eH^0(X(U_a), \ \mathcal{O}), \mathcal{O}) \\ &\overset{\langle\ ,\ \rangle_{U_a}}{=} eH^0(X(U_a), \ \mathcal{O})^+.\end{aligned}$$

Denote by λ_a this identification of $\mathrm{Hom}_{\mathcal{O}[[\mathbf{G}(U)]]}(M_\infty^+(U), \mathcal{O}[[\mathbf{G}(U_a)]])$ with $eH^0(X(U_a), \mathcal{O})^+$. For $b \geqslant a$ we have a commutative diagram of $\mathbf{T}_\infty(U, \mathcal{O})$-modules

$$\begin{array}{ccc}\mathrm{Hom}_{\mathcal{O}[[\mathbf{G}(U)]]}(M_\infty^+(U), \ \mathcal{O}[[\mathbf{G}(U_b)]]) & \xrightarrow{\lambda_b} & eH^0(X(U_b), \ \mathcal{O})^+ \\ \downarrow & & \downarrow \mathrm{tr}(U_b, U_a) \\ \mathrm{Hom}_{\mathcal{O}[[\mathbf{G}(U)]]}(M_\infty^+(U), \mathcal{O}[[\mathbf{G}(U_a)]]) & \xrightarrow{\lambda_a} & eH^0(X(U_a), \ \mathcal{O})^+\end{array}$$

where the left vertical arrow is induced from the natural projection $\mathcal{O}[[\mathbf{G}(U_b)]] \longrightarrow \mathcal{O}[[\mathbf{G}(U_a)]]$. We obtain therefore an identification

$$\lambda_\infty : \mathrm{Hom}_{\mathcal{O}[[\mathbf{G}(U)]]}(M_\infty^+(U), \ \mathcal{O}[[\mathbf{G}(U)]]) \simeq M_\infty(U)$$

satisfying $\lambda_\infty(tm) = t\lambda_\infty(m)$ for all $t \in \mathbf{T}_\infty(U, \mathcal{O})$.

Recall that there is an isomorphism $\mathbf{G}(U) \simeq (\mathcal{O}_F \otimes \mathbf{Z}_p)^\times \times Z(U)$ inducing an identification $\mathcal{O}[[\mathbf{G}(U)]] = \Lambda'_{\mathcal{O}}[[Z_0]]$ with Z_0 a finite group. Composing λ_∞ with the isomorphism

$$\mathrm{Hom}_{\Lambda'_{\mathcal{O}}}(M_\infty^+(U), \Lambda'_{\mathcal{O}}) \simeq \mathrm{Hom}_{\mathcal{O}[[\mathbf{G}(U)]]}(M_\infty^+(U), \ \mathcal{O}[[\mathbf{G}(U)]])$$

coming from the trace from $\mathcal{O}[[\mathbf{G}(U)]]$ to $\Lambda'_{\mathcal{O}}$ induces an isomorphism

$$\beta_\infty(U) : \mathrm{Hom}_{\Lambda'_{\mathcal{O}}}(M_\infty^+(U), \Lambda'_{\mathcal{O}}) \simeq M_\infty(U).$$

Moreover, if $\varphi : M_\infty^+(U) \longrightarrow M_\infty^+(V)$ is any map compatible with the canonical map $\mathbf{G}(U) \longrightarrow \mathbf{G}(V)$ then φ can be written as $\varphi = \varprojlim_a \varphi_a$ with $\varphi_a : eH^0(X(U_a), \ \mathcal{O}) \longrightarrow$

$e\mathrm{H}^0(\mathrm{X}(\mathrm{V}_a), \mathcal{O})$, and there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\Lambda'_{\mathcal{O}}}(\mathrm{M}^+_\infty(\mathrm{U}), \Lambda'_{\mathcal{O}}) & \xrightarrow[\sim]{\beta_\infty(\mathrm{U})} & \mathrm{M}_\infty(\mathrm{U}) \\ \uparrow \circ\varphi & & \uparrow \widehat{\varphi} = \varprojlim_a \widehat{\varphi}_a \\ \mathrm{Hom}_{\Lambda'_{\mathcal{O}}}(\mathrm{M}^+_\infty(\mathrm{V}), \Lambda'_{\mathcal{O}}) & \xrightarrow[\sim]{\beta_\infty(\mathrm{V})} & \mathrm{M}_\infty(\mathrm{V}) \end{array}$$

where $\widehat{\varphi}_a$ is the adjoint of φ_a with respect to the pairings $\langle\ ,\ \rangle_{\mathrm{U}_a}$ and $\langle\ ,\ \rangle_{\mathrm{V}_a}$.

Now suppose that $\mathfrak{p} \subseteq \mathbf{T}_\infty(\mathrm{U}, \mathcal{O})$ is a prime. Let $\mathrm{P} = \Lambda'_{\mathcal{O}} \cap \mathfrak{p}$. It is easily deduced from the above that $\beta_\infty(\mathrm{U})$ induces an identification

$$\textbf{(3.16)} \qquad \widehat{\mathrm{M}_\infty(\mathrm{U})_\mathfrak{p}} \simeq \mathrm{Hom}_{\widehat{\Lambda}'_{\mathcal{O},\mathrm{P}}}(\widehat{\mathrm{M}^+_\infty(\mathrm{U})_\mathfrak{p}}, \widehat{\Lambda}'_{\mathcal{O},\mathrm{P}})$$

of $\widehat{\mathbf{T}}_\infty(\mathrm{U}, \mathcal{O})_\mathfrak{p}$-modules.

Recall that we defined in §3.3 an injection $\Lambda'_{\mathcal{O}} \hookrightarrow \Lambda_{\mathcal{O}}$ which identifies $\Lambda_{\mathcal{O}}$ with $\Lambda'_{\mathcal{O}}[\mathrm{Z}_1]$ for some finite group Z_1. Suppose that $\mathfrak{m}$ is a permissible maximal ideal of $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})$. By Lemma 3.10 both $\mathrm{M}_\infty(\mathrm{U})_\mathfrak{m}$ and $\mathrm{M}^+_\infty(\mathrm{U})_\mathfrak{m}$ are free $\Lambda_{\mathcal{O}}$-modules, so composing with the trace map from $\Lambda_{\mathcal{O}}$ to $\Lambda'_{\mathcal{O}}$ induces an isomorphism

$$\mathrm{Hom}_{\Lambda_{\mathcal{O}}}(\mathrm{M}^+_\infty(\mathrm{U})_\mathfrak{m}, \Lambda_{\mathcal{O}}) \simeq \mathrm{Hom}_{\Lambda'_{\mathcal{O}}}(\mathrm{M}^+_\infty(\mathrm{U})_\mathfrak{m}, \Lambda'_{\mathcal{O}})$$

of $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})_\mathfrak{m}$-modules. Combining this with (3.16) yields an isomorphism

$$\textbf{(3.17)} \qquad \mathrm{M}_\infty(\mathrm{U})_\mathfrak{m} \simeq \mathrm{Hom}_{\Lambda_{\mathcal{O}}}(\mathrm{M}^+_\infty(\mathrm{U})_\mathfrak{m}, \Lambda_{\mathcal{O}})$$

of $\mathbf{T}_\infty(\mathrm{U}, \mathcal{O})_\mathfrak{m}$-modules. This will be important in our later computation of various congruences.

3.8. Congruence maps

In this subsection we prove a number of results that will be helpful in our analysis of "congruences" between Hecke rings in §8. As always, $\mathrm{U} \subseteq \mathrm{GL}_2(\mathcal{O}_\mathrm{F} \otimes \widehat{\mathbf{Z}})$ is a compact open subgroup such that $\mathrm{U} = \Pi \mathrm{U}_w$ and $\mathrm{U}_0(\mathfrak{n}) \supseteq \mathrm{U} \supseteq \mathrm{U}(\mathfrak{n})$ for some $\mathfrak{n}$. Let $w \nmid p$ be such that $\mathrm{U}_w = \mathrm{GL}_2(\mathcal{O}_{\mathrm{F},w})$, let $\ell = \ell_w$, and let $\lambda = \lambda^{(\ell)}$ be as in the definition of $\mathrm{T}(\ell)$. For any $f : \mathrm{G}^\mathrm{D}(\mathbf{A}_f) \longrightarrow \mathrm{R}$ (R an $\mathcal{O}$-module) put $(\alpha f)(g) = f\left(g\begin{pmatrix}1 & \\ & \lambda\end{pmatrix}\right)$. Let $\mathrm{V} = \mathrm{U} \cap \mathrm{U}_0(\ell)$.

Consider the map $\xi_1 : \mathrm{H}^0(\mathrm{X}(\mathrm{U}), \mathrm{R})^2 \longrightarrow \mathrm{H}^0(\mathrm{X}(\mathrm{V}), \mathrm{R})$ given by $\xi_1(f, g) = f + \alpha g$. The following is the analog of Ihara's Lemma (cf. [Ri]) in our setting.

Lemma **3.26.** — *The kernel of* ξ_1 *is annihilated by* $\left[\mathrm{U}\begin{pmatrix}1 & \\ & \lambda^{(q)}\end{pmatrix}U\right] - 1 - \mathrm{Nm}(q)$ *for any prime ideal* q *of* F *that splits completely in the ray class field of conductor* $\mathfrak{n} \cdot \infty$.

Proof. — Our proof of this lemma is a straight-forward generalization of [DT, Lemma 2, p. 445]. Put $\delta = \begin{pmatrix} 1 & \\ & \lambda^{-1} \end{pmatrix}$. Suppose that $(f_1, f_2) \in \ker(\xi_1)$. We first claim that $f_1(gu) = f_1(g)$ for all $u \in \delta^{-1}\mathrm{GL}_2(\mathcal{O}_{\mathrm{F},w})\delta$. This is an easy calculation: if $u = \delta^{-1}u' \cdot \delta \in \delta^{-1}\mathrm{GL}_2(\mathcal{O}_{\mathrm{F},w})\delta$, then

$$\begin{aligned} f_1(gu) &= -f_2(gu\delta^{-1}) \\ &= -f_2(g\delta^{-1}u') \\ &= -f_2(g\delta^{-1}) \\ &= f_1(g). \end{aligned}$$

As $\mathrm{SL}_2(\mathcal{O}_{\mathrm{F},w})$ and $\delta^{-1}\mathrm{SL}_2(\mathcal{O}_{\mathrm{F},w})\delta$ generate $\mathrm{SL}_2(\mathrm{F}_w)$ it follows that

(3.18) $$f_1(gu) = f_1(g) \quad \text{for all} \quad u \in \mathrm{U} \cdot \mathrm{SL}_2(\mathrm{F}_w).$$

Now let q be a prime that splits in the ray class field of conductor $\mathfrak{n} \cdot \infty$. It follows from class field theory that such a prime has a uniformizer $\pi \in \mathrm{F}$ that is totally positive and satisfies $\pi \equiv 1 \bmod \mathfrak{n}$. Suppose now that $\gamma \in \mathrm{G}^{\mathrm{D}}(\mathbf{A}_f)$ is any element such that $\nu_{\mathrm{D}}(\gamma) = \pi^{-1}$. For any $g \in \mathrm{G}^{\mathrm{D}}(\mathbf{A}_f)$, $\delta_0 g\gamma g^{-1} \in \mathrm{G}_1^{\mathrm{D}}(\mathbf{A}_f)$, where $\mathrm{G}_1^{\mathrm{D}} \subseteq \mathrm{G}^{\mathrm{D}}$ is the kernel of the reduced norm ν_{D} and $\delta_0 \in \mathrm{D}^\times$ is such that $\nu_{\mathrm{D}}(\delta_0) = \pi$. Such a δ_0 exists as π is totally positive (cf. [We, XI, §3, Proposition 3]). As $\mathrm{G}_1^{\mathrm{D}}$ is a twisted form of SL_2 for which $\mathrm{G}_1^{\mathrm{D}}(\mathrm{F}_w) = \mathrm{SL}_2(\mathrm{F}_w)$, it follows from strong approximation that $\delta_0 g\gamma g^{-1} = \delta' gug^{-1}$ for some $\delta' \in \mathrm{D}^\times$ and $u \in \mathrm{U} \cdot \mathrm{SL}_2(\mathrm{F}_w)$. We have then by (3.18) that

(3.19) $$f_1(g\gamma) = f_1(\delta_0^{-1}\delta' gu) = f_1(gu) = f_1(g).$$

As $\mathrm{U}\begin{pmatrix} 1 & \\ & \lambda^{(q)} \end{pmatrix}\mathrm{U} = \coprod_{i=1}^{\mathrm{Nm}(q)+1} \mathrm{U}g_i$ with $\nu_{\mathrm{D}}(g_i) = \pi$, it follows from (3.19) and the definition of $\left[\mathrm{U}\begin{pmatrix} 1 & \\ & \lambda^{(q)} \end{pmatrix}\mathrm{U}\right]$ that $\left[\mathrm{U}\begin{pmatrix} 1 & \\ & \lambda^{(q)} \end{pmatrix}\mathrm{U}\right] f_1 = (1 + \mathrm{Nm}(q)) f_1$. The lemma follows. □

Now put $\mathrm{U}^{(r)} = \mathrm{U} \cap \mathrm{U}_1(\ell^r)$, $r \geqslant 0$.

Lemma **3.27.** — *For $r \geqslant 1$ the sequence*

$$\mathrm{H}^0(\mathrm{X}(\mathrm{U}^{(r-1)}), \mathrm{R}) \xrightarrow{\delta} \mathrm{H}^0(\mathrm{X}(\mathrm{U}^{(r)}), \mathrm{R})^2 \xrightarrow{\gamma} \mathrm{H}^0(\mathrm{X}(\mathrm{U}^{(r+1)}), \mathrm{R}),$$

with $\delta(f) = (f, -\alpha f)$ and $\gamma(f_1, f_2) = \alpha f_1 + f_2$, is exact.

Proof. — To establish exactness, it suffices to prove that if (f_1, f_2) is in the kernel of γ then $f_1 \in \mathrm{H}^0(\mathrm{X}(\mathrm{U}^{(r-1)}), \mathrm{R})$.

For any function $f : \mathrm{G}^{\mathrm{D}}(\mathbf{A}_f) \longrightarrow \mathrm{R}$ put $\alpha^{-1}f(g) = f\left(g\begin{pmatrix} 1 & \\ & \lambda^{-1} \end{pmatrix}\right)$. Suppose that (f_1, f_2) is in the kernel of γ. As $\alpha f_1 = -f_2$ we also have $f_1 = -\alpha^{-1} f_2$. Now observe that $\alpha^{-1}f_2(gu) = \alpha^{-1}f_2(g)$ for all $u \in U' = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U : a-1 \in \ell^r, c \in \ell^{r-1}, b \in \ell\right\}$. It follows that $f_1 \in H^0(X(W), R)$ where W is the subgroup of U generated by $U^{(r)}$ and U'. This subgroup is just $U^{(r-1)}$. □

Now consider the map $\xi : H^0(X(U), R)^3 \longrightarrow H^0(X(U^{(2)}), R)$ given by $\xi(f_1, f_2, f_3) = f_1 + \alpha f_2 + \alpha^2 f_3$. As a consequence of Lemmas 3.26 and 3.27 we obtain the following.

Lemma **3.28.** — *The kernel of* ξ *is annihilated by* $\left[U\begin{pmatrix} 1 & \\ & \lambda^{(r)} \end{pmatrix}U\right] - 1 - \mathrm{Nm}(r)$ *for any prime ideal* r *of* F *that splits completely in the ray class field of conductor* $\mathfrak{n} \cdot \infty$.

Proof. — We can write ξ as the composite

$$H^0(X(U), R)^3 \xrightarrow{\beta} H^0(X(U), R)^4 \xrightarrow{\xi_1 \oplus \xi_1} H^0(X(U^{(1)}), R)^2 \xrightarrow{\gamma} H^0(X(U^{(2)}), R)$$

where $\beta(f_1, f_2, f_3) = (0, f_3, f_1, f_2)$. It follows from Lemma 3.27 that $\{f, 0, 0, -f\} \subseteq H^0(X(U), R)^4$ surjects onto the kernel of γ. If $\beta(f_1, f_2, f_3) \in \ker(\gamma \circ (\xi_1 \oplus \xi_1)$, then there exists some $f \in H^0(X(U), R)$ such that $(-f, f_3, f_1, f_2 + f) \in \ker(\xi_1 \oplus \xi_1)$. Therefore by Lemma 3.26, $f, f_3, f_1, f_2 + f$ are annihilated by the operators in question. This proves the lemma. □

We conclude our discussion of congruence maps with an important application of Lemma 3.26. Let $U \subseteq GL_2(\mathcal{O}_F \otimes \widehat{\mathbf{Z}})$ be as at the start of this subsection, only now we assume that $(\mathfrak{n}, p) = 1$. Suppose that $\mathfrak{p} \subseteq \mathbf{T}_\infty(U, \mathcal{O})$ is a prime such that

- $p \in \mathfrak{p}$
- $\rho_\mathfrak{p}$ is irreducible and not dihedral.

For simplicity we shall also assume that

- $\mathfrak{p}$ is contained in a permissible maximal ideal.

Suppose that ℓ is a prime ideal of F such that

- $\ell \nmid \mathfrak{n}p$
- $p \nmid (\mathrm{Nm}(\ell) - 1)$
- the ratio of the eigenvalues of $\rho_\mathfrak{p}(\mathrm{Frob}_\ell)$ does not equal $\mathrm{Nm}(\ell)$ or $\mathrm{Nm}(\ell)^{-1}$.

Put $U^{(0)} = U \cap U_0(\ell)$ and $U^{(1)} = U \cap U_1(\ell)$.

Lemma **3.29.**

(i) $\mathbf{T}_\infty(U^{(1)}, \mathcal{O})_\mathfrak{p} \simeq \mathbf{T}_\infty(U^{(0)}, \mathcal{O})_\mathfrak{p} \simeq \mathbf{T}_\infty(U, \mathcal{O})_\mathfrak{p}$.

(ii) $\widehat{M}_\infty(U^{(1)})_\mathfrak{p} \simeq \widehat{M}_\infty(U^{(0)})_\mathfrak{p} \simeq \widehat{M}_\infty(U)^2_\mathfrak{p}$ *and* $\widehat{M}^+_\infty(U^{(1)})_\mathfrak{p} \simeq \widehat{M}^+_\infty(U^{(0)})_\mathfrak{p} \simeq \widehat{M}^+_\infty(U)^2_\mathfrak{p}$ *as* $\widehat{\mathbf{T}}_\infty(U^{(1)}, \mathcal{O})_\mathfrak{p}$*-modules.*

Proof. — Let $\mathfrak{m}$ be the permissible maximal ideal of $\mathbf{T}_\infty(U, \mathcal{O})$ containing $\mathfrak{p}$. Write $\mathfrak{m}$ for the inverse image of this maximal ideal in $\mathbf{T}_\infty(U^{(0)}, \mathcal{O})$ and $\mathbf{T}_\infty(U^{(1)}, \mathcal{O})$.

By Corollary 3.12 we have surjections $\mathbf{T}_\infty(U^{(1)}, \mathcal{O})_\mathfrak{m} \twoheadrightarrow \mathbf{T}_\infty(U^{(0)}, \mathcal{O})_\mathfrak{m} \twoheadrightarrow \mathbf{T}_\infty(U, \mathcal{O})_\mathfrak{m}$. To prove part (i) it suffices to show that every minimal prime of $\mathbf{T}_\infty(U^{(1)}, \mathcal{O})_\mathfrak{m}$ contained in $\mathfrak{p}$ is the inverse image of a prime of $\mathbf{T}_\infty(U, \mathcal{O})_\mathfrak{m}$. Let $Q \subseteq \mathbf{T}_\infty(U^{(1)}, \mathcal{O})_\mathfrak{m}$ be such a prime. An analysis of the possibilities for $\rho_Q|_{D_\ell}$ shows that ρ_Q must be unramified at ℓ. It then follows from Proposition 3.14 that Q is the inverse image of a prime of $\mathbf{T}_\infty(U, \mathcal{O})_\mathfrak{m}$.

We now prove part (ii). For each $a > 0$, let $\xi_a : eH^0(X(U_a), \mathcal{O})^2 \longrightarrow eH^0(X(U_a^{(1)}), \mathcal{O})$ be given by $\xi_a(f, g) = f + \alpha g$ where α is as in Lemma 3.26. Let $\xi = \varprojlim_a \xi_a$: $M_\infty^+(U)^2 \longrightarrow M_\infty^+(U^{(1)})$. Also for each $a > 0$, let $I_a = \ker\{\mathbf{T}_2(U_a^{(0)}, \mathcal{O}) \longrightarrow \mathbf{T}_2(U_a, \mathcal{O})\}$. Then $I = \varprojlim_a I_a \subseteq \mathbf{T}_\infty(U^{(0)}, \mathcal{O})$ is just $\ker\{\mathbf{T}_\infty(U^{(0)}, \mathcal{O}) \longrightarrow \mathbf{T}_\infty(U, \mathcal{O})\}$.

We claim that

(3.20) $$M_\infty^+(U)^2_\mathfrak{p} \cong M_\infty^+(U^{(0)})[I]_\mathfrak{p}.$$

For this we note that $M_\infty^+(U^{(0)})[I] = \varprojlim_a eH^0(X(U_a^{(0)}), \mathcal{O})[I_a]$. Recall that we have fixed an identification of $\overline{\mathbf{Q}}_p$ with $\mathbf{C}$ (see §3.2). The map ξ_a extends to a map $\xi_a \otimes_\mathcal{O} \mathbf{C} : eH^0(X(U_a), \mathbf{C})^2 \longrightarrow eH^0(X(U_a^{(0)}), \mathbf{C})$. By the Jacquet-Langlands correspondence (see §3.2) we have $\mathbf{T}_2(U_a, \mathcal{O})$-equivariant isomorphisms

$$eH^0(X(U_a^{(0)}), \mathbf{C}) \simeq \bigoplus_{\pi \in \Pi_2^{\mathrm{ord}}(U_a^{(0)})} V_\pi^{U_a^{(0)}} \quad \text{and} \quad eH^0(X(U_a), \mathbf{C}) \simeq \bigoplus_{\pi \in \Pi_2^{\mathrm{ord}}(U_a)} V_\pi^{U_a}.$$

It is easy to see that $V_\pi^{U_a^{(0)}}[I_a] \neq 0$ if and only if $\pi \in \Pi_2^{\mathrm{ord}}(U_a)$. On the other hand, if $\pi \in \Pi_2^{\mathrm{ord}}(U_a)$, then $V_\pi^{U_a^{(0)}} = V_\pi^{U_a} + \alpha(V_\pi^{U_a})$, whence

$$eH^0(X(U_a), \mathbf{C})[I_a] = \mathrm{im}(\xi_a \otimes \mathbf{C}).$$

Let K be the field of fractions of $\mathcal{O}$. It follows that

(3.21) $$eH^0(X(U_a), K)[I_a] = \mathrm{im}(\xi_a \otimes K).$$

Now consider the commutative diagram

$$\begin{array}{ccccccc}
\varprojlim_a eH^0(X(U_a), \mathcal{O})^2 & \xrightarrow{\varprojlim_a \xi_a} & \varprojlim eH^0(X(U_a^{(0)}), \mathcal{O})[I_a] & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
0 \longrightarrow \varprojlim_a \mathrm{im}(\xi_a \otimes K) & \xrightarrow{\sim} & \varprojlim eH^0(X(U_a^{(0)}), K)[I_a] & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

where the vertical arrows are the natural ones. Applying the snake lemma we find that C embeds into a quotient of $\varprojlim_a \ker(\xi_a \otimes K/\mathcal{O})$.

Now, for each $a > 0$ let F_a be the ray class field of conductor $\ell \mathfrak{n} p^a \cdot \infty$. Let $F_\infty = \cup F_a$. Let F' be the maximal extension of F unramified away from places dividing $\ell \mathfrak{n} p \cdot \infty$. Any element $\sigma \in \mathrm{Gal}(F'/F_\infty)$ is the limit of a sequence of Frobenii $\{\mathrm{Frob}_{r_a}\}$ with r_a splitting completely in F_a. It follows that $\mathrm{trace}\, \rho_\mathfrak{m}(\sigma)$ is the limit of the sequence $\{T(r_a)\}$ and $\varepsilon(\sigma)$ is the limit of $\{\mathrm{Nm}(r_a)\}$. It then follows from Lemma 3.26 that $\mathrm{trace}\, \rho_\mathfrak{m}(\sigma) - 1 - \varepsilon(\sigma)$ annihilates $C_\mathfrak{m}$ for all $\sigma \in \mathrm{Gal}(F'/F_\infty)$. Thus if $C_\mathfrak{p} \neq 0$, then it must be that $\mathrm{trace}\, \rho_\mathfrak{m}(\sigma) - 1 - \varepsilon(\sigma)$ is in $\mathfrak{p}$ for all $\sigma \in \mathrm{Gal}(\overline{F}/F_\infty)$. It is easily deduced from this that $\rho|_{\mathrm{Gal}(\overline{F}/F_\infty)}$ is reducible, and hence $\rho_\mathfrak{p}$ is either reducible or dihedral, contradicting our assumptions on $\rho_\mathfrak{p}$. Therefore $C_\mathfrak{p} = 0$. The same argument shows that $\ker(\xi)_\mathfrak{p} = 0$. This proves (3.20). It follows from part (i) that $M^+_\infty(U^0)_\mathfrak{p} = M^+_\infty(U^0)_\mathfrak{p}[I] = M^+_\infty(U^0)[I]_\mathfrak{p}$, whence

$$\textbf{(3.22)} \qquad M^+_\infty(U^{(0)})_\mathfrak{p} \simeq M^+_\infty(U)^2_\mathfrak{p}.$$

We next prove that

$$\textbf{(3.23)} \qquad M^+_\infty(U^{(1)})_\mathfrak{p} \simeq M^+_\infty(U^{(0)})_\mathfrak{p}.$$

For this we note that $M^+_\infty(U^{(0)}) = M^+_\infty(U^{(1)})[S(\ell) - 1]$. By part (i),

$$M^+_\infty(U^{(1)})_\mathfrak{p} = M^+_\infty(U^{(1)})_\mathfrak{p}[S(\ell) - 1] = M^+_\infty(U^{(1)})[S(\ell) - 1]_\mathfrak{p},$$

from which (3.23) follows. A similar argument shows that $M_\infty(U^{(1)})_\mathfrak{p} \simeq M_\infty(U)_\mathfrak{p}$. □

4. The Theorems

4.1. Pro-modularity and primes of $R_\mathscr{D}$

We assume throughout §4 that F, χ, and k are as in §2 and that the degree of F is even unless indicated otherwise. In this subsection we also assume that $L_p(F, -1, \chi\omega)$ is not a unit (so $\mathbf{T}_\mathscr{D}$ exists for any $\mathscr{D}$).

Suppose that $\mathscr{D} = (\mathcal{O}, \Sigma, c, \mathcal{M})$ is a deformation datum for F. Let $\mathscr{D}^{\mathrm{ps}} = (\mathcal{O}, \Sigma)$. Let $\mathfrak{q}$ be a prime of $R_\mathscr{D}$. There is a map $\varphi_\mathfrak{q} : R_{\mathscr{D}^{\mathrm{ps}}} \longrightarrow R_\mathscr{D}/\mathfrak{q}$ corresponding to the pseudo-deformation associated to $\rho_\mathscr{D} \bmod \mathfrak{q}$. The prime $\mathfrak{q}$ is *pro-modular* if $\varphi_\mathfrak{q}$ factors through $\pi_\mathscr{D} : R_{\mathscr{D}^{\mathrm{ps}}} \twoheadrightarrow \mathbf{T}_\mathscr{D}$. That is, $\mathfrak{q}$ is pro-modular if there is a homomorphism $\theta_\mathfrak{q} : \mathbf{T}_\mathscr{D} \longrightarrow R_\mathscr{D}/\mathfrak{q}$ such that

$$\textbf{(4.1)} \qquad \varphi_\mathfrak{q} = \theta_\mathfrak{q} \circ \pi_\mathscr{D}.$$

(Throughout this section, if a deformation datum $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$ is given, then $\mathscr{D}^{\mathrm{ps}}$ will denote the pseudo-datum $\mathscr{D}^{\mathrm{ps}} = (\mathscr{O}, \Sigma)$.) Note that in (4.1) $\theta_{\mathfrak{q}}(T(\ell)) = \operatorname{trace} \rho_{\mathscr{D}}(\mathrm{Frob}_\ell) \bmod \mathfrak{q}$ for all $\ell \notin \Sigma$. Similarly, a deformation $\rho : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(A)$ of type-$\mathscr{D}$ is a *pro-modular deformation* if the kernel of the corresponding map $R_{\mathscr{D}} \longrightarrow A$ is a pro-modular prime.

It is immediate from the above definition that if $\mathfrak{q}$ is a pro-modular prime of $R_{\mathscr{D}}$ and if $\mathfrak{p} \supseteq \mathfrak{q}$ is another prime ideal, then $\mathfrak{p}$ is also pro-modular. In particular, if a minimal prime of $R_{\mathscr{D}}$ is pro-modular, then so is every prime ideal on the corresponding irreducible component of $\operatorname{spec}(R_{\mathscr{D}})$. In this case we say that the component is pro-modular.

4.2. Good data and properties (P1) and (P2)

Our primary goal is to show that for certain "good" deformation data $\mathscr{D}$ the components of $\operatorname{spec}(R_{\mathscr{D}})$ are all pro-modular provided the data have certain properties (labeled (P1) and (P2) below). In this subsection we describe these "good" data and the relevant properties.

Let $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$ be a deformation datum for F. The pair $(F, \mathscr{D})$ is *good* if

- the degree d of F is even
- $L_p(F, -1, \chi\omega) \in \mathscr{O}$ but $L_p(F, -1, \chi\omega) \notin \mathscr{O}^\times$
- $d > 2 + \delta_F + 8 \cdot (\#\Sigma + \dim_k H_{\Sigma_0}(F, k))$
- for each $v_i | p$ the degree d_{v_i} of F_{v_i} over $\mathbf{Q}_p$ satisfies

$$d_{v_i} > 2 + 2t + 7 \cdot (\#\Sigma + \dim_k H_{\Sigma_0}(F, k))$$

- if $\rho_c|_{I_w} \neq 1$ and $w \nmid p$, then either $\chi|_{I_w} \neq 1$ or $\chi|_{D_w} = 1$.

As before, $t = \#\mathscr{P}$, where $\mathscr{P} = \{v_i\}$ is the set of places of F over p, Σ_0 is the set of finite places at which χ is ramified together with $\mathscr{P}$, and

$$H_{\Sigma_0}(F, k) = \ker\{H^1(F_{\Sigma_0}/F, k(\chi^{-1})) \xrightarrow{\mathrm{res}} \bigoplus_{i=1}^{t} H^1(D_{v_i}, k(\chi^{-1}))\}.$$

Note that if $\mathscr{D}$ is good and if $\mathscr{D}' = (\mathscr{O}, \Sigma', c, \mathscr{M}')$ is another datum with $\Sigma' \subseteq \Sigma$, then $(F, \mathscr{D}')$ is also good. However, being good does not behave well with respect to change of fields, meaning that if $(F, \mathscr{D})$ is good and if L/F is permissible for $\mathscr{D}$ (as defined before Remark 2.1), then it can happen that $(L, \mathscr{D}_L)$ is not good. On the other hand, it can also happen that $(F, \mathscr{D})$ is not good but $(L, \mathscr{D}_L)$ is. This will be a key ingredient in our reduction in §4.6 of Theorems A and B to the Main Theorem.

Let $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$ be a deformation datum for F. Let $\mathfrak{p}$ be a dimension one prime of $\mathbf{T}_{\mathscr{D}}$. Let $\rho_{\mathfrak{p}}$ be the representation described in §3.3. Let A be the integral closure of $\mathbf{T}_{\mathscr{D}}/\mathfrak{p}$ in its field of fractions K. If $\rho_{\mathfrak{p}}$ is irreducible, then Lemma 2.13

associates to $\rho_{\mathfrak{p}}$ a representation $\rho : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(A)$ such that $\rho \otimes \overline{K} \simeq \rho_{\mathfrak{p}}$ and ρ is a deformation of some $\rho_{c'}$ for some cocycle $0 \neq c' \in H^1(F_\Sigma/F, k'(\chi^{-1}))$, k' some finite extension of k. We claim that c' is admissible and that ρ is a deformation of type-$(\mathcal{O}', \Sigma, c', \emptyset)$, where $\mathcal{O}'$ has residue field k'. To see this, let v_i be one of the places over p. Choose $\sigma_i \in D_i$ such that $\chi(\sigma_i) \neq 1$ and choose a basis for ρ such that $\rho(\sigma_i) = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ with $\beta \bmod \mathfrak{m}_A = \chi(\sigma_i)$. As $\rho \otimes \overline{K} \simeq \rho_{\mathfrak{p}}$ it follows from (3.4) that with respect to this basis either $\rho|_{D_i}$ is split, $\rho|_{D_i}$ is non-split and $\rho|_{D_i} = \begin{pmatrix} \phi_1 & * \\ & \phi_2 \end{pmatrix}$, or $\rho|_{D_i}$ is non-split and $\rho|_{D_i} = \begin{pmatrix} \phi_1 & \\ * & \phi_2 \end{pmatrix}$. If $\rho|_{D_i}$ is split then clearly $0 = \mathrm{res}_{v_i}(c') \in H^1(D_i, k'(\chi^{-1}))$ and $\rho|_{D_i}$ satisfies the desired criteria. If $\rho|_{D_i} = \begin{pmatrix} \phi_1 & * \\ & \phi_2 \end{pmatrix}$, then $\phi_2(\lambda_{v_i}) \bmod \mathfrak{m}_A = \chi(\lambda_{v_i}) \neq 1$. (Here λ_{v_i} is the uniformizer of $\mathcal{O}_{F, v_i}$ chosen for the definition of $T_0(\mathfrak{p}_i)$ – see §3.2). However, as $\rho \otimes \overline{K} \simeq \rho_{\mathfrak{p}}$, it follows from (3.4) that $T_0(\mathfrak{p}_i) \bmod \mathfrak{p} = \phi_2(\lambda_{v_i})$ and by the permissibility of the maximal ideal of $\mathbf{T}_{\mathcal{D}}$, $T_0(\mathfrak{p}_i) \bmod \mathfrak{m}_A = 1$. This contradiction shows that if $\rho|_{D_i}$ is non-split, then $\rho|_{D_i} = \begin{pmatrix} \phi_1 & \\ * & \phi_2 \end{pmatrix}$ with $\phi_2 \bmod \mathfrak{m}_A = \chi$. One sees immediately that $\mathrm{res}_{v_i}(c') = 0$ and that $\rho|_{D_i}$ satisfies the desired hypotheses. Therefore c' is admissible and ρ is a deformation of type-$(\mathcal{O}', \Sigma, c', \emptyset)$. We say that the prime $\mathfrak{p}$ is *nice for* $\mathcal{D}$ if

- $\mathfrak{p}$ is a dimension one prime of $\mathbf{T}_{\mathcal{D}}$,
- $\rho_{\mathfrak{p}}$ is irreducible,
- $\mathfrak{p}$ is the inverse image of a prime of $\mathbf{T}_{\mathcal{D}_c}$ (where $\mathcal{D}_c$ is the deformation datum defined in §2.3),
- c' is a scalar multiple of c,
- some conjugate of ρ is a nice deformation of type-$(\mathcal{O}', \Sigma, c, \mathcal{M})$ in the sense of §2.3.

A prime $\mathfrak{p}$ of $R_{\mathcal{D}}$ is *good* if $\rho_{\mathcal{D}} \bmod \mathfrak{p}$ is nice in the sense of §2.3. Such a prime is *nice* if it is also the inverse image of a pro-modular prime of $R_{\mathcal{D}_c}$.

If $\mathfrak{p}$ is nice for $\mathcal{D}$, then the universality of $R_{\mathcal{D}}$ yields a unique map $R_{\mathcal{D}} \longrightarrow A$ inducing a conjugate of ρ. We denote by $\mathfrak{p}_{\mathcal{D}}$ the kernel of this map. This is a nice prime. The first of the aforementioned properties of $\mathcal{D}$ is that

(P1) if $\mathfrak{p} \subseteq \mathbf{T}_{\mathcal{D}}$ is any prime that is nice for $\mathcal{D}$, then any prime $Q \subseteq \mathfrak{p}_{\mathcal{D}} \subseteq R_{\mathcal{D}}$ is pro-modular.

The second important property of $\mathcal{D}$ is that

(P2) there exists a pro-modular prime of $R_{\mathcal{D}_c}$ whose corresponding deformation is nice in the sense of §2.3.

4.3. The key proposition

The following proposition is the key ingredient in our proof of the Main Theorem.

Proposition **4.1.** — *Let* $\mathcal{D}$ *be a deformation datum for* F. *If* $(\mathrm{F}, \mathcal{D})$ *is good, and if* (P1) *and* (P2) *hold for* $\mathcal{D}$ *and* $\mathcal{D}_c$, *then every prime of* $\mathrm{R}_{\mathcal{D}}$ *is pro-modular.*

Proof. — Let $\mathcal{C}_{\mathcal{D}}$ be the set of irreducible components of $\mathrm{spec}(\mathrm{R}_{\mathcal{D}})$ and let $\mathcal{C}_{\mathcal{D}}^{\mathrm{mod}} \subseteq \mathcal{C}_{\mathcal{D}}$ be the subset consisting of pro-modular components. The assertion of the proposition is equivalent to $\mathcal{C}_{\mathcal{D}} = \mathcal{C}_{\mathcal{D}}^{\mathrm{mod}}$.

We begin by proving the proposition for the case $\mathcal{D} = \mathcal{D}_c$. (Note that since $(\mathrm{F}, \mathcal{D})$ is good, so is $(\mathrm{F}, \mathcal{D}_c)$.) The proof consists of two steps. In the first, we show that any component of $\mathrm{spec}(\mathrm{R}_{\mathcal{D}_c})$ containing a nice prime is itself pro-modular. As a consequence of this and of (P2) we have that $\mathcal{C}_{\mathcal{D}_c}^{\mathrm{mod}} \neq \emptyset$. In the second step we combine step one with our analysis of the structure of the ring $\mathrm{R}_{\mathcal{D}_c}$ to conclude that $\mathcal{C}_{\mathcal{D}_c}^{\mathrm{mod}} = \mathcal{C}_{\mathcal{D}_c}$.

Suppose that $\mathfrak{p}$ is a nice prime of $\mathrm{R}_{\mathcal{D}_c}$. By the definition of pro-modularity of $\mathfrak{p}$ there is a unique map $\theta_{\mathfrak{p}} : \mathbf{T}_{\mathcal{D}_c} \longrightarrow \mathrm{R}_{\mathcal{D}_c}/\mathfrak{p}$ inducing the pseudo-deformation associated to $\rho_{\mathcal{D}_c} \bmod \mathfrak{p}$. Call the kernel of this map $\mathfrak{p}_1$. Clearly, $\mathfrak{p}_1$ is nice for $\mathcal{D}_c$. It follows from (P1) that if $\mathrm{Q} \subseteq \mathfrak{p}$ is any prime of $\mathrm{R}_{\mathcal{D}_c}$ then Q is pro-modular. In particular, any minimal prime of $\mathrm{R}_{\mathcal{D}_c}$ contained in $\mathfrak{p}$ is pro-modular. This completes step one. Combining this with (P2), which asserts the existence of a nice prime of $\mathrm{R}_{\mathcal{D}_c}$, yields $\mathcal{C}_{\mathcal{D}_c}^{\mathrm{mod}} \neq \emptyset$.

The next step is to prove that $\mathcal{C}_{\mathcal{D}_c} = \mathcal{C}_{\mathcal{D}_c}^{\mathrm{mod}}$. Put $\mathcal{C}'_{\mathcal{D}_c} = \mathcal{C}_{\mathcal{D}_c} \backslash \mathcal{C}_{\mathcal{D}_c}^{\mathrm{mod}}$. If $\mathcal{C}'_{\mathcal{D}_c} = \emptyset$, then there is nothing to prove, so assume otherwise. It follows from Proposition 2.4 and Corollary A.2 that there are components $\mathrm{C}_1 \in \mathcal{C}_{\mathcal{D}_c}^{\mathrm{mod}}$ and $\mathrm{C}_2 \in \mathcal{C}'_{\mathcal{D}_c}$ such that $\mathrm{C}_1 \cap \mathrm{C}_2$ contains a prime Q of dimension $d - 2t + \delta_{\mathrm{F}} - 3 \cdot \#\mathcal{M}_c$. Let I_1 be the ideal generated by the set $\{p; \det \rho_{\mathcal{D}_c}(\gamma_i) - 1 \mid i = 1, \ldots, \delta_{\mathrm{F}}\}$. Let Q_1 be a minimal prime of $\mathrm{R}_{\mathcal{D}_c}/(\mathrm{Q}, \mathrm{I}_1)$. The dimension of Q_1 is at least $d - 2t - 3 \cdot \#\mathcal{M}_c - 1 > 1 + \delta_{\mathrm{F}} + (\#\Sigma + \dim_k \mathrm{H}_{\Sigma_0}(\mathrm{F}, k))$, the inequality by (G). It follows from Lemma 2.6 that $\rho_{\mathcal{D}_c} \bmod \mathrm{Q}_1$ is irreducible.

Since $\mathrm{Q}_1 \in \mathrm{C}_1$, Q_1 is pro-modular. The prime Q_1 determines a prime $\mathrm{Q}_1^{\mathrm{mod}}$ of $\mathbf{T}_{\mathcal{D}_c}$. The prime $\mathrm{Q}_1^{\mathrm{mod}}$ is the kernel of $\theta_{\mathrm{Q}_1} : \mathbf{T}_{\mathcal{D}_c} \longrightarrow \mathrm{R}_{\mathcal{D}_c}/\mathrm{Q}_1$. Moreover, since $\rho_{\mathcal{D}} \bmod \mathrm{Q}_1$ is irreducible as remarked in the preceding paragraph, it follows from Proposition 2.12 that $\dim \mathbf{T}_{\mathcal{D}_c}/\mathrm{Q}_1^{\mathrm{mod}} \geqslant \dim \mathrm{R}_{\mathcal{D}}/\mathrm{Q}_1$. Recall that $\mathbf{T}_{\mathcal{D}_c}$ is an integral extension of $\Lambda_{\mathcal{O}} = \mathcal{O}[[\mathrm{Y}_1^{(1)}, \ldots, \mathrm{Y}_{d_t}^{(t)}, \mathrm{T}_1, \ldots, \mathrm{T}_{\delta_{\mathrm{F}}}]]$ (cf. Corollary 3.4). By construction $\mathrm{Q}_1^{\mathrm{mod}} \cap \Lambda_{\mathcal{O}}$ contains $\mathrm{T}_1, \ldots, \mathrm{T}_{\delta_{\mathrm{F}}}$. If $\mathrm{Q}_1^{\mathrm{mod}} \cap \Lambda_{\mathcal{O}}$ also contained $\mathrm{Y}_1^{(i)}, \ldots, \mathrm{Y}_{d_i}^{(i)}$ it would follow that the dimension of $\mathrm{Q}_1^{\mathrm{mod}}$ would be at most $d - d_i$. Comparing this with the lower bound for the dimension of Q_1 obtained earlier and recalling that the dimension of Q_1 is at most that of $\mathrm{Q}_1^{\mathrm{mod}}$, one finds that $d_i \leqslant 2t + 3 \cdot \#\mathcal{M}_c + 1$ which contradicts (G). Thus, after possibly reordering the $\mathrm{Y}_j^{(i)}$'s we may assume that $\mathrm{Y}_1^{(i)} \notin \mathrm{Q}_1$ for each $i = 1, \ldots, t$.

Fix now a basis for $\rho_{\mathscr{D}}$ for which $\rho_{\mathscr{D}}(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Write $\rho_{\mathscr{D}}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$. As $\rho_{\mathscr{D}} \bmod Q_1$ is irreducible, there is some σ_0 for which $c_{\sigma_0} \notin Q_1$. Let $\mathfrak{p} \supseteq Q_1$ be a prime of dimension one not containing $c_{\sigma_0}, Y_1^{(1)}, \ldots, Y_1^{(t)}$. Such a $\mathfrak{p}$ always exists. As $\mathfrak{p} \in C_1$ it is pro-modular. We claim that it is also good. By construction $\mathfrak{p}$ contains p, and it is, of course, a prime of $R_{\mathscr{D}_c}$, so it remains to check the conditions at each D_i. Let $A = R_{\mathscr{D}_c}/\mathfrak{p}$ and let $\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(A)$ be the deformation $\rho_{\mathscr{D}} \bmod \mathfrak{p}$. Consider $\rho|_{D_i} \simeq \begin{pmatrix} \psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$. By definition $\psi_2^{(i)}(y_1^{(i)})$ equals $1 + Y_1^{(i)}$, which has infinite order in A. Thus $\psi_2^{(i)}$ is a character of infinite order. On the other hand, $\det \rho(\gamma_j) = 1$ for $j = 1, \ldots, \delta_F$, so, as char $A = p$, $\det \rho = \chi$. It follows that $\psi_1^{(i)} = \chi \cdot \psi_2^{(i)-1}$, whence $\psi_1^{(i)}/\psi_2^{(i)}$ has infinite order. Therefore $\mathfrak{p}$ is a nice prime of $R_{\mathscr{D}_c}$. As $\mathfrak{p} \in C_2$ it follows from step one that $C_2 \in \mathscr{C}^{\mathrm{mod}}_{\mathscr{D}_c}$ contradicting the assumption that $C_2 \in \mathscr{C}'_{\mathscr{D}_c}$. This proves that $\mathscr{C}_{\mathscr{D}_c} = \mathscr{C}^{\mathrm{mod}}_{\mathscr{D}_c}$.

We now prove the proposition in its full generality. We first show that any component of $\mathrm{spec}(R_{\mathscr{D}})$ containing a good prime is pro-modular. For this we use the proposition in the case $\mathscr{D} = \mathscr{D}_c$. We then combine this with our previous analysis of $R_{\mathscr{D}}$ to conclude that $\mathscr{C}_{\mathscr{D}} = \mathscr{C}^{\mathrm{mod}}_{\mathscr{D}}$.

Suppose that $\mathfrak{p}$ is a good prime of $R_{\mathscr{D}}$. It follows that $\mathfrak{p}$ is the inverse image of a prime $\mathfrak{p}_1$ of $R_{\mathscr{D}_c}$ under the canonical map $R_{\mathscr{D}} \twoheadrightarrow R_{\mathscr{D}_c}$. By the proposition in the case $\mathscr{D} = \mathscr{D}_c$, $\mathfrak{p}_1$ is a pro-modular prime. Thus there is a map $\theta_{\mathfrak{p}_1} : \mathbf{T}_{\mathscr{D}_c} \longrightarrow R_{\mathscr{D}_c}/\mathfrak{p}_1 = R_{\mathscr{D}}/\mathfrak{p}$ inducing the pseudo-deformation associated to $\rho_{\mathscr{D}_c} \bmod \mathfrak{p}_1 = \rho_{\mathscr{D}} \bmod \mathfrak{p}$. Composing $\theta_{\mathfrak{p}_1}$ with the canonical map $\mathbf{T}_{\mathscr{D}} \twoheadrightarrow \mathbf{T}_{\mathscr{D}_c}$ yields a map $\theta_{\mathfrak{p}} : \mathbf{T}_{\mathscr{D}} \longrightarrow R_{\mathscr{D}}/\mathfrak{p}$ inducing the pseudo-deformation associated to $\rho_{\mathscr{D}} \bmod \mathfrak{p}$. Let $\mathfrak{p}_2$ be the kernel of $\theta_{\mathfrak{p}}$. It follows from the definition of $\mathfrak{p}_2$ that it is nice for $\mathscr{D}$, whence by (P1) any prime $Q \subseteq \mathfrak{p}_{2,\mathscr{D}} \subseteq R_{\mathscr{D}}$ is pro-modular. As $\mathfrak{p} = \mathfrak{p}_{2,\mathscr{D}}$, it follows that any component of $\mathrm{spec}(R_{\mathscr{D}})$ containing $\mathfrak{p}$ is also pro-modular.

In our final step we complete the proof of the proposition in its full generality. Let Q be a minimal prime of $R_{\mathscr{D}}$. Let $I_2 \subseteq R_{\mathscr{D}}$ be the ideal defined as follows. Choose a basis for $\rho_{\mathscr{D}}$ such that $\rho_{\mathscr{D}}(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Write $\rho_{\mathscr{D}}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$ and $\rho_c(\sigma) = \begin{pmatrix} 1 & u_\sigma \\ & \chi(\sigma) \end{pmatrix}$. For each place $v \in \Sigma \backslash \mathscr{P}$ fix a generator $\tau_v \in I_v$ of the pro-p-part of tame inertia at v. Let I_2 be the ideal generated by the set

$$\{p; a_{\tau_v} - 1,\ b_{\tau_v} - u_{\tau_v},\ c_{\tau_v},\ d_{\tau_v}; \det \rho_{\mathscr{D}}(\gamma_j) - 1 \mid v \in \Sigma \backslash \mathscr{P},\ j = 1, \ldots, \delta_F\}.$$

Let Q_2 be a minimal prime of $R_{\mathscr{D}}/(Q, I_2)$. By Proposition 2.4 the dimension of Q_2 is at least $d - 7 \cdot \#\Sigma - 1$. It follows from this and from (G) that the dimension of Q_2 is at least $\delta_F + \#\Sigma + \dim_k H_{\Sigma_0}(F) + 1$ from which it follows by Lemma 2.6 that $\rho_{\mathscr{D}} \bmod Q_2$ is not reducible. Moreover, it is clear from the fact that $Q_2 \supseteq I_2$ that $\rho_{\mathscr{D}} \bmod Q_2$ is a deformation of type-$\mathscr{D}_c$. It follows from the proposition in the case $\mathscr{D} = \mathscr{D}_c$ that Q_2 is pro-modular. Arguing as in step two of the proof in the case $\mathscr{D} = \mathscr{D}_c$ shows that Q_2 is contained in a good prime. As $Q \subseteq Q_2$, the same is true of Q. The conclusion of the preceding paragraph now implies that Q is pro-modular. Therefore, every minimal prime of $R_{\mathscr{D}}$ is pro-modular. This completes the proof of the proposition. □

4.4. Conditions under which (P2) holds

In this subsection we establish the following criteria for (P2) to hold for a given deformation datum $\mathscr{D}$.

Proposition **4.2.** — *Let* $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$ *be a deformation datum. If* $(F, \mathscr{D})$ *is a good pair, and if* (P1) *holds for each datum* $(\mathscr{O}', \Sigma', c', \mathscr{M}')$ *with* $\Sigma' \subseteq \Sigma$ *and* $\mathscr{O}' \supseteq \mathscr{O}$, *then* (P2) *holds for* $\mathscr{D}$.

Proof. — The proof of this proposition consists roughly of three steps. In the first we prove that (P2) holds for some deformation datum $\mathscr{D}_0 = (\mathscr{O}', \Sigma_0, c_0, \mathscr{M}_0)$ with $\mathscr{O}' \supseteq \mathscr{O}$. From this, together with the hypotheses of the proposition and Proposition 4.1, we obtain that if $\mathscr{D}' = (\mathscr{O}', \Sigma', c_0, \mathscr{M}')$ with $\Sigma' \subseteq \Sigma$ then every prime of $R_{\mathscr{D}'}$ is pro-modular. In the second step we combine step one with the existence of suitable reducible deformations to show that there exists a prime $\mathfrak{p}_1$ of $\mathbf{T}_{\mathscr{D}_1}$ (where $\mathscr{D}_1 = (\mathscr{O}', \Sigma_c, c_0, \mathscr{M}_1)$ for a suitable $\mathscr{M}_1$) such that the pseudo-deformation associated to $\mathfrak{p}_1$ comes from the pseudo-deformation associated to a deformation ρ_1 of type-$(\mathscr{O}, \Sigma_c, c, \emptyset)$. In the third step we prove that ρ_1 is actually of type-$\mathscr{D}_c$ and that $\mathfrak{p}_1$ is essentially the inverse image of a prime of $\mathbf{T}_{\mathscr{D}_c}$, thereby proving that (P2) holds for $\mathscr{D}$.

We now prove that (P2) holds for some deformation datum $\mathscr{D}_0 = (\mathscr{O}', \Sigma_0, c_0, \mathscr{M}_0)$. (Recall that Σ_0 is the set of finite places at which χ is ramified together with the places $v_1, \dots, v_t$ over p and that $\mathscr{M}_0 = \Sigma_0 \backslash \{v_1, \dots, v_t\}$.) Let $U^\chi \subseteq GL_2(\mathscr{O}_F \otimes \hat{\mathbf{Z}})$ be as in §3.4. Since the pair $(F, \mathscr{D})$ is good, $L_p(F, -1, \chi\omega)$ is not a unit in $\mathscr{O}$, so it follows from Proposition 3.14 that $\mathbf{T}_\infty(U^\chi, \mathscr{O})$ has a permissible maximal ideal $\mathfrak{m}$. Recall that by Corollary 3.4, $\mathbf{T}_\chi = \mathbf{T}_\infty(U^\chi, \mathscr{O})_\mathfrak{m}$ is an integral extension of $\Lambda_\mathscr{O} = \mathscr{O}[[Y_1^{(1)}, \dots, Y_{d_t}^{(t)}, T_1, \dots, T_{\delta_F}]]$. Let $Q \subseteq \mathbf{T}_\chi/(p, T_1, \dots, T_{\delta_F})$ be a minimal prime. By its choice, the dimension of Q is at least d. Let $R = \mathbf{T}_\chi/Q$. The pseudo-representation associated to ρ_Q determines a pseudo-deformation into R of type-$(\mathscr{O}, \Sigma_0)$. We denote this pseudo-deformation by $\widetilde{\rho_Q} = \{a(\sigma), d(\sigma), x(\sigma, \tau)\}$. We claim that $x(\sigma, \tau)$ is not identically zero. If it were then $\rho : \mathrm{Gal}(F_{\Sigma_0}/F) \longrightarrow GL_2(R)$ defined by $\rho(\sigma) = \begin{pmatrix} a(\sigma) & \\ & d(\sigma) \end{pmatrix}$ would be a diagonal deformation of type-$(\mathscr{O}, \Sigma_0)$ (see §2.3). Therefore, there would be a map $\gamma : R^{\mathrm{diag}}_{(\mathscr{O}, \Sigma)} \longrightarrow R$ inducing ρ. Since it follows from Lemma 3.11 that R is generated (pro-finitely) by the set $\{\mathrm{trace}\, \rho(\sigma)\} = \{\mathrm{trace}\, \rho_Q(\sigma)\}$, γ must be surjective. Thus the kernel of γ would be a prime $\mathfrak{q}$ of $R^{\mathrm{diag}}_{(\mathscr{O}, \Sigma)}$ of dimension at least d. However by the choice of Q, $\det \rho$ (and hence $\det \rho^{\mathrm{diag}}_{(\mathscr{O}, \Sigma)} \bmod \mathfrak{q}$) has finite order. Lemma 2.9 would now imply that $d \leqslant 1 + \delta_F$, but this contradicts (G). This contradiction implies that there exists some σ_0 and τ_0 such that $x(\sigma_0, \tau_0) \neq 0$. Now let $\mathfrak{p} \supseteq Q$ be a dimension one prime of $\mathbf{T}_\chi$ not containing $x(\sigma_0, \tau_0)$, $Y_1^{(1)}, ..., Y_1^{(t)}$. Let A be the normalization of $\mathbf{T}_\chi/\mathfrak{p}$ (this is a complete DVR with residue field k' a finite extension of k). Let $\mathcal{O}' = \mathcal{O} \otimes_{W(k)} W(k')$. Let $\varphi = \widetilde{\rho_Q} \bmod \mathfrak{p}$ be the induced pseudo-deformation into A of type-$(\mathcal{O}', \Sigma_0)$. This is nothing more than the pseudo-representation associated to $\rho_\mathfrak{p}$. By Corollary 2.14 there exists a cocycle $0 \neq c_0 \in H^1(F_{\Sigma_0}/F, k'(\chi^{-1}))$ and a deformation $\rho_\varphi : \mathrm{Gal}(F_{\Sigma_0}/F) \longrightarrow GL_2(A)$ of ρ_{c_0} whose associated pseudo-deformation is φ. Let K be the field of fractions of A (equivalently, the field of fractions of $\mathbf{T}_\chi/\mathfrak{p}$).

Comparing traces we find that $\rho_\varphi \otimes \overline{K} \simeq \rho_\mathfrak{p}$. Arguing as in the second full paragraph of §4.2 (the paragraph describing primes of $\mathbf{T}_\mathscr{D}$ that are nice for $\mathscr{D}$) shows that c_0 is admissible and that ρ_φ is a deformation of type-$(\mathcal{O}', \Sigma_0, c_0, \emptyset)$. We claim that it is in fact a nice deformation of type-$\mathscr{D}_0$, where $\mathscr{D}_0 = (\mathcal{O}', \Sigma_0, c_0, \mathscr{M}_{c_0})$. Recall that $\mathscr{M}_{c_0}$ is nothing more than the set of finite places other than $v_1, ..., v_t$ at which χ is ramified. Therefore if $w \in \mathscr{M}_{c_0}$, then one sees easily that $\rho_\varphi|_{I_w} \simeq \begin{pmatrix} \phi_1 & * \\ & \tilde{\chi}\phi_2 \end{pmatrix}$ with ϕ_1 and ϕ_2 finite characters of p-power order. However, since the characteristic of A is p it must be that $\phi_1 = 1 = \phi_2$. This shows that ρ_φ is of type-$\mathscr{D}_0$. Moreover, since $\rho_\varphi \otimes K \simeq \rho_\mathfrak{p}$ it follows from (3.4) that $\rho_\varphi|_{D_i} \simeq \begin{pmatrix} \phi_1^{(i)} & * \\ & \phi_2^{(i)} \end{pmatrix}$ with $\phi_1^{(i)} \cdot \phi_2^{(i)}$ a character of finite order and $\phi_2^{(i)}(y_1^{(i)}) = 1 + Y_1^{(i)}$, which is an element of infinite order in $A^\times$. To conclude that ρ_φ is a nice deformation it remains to check that the corresponding prime of $R_{\mathscr{D}_0}$ is of dimension one. This follows from Lemma 2.12. By construction ρ_φ is a pro-modular deformation of type-$\mathscr{D}_0$ (since $\mathbf{T}_{\mathscr{D}_0} = \mathbf{T}_\chi \otimes_\mathcal{O} \mathcal{O}'$). It follows that the prime of $R_{\mathscr{D}_0}$ corresponding to ρ_φ is a nice prime. This completes step one.

If the cocycle c is a scalar multiple of c_0, then (P2) holding for $\mathscr{D}_0$ easily implies that (P2) holds for $\mathscr{D}_c$ and hence also for $\mathscr{D}$, as was to be proved. For let ρ_φ be the deformation of type-$\mathscr{D}_0$ described in the preceding paragraph. There exists a conjugate ρ'_φ of ρ_φ that takes values in $GL_2(B)$ with $B \subseteq A$ an $\mathcal{O}$-subalgebra with residue field k and that is a deformation of type-$(\mathcal{O}, \Sigma_0, c, \mathscr{M}_{c_0})$. Since $(\mathcal{O}, \Sigma_0, c, \mathscr{M}_{c_0}) = \mathscr{D}_c$, this shows that ρ'_φ is a nice, pro-modular deformation of type-$\mathscr{D}_c$ (since $\mathbf{T}_{\mathscr{D}_c} \otimes_\mathcal{O} \mathcal{O}' = \mathbf{T}_{\mathscr{D}_0}$).

Suppose from now on that c is not a scalar multiple of c_0. Let $\mathscr{D}_1 = (\mathcal{O}', \Sigma_c, c_0, \mathscr{M}_1)$ with $\mathscr{M}_1$ the set of finite places $w \in \Sigma_c$ other than $v_1, ..., v_t$ such that $\chi|_{I_w} \neq 1$. Since $(F, \mathscr{D})$ is good so is $(F, \mathscr{D}_1)$. Having shown that (P2) holds for $\mathscr{D}_0$ we see that (P2) also holds for $\mathscr{D}_1$. Combining this with the hypothesis that (P1) holds for $\mathscr{D}_0$ and $\mathscr{D}_1$, and with Proposition 4.1, yields that every prime of $R_{\mathscr{D}_1}$ is pro-modular.

As both c and c_0 are classes in $H_{\Sigma_c}(F, k')$ they can both be viewed as $\mathrm{Gal}(F(\chi)/F)$-equivariant homomorphisms $\mathrm{Gal}(F_0(\chi)/F(\chi)) \longrightarrow k'(\chi^{-1})$, where $F_0(\chi)$ is the minimal field over which every cocycle $c' \in H_{\Sigma_c}(F)$ becomes trivial. Fix $\mathrm{Gal}(F(\chi)/F)$-generators $\sigma_1, ..., \sigma_s$ of $\mathrm{Gal}(F_0(\chi)/F(\chi))$. Then any cocycle in $H_\Sigma(F, k')$ is determined completely by its values on the σ_i's. Let $\{(\alpha_{1,j}, ..., \alpha_{s,j}) \in k'^s, 1 \leqslant j \leqslant s-2\}$ be $s-2$ linearly independent vectors such that $\sum_{i=1}^{s} \alpha_{i,j} c_0(\sigma_i) = 0$ and $\sum_{i=1}^{s} \alpha_{i,j} c(\sigma_i) = 0$. Note that

$$s - 2 = \dim_{k'} \mathrm{H}_{\Sigma_c}(\mathrm{F}, k') - 2 \leqslant \#\Sigma_c + \dim_k \mathrm{H}_{\Sigma_0}(\mathrm{F}, k). \tag{4.2}$$

Fix a lift $\tilde{\alpha}_{i,j}$ of each $\alpha_{i,j}$ to $\mathcal{O}'$.

Fix now a basis of $\rho_{\mathscr{D}_1}$ such that $\rho_{\mathscr{D}_1}(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Write $\rho_{\mathscr{D}_1}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$. Let $\mathrm{I} \subseteq \mathrm{R}_{\mathscr{D}_1}$ be the ideal generated by $\{p; \sum_{i=1}^{s} \tilde{\alpha}_{i,j} b_{\sigma_i}; \det\rho_{\mathscr{D}_1}(\gamma_\ell) \mid j = 1, \dots, s-2; \ell = 1, \dots, \delta_{\mathrm{F}}\}$. It follows from (G), (4.2), and Proposition 2.4 that any minimal prime of $\mathrm{R}_{\mathscr{D}_1}/\mathrm{I}$ has dimension at least

$$\begin{aligned} \dim \mathrm{R}_{\mathscr{D}_1} & -(\dim_k \mathrm{H}_{\Sigma_c}(\mathrm{F}, k) - 2) - \delta_{\mathrm{F}} - 1 \\ & \geqslant d + 7 - 3 \cdot \#\mathscr{M}_1 - 4 \cdot \dim_k \mathrm{H}_{\Sigma_c}(\mathrm{F}, k) - 2t \\ & \geqslant d + 7 - 7 \cdot (\#\Sigma_c + \dim_k \mathrm{H}_{\Sigma_0}(\mathrm{F}, k)) \\ & > \delta_{\mathrm{F}} + \dim_k \mathrm{H}_{\Sigma_c}(\mathrm{F}, k). \end{aligned} \tag{4.3}$$

Comparing this estimate with that in Lemma 2.6 shows that any minimal prime of $\mathrm{R}_{\mathscr{D}_1}/\mathrm{I}$ corresponds to an irreducible (pro-modular) deformation.

Now, there exists a reducible deformation $\rho : \mathrm{Gal}(\mathrm{F}_{\Sigma_c}/\mathrm{F}) \longrightarrow \mathrm{GL}_2(k'[[x]])$ of ρ_{c_0} given as follows. Let $\tilde{c}$ and $\tilde{c}_0$ be cocycle representatives of c and c_0 such that $\tilde{c}(z_1) = 0 = \tilde{c}_0(z_1)$. Define ρ by

$$\rho(\sigma) = \begin{pmatrix} 1 & \chi(\sigma) \cdot (\tilde{c}_0(\sigma) + \tilde{c}(\sigma)\mathrm{X}) \\ & \chi(\sigma) \end{pmatrix}.$$

Clearly, ρ is a deformation of type-$\mathscr{D}_1$, so ρ corresponds to a dimension one prime $\mathfrak{p}$ of $\mathrm{R}_{\mathscr{D}_1}/\mathrm{I}$. Let Q be a minimal prime of $\mathrm{R}_{\mathscr{D}_1}/\mathrm{I}$ contained in $\mathfrak{p}$. As we observed in the preceding paragraph, $\rho_{\mathscr{D}_1} \bmod Q$ is irreducible. Let Q^{tr} be the inverse image of Q under $r_{\mathscr{D}_1} : \mathrm{R}_{\mathscr{D}_1^{\mathrm{ps}}} \longrightarrow \mathrm{R}_{\mathscr{D}_1}$. Let $\mathrm{A}^{\mathrm{tr}} = \mathrm{R}_{\mathscr{D}_1^{\mathrm{ps}}}/\mathrm{Q}^{\mathrm{tr}}$ and let A be the integral closure of A^{tr} in its field of fractions L. The ring A is a Krull domain [N, (33.10)]. Let K be the field of fractions of $\mathrm{R}_{\mathscr{D}_1}/\mathrm{Q}$.

Choose $\beta_1, \dots, \beta_s \in k'$ such that $\sum_{i=1}^{s} \beta_i c(\sigma_i) = 0$ but $\sum_{i=1}^{s} \beta_i c_0(\sigma_i) \neq 0$. Fix a lift $\tilde{\beta}_i$ of each β_i to $\mathcal{O}'$. Choose a basis for $\rho_{\mathscr{D}_1}$ such that $\rho_{\mathscr{D}_1}(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $\sum \tilde{\beta}_i b_{\sigma_i} = u_0 \in \mathcal{O}'^{\times}$, where $\rho_{\mathscr{D}_1}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$. Put $x(\sigma, \tau) = b_\sigma c_\tau$. Suppose that P is a height one prime of A for which $\sum \tilde{\beta}_i x(\sigma_i, \sigma_{\mathrm{P}}) \notin \mathrm{P}$ for some $\sigma_{\mathrm{P}} \in \mathrm{Gal}(\mathrm{F}_{\Sigma_c}/\mathrm{F})$. Then, since $\sum \tilde{\beta}_i x(\sigma_i, \sigma_{\mathrm{P}}) = u_0 c_{\sigma_{\mathrm{P}}}$, $c_{\sigma_{\mathrm{P}}} \notin \mathrm{P}$. It follows that $b_\sigma \in \mathrm{A}_{\mathrm{P}}$ for all $\sigma \in \mathrm{Gal}(\mathrm{F}_{\Sigma_c}/\mathrm{F})$. In particular, the matrix entries of each $\rho_{\mathscr{D}_1}(\sigma) \bmod \mathrm{Q}$ are in A_{P}. Thus, if such a σ_{P}

existed for each height one prime P of A, then $\rho_{\mathscr{D}_1} \bmod Q$ would have matrix entries in $\bigcap_P A_P = A$. It would then follow from Lemma 2.6 that any dimension one prime of $R_{\mathscr{D}_1}/Q$ pulls back to a prime of A, and hence to one of A^{tr}, of dimension at least one. However, this is impossible as the non-maximal prime $\mathfrak{p}$ of $R_{\mathscr{D}_1}/Q$ pulls back to the maximal ideal of A^{tr}. Therefore, there must exist a height one prime P_0 of A^{tr} for which $\{\sum_{i=1}^{s} \beta_i x(\sigma_i, \tau) : \tau \in \mathrm{Gal}(F_{\Sigma_c}/F)\} \subseteq P_0$.

Suppose that $x(\sigma, \tau) \in P_0$ for all σ and τ. It would follow that the representation ρ_{P_0} defined by

$$\rho_{P_0}(\sigma) = \begin{pmatrix} a_\sigma & \\ & d_\sigma \end{pmatrix} \in GL_2(A^{tr}/P_0)$$

would be a diagonal deformation of $\rho_0 = \begin{pmatrix} 1 & \\ & \chi \end{pmatrix}$ of type-$(\mathscr{O}', \Sigma_c)$ having determinant equal to χ. As A^{tr}/P_0 is (pro-finitely) generated by the traces of ρ_{P_0} it follows that the natural map $R^{diag}_{(\mathscr{O}', \Sigma_c)} \longrightarrow A^{tr}/P_0$ would be a surjection, whence by Lemma 2.9 (ii) the dimension of A^{tr}/P_0 would be at most $1 + \delta_F$. However

$$\begin{aligned} \dim A^{tr}/P_0 = \dim A^{tr} - 1 &\geqslant \dim R_{\mathscr{D}_1}/Q - 1 \\ &\geqslant \delta_F + \dim_{k'} H_{\Sigma_c}(F, k') \\ &> \delta_F + 1, \end{aligned}$$

the first inequality coming from Proposition 2.12, the second from (4.3), and the last from the fact that c and c_0 span a two-dimensional k'-subspace of $H_{\Sigma_c}(F, k')$ by hypothesis. This contradiction implies that there is some σ' and τ' for which $x(\sigma', \tau') \notin P_0$.

We next claim that after possibly renumbering $Y_1^{(i)}, \ldots, Y_{d_i}^{(i)}$ we can assume that $Y_1^{(i)} \notin P_0$. For this we recall that since Q is a prime of $R_{\mathscr{D}_1}$ it is pro-modular, so $A^{tr} = \mathbf{T}_{\mathscr{D}_1}/Q^{mod}$ for some prime $Q^{mod} \subseteq \mathbf{T}_{\mathscr{D}_1}$ such that $\rho^{mod}_{\mathscr{D}} \bmod Q^{mod}$ is the pseudo-representation associated to $\rho_{\mathscr{D}_1} \bmod Q$. Recall also that $\mathbf{T}_{\mathscr{D}_1}$ is an integral extension of $\Lambda_{\mathscr{O}}$. By the choice of Q, Q^{mod} contains $(T_1, \ldots, T_{\delta_F}, p)$. Hence so does P_0. If P_0 also contained $(Y_1^{(i)}, \ldots, Y_{d_i}^{(i)})$ then the dimension of P_0 would be at most $d - d_i - 1$. Hence the dimension of Q^{mod} (and hence of A^{tr}) would be at most $d - d_i$. However, as the dimension of Q is at least $d + 7 - 3 \cdot \#\mathscr{M}_1 - 4 \cdot \dim_k H_{\Sigma_c}(F) - 2t$, it would then follow from Proposition 2.12 that $d_i \leqslant 7 \cdot (\#\Sigma_c + \dim_k H_{\Sigma_0}(F)) + 2t$, contradicting (G). This proves the claim.

Now let $\mathfrak{p}_1$ be a dimension one prime of A^{tr} containing P_0 but not containing $Y_1^{(1)}, \ldots, Y_1^{(t)}$, or $x(\sigma', \tau')$. Let B be the integral closure of $A^{tr}/\mathfrak{p}_1$ in its field of fractions L. Let k' be the residue field of B. By Corollary 2.14 there is a representation ρ_1 :

$\mathrm{Gal}(F_{\Sigma_c}/F) \longrightarrow GL_2(B)$ whose associated pseudo-deformation comes from $\rho_{\mathscr{D}_1^{ps}} \bmod \mathfrak{p}_1$ and for which $\rho_1 \bmod \mathfrak{m}_B = \rho_{c_1}$ for some cocycle $0 \neq c_1 \in H^1(F_{\Sigma_c}/F, k''(\chi^{-1}))$. Recall that since Q is a pro-modular prime of $R_{\mathscr{D}_1}$ there is a map $\mathbf{T}_{\mathscr{D}_1} \twoheadrightarrow A^{tr}$ inducing the pseudo-deformation $\rho_{\mathscr{D}_1^{ps}} \bmod Q^{tr}$. Thus $\mathfrak{p}_1$ corresponds to a prime of $\mathbf{T}_{\mathscr{D}_1}$, which we also denote by $\mathfrak{p}_1$. It is clear that $\rho_1 \otimes \overline{L} \simeq \rho_{\mathfrak{p}_1}$. Arguing as in the paragraph describing primes of $\mathbf{T}_{\mathscr{D}}$ that are nice for $\mathscr{D}$ shows that c_1 is admissible and that ρ_1 is a deformation of type-$(\mathscr{O}'', \Sigma_c, c_1, \emptyset)$, where $\mathscr{O}'' = \mathscr{O} \otimes_{W(k)} W(k'')$.

We next claim that c_1 and c differ by a scalar, or, in other words, $\rho_c \simeq \rho_{c_1}$. Recall that $\{\sum \tilde{\beta}_i x(\sigma_i, \tau), \sum \tilde{\alpha}_{i,j} x(\sigma_i, \tau) \mid \tau \in \mathrm{Gal}(\overline{F}/F); j = 1, \dots, s-2\}$ is contained in $\mathfrak{p}_1$. Suppose that $\sum \alpha_{i,j} c_1(\sigma_i) \neq 0$ for some j. Fix a basis for ρ_1 for which $\rho_1(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Write $\rho_1(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$. From our supposition it follows that $b_j = \sum \alpha_{i,j} b_{\sigma_i}$ is a unit in B. But we also have $(\sum \alpha_{i,j} b_{\sigma_i}) c_\sigma = \sum \alpha_{i,j} x(\sigma_i, \sigma) = 0$ in B. It follows that $c_\sigma = 0$ for all σ and hence that $x(\sigma, \tau) = 0$ in A for all σ and τ, contradicting the assumption that $x(\sigma', \tau') \notin \mathfrak{p}_1$. A similar argument shows that $\sum \beta_i c_1(\sigma_i) = 0$. It follows that c_1 restricted to $\mathrm{Gal}(F_{\Sigma_c}/F(\chi))$ is a scalar multiple of c. This proves the claim since restriction determines an isomorphism $H^1(F_{\Sigma_c}/F, k(\chi^{-1})) \simeq \mathrm{Hom}(\mathrm{Gal}(F_{\Sigma_c}/F(\chi)), k(\chi^{-1}))^{\mathrm{Gal}(F_{\Sigma_c}/F)}$.

Therefore, after possibly replacing ρ_1 by a conjugate, we may assume that $c_1 = c$ and that ρ_1 takes values in $GL_2(B')$ with B' a $\mathscr{O}$-subalgebra with residue field k and hence that ρ_1 is a deformation of type-$(\mathscr{O}, \Sigma_c, c, \emptyset)$. This completes step two.

We now prove that ρ_1 is a nice deformation of type-$\mathscr{D}_c$. The only thing needing proof is that ρ_1 is actually of type-$\mathscr{D}_c$, for the desired properties of $\rho_1|_{D_i}$ follow from the isomorphism $\rho_1 \otimes \overline{L} \simeq \rho_{\mathfrak{p}_1}$, and that the corresponding prime of $R_{\mathscr{D}_c}$ will have dimension one will follow from Lemma 2.12. As $\mathscr{M}_c$ consists only of those finite places other than $v_1, \dots, v_t$ at which ρ_c is ramified one finds that each $w \in \mathscr{M}_c$ satisfies exactly one of the following possibilities:

$$\textbf{(4.4)} \quad \begin{array}{ll} \text{(i)} & \chi|_{I_w} \neq 1 \\ \text{(ii)} & \chi|_{I_w} = 1, \quad \chi|_{D_w} = \omega^{-1} = 1. \end{array}$$

If $w \in \mathscr{M}_c$ satisfies (i) then it is easily seen that $\rho_1|_{I_w} \cong \begin{pmatrix} 1 & \\ & \tilde{\chi} \end{pmatrix}$ (here we have used that char $(B') = p$). If $w \in \mathscr{M}_c$ satisfies (ii) then $\rho_c|_{D_w}$ is type A. We want to show that the same is true for ρ_1. Since $\det \rho_1 = \tilde{\chi}$ there are two possibilities for $\rho_1 \otimes \overline{L}|_{D_w}$: it is either type A or type B$'$. If it were type B$'$ then $\rho_1 \otimes \overline{L}|_{I_w} \simeq \begin{pmatrix} \phi & \\ & \phi^{-1} \end{pmatrix}$ with ϕ a finite character of p-power order. However, since char$(\overline{L}) = p$ any such ϕ must in fact be trivial, from which it would follow that ρ_1 is unramified at w, contradicting the assumption that ρ_c (and hence ρ_1) is ramified at w. Thus it must be that $\rho_1 \otimes \overline{L}|_{D_\omega}$ is type A. It is now straightforward to show that since $\rho_c|_{D_w} = \begin{pmatrix} 1 & * \\ & \chi \end{pmatrix}$ is non-split,

$\rho_1|_{D_w} \simeq \begin{pmatrix} 1 & * \\ & \tilde{\chi} \end{pmatrix}$ as well. For let $u \in I_w$ be a generator of the pro-p-part of tame inertia. It follows that $\rho_c(u) = \begin{pmatrix} 1 & b_0 \\ & 1 \end{pmatrix}$ for some $0 \neq b_0 \in k$. Let V be the underlying representation space for ρ_1 (so V is a free B′-module of rank 2). Let $U \subseteq V$ be the B′-submodule annihilated by $u - 1$. That $U \neq 0$ follows easily from $\rho_1 \otimes \overline{L}$ being type A at w. That $U \neq V$ follows from the fact that $\rho_c(u) \neq 1$. It follows that $U \simeq B'$ and $V/U \simeq B'$. Let $e_1, e_2 \in V$ be such that e_1 generates U as a B-module and e_2 generates V/U as a B′-module. Thus e_1 and e_2 form a basis of V as a B′-module. With respect to this basis u acts via a matrix of the form $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$. Since the reduction $\bar{e}_1$ and $\bar{e}_2$, of e_1 and e_2 mod $\mathfrak{m}_B$ form a basis for $\overline{V} = V \bmod \mathfrak{m}_{B'}$, the k-space underlying ρ_c, and since u does not act trivially on $\overline{V}$, b is a unit in B′. After possibily scaling e_1 and e_2 by units in B′ we may assume that b reduces to b_0. In this case e_1 and e_2 form a basis for the deformation ρ_1 with respect to which $\rho_1(u) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ $(b \neq 0)$. It now follows from the well-known action of D_w on tame inertia that $\rho_1|_{D_w} = \begin{pmatrix} 1 & * \\ & \tilde{\chi} \end{pmatrix}$. (Here we have used that $\mathrm{char}(B') = p$ and that $\det \rho_1 = \tilde{\chi}$). This completes the proof that ρ_1 is of type-$\mathscr{D}_c$.

Let $\mathfrak{p}_2 \subseteq R_{\mathscr{D}_c}$ be the prime corresponding to ρ_1. We have shown that $\rho_{\mathscr{D}_c} \bmod \mathfrak{p}_2$ is nice. We now show that $\mathfrak{p}_2$ is pro-modular. Put $\mathscr{D}_2 = (\mathscr{O}, \Sigma_c, c, \mathscr{M}_1)$. Clearly $\mathbf{T}_{\mathscr{D}_2} \otimes_{\mathscr{O}} \mathscr{O}' = \mathbf{T}_{\mathscr{D}_1}$. From the choice of $\mathfrak{p}_1$ and the definitions of $\mathfrak{p}_2$ and ρ_1 we have a commutative diagram

$$\textbf{(4.5)} \qquad \begin{array}{ccccccc} R^{ps}_{\mathscr{D}_2} & \twoheadrightarrow & \mathbf{T}_{\mathscr{D}_2} \longrightarrow \mathbf{T}_{\mathscr{D}_1} & \twoheadrightarrow & A^{tr} & \twoheadrightarrow & A^{tr}/\mathfrak{p}_1 \\ \downarrow r_{\mathscr{D}_2} & & & & & & \downarrow \\ R_{\mathscr{D}_2} & \twoheadrightarrow & R_{\mathscr{D}_c} & \longrightarrow & B' & \hookrightarrow & B \end{array}$$

where the map $\mathbf{T}_{\mathscr{D}_1} \twoheadrightarrow A^{tr}$ induces the pseudo-deformation associated to $\rho_{\mathscr{D}_1} \bmod Q$ and $R_{\mathscr{D}_c} \longrightarrow B'$ corresponds to ρ_1. Denote by $\mathfrak{p}_3$ the kernel of the map $\mathbf{T}_{\mathscr{D}_2} \twoheadrightarrow A^{tr}/\mathfrak{p}_1$ in (4.5). We need to show that $\mathfrak{p}_3$ is the inverse image of a prime of $\mathbf{T}_{\mathscr{D}_c}$ under the canonical surjection $\mathbf{T}_{\mathscr{D}_2} \twoheadrightarrow \mathbf{T}_{\mathscr{D}_c}$.

Let $Q_1 \subseteq \mathfrak{p}_3$ be a minimal prime of $\mathbf{T}_{\mathscr{D}_2}$. We describe the possibilities for $\rho_{Q_1}|_{D_w}$ when $w \in \mathscr{M}_c$. First, note that ρ_{Q_1} is ramified at every place in $\mathscr{M}_c$ since $\rho_{\mathfrak{p}_3} \simeq \rho_{\mathfrak{p}_1}$ is. Second, recall that by (G) every place $w \in \mathscr{M}_c$ satisfies one of the two possibilities listed in (4.4). If w satisfies (4.4i) then $w \in \mathscr{M}_1$ (by the definition of $\mathscr{M}_1$), and it is easy to see that $\rho_{Q_1}|_{I_w} \simeq \begin{pmatrix} \phi_1 & \\ & \tilde{\chi}\phi_2 \end{pmatrix}$ with ϕ_1 and ϕ_2 finite characters of p-power order. As $w \in \mathscr{M}_1$, $U_{\mathscr{D}_2, w} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, c \in \ell_w^{\max(1, r(\ell_w))},\ a \bmod \ell_w^{\max(1, r(\ell_w))} \in \Delta_{\ell_w} \right\}$. (See §3.5, especially for notation.) It follows from Lemma 3.16 that ϕ_1 and ϕ_2 are in fact trivial. This shows that

$$\textbf{(4.6a)} \qquad \text{if} \quad \chi|_{I_w} \neq 1, \quad \text{then} \quad \rho_{Q_1}|_{I_w} \simeq \begin{pmatrix} 1 & \\ & \tilde{\chi} \end{pmatrix}.$$

Finally, suppose that $w \in \mathcal{M}_c$ satisfies (4.4ii). In particular $\omega(\mathrm{Frob}_w) = 1$. A straightforward analysis of the possibilities for $\rho_{Q_1}|_{D_w}$ using that $\rho_{Q_1}|_{I_w}$ factors through the pro-p-part of tame inertia at w shows that there exists a finite character ϕ of D_w of p-power order such that $\rho_{Q_1}|_{D_w} \otimes \phi$ is either type A, type B, or type C. As type C can only occur if $\omega(\mathrm{Frob}_w) = -1$, this case is impossible. It follows that if $\chi|_{D_w} = \omega^{-1} = 1$, then either

(4.6*b*)
(1) $\rho_{Q_1}|_{I_w} \simeq \begin{pmatrix} \phi_1 & \\ & \phi_2 \end{pmatrix}$, ϕ_1 and ϕ_2 finite characters of p-power order, or
(2) $\rho_{Q_1}|_{I_w} \simeq \begin{pmatrix} \phi & * \\ & \phi \end{pmatrix}$, ϕ a finite character of p-power order.

Now write $\det \rho_{Q_1} = \tilde{\chi} \cdot \phi \cdot \psi$ where ϕ is finite of p-power order and ψ has infinite order and factors through a free $\mathbf{Z}_p$-extension of F (hence ψ is ramified only at places in $\mathcal{P}$). It follows from (4.6 *a, b*) that ϕ is ramified only at places in $\mathcal{M}_c \backslash \mathcal{M}_1$ and in $\mathcal{P}$. Fix a character $\phi_1 : \mathrm{Gal}(F_{\Sigma_c}/F) \longrightarrow (\mathbf{T}_{\mathcal{D}_2}/Q_1)^\times$ ramified only at places in $\Sigma_c \backslash \mathcal{M}_1$ and such that $\phi_1^2 = \phi^{-1}$. By Lemma 3.17 there are primes $Q_2 \subseteq \mathfrak{p}_4 \subseteq \mathbf{T}_{\mathcal{D}_2}$ with Q_2 minimal such that $\rho_{Q_2} \simeq \rho_{Q_1} \otimes \phi_1$ and $\rho_{\mathfrak{p}_4} \simeq \rho_{\mathfrak{p}_3} \otimes \phi_1$. As $\rho_{\mathfrak{p}_3} \otimes \phi_1 = \rho_{\mathfrak{p}_3}$, it follows that $\mathfrak{p}_4 = \mathfrak{p}_3$. Thus Q_2 is contained in $\mathfrak{p}_3$. It follows from (4.6 *a, b*) and the definition of ϕ_1 that

(4.7) (i) if $\chi|_{I_w} \neq 1$, then $\rho_{Q_2}|_{I_w} \simeq \begin{pmatrix} 1 & \\ & \tilde{\chi} \end{pmatrix}$,
(ii) if $\chi|_{D_w} = \omega^{-1} = 1$, then either
a) $\rho_{Q_2}|_{D_w} \simeq \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$, or
b) $\rho_{Q_2}|_{D_w} \simeq \begin{pmatrix} \phi & \\ & \phi^{-1} \end{pmatrix}$ with ϕ a finite character of p-power order.

Next we introduce some new subgroups of $\mathrm{GL}_2(\mathcal{O}_F \otimes \hat{\mathbf{Z}})$. Write $U_{\mathcal{D}_c} = \prod_w U_{\mathcal{D}_c, w}$. Let $\mathcal{W}$ be the set of places $w \in \mathcal{M}_c \backslash \mathcal{M}_1$ for which (4.7ii*b*) holds. For $w \in \mathcal{W}$ define

$$U_w = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{F,w}) \;\middle|\; c \in \ell_w,\; ad^{-1} \bmod \ell_w \in \Delta'_w \right\}$$

(see §3.2 for the definition of Δ'_w). Put

$$U' = \prod_{w \notin \mathcal{W}} U_{\mathcal{D}_c, w} \times \prod_{w \in \mathcal{W}} U_w \quad \text{and} \quad U'' = U_{\mathcal{D}_2} \cap U'. \tag{4.8}$$

Let $\mathfrak{m}'$ and $\mathfrak{m}''$ be the permissible maximal ideals of $\mathbf{T}_\infty(U', \mathcal{O})$ and $\mathbf{T}_\infty(U'', \mathcal{O})$, respectively, obtained by pulling back the permissible maximal ideal of $\mathbf{T}_\infty(U_{\mathcal{D}_c}, \mathcal{O})$

via the canonical projections. There is a commutative diagram

$$\begin{array}{ccccc} & & \mathbf{T}' & & \\ & & \| & & \\ & \nearrow & \mathbf{T}_\infty(\mathrm{U}', \mathcal{O})_{\mathfrak{m}'} & \searrow & \\ \mathbf{T}'' = \mathbf{T}_\infty(\mathrm{U}'', \mathcal{O})_{\mathfrak{m}''} & & & & \mathbf{T}_{\mathcal{D}_c} \\ & \searrow & \mathbf{T}_{\mathcal{D}_2} & \nearrow & \end{array} \tag{4.9}$$

where all the maps are the canonical ones and are surjective. Let $Q_2'' \subseteq \mathfrak{p}_3'' \subseteq \mathbf{T}''$ be the inverse images of Q_2 and $\mathfrak{p}_3$. It follows from (4.7), (4.8), and Lemma 3.15 that there are primes $Q_2' \subseteq \mathfrak{p}_3' \subseteq \mathbf{T}'$ whose inverse images in $\mathbf{T}''$ are just Q_2'' and $\mathfrak{p}_3''$, respectively. It now follows from Proposition 3.20 that $\mathfrak{p}_3'$ is the inverse image of a prime $\mathfrak{p}_c$ of $\mathbf{T}_{\mathcal{D}_c}$. By the commutativity of (4.9) and the fact that $\mathfrak{p}_3''$ is the inverse image of both $\mathfrak{p}_3$ and $\mathfrak{p}_3'$ it follows that $\mathfrak{p}_3$ is the inverse image of $\mathfrak{p}_c$. This proves that the map $\mathbf{T}_{\mathcal{D}_2} \longrightarrow \mathrm{A}^{\mathrm{tr}}/\mathfrak{p}_1$ in (4.5) factors through the canonical surjection $\mathbf{T}_{\mathcal{D}_2} \twoheadrightarrow \mathbf{T}_{\mathcal{D}_c}$, completing the proof that the deformation ρ_1 is nice and pro-modular of type-$\mathcal{D}_c$. This completes the third and final step in the proof of the proposition. □

4.5. The Main Theorem

We now state and prove our Main Theorem. In this subsection and the next, we forego the convention that F has even degree. We will, however, assume property (P1) for certain fields. That this property holds is proven in §5-8 (see Proposition 8.4) which are independent of §4.

Main Theorem. — Suppose that F *is a totally real field and that* $\mathcal{D} = (\mathcal{O}, \Sigma, c, \mathcal{M})$ *is a deformation datum for* F. *Suppose also that* $\rho : \mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F}) \longrightarrow \mathrm{GL}_2(\mathcal{O})$ *is a deformation of type-*$\mathcal{D}$ *such that*

- ρ *is irreducible*
- $\det\rho = \psi\varepsilon^\mu$ *with* $\mu \geqslant 1$ *an integer and* ψ *a finite character*
- $\rho|_{\mathrm{D}_i} \simeq \begin{pmatrix} \widetilde{\chi}\psi_1^{(i)} & * \\ & \psi_2^{(i)} \end{pmatrix}$ *with* $\psi_2^{(i)}|_{\mathrm{I}_i}$ *of finite order for each* $i = 1, \ldots, t$.

If there exists an extension L/F of totally real fields such that

(i) *the Galois closure of* L *over* F *is solvable*
(ii) L *has even degree over* $\mathbf{Q}$
(iii) L *is permissible for* $\mathcal{D}$
(iv) $(\mathrm{L}, \mathcal{D}_{\mathrm{L}})$ *is a good pair in the sense of* §4.2,

then $\rho \otimes \overline{\mathbf{Q}}_p$ *is a representation associated to a Hilbert modular newform.*

Proof. — Let $\rho_1 = \rho|_{\mathrm{Gal}(\overline{\mathrm{L}}/\mathrm{L})}$. As L is permissible for $\mathscr{D}$, ρ_1 is a deformation of type-$\mathscr{D}_{\mathrm{L}}$. Since L has even degree over $\mathbf{Q}$, it follows from Proposition 8.4 that (P1) holds for any deformation datum for L. As the pair $(\mathrm{L}, \mathscr{D}_{\mathrm{L}})$ is good it then follows from Propositions 4.1 and 4.2 that ρ_1 is a pro-modular deformation. In particular, there is a map $\lambda : \mathbf{T}_{\mathscr{D}_{\mathrm{L}}} \longrightarrow \mathscr{O}$ inducing the pseudo-representation associated to ρ_1. By the conditions imposed on ρ in the statement of the theorem, the map λ satisfies the hypotheses of Proposition 3.7. Thus there is a Hilbert modular newform f_1 (over L) such that

(4.10) $$\mathrm{trace}\,\rho_1(\mathrm{Frob}_\ell) = (\text{eigenvalue of } \mathrm{T}(\ell) \text{ acting on } f_1)$$

for $\ell \notin \Sigma_{\mathrm{L}}$. Let $\rho_{f_1} : \mathrm{Gal}(\overline{\mathrm{L}}/\mathrm{L}) \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be the representation associated to f_1. (If π is the automorphic representation associated to f_1, then ρ_{f_1} is just ρ_π, the latter being the representation described in (3.2)). This representation satisfies

(4.11) $$\mathrm{trace}\,\rho_{f_1}(\mathrm{Frob}_\ell) = (\text{eigenvalue of } \mathrm{T}(\ell) \text{ acting on } f_1)$$

for $\ell \notin \Sigma_{\mathrm{L}}$. As ρ is irreducible by assumption and odd, ρ_1 (and hence $\rho_1 \otimes \overline{\mathbf{Q}}_p$) is also irreducible. It therefore follows from (4.10) and (4.11) that $\rho_1 \otimes \overline{\mathbf{Q}}_p \simeq \rho_{f_1}$. Now, as the Galois closure of L/F is solvable, it follows from the known cases of base change for (holomorphic) Hilbert modular forms (cf. [GL]) that there is a newform f over F such that $\rho_f|_{\mathrm{Gal}(\overline{\mathrm{L}}/\mathrm{L})} \simeq \rho_{f_1}$ (here $\rho_f : \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F}) \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is the representation associated to f). Since $\rho_f|_{\mathrm{Gal}(\overline{\mathrm{L}}/\mathrm{L})} \simeq \rho \otimes \overline{\mathbf{Q}}_p|_{\mathrm{Gal}(\overline{\mathrm{L}}/\mathrm{L})}$ and these are irreducible, it is easy to see that there is some finite character $\phi : \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F}) \longrightarrow \overline{\mathbf{Q}}_p^\times$ such that $\rho_f \otimes \phi \cong \rho \otimes \overline{\mathbf{Q}}_p$. As $\rho_f \otimes \phi$ is the representation associated to the newform corresponding to the twist of f by ϕ, this proves the theorem. □

In the next subsection we will deduce the following theorems from this one.

Theorem A. — *Let* F *be a totally real abelian extension of* $\mathbf{Q}$. *Suppose that* p *is an odd prime and that* $\rho : \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F}) \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ *is a continuous, irreducible representation unramified away from a finite number of places of* F. *Suppose also that the reduction of* ρ *satisfies* $\bar{\rho}^{ss} \cong \chi_1 \oplus \chi_2$. *If*

- *the splitting field* $\mathrm{F}(\chi_1/\chi_2)$ *of* χ_1/χ_2 *is abelian over* $\mathbf{Q}$,
- $(\chi_1/\chi_2)(z) = -1$ *for each complex conjugation* z,
- $(\chi_1/\chi_2)|_{\mathrm{D}_v} \neq 1$ *for each* $v|p$,
- $\rho|_{D_v} \simeq \begin{pmatrix} \psi_1^{(v)}\tilde{\chi}_1 & * \\ & \psi_2^{(v)}\tilde{\chi}_2 \end{pmatrix}$ *with* $\psi_2^{(v)}$ *factoring through a pro-p-group of* F_v *and* $\psi_2^{(v)}|_{\mathrm{I}_v}$ *of finite order for each* $v|p$,
- $\det\rho = \psi\varepsilon^{k-1}$ *with* $k \geqslant 2$ *an integer and* ψ *a character of finite order,*

then ρ *is a representation associated to a Hilbert modular newform.*

A critical ingredient in the proof of Theorem A is a result of Washington on the boundedness of the p-part of the class group of a cyclotomic $\mathbf{Z}_\ell$-extension of an abelian number field (cf. [Wa]). A similar result for any totally real field would yield the same theorem but with the omission of the hypotheses that F and $F(\chi_1/\chi_2)$ be abelian.

For our next theorem, we make the following hypothesis, which plays a role similar to that of Washington's theorem in the proof of Theorem A. We believe that this hypothesis will be easier to establish than the analog of Washington's theorem, though the latter would yield a stronger result.

Hypothesis H. — *There exists* $0 < \varepsilon < \frac{1}{4}$ *and a constant* $c(\varepsilon) > 0$ *such that given a totally real field* K *and a finite set* S *of finite places of* K *there is an imaginary quadratic extension* L *of* K *having prescribed behavior at each place in* S *and such that the relative class group of* L/K *has p-rank at most* $c(\varepsilon)\deg(K/\mathbf{Q})^{1-\varepsilon}$.

Theorem B. — *Let* F *be a totally real extension of* $\mathbf{Q}$. *Assume Hypothesis* H *for all solvable, totally real extensions of* F. *Suppose that p is an odd prime and that* $\rho : \mathrm{Gal}(\overline{F}/F) \longrightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ *is a continuous, irreducible representation unramified away from a finite number of places of* F. *Suppose also that the reduction of* ρ *satisfies* $\overline{\rho}^{ss} \cong \chi_1 \oplus \chi_2$. *If*

- $(\chi_1/\chi_2)(z) = -1$ *for each complex conjugation z,*
- $(\chi_1/\chi_2)|_{D_v}$ *has even order for each* $v|p$,
- $\rho|_{D_v} \simeq \begin{pmatrix} \psi_1^{(v)}\tilde{\chi}_1 & * \\ & \psi_2^{(v)}\tilde{\chi}_2 \end{pmatrix}$ *with* $\psi_2^{(v)}$ *factoring through a pro-p-extension of* F_v *and* $\psi_2^{(v)}|_{I_v}$ *of finite order for each* $v|p$,
- $\det\rho = \psi\varepsilon^{k-1}$ *with* $k \geqslant 2$ *an integer and* ψ *a character of finite order,*

then ρ *is a representation associated to a Hilbert modular newform.*

4.6. Proofs of Theorems A and B

We now prove Theorems A and B. In both cases this is done by reducing to a situation to which the Main Theorem applies.

Proof of Theorem A. — Put $\rho_1 = \rho \otimes \tilde{\chi}_2^{-1}$ and $\chi = \chi_1/\chi_2$. Let Σ be the set of finite places at which ρ_1 is ramified together with the places over p. There exists a finite extension K of $\mathbf{Q}_p$ such that for some choice of basis ρ_1 takes values in $\mathrm{GL}_2(\mathcal{O})$ with $\mathcal{O}$ the ring of integers of K. Such a basis can be chosen so that the reduction of ρ_1 modulo the maximal ideal (λ) of $\mathcal{O}$, $\overline{\rho}_1 = \rho_1 \bmod \lambda$, satisfies $\overline{\rho}_1 = \begin{pmatrix} 1 & * \\ & \chi \end{pmatrix}$ and is non-split. Let k be the residue field of $\mathcal{O}$. It follows that $\overline{\rho}_1 \simeq \rho_c$ for some cocycle $0 \neq c \in H^1(F_\Sigma/F, k(\chi^{-1}))$. The hypotheses on $\rho|_{D_v}$ ensure that c is an admissible cocycle. Thus after possibly replacing ρ_1 by a conjugate we may assume that ρ_1 is a deformation of type-$\mathscr{D}$, where $\mathscr{D} = (\mathscr{O}, \Sigma, c, \emptyset)$. Clearly ρ_1 satisfies the hypotheses of the Main Theorem. The conclusion of Theorem A will thus follow from that of the Main Theorem if we exhibit an extension L/F of totally real fields that (i) has solvable Galois closure over F, (ii) has even degree over $\mathbf{Q}$, (iii) is permissible for $\mathscr{D}$, and (iv) is such that $(\mathrm{L}, \mathscr{D}_{\mathrm{L}})$ is a good pair. We will construct such an L.

Let E/F be any even extension that is permissible for $\mathscr{D}$ and is such that $\mathrm{E}/\mathbf{Q}$ is abelian, each place $v|p$ of F splits in E, and if w is a place of F at which ρ_c is ramified and $\chi|_{D_w}$ is unramified then $\chi|_{\mathrm{D}_{w'}} = 1$ for each place $w'|w$ of E. It is easy to find such fields: take for example, $\mathrm{E} = \mathrm{F}\cdot\mathrm{E}'$ where E′ is a real cyclic extension of $\mathbf{Q}$ of sufficiently divisible degree in which p splits completely and all primes $q \neq p$ divisible by a place in Σ are inert. Choose an odd rational prime ℓ such that $\ell \nmid \#k^\times$ and ℓ is not divisible by any of the places in Σ. For each positive integer n let E_n be the cyclotomic $\mathbf{Z}/\ell^n$-extension of E. It is easily checked that E_n is permissible for $\mathscr{D}$. Let Σ_n be the set of places of E_n dividing those in Σ, and let $\mathscr{P}_n$ be the set of places of E_n dividing p. Let r_n denote the p-rank of the χ^{-1}-isotypical piece of the p-part of the class group of $\mathrm{E}_n(\chi)$ and let p^{c_n} denote the order of the p-part of the class group of E_n. From the theory of cyclotomic extensions we know that there exist integers s and t such that

$$\textbf{(4.12)} \qquad \#\Sigma_n = s \quad \text{and} \quad \#\mathscr{P}_n = t \quad \text{for} \quad n \gg 0.$$

Similarly, it follows from [Wa] that there exist r and c such that

$$\textbf{(4.13)} \qquad c_n = c \quad \text{and} \quad r_n = r \quad \text{for} \quad n \gg 0.$$

As E_n/F is a Galois extension, we also have that

$$\textbf{(4.14)} \qquad \deg \mathrm{E}_{n,v}/\mathbf{Q}_p \geqslant \ell^n/t \quad \forall v|p.$$

Let p^{e_n} be the number of p-th power roots of unity in $\mathrm{E}_n(\zeta_p)$, ζ_p a primitive p-th root of unity. As the degree of E_n/E is a power of ℓ, there is an integer e such that

$$\textbf{(4.15)} \qquad e_n = e \quad \forall n.$$

For each $n \geqslant 1$ choose a set S_n of $e_n + c_n + 1$ finite places of E_n disjoint from Σ_n and such that

(4.16)
- $p^{e_n+c_n+1} \mid (\mathrm{Nm}(w) - 1) \quad \forall w \in \mathrm{S}_n$
- $\chi(\mathrm{Frob}_w) = 1 \quad \forall\ w \in \mathrm{S}_n$
- there exists an abelian p-extension $\mathrm{L}_n/\mathrm{E}_n$ of degree at most $p^{e_n+2c_n+1}$, unramified away from S_n, and such that the subgroup of $\mathrm{Gal}(\mathrm{L}_n/\mathrm{E}_n)$ generated by $\{\mathrm{I}_w : w \in \mathrm{S}_n\}$ is isomorphic to $(\mathbf{Z}/p)^{e_n+c_n+1}$.

Note that L_n is necessarily ramified at each place in S_n. The existence of such a set S_n follows easily from Class Field Theory.

Let $E_n \subseteq H_n \subseteq L_n$ be the maximal unramified subextension of L_n. It follows that

$$L(L_n, -1, \chi\omega^{-1}) = \prod_{\phi} L(H_n, -1, \chi\omega^{-1}\phi)$$

where ϕ runs over the characters of $\mathrm{Gal}(L_n/H_n) \simeq (\mathbf{Z}/p)^{e_n+c_n+1}$. Let p^{ℓ_n} be the number of p-th power roots of unity in $H_n(\zeta_p)$. Note that $\ell_n \leqslant e_n + c_n$. It follows from well-known congruences for p-adic L-functions that if $\chi\omega^{-1} \neq 1$ then $L(H_n, -1, \chi\omega^{-1}\phi) \in \mathbf{Z}_p[\chi\omega^{-1}\phi]$ for all ϕ and if $\chi\omega^{-1} = 1$ then $L(H_n, -1, \chi\omega^{-1}\phi) \in \mathbf{Z}_p[\phi]$ for ϕ non-trivial and $p^{\ell_n} L(H_n, -1, \chi\omega^{-1}) \in \mathbf{Z}_p$ (cf. [Co], [D-R], or [Se]). Here, for any character θ, $\mathbf{Z}_p[\theta]$ denotes the ring obtained by adjoining the values of θ to $\mathbf{Z}_p$. We also have by our choice of S_n that if ϕ is non-trivial and if π_ϕ is a uniformizer of $\mathbf{Z}_p[\chi\omega^{-1}\phi]$ then

$$\begin{aligned} L(H_n, -1, \chi\omega^{-1}\phi) &\equiv L(H_n, -1, \chi\omega^{-1}) \prod_{w\in S_n(\phi)} (1 - \chi\omega^{-1}\phi(\mathrm{Frob}_w)\mathrm{Nm}(w)) \\ &\equiv 0 \bmod \pi_\phi, \end{aligned}$$

where $S_n(\phi)$ is the set of places of H_n at which ϕ is ramified. Combining this with the earlier expression for $L(L_n, -1, \chi\omega^{-1})$ we obtain that $L(L_n, -1, \chi\omega^{-1}) \in \mathbf{Z}_p[\chi\omega^{-1}]$ and $L(L_n, -1, \chi\omega^{-1}) \equiv 0 \bmod \lambda$, λ a uniformizer of $\mathbf{Z}_p[\chi\omega^{-1}]$. We have thus shown that

$$\textbf{(4.17)} \qquad L_p(L_n, -1, \chi\omega) \in \mathscr{O} \backslash \mathscr{O}^\times.$$

Since E_n is permissible for $\mathscr{D}$, the field L_n is as well. Moreover, it is a simple exercise in p-groups to show that

$$\textbf{(4.18)} \qquad \begin{aligned} \dim_k H_{\Sigma_0}(L_n, k) &\leqslant \#\mathrm{Gal}(L_n/E_n) \cdot \dim_k H_{\Sigma_0}(L_n, k)^{\mathrm{Gal}(L_n/E_n)} \\ &\leqslant \#\mathrm{Gal}(L_n/E_n) \cdot \dim_k H_{\Sigma_0 \cup S_n}(E_n, k) \\ &\leqslant p^{e_n+2c_n+1}(r_n + e_n + c_n + 1). \end{aligned}$$

Now choose n_0 so large that

$$\textbf{(4.19)} \qquad \begin{aligned} &\ell^{n_0}/2 > 2 + 8(p^{e+2c+1}(s + r + e + c + 1)) \\ &\ell^{n_0} > t(2 + p^{e+2c+1}(t + 7(s + r + e + c + 1))) \end{aligned}$$

and such that the equalities in (4.12) and (4.13) hold.

Let $L = L_{n_0}$. By construction the Galois closure of L/F is solvable and the degree of L is even. As noted above, L is permissible for $\mathscr{D}$. It remains to verify that $(L, \mathscr{D}_L)$ is a good pair. For this let d_L be the degree of L. By construction $d_L \geqslant \ell^{n_0}$. Also, by

[Wal], $\delta_L \leqslant \frac{d_L}{2}$. It follows that

$$(4.20)\qquad \begin{aligned} d_L &\geqslant \delta_L + \ell^{n_0}/2 > 2 + 8(p^{e+2c+1}(s+e+1c+1+r)) + \delta_L \\ &\geqslant 2 + \delta_L + 8(\#\Sigma_L + \dim_k H_{\Sigma_0}(L, k)), \end{aligned}$$

the last inequality following from (4.12), (4.13), (4.15), and (4.18). Suppose that v is a place of L dividing p. Let d_v be the degree of L_v. The number of such places v is at most $p^{e+2c+1}t$, so it follows from (4.14) and (4.19) that

$$\begin{aligned} d_v &\geqslant \ell^{n_0}/t > 2 + p^{e+2c+1}t + 7\cdot(p^{e+2c+1}(s+r+e+1c+1)) \\ &\geqslant 2 + p^{e+2c+1}t + 7(\#\Sigma_L + \dim_k H_{\Sigma_0}(L, k)). \end{aligned}$$

That $(L, \mathscr{D}_L)$ is good now follows from this, from (4.17) and (4.20), and from the choice of E. □

Proof of Theorem B. — Let $\chi = \chi_1/\chi_2$. It follows from base change that it suffices to prove the theorem with F replaced by the maximal totally real subfield F^+ of $F(\chi)$. By the hypotheses in the theorem $\chi|_{D_v} \neq 1$ for each place $v|p$ of F^+, and therefore we may assume that χ is quadratic.

Put $\rho_1 = \rho \otimes \tilde{\chi}_2^{-1}$. Let Σ be the set of finite places of F at which ρ_1 is ramified together with those dividing p. As in the proof of Theorem A, there exists a finite extension K of $\mathbf{Q}_p$ with integer ring $\mathscr{O}$ such that, for a suitable choice of basis, ρ_1 takes values in $GL_2(\mathscr{O})$ and is a deformation of type-$\mathscr{D}$ for some $\mathscr{D} = (\mathscr{O}, \Sigma, c, \emptyset)$. The conclusion of the theorem will follow from the Main Theorem if we can find an extension L/F that (i) has solvable Galois closure over F, (ii) has even degree over $\mathbf{Q}$, (iii) is permissible for $\mathscr{D}$, and (vi) is such that $(L, \mathscr{D}_L)$ is a good pair.

Arguing as in the proof of Theorem A shows that we can find a solvable permissible extension E/F that has even degree over $\mathbf{Q}$ and is such that

(4.21)
- $L_p(E, -1, \chi\omega) \in \mathscr{O}$ and is not a unit,
- if $\rho_c|_{I_w} \neq 1$ and $w \nmid p$, then either $\chi|_{I_w} \neq 1$ or $\chi|_{D_w} = 1$, for w a place of E.

Let Σ' be the primes of E above those in Σ.

We now construct a solvable permissible extension L of E such that $(L, \mathscr{D}_L)$ is good. By Hypothesis H there is a totally imaginary quadratic character ψ over E such that

- if $w \in \Sigma'$ and $w \nmid p$ then ψ is unramified at w and $\chi\psi|_{D_w} \neq 1$,
- if $v|p$ and $\chi|_{I_v} \neq 1$ then ψ is unramified at v and $\psi(\mathrm{Frob}_v) \neq 1$,
- if $v|p$ and $\chi|_{I_v} = 1$ then ψ is ramified at v, and
- the p-rank of the relative class group of $E(\psi)/E$ is at most $c(\varepsilon)2^{1-\varepsilon}\deg(E/\mathbf{Q})$.

Let L_1 be the splitting field of the character $\chi\psi$ over E. This is a totally real quadratic extension of E and clearly permissible for $\mathscr{D}$. Let Σ_1 be the set of places of L_1 above those in Σ'. It follows from the choice of ψ that $\#\Sigma_1 = \#\Sigma'$. It is also relatively easy to see that

$$\dim_k H_{\Sigma_1}(L_1, k) \leqslant \dim_k H_{\Sigma_0}(E, k) + c(\varepsilon)2^{1-\varepsilon} \deg(E/\mathbf{Q}) + \#\Sigma'.$$

Proceeding inductively, one constructs in the same manner for each $n > 1$ a totally real quadratic extension L_n of L_{n-1} such that

(4.22)
- L_n is permissible,
- $\#\Sigma_n = \#\Sigma'$, where Σ_n is the set of places of L_n over those in Σ',
- $\dim_k H_{\Sigma_n}(L_n) < \dim_k H_{\Sigma'}(E, k) + c(\varepsilon) \sum_{i=1}^{n} 2^{i(1-\varepsilon)} \deg(E/\mathbf{Q}) + \#\Sigma'$.

Now choose n_0 so large that

$$2^{n_0-1} > 2 + 17 \cdot \#\Sigma' + 8 \cdot \dim_k H_{\Sigma_0}(E, k) + c(\varepsilon)2^{(n_0+1)(1-\varepsilon)+6} \deg(E/\mathbf{Q}). \tag{4.23}$$

Put $L = L_{n_0}$ and $\Sigma_L = \Sigma_{n_0}$. Let d_L be the degree of L over $\mathbf{Q}$. By [Wal] $\delta_L \leqslant \frac{d_L}{2}$, so by (4.22) and (4.23) we have that

$$\begin{aligned} d_L &\geqslant \delta_L + 2^{n_0-1} \deg(E/\mathbf{Q}) \\ &> \delta_L + 2 + 9 \cdot \#\Sigma_L + 8 \cdot \dim_k H_{\Sigma_L}(L, k). \end{aligned} \tag{4.24}$$

If v is any place of L dividing p, then $d_v = \deg(L_v/\mathbf{Q}_p)$ is at least 2^{n_0}. It follows from (4.22) and (4.23) that

$$d_v > 2 + 9 \cdot \#\Sigma_L + 8 \cdot \dim_k H_{\Sigma_L}(L, k). \tag{4.25}$$

Since L has even degree by construction, combining (4.25) with (4.24) and (4.21) shows that $(L, \mathscr{D}_L)$ is good. □

5. A formal patching argument

In the next four sections we give the proof of property (P1) (see Proposition 8.4). These sections do not make use of any results from §4.

In this section we will describe a formal patching argument which is a variant on the patching argument in [TW] and its refinement in [D2]. The extra complexity in our case is caused by the fact that we are considering the localizations of deformation rings and Hecke rings and not the original rings themselves. In particular the residue fields are not finite. We will, in section §7, apply our patching argument to localizations of deformation rings (in contrast to [TW] where it is applied to Hecke rings), but in this section we will just axiomatize what is assumed (and later proved) about these rings and consider only the formal aspects of the argument.

Let k be a finite field of characteristic p and let $A = k[[T]]$. Let K be the field of fractions of A. Let $\mathscr{L} = \{N\}$ be a sequence of strictly increasing odd integers together with zero. Let n be a fixed positive integer.

We introduce rings A_N, B_N (for each $N \in \mathscr{L}$) given by

$$A_N = A[[s_1, \dots, s_n]] / (s_1^{N+1}, \dots, s_n^{N+1}), \quad A_0 = A$$
$$B_N = A[[t_1, \dots, t_n]] / (t_1^{(N+1)/2}, \dots, t_n^{(N+1)/2}), \quad B_0 = A.$$

There is a homomorphism $B_N \longrightarrow A_N$ given by $t_i \longmapsto (1+s_i) + (1+s_i)^{-1} - 2$ which we use to identify B_N as a subring of A_N. We assume that we are given a ring $R^{(N)}$ for each $N \in \mathscr{L}$ of the form

(5.1) $$R^{(N)} = A[[x_1, \dots, x_m]]/\mathfrak{a}^{(N)}$$

with m independent of N. Furthermore we assume that $R^{(N)}$ has the following properties:

(5.2) (i) $R^{(N)}$ is finite and free as an A-module,
(ii) $\mathfrak{a}^{(N)} \subseteq (x_1, \dots, x_m)$,
(iii) $\exists$ a surjective map $R^{(N)} \twoheadrightarrow R^{(0)}$ of A-algebras,
(iv) $R^{(N)}$ is a B_N-algebra for $N > 0$.

Now letting $\mathfrak{p}^{(N)}$ be the prime of $R^{(N)}$ corresponding to $(x_1, \dots, x_m)$ (which we usually abbreviate to $\mathfrak{p}$ if the N is clear from the context) we assume two further (and less formal) properties of $R^{(N)}$:

(5.3) (i) $\exists\, d(0) > 0$ such that $\mathfrak{p}^{d(0)} = 0$ in $R^{(0)}$,
(ii) $\mathfrak{p}^{(N)}/(\mathfrak{p}^{(N)})^2 \cong A^n \oplus \mathrm{Tor}_{(N)}$,

where the free summand A^n is spanned by $x_1, \dots, x_n$ and $\mathrm{Tor}_{(N)}$ is a finite group whose order is bounded independent of N.

For each odd $0 \leqslant a \leqslant N$ together with zero we assume given a ring $R_a^{(N)}$ which has the following properties:

(5.4) (i) $R_a^{(N)}$ is finite and free as an A-module,
(ii) $R_N^{(N)} = R^{(N)}$, $R_0^{(N)} = R^{(0)}$,
(iii) there are surjective maps of B_N-algebras

$$R_0^{(N)} \twoheadleftarrow R_1^{(N)} \twoheadleftarrow R_3^{(N)} \twoheadleftarrow R_5^{(N)} \dots,$$

(iv) $R_a^{(N)}$ is a B_a-algebra (compatible with $B_N \twoheadrightarrow B_a$) such that if $a > 1$, then

$$(R_a^{(N)} \otimes_A K)/(t_1^{\frac{a-1}{2}}, \dots, t_n^{\frac{a-1}{2}}) \simeq R_{a-2}^{(N)} \otimes_A K,$$

(v) $R_a^{(N)} \otimes_A K$ is an $A_a \otimes_A K$-algebra satisfying (via the map in (iii))

$$R_a^{(N)} \otimes_A K/(s_1, \dots, s_n) \twoheadrightarrow R_0^{(N)} \otimes_A K.$$

Letting $\mathfrak{p}_a^{(N)}$ denote the prime corresponding to $(x_1, \ldots, x_m)$ (which we again write as $\mathfrak{p}$ if a and N are clear from the context) we assume two further properties:

(5.5) (i) $\exists\, d(a) > 0$ independent of N such that $\mathfrak{p}^{d(a)} = 0$ in $R_a^{(N)}$,

(ii) $\mathfrak{p}_a^{(N)}/(\mathfrak{p}_a^{(N)})^2 \cong A^n \oplus \mathrm{Tor}_{(N,\, a)}$,

where the free summand A^n is spanned by $x_1, \ldots, x_n$ and $\mathrm{Tor}_{(N,\, a)}$ is a finite group whose order is bounded independent of N and a.

Associated to the rings $R_a^{(N)}$ are certain subrings $R_a^{\mathrm{tr}(N)}$ (of "traces" in the application) which are assumed to satisfy the following conditions. First we assume given an A-subalgebra $R^{\mathrm{tr}(N)} \subseteq R^{(N)}$ satisfying

(5.6) (i) $R^{\mathrm{tr}(N)} = A\,[[y_1, \ldots, y_m]]/\mathfrak{b}^{(N)}$ (same m as in (5.1)) where $\mathfrak{b}^{(N)} \subseteq (y_1, \ldots, y_m)$,

(ii) $R^{\mathrm{tr}(N)}$ is a B_N-algebra for $N > 0$ compatible with the algebra structure on $R^{(N)}$.

Setting $\mathfrak{q}^{(N)} = \mathfrak{p}^{(N)} \cap R^{\mathrm{tr}(N)}$ (thus the prime corresponding to $(y_1, \ldots, y_m)$) we assume in addition the property

(5.7) $\quad \mathrm{coker} : \mathfrak{q}^{(N)}/(\mathfrak{q}^{(N)})^2 \longrightarrow \mathfrak{p}^{(N)}/(\mathfrak{p}^{(N)})^2$ has order bounded independent of N.

For $0 \leqslant a \leqslant N$, a odd or zero, we set

$$R_a^{\mathrm{tr}(N)} = \mathrm{im}\,\{R^{\mathrm{tr}(N)} \longrightarrow R_a^{(N)}\}.$$

Observe that $R_a^{\mathrm{tr}(N)}$ inherits a B_a-algebra structure from $R_a^{(N)}$. Then we deduce from (5.7) that

(5.8) $\quad \mathrm{coker}\{\mathfrak{q}_a^{(N)}/(\mathfrak{q}_a^{(N)})^2 \longrightarrow \mathfrak{p}_a^{(N)}/(\mathfrak{p}_a^{(N)})^2\}$ has order bounded independent of N and a, where $\mathfrak{q}_a^{(N)} = \mathfrak{p}^{(N)} \cap R_a^{\mathrm{tr}(N)}$.

Associated to the rings we have described we will assume given a set of modules as follows. First we assume given an integer r, independent of N. Then we assume we are given $M^{(N)}$, a finite $R^{\mathrm{tr}(N)}$-module, satisfying the hypotheses that

(5.9) (i) $M^{(N)}$ is a free A-module of rank equal to the rank of A_N^r,

(ii) $M^{(N)}$ is an A_N-module compatible with the B_N-structure via $R^{\mathrm{tr}(N)}$,

(iii) there is a map $M^{(N)} \longrightarrow M^{(0)}$ of $R^{\mathrm{tr}(N)}$-modules.

For $0 \leqslant a \leqslant N$ (a odd or $a = 0$) we assume that we are given a $R^{\mathrm{tr}(N)}$-module quotient of $M^{(N)}$ denoted $M_a^{(N)}$ satisfying

(5.10) (i) $M_a^{(N)}$ is a free A-module of rank equal to the rank of A_a^r,

(ii) $M_N^{(N)} = M^{(N)}$, $M_0^{(N)} \subseteq M^{(0)}$ and there exists some $z \in R^{\mathrm{tr}(0)}$ independent of N, $\mathrm{ord}_T(z \bmod \mathfrak{q}^{(0)}) \neq 0$, such that $z \cdot M^{(0)} \subseteq M_0^{(N)}$,

(iii) there are surjective maps of $R^{\mathrm{tr}(N)}$-modules $M_0^{(N)} \twoheadleftarrow M_1^{(N)} \twoheadleftarrow M_3^{(N)} \ldots$,

(iv) $M_a^{(N)}$ is an $R_a^{tr(N)}$-module (compatible with the $R^{tr(N)}$-structure),

(v) $M_a^{(N)}$ is an A_a-module (compatible with the B_a-structure induced in (iv)) in such a way that the maps in (iii) are compatible with $A_0 \twoheadleftarrow A_1 \twoheadleftarrow A_3 \ldots$, and the actions of $R_a^{tr(N)}$ and A_a commute on $M_a^{(N)}$,

(vi) $M_a^{(N)} \otimes_A K$ is a free $A_a \otimes_A K$-module and $M_a^{(N)} \otimes_A K/(s_1, \ldots, s_n) \simeq M_0^{(0)} \otimes_A K$.

Furthermore, we assume there exists $x^{(N)} \in R^{tr(N)}$ such that

(5.11) (i) $x^{(N)}$ annihilates $\ker\{M_a^{(N)}/(s_1, \ldots, s_n) \twoheadrightarrow M_0^{(N)}\}$,

(ii) $\mathrm{ord}_T(x^{(N)} \bmod \mathfrak{q}^{(N)}) = t < \infty$ with t independent of N.

We now derive some simple properties of the above rings and modules.

Lemma **5.1.** — $\mathrm{rank}_A R_a^{(N)} \leqslant \ell(a)$ *where* $\ell(a)$ *depends only on* a.

Proof. — This follows immediately from (5.5i). □

Lemma **5.2.** — *There exists an* $E(a)$ *independent of* N *such that*

$$T^{E(a)} R_a^{(N)} \subseteq R_a^{tr(N)}.$$

In particular $R_a^{tr(N)} \otimes_A K = R_a^{(N)} \otimes_A K$.

Proof. — Since $\mathfrak{p}^{d(a)} = 0$ in $R_a^{(N)}$ by (5.5) it follows that it is enough to check that $\mathrm{coker}\Phi_r$, where Φ_r is the natural map

$$\Phi_r : (\mathfrak{q}_a^{(N)})^r/(\mathfrak{q}_a^{(N)^2})^{r+1} \longrightarrow (\mathfrak{p}_a^{(N)})^r/(\mathfrak{p}_a^{(N)})^{r+1},$$

is finite and bounded independent of N for $r < d(a)$. For $r = 1$ this is given by (5.8). A similar bound follows for $r = 2$ by picking generators for $\mathrm{im}(\Phi_1)$, lifting them to elements, say $z_1, \ldots, z_s$, in $(\mathfrak{q}_a^{(N)})/(\mathfrak{q}_a^{(N)})^2$ and considering the map

$$(\mathfrak{q}_a^{(N)}/(\mathfrak{q}_a^{(N)})^2)^s \longrightarrow (\mathfrak{q}_a^{(N)})^2/((\mathfrak{q}_a^{(N)})^2 \cap (\mathfrak{p}_a^{(N)})^3), \quad (a_1, \ldots, a_s) \longmapsto \sum a_i z_i$$

which is surjective. The property for $r = 2$ can now be deduced from the property for $r = 1$, and we proceed by induction up to $r = d(a) - 1$. □

From this lemma we deduce immediately that for any c the kernel and cokernel of

$$\textbf{(5.12)} \qquad R_a^{tr(N)}/T^c \longrightarrow R_a^{(N)}/T^c$$

is annihilated by $T^{E(a)}$.

Now let t, $x^{(N)}$ be as in (5.11) and let z be as in (5.10ii). Let $d_1 = \mathrm{ord}_T(z \bmod \mathfrak{q}^{(0)})$.

Lemma **5.3.** — $T^{d(a)(t+d_1)}$ *annihilates both the kernel and the cokernel of the map* $M_a^{(N)}/(T^c, s_1, \ldots, s_r) \longrightarrow M^{(0)}/T^c$ *for any* c.

Proof. — Let $\tilde{z}$ be a lift of z to $R^{tr(N)}$. By (5.11) and the definition of d_1, $x^{(N)} \cdot \tilde{z} - uT^{t+d_1} \in \mathfrak{q}^{(N)}$ for some unit $u \in A^\times$. So

$$T^{t+d_1} = u^{-1}x^{(N)}\tilde{z} + v, \quad v \in \mathfrak{q}^{(N)}.$$

Hence $T^{d(a)(t+d_1)} = w \cdot x^{(N)}\tilde{z} + v^{d(a)}$ for some $w \in R^{tr(N)}$. By (5.5) $v^{d(a)} = 0$, so the result follows from the defining properties of $x^{(N)}$ and z. □

Next we introduce level structures which we will use to make a patching argument similar to the one in [TW]. A level-(a, c) structure $J(N, a, c)$ is a collection of data comprising

(i) B_a-algebras, $R_{a,c}^{tr(N)} = R_a^{tr(N)}/T^c$, $R_{a,c}^{(N)} = R_a^{(N)}/T^c$,

(ii) an A_a-module $M_{a,c}^{(N)} = M_a^{(N)}/T^c$ that is also an $R_{a,c}^{tr(N)}$-module,

(iii) a map of B_a-algebras

$$R_{a,c}^{tr(N)} \longrightarrow R_{a,c}^{(N)},$$

(iv) a map of $R_{a,c}^{tr(N)}$-modules

$$M_{a,c}^{(N)}/(s_1, \ldots, s_n) \longrightarrow M^{(0)}/T^c$$

compatible with the actions of A_a and A via $A_a \twoheadrightarrow A$,

(v) elements $\{x_1, \ldots, x_m\}$ of $R_{a,c}^{(N)}$ such that $R_{a,c}^{(N)}/(x_1, \ldots, x_m) \simeq A$,

(vi) elements $\{y_1, \ldots, y_m\}$ of $R_{a,c}^{tr(N)}$ such that $R_{a,c}^{tr(N)}/(y_1, \ldots, y_m) \simeq A$.

Let $\mathscr{L}_0 = \mathscr{L} = \{N\}$ as at the beginning of the section. Let $\mathscr{L}_1(0) = \mathscr{L}_0$ and define $\mathscr{L}_1(c) \subset \mathscr{L}_1(c-1)$ inductively as follows. We require that $\mathscr{L}_1(c)$ should be an infinite strictly increasing subsequence of integers from $\mathscr{L}_1(c-1)$ with the property that $J(N', 1, c) = J(N'', 1, c)$ for $N', N'' \in \mathscr{L}_1(c)$. The equality here signifies that the $J(N, a, c)$-structures for $N = N', N''$ can be identified (non-canonically). Since the total number of such non identifiable structures for fixed a and c is finite a choice of $\mathscr{L}_1(c)$ can be made. Then define

$$\mathscr{L}_1 = \{N_i : N_i \in \mathscr{L}_1(i)\}$$

again with the N_i strictly increasing. Finally we can define $\mathscr{L}_a$ for an odd $a > 13$ inductively by $\mathscr{L}_a(0) = \mathscr{L}_{a-2}$ and defining $\mathscr{L}_a(c)$ inductively in the same manner as $\mathscr{L}_1(c)$. We set

$$R_a^{tr} = \varprojlim_{N_c \in \mathscr{L}_a} R_{a,c}^{tr(N)}, \quad R_a = \varprojlim_{N_c \in \mathscr{L}_a} R_{a,c}^{(N_c)}, \quad M_a = \varprojlim_{N_c \in \mathscr{L}_a} M_{a,c}^{(N_c)}.$$

Lemma **5.4.**

a) $R_a^{tr} \otimes_A K = R_a \otimes_A K$, $R_0 \otimes_A K = R^{(0)} \otimes_A K$.

b) $R_a \otimes_A K$ *is a quotient of* $K[[x_1, \ldots, x_n]]$.

c) $M_a \otimes_A K$ *is a free* $A_a \otimes_A K$*-module of rank* r *and* $(M_a \otimes_A K) / (s_1, \ldots, s_n) = M^{(0)} \otimes_A K$.

Proof

a) By Lemma 5.2 for any c we have natural maps of A-modules

$$R_{a,c}^{tr(N_c)} \longrightarrow R_{a,c}^{(N_c)} \longrightarrow R_{a,c}^{tr(N_c)}$$

whose composite is multiplication by $T^{E(a)}$ with $E(a)$ independent of c. Taking projective limits and tensoring with K gives the isomorphism.

b) By construction there are elements $\{x_1, \ldots, x_m\}$ of R_a such that $R_a/(x_1, \ldots, x_m) \simeq A$. Let $\mathfrak{p}_a = (x_1, \ldots, x_m) \subseteq R_a$. Letting $\mathfrak{p}_{a,c}^{(N_c)}$ denote the ideal generated by $\{x_1, \ldots, x_m\}$ in $R_{a,c}^{(N_c)}$ we see easily that

$$\varprojlim \mathfrak{p}_{a,c}^{(N_c)} = \mathfrak{p}_a, \qquad \varprojlim \mathfrak{p}_{a,c}^{(N_c)}/(\mathfrak{p}_{a,c}^{(N_c)})^2 \simeq \mathfrak{p}_a/\mathfrak{p}_a^2.$$

Then by (5.5) we deduce that $\mathfrak{p}_a/\mathfrak{p}_a^2 \simeq A^n \oplus T$, where T is a finite group and $x_1, \ldots, x_n$ span the free summand. Hence

$$(\mathfrak{p}_a \otimes_A K)/(\mathfrak{p}_a \otimes_A K)^2 \simeq K^n$$

by (5.5) which ensures that $R_a, \mathfrak{p}_a, \mathfrak{p}_a^2$ are all finite A-modules. Part *b*) follows by Nakayama's lemma.

c) By Lemma 5.3,

$$(M_a \otimes_A K)/(s_1, \ldots, s_n) \simeq M^{(0)} \otimes_A K. \tag{5.13}$$

By construction,

$$\dim_K(M_a \otimes_A K) \geqslant \mathrm{rank}_A(M_a^{(N_c)})$$

for large enough c and $N_c \in \mathscr{L}_a$. By (5.10i) the right-hand side has rank equal to $\mathrm{rank}_A(A_a^r)$. But r is the K-rank of the right-hand side of (5.13). The result follows. □

Lemma **5.5.** — *For odd* $a > 1$, *there are surjections*

a) $R_a \twoheadrightarrow R_{a-2}$ *of* $A[[x_1, \ldots, x_m]]$*-algebras and* B_a*-algebras.*

b) $R_a^{tr} \twoheadrightarrow R_{a-2}^{tr}$ *of* $A[[y_1, \ldots, y_m]]$*-algebras and* B_a*-algebras.*

c) $M_a \twoheadrightarrow M_{a-2}$ *of* R_a^{tr}*-algebras compatible with b) and of* A_a*-algebras. Here the* B_a *action via* R_a^{tr} *and* A_a *are the same, and the* R_a^{tr} *action commutes with the* A_a *action.*

The same holds for $a = 1$ *with* $a - 2$ *replaced by* 0.

Proof. — We have that $\mathscr{L}_{a-2} = \{N_i\}$ and $\mathscr{L}_a = \{N_{j_i}\} \subseteq \mathscr{L}_{a-2}$. Note that $j_i \geqslant i$. By our choice of $\mathscr{L}_a$ and $\mathscr{L}_{a-2}$,

$$R_{a,i}^{(N_{j_i})} \twoheadrightarrow R_{a-2,i}^{(N_{j_i})} = R_{a-2,i}^{(N_i)}$$

whence taking limits yields $R_a \twoheadrightarrow R_{a-2}$. The same works also for R_a^{tr} and M_a. □

Now we set, taking limits over odd integers a,

$$R'_a = R_a \otimes_A K = R_a^{tr} \otimes_A K, \qquad M'_a = M_a \otimes_A K,$$

$$R_\infty = \varprojlim_a R'_a, \qquad M_\infty = \varprojlim_a M'_a.$$

Thus M'_a is an R'_a-module and M_∞ is an R_∞-module.

Lemma **5.6.**
(i) $R_\infty = K\,[[x_1, \dots, x_n]]$.
(ii) M_∞ *is a free* R_∞*-module.*

Proof. — By Lemma 5.4 *c)* there is a map $K[[s_1, \dots, s_n]]^r \twoheadrightarrow M_\infty$ which is seen to be an isomorphism. Consequently M_∞ is also a free $K[[t_1, \dots, t_n]]$-module. Thus $K[[t_1, \dots, t_n]] \hookrightarrow R_\infty/\mathrm{Ann}_{R_\infty}(M_\infty)$. On the other hand there is a surjection $K[[x_1, \dots, x_n]] \twoheadrightarrow R_\infty$. By Krull's dimension theorem we deduce part (i). (Note that $R_\infty/\mathrm{Ann}_{R_\infty}(M_\infty)$ is a finite $K[[t_1, \dots, t_n]]$-module.) Then part (ii) follows from the Auslander-Buchsbaum Theorem ([Mat, Theorem 19.1]) since $\mathrm{depth}_{R_\infty}(M_\infty) \geqslant n$ ($\{t_1, \dots, t_n\}$ is a regular R_∞-sequence for M_∞). □

Proposition **5.7.** — *If* $N \gg 1$ *then*

$$\dim_K(R_1^{(N)} \otimes_A K) \geqslant 2^n \dim_K(R_0^{(N)} \otimes_A K).$$

Proof. — Fix a projection $K[[x_1, \dots, x_n]] \twoheadrightarrow R_N^{(N)} \otimes_A K$. Choose maps

$$K[[t'_1, \dots, t'_n]] \longrightarrow K[[s'_1, \dots, s'_n]] \longrightarrow K[[x_1, \dots, x_n]]$$

such that $t'_i \longmapsto (1 + s'_i) + (1 + s'_i)^{-1} - 2$ and such that the images of $K[[s'_1, \dots, s'_n]]$ and $K[[t'_1, \dots, t'_n]]$ in $R_N^{(N)} \otimes_A K$ are just those of $A_N \otimes_A K$ and $B_N \otimes_A K$ respectively in the sense that the images of t_i' and t_i are the same, and so are the images of s_i' and s_i. It follows from (5.10 vi) that $B_N \otimes_A K \longrightarrow R_N^{(N)} \otimes_A K$ is injective, for by (5.10) the action of $B_N \otimes_A K$ on $M_N^{(N)} \otimes_A K$ factors through its image in $R_N^{(N)} \otimes_A K$. So if

$$\frac{N+1}{2} \geqslant n^{n-1}(\dim_K(R_N^{(N)} \otimes_A K/(t_1', \ldots, t_n'))^n = n^{n-1} \dim_K(R_1^{(n)} \otimes_A K)^n$$

then the hypotheses of Lemma 4.1 of [DRS] hold and using (5.4iv) we get that

$$\textbf{(5.14)} \qquad K[[x_1, \ldots, x_n]] \,/\, (t_1', \ldots, t_n') \simeq R_1^{(N)} \otimes_A K.$$

Applying the Auslander-Buchsbaum Theorem again we find that $K[[x_1, \ldots, x_n]]$ is a free $K[[t_1', \ldots, t_n']]$-module of some rank d. Applying the same theorem yet again we find that $K[[x_1, \ldots, x_n]]$ is a free $K[[s_1', \ldots, s_n']]$-module of rank $d/2^n$. It follows that

$$\begin{aligned} d &= 2^n \dim_K(K[[x_1, \ldots, x_n]] \,/\, (s_1', \ldots, s_n')) \\ &\geqslant 2^n \dim_K(R_1^{(N)} \otimes_A K/(s_1, \ldots, s_n)) \\ &\geqslant 2^n \dim_K(R_0^{(N)} \otimes_A K), \end{aligned}$$

the last inequality by (5.4v). Combined with (5.14) this proves the proposition. □

Proposition **5.8.** — $M^{(0)} \otimes_A K \simeq (R^{(0)} \otimes_A K)^e$ *where* $e = \mathrm{rk}_{R_\infty}(M_\infty)$.

Proof. — We set $\tilde{R} := R_\infty/(t_1, \ldots, t_n)$ and

$$\widetilde{M} := M_\infty/(t_1, \ldots, t_n) = M_\infty/(s_1^2, \ldots, s_n^2).$$

Thus $\widetilde{M}$ is a free $\tilde{R}$-module of rank e. Now $\tilde{R} \twoheadrightarrow R_1'$ since the t_i's are zero in R_1', so

$$\textbf{(5.15)} \qquad \dim_K \tilde{R} \geqslant \dim_K R_1' \geqslant \dim_K(R_1^{(N)} \otimes_A K),$$

for any sufficiently large $N \in \mathscr{L}_1$. (More precisely if T^i annihilates the A-torsion submodule of R_1' we can take $N \geqslant N_{i+1} \in \mathscr{L}(i+1)$.)

Now let $\mu_S(X)$ for any ring S and S-module X denote the minimum number of generators of X as an S-module. Let

$$e_1 = \mu_{R^{(0)} \otimes_A K}(M^{(0)} \otimes_A K).$$

Then we have the inequality

$$e := \mathrm{rk}_{R_\infty}(M_\infty) = \mu_{\tilde{R}}(\widetilde{M}) \geqslant \mu_{\tilde{R}}(\widetilde{M}/(s_1, \ldots, s_n)) \geqslant e_1.$$

Also we have the inequality

$$\textbf{(5.16)} \qquad 2^n e_1 \dim_K(R^{(0)} \otimes_A K) \geqslant 2^n \dim_K(M^{(0)} \otimes_A K).$$

Now as remarked at the beginning of the proof of Lemma 5.6, M_∞ is a free $K[[s_1, \ldots, s_n]]$-module of rank r, where $r = \dim_K(M^{(0)} \otimes_A K)$, whence

$$(5.17) \qquad 2^n \dim_K(M^{(0)} \otimes_A K) = \dim_K(\widetilde{M}) = e \dim_K(\widetilde{R}).$$

Combining the inequalities (5.15), (5.16), (5.17) with Proposition 5.7 gives $e_1 \geqslant e$. Since also $e \geqslant e_1$ we have equality and all the inequalities just cited are equalities. In particular

$$e_1 = e, \quad e \dim_K(R^{(0)} \otimes_A K) = \dim_K(M^{(0)} \otimes_A K).$$

It follows that $M^{(0)} \otimes_A K \simeq (R^{(0)} \otimes_A K)^e$ as claimed. □

Proposition **5.9.** — $R^{(0)} \otimes_A K$ *is a complete intersection as a* K-*algebra.*

Proof. — We recall what we have proved so far. By Lemma 5.6,

$$(5.18) \qquad M_\infty \simeq R_\infty^e \text{ and } R_\infty \text{ is a power series ring over K of dimension } n.$$

By construction we have elements $\{s_1, \ldots, s_n\}$ acting on M_∞, and

$$(5.19) \qquad s_1, \ldots, s_n \in \mathrm{End}_{R_\infty}(M_\infty).$$

By Lemma 5.4 *c)*, we have

$$(5.20) \qquad M_\infty / (s_1, \ldots, s_n) M_\infty \simeq M^{(0)} \otimes_A K.$$

The action of R_∞ on $M^{(0)} \otimes_A K$ is via $R^{(0)} \otimes_A K$ and

$$(5.21) \qquad M^{(0)} \otimes_A K \simeq (R_0^{(0)} \otimes_A K)^e$$

by Proposition 5.8.

Now let $\mathfrak{a} = \ker : R_\infty \longrightarrow R^{(0)} \otimes_A K$. Let

$$N = \Sigma s_i M_\infty \subseteq M_\infty.$$

Then

$$M_\infty / N \simeq M^{(0)} \otimes_A K \simeq R_\infty^e / \mathfrak{a}^e$$

by (5.20) and (5.21). Since $M_\infty \simeq R_\infty^e$ it follows that $N \simeq \mathfrak{a}^e$ as R_∞-modules. (Consider the map $\varphi_\infty : M_\infty \simeq R_\infty^e$ of (5.18). Then the above isomorphism easily implies that $\mathfrak{a}^e \simeq \varphi_\infty(N)$.)

Let $w_1, \ldots, w_e$ be an R_∞-basis of M_∞. Then N is generated as an R_∞-module by the set $\{s_i w_j : 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant e\}$. In particular a set of minimal generators has cardinality $\leqslant en$. Let $\{x_1, \ldots, x_t\}$ be a minimal set of generators of $\mathfrak{a}$. Then $\{x_i w_j : 1 \leqslant i \leqslant t, \ 1 \leqslant j \leqslant e\}$ is a minimal set of generators of $\mathfrak{a} M_\infty \simeq \mathfrak{a}^e \simeq N$. It follows that $et \leqslant en$, whence $t \leqslant n$. However as $R^{(0)} \otimes_A K$ has dimension zero it follows that $t = n$ and that $R^{(0)} \otimes_A K$ is a complete intersection. □

Remark **5.10.** — The circuitous route to this proposition via a counting argument is forced on us by the lack of a natural $K[[s_1, \ldots, s_n]]$-algebra structure on R_∞. Only the elements $\{t_1, \ldots, t_n\}$ are naturally defined in R_∞. The structure assumed in (5.4v) is an artifice which is not assumed to be related to the action of $A_a \otimes_A K$ on $M_a^{(N)} \otimes_A K$, except for the compatibility with the subring $B_a \otimes_A K$.

6. Estimates of cohomology groups

In this section we consider a representation

$$\textbf{(6.1)} \qquad \rho : G_\Sigma = \mathrm{Gal}(F_\Sigma/F) \longrightarrow \mathrm{GL}_2(A)$$

where $A \simeq k[[\lambda]]$. Here we are using the notation and assumptions of §2.1 so that, in particular, F is a totally real field. We let K be the field of fractions of A, and we recall that if ρ is ramified at $w \nmid p$ then we distinguish the following possibilities for $\rho|_{I_w}$:

$$\text{type A} \quad \rho \otimes \overline{K}|_{I_w} \simeq \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}, \quad * \neq 0$$
$$\text{type B} \quad \rho \otimes \overline{K}|_{I_w} \simeq \begin{pmatrix} 1 & \\ & \psi_q \end{pmatrix}, \quad \psi_q \text{ non-trivial of finite order.}$$

Throughout this section we make the following assumptions on ρ:

(6.2) (i) $\rho \otimes \overline{K}$ is irreducible and of type A, type B or unramified at each prime $w \nmid p$,
(ii) $\overline{\rho} := \rho \bmod \lambda = \rho_c$ for some c as in (2.1),
(iii) Σ contains the primes dividing p and all primes at which $\overline{\rho}$ is ramified,
(iv) ρ is of type A or type B precisely where $\overline{\rho}$ is,
(v) $\det \rho = \chi$, with χ as in §2.1,
(vi) $\rho|_{D_v} \simeq \begin{pmatrix} \chi_1 & * \\ & \chi_2 \end{pmatrix}$ with χ_1/χ_2 of infinite order for each $v | p$.

Lemma **6.1.** — *The* $\mathrm{Gal}(F_\Sigma/F)$*-module* $W = \mathrm{ad}^0\rho \otimes_A \overline{K}$ *is irreducible. In particular* $\rho \otimes \overline{K}$ *is not "dihedral" (i.e., is not induced from a character over a quadratic extension).*

Proof. — By condition (vi) we see that there is an element $\sigma \in D_v \subset G_\Sigma$ such that $(\chi_1/\chi_2)(\sigma)$ has infinite order. Here we may choose v dividing p such that $\chi|_{D_v}$ is non-trivial by condition (ii). It follows from the existence of σ and the self-duality $W \simeq \mathrm{Hom}_{\overline{K}}(W, \overline{K})$ that any invariant subspace of W has a complement. So if W is reducible then either $W \simeq Y_1 \oplus Y_2$ with Y_2 of dimension 2 and irreducible, or $W \simeq Y_1 \oplus Y_2 \oplus Y_3$. The self-duality also shows that in the former case Y_1 is acted on by G_Σ via a quadratic character, possibly trivial, and in the second case that there is also a unique subspace, Y_1 say, on which G_Σ acts via a quadratic character.

Now let z be a complex conjugation and pick a basis for ρ such that $\rho(z) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Suppose first that z acts trivially on Y_1. Then we may identify Y_1 with $\left\{\begin{pmatrix} -a & \\ & a \end{pmatrix}\right\} \subseteq M_2(\overline{K})$ and we see that $\mathrm{im}\rho \subseteq \left\{\begin{pmatrix} * & \\ & * \end{pmatrix}, \begin{pmatrix} & * \\ * & \end{pmatrix}\right\} \subseteq GL_2(\overline{K})$. In particular, either $\mathrm{im}\,\rho$ is abelian, which contradicts assumption (i), or $\mathrm{im}\,\rho$ has a subgroup H of index 2 for which the action is abelian. In the latter case, H acts via two characters ψ and ψ^τ ($\psi^\tau(\sigma) = \psi(\tau^{-1}\sigma\tau)$) for any $\tau \notin H$. Thus $\rho = \mathrm{Ind}_H^{G_\Sigma}\psi$.

Now suppose that z acts non-trivially on Y_1. This time $Y_1 \subseteq \left\{\begin{pmatrix} & a \\ b & \end{pmatrix}\right\} \subseteq M_2(\overline{K})$. An easy calculation shows that if $\sigma = 1$ on Y_1 then $\rho(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$ with $a_\sigma = d_\sigma$. Using that $H_1 = \{\sigma : \sigma = 1 \text{ on } Y_1\}$ is a group we check that $\rho(H_1)$ is abelian. Thus just as above, $\rho = \mathrm{Ind}_{H_1}^{G_\Sigma}\psi$ for some character ψ.

Now consider ρ restricted to D_v. Then the quadratic field associated to ρ (i.e. the fixed field of H or H_1 in the two cases) is not split at v as otherwise $\rho|_{D_v} = \psi \oplus \psi|_{D_v}$ and this contradicts assumption (vi). So letting H_v be $H \cap D_v$ or $H_1 \cap D_v$ in the two cases, we see that $\rho|_{D_v} = \mathrm{Ind}_{H_v}^{D_v}\psi_v$ where $\psi_v = \psi|_{D_v}$. Again this contradicts assumption (vi) since if the ratio of the two characters on H_v had infinite order then $\rho|_{D_v}$ would be irreducible. □

Now let F′ be the splitting field of $\det\rho$ adjoin all p-power roots of unity, and let F^+ be the subfield of F′ fixed by the complex conjugation z_1.

Lemma **6.2.** — *The restriction of* $\rho \otimes \overline{K}$ *to* $\mathrm{Gal}(F_\Sigma/F^+)$ *is neither reducible nor dihedral.*

Proof. — Let V be the representation space for $\rho \otimes \overline{K}$. Suppose first that $\rho \otimes \overline{K}$ restricted to $\mathrm{Gal}(F_\Sigma/F^+)$ has an invariant subspace V_0. Then since $\mathrm{Gal}(F_\Sigma/F^+)$ is normal in G_Σ we see that for any $\sigma \in G_\Sigma$, σV_0 is also invariant. As z_1 acts by ± 1 on V_0 and by the opposite sign on V/V_0 there are at most two invariant subspaces. So either V_0 is invariant by G_Σ or $\rho \otimes \overline{K}$ is dihedral, but in each case this contradicts Lemma 6.1.

Suppose next that $\rho \otimes \overline{K}$ restricted to $\mathrm{Gal}(F_\Sigma/F^+)$ is dihedral. Then there is a subgroup $H \subset \mathrm{Gal}(F_\Sigma/F^+)$ of index 2 which has two fixed spaces. From the form of ρ_c and the definition of F′ we see that the splitting field of $\rho \otimes \overline{K}$ generates an extension of F′ which is pro-p, whence $H = \mathrm{Gal}(F_\Sigma/F')$. So H acts on the two fixed spaces via a character ψ and its inverse ψ^{-1} (and the two spaces are unique if $\psi \not\equiv 1$ as ψ cannot be of order 2). So H acts on W via the characters $\{1, \psi^2, \psi^{-2}\}$. Either ψ is trivial, in which case G_Σ acts on W via the abelian group $\mathrm{Gal}(F'/F)$, or the subspace of W corresponding to the character 1 is invariant under G_Σ as $\mathrm{Gal}(F_\Sigma/F') \triangleleft G_\Sigma$. In either case we get a contradiction to Lemma 6.1. □

Lemma **6.3.**

(i) *There exists* $\sigma \in \mathrm{Gal}(F_\Sigma/F')$ *such that the eigenvalues of* $\rho(\sigma)$ *have infinite order and are in* A.

(ii) *There exists* $\sigma \in \mathrm{Gal}(F_\Sigma/F^+)\backslash \mathrm{Gal}(F_\Sigma/F')$ *such that the eigenvalues of* $\rho(\sigma)$ *have infinite order and are in* A.

Proof. — First we prove parts (i) and (ii) without requiring that the eigenvalues are in A.

(i) If $\sigma \in \mathrm{Gal}(F_\Sigma/F')$ has eigenvalues of finite order then the eigenvalues must be 1 as the image of $\mathrm{Gal}(F_\Sigma/F')$ is a pro-p group and $\overline{K}$ has characteristic p. Assume no σ as in the Lemma exists. Pick a $\tau \in \mathrm{Gal}(F_\Sigma/F')$ such that $\rho(\tau) \neq 1$, which can be done as ρ is not abelian. Pick a basis for $\rho \otimes \overline{K}$ such that $\rho(\tau) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \neq 0$. Then for any $\sigma \in \mathrm{Gal}(F_\Sigma/F')$ we have $\mathrm{trace}\,\rho(\sigma\tau) = 2$, so if $\rho(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$, then

$$a_\sigma + ac_\sigma + d_\sigma = 2 = a_\sigma + d_\sigma.$$

It follows that $c_\sigma = 0$ for all $\sigma \in \mathrm{Gal}(F_\Sigma/F')$, contradicting Lemma 6.2. Thus there exists a $\sigma \in \mathrm{Gal}(F_\Sigma/F')$ such that $\rho(\sigma)$ has eigenvalues of infinite order.

(ii) Assume otherwise. Then as in part (i), we see that there are only finitely many possibilities for the trace of $\rho(\sigma)$ with $\sigma \in S = \mathrm{Gal}(F_\Sigma/F^+)\backslash \mathrm{Gal}(F_\Sigma/F')$. Fix a $\tau \in \mathrm{Gal}(F_\Sigma/F')$ such that $\rho(\tau)$ has eigenvalues of infinite order. Choose a basis for $\rho \otimes \overline{K}$ such that $(\rho \otimes \overline{K})(\tau) = \begin{pmatrix} \beta & \\ & \beta^{-1} \end{pmatrix}$. For any $\sigma \in S$, if $\rho(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$, then we have

$$\mathrm{trace}\ \rho(\tau^n\sigma) = \beta^n a_\sigma + \beta^{-n} d_\sigma.$$

Since there are supposed to be only finitely many choices for the trace, $a_\sigma = d_\sigma = 0$ for all $\sigma \in S$. It follows easily that $\rho \otimes \overline{K}|_{\mathrm{Gal}(F_\Sigma/F^+)}$ is dihedral, contradicting Lemma 6.2.

To complete the proof of the lemma, note that in (ii) the eigenvalues will necessarily be in A. This follows from Hensel's lemma using that the two eigenvalues are distinct modulo λ. Then part (i) follows also by taking the square of any σ obtained in part (ii). □

Lemma **6.4.** — *If* G *is a normal subgroup of* $\mathrm{Gal}(F_\Sigma/F^+)$ *of finite index then* $\rho \otimes \overline{K}|_G$ *is irreducible.*

Proof. — Suppose that V is the representation space for $\rho \otimes \overline{K}|_{\mathrm{Gal}(F_\Sigma/F^+)}$ and V_0 is a subspace invariant by G. By Lemma 6.3(ii) there exists an element of G whose eigenvalues are β, β^{-1} with β of infinite order. Arguing as in Lemma 6.2, we deduce that either V_0 is invariant under $\mathrm{Gal}(F_\Sigma/F^+)$ or the representation is dihedral, contradicting Lemma 6.2. □

Let $\mathscr{T} = \mathrm{ad}^0\rho = \{ f \in \mathrm{ad}\,\rho : \mathrm{trace}\, f = 0 \}$ where as usual we identify $\mathrm{ad}\,\rho$ with $\mathrm{Hom}_A(\mathscr{U}, \mathscr{U})$, $\mathscr{U}$ being the representation space for ρ (more precisely $\mathscr{U}$ is a free A-module of rank 2). Let $\mathscr{T}_n = \mathscr{T}/\lambda^n$.

Lemma **6.5.** — *There exists an integer* N_2 *with the following property. If* $M \subset \mathscr{T}_n$ *is a submodule for some* n *and* $\lambda^a m \neq 0$ *for some* $a > N_2$ *and* $m \in M$, *then* $\lambda^{n-(a-N_2)}\mathscr{T}_n \subset M$. *The same holds if* M *is a* G*-submodule of* $\mathscr{T}_n$ *for* G *a normal subgroup of* $\mathrm{Gal}(F_\Sigma/F^+)$ *of finite index,* N_2 *depending only on* G.

Proof. — Suppose $x \in \mathrm{ad}^0\rho - \lambda\mathrm{ad}^0\rho$. Then by Lemma 6.4, $A[G]x \supset \lambda^r\mathrm{ad}^0\rho$ for some minimal $r = r(x)$. Define a function $f : \mathrm{ad}^0\rho - \lambda\mathrm{ad}^0\rho \longrightarrow \mathbf{Z}$ by $f(x) = r(x)$. Then f is continuous and hence $\mathrm{im} f$ is finite. Let N_2 be the greatest value of $\mathrm{im} f$.

Now $\mathscr{T}_n = \mathrm{ad}^0\rho/\lambda^n$ and we pick s maximal such that $\lambda^s y = m$ for some $y \in \mathscr{T}_n$. So $a + s < n$. By the definition of N_2 we see that $\lambda^{N_2}\mathscr{T}_n \subset \rho(G)y$, whence

$$\lambda^{n-a+N_2}\mathscr{T}_n \subset \lambda^{s+N_2}\mathscr{T}_n \subset \rho(G)\lambda^s y \subset M$$

which completes the proof. □

Remark **6.6.** — When combined with Lemma 6.3 this shows in particular that $\#(\mathscr{T}_n)^G$ is bounded independent of n.

As above, let $\mathscr{U}$ be the representation space for ρ. This is a free A-module of rank 2 having for each $v|p$ a filtration $0 \subset \mathscr{U}_{1,v} \subset \mathscr{U}$ such that $\mathscr{U}_{1,v}$ is a free A-module on which D_v acts via a character reducing to χ modulo λ. The quotient $\mathscr{U}_{2,v} = \mathscr{U}/\mathscr{U}_{1,v}$ is a free A-module on which D_v acts via a character reducing to 1 modulo λ. If $\rho \otimes \overline{K}$ is type A at w, then there is a filtration $0 \subset \mathscr{U}_1^w \subset \mathscr{U}$ such that both $\mathscr{U}_1^w$ and the quotient $\mathscr{U}_2^w = \mathscr{U}/\mathscr{U}_1^w$ are free A-modules on which I_w acts trivially. If $\rho \otimes \overline{K}$ is type B at w, then $\mathscr{U}$ decomposes as $\mathscr{U} = \mathscr{U}_1^w \oplus \mathscr{U}_2^w$ with I_w acting on the first factor via $\tilde{\chi}$ and acting trivially on the second factor. Also as above, let $\mathscr{T} = \{ f \in \mathrm{ad}\,\rho : \mathrm{trace}(f) = 0 \}$. Let $\mathscr{T}_v^{\mathrm{ord}} = \{ f \in \mathscr{T} : f(\mathscr{U}) \subseteq \mathscr{U}_{1,v} \}$. Similarly, if $\rho \otimes \overline{K}$ is type A or type B at w, then let $\mathscr{T}^w = \{ f \in \mathscr{T} : f(\mathscr{U}) \subseteq \mathscr{U}_1^w \}$. We write $\mathscr{T}_n$, $\mathscr{T}_{n,v}^{\mathrm{ord}}$, and $\mathscr{T}_n^w$ for $\mathscr{T}/\lambda^n$, $\mathscr{T}_v^{\mathrm{ord}}/\lambda^n$, and $\mathscr{T}^w/\lambda^n$, respectively. Let

$$H_v(\mathscr{T}_n) = H^1(I_v, \mathscr{T}_n/\mathscr{T}_{n,v}^{\mathrm{ord}}),$$

and let

$$H_w(\mathscr{T}_n) = \begin{cases} H^1(I_w, \mathscr{T}_n/\mathscr{T}_n^w) & \text{if } \rho \otimes \overline{K} \text{ is type A or type B at } w, \\ 0 & \text{otherwise.} \end{cases}$$

For each $w \in \Sigma$, put

$$L_w(\mathcal{T}_n) = \ker\{\, H^1(D_w, \mathcal{T}_n) \longrightarrow H_w(\mathcal{T}_n)\,\}.$$

We define a Selmer group for $\mathcal{T}_n$ by

$$H_\Sigma(\mathcal{T}_n) = \{\, \alpha \in H^1(F_\Sigma/F, \mathcal{T}_n) : \mathrm{res}_w \alpha \in L_w(\mathcal{T}_n) \text{ for each } w \in \Sigma\,\}.$$

For each place $w \in \Sigma$, denote by $L_w^*(\mathcal{T}_n)$ the orthogonal complement of $L_w(\mathcal{T}_n)$ under local duality (so $L_w^*(\mathcal{T}_n) \subseteq H^1(D_w, \mathcal{T}_n(1))$), and put

$$H_\Sigma^*(\mathcal{T}_n) = \{\, \alpha \in H^1(F_\Sigma/F, \mathcal{T}_n(1)) : \mathrm{res}_w \alpha \in L_w^*(\mathcal{T}_n) \text{ for each } w \in \Sigma\,\}.$$

By the argument for [W1, Proposition 1.6], which generalizes easily to the case of an arbitrary totally real field F,

$$\textbf{(6.3)} \qquad \frac{\#H_\Sigma(\mathcal{T}_n)}{\#H_\Sigma^*(\mathcal{T}_n)} = h_\infty(\mathcal{T}_n) \prod_{w\in\Sigma} h_w(\mathcal{T}_n),$$

where

$$h_\infty(\mathcal{T}_n) = \frac{\#H^0(F_\Sigma/F, \mathcal{T}_n)\cdot(\#H^0(\mathbf{R}, \mathcal{T}_n(1)))^{[F:\mathbf{Q}]}}{\#H^0(F_\Sigma/F, \mathcal{T}_n(1))}$$

$$h_w(\mathcal{T}_n) = \frac{\#H^0(D_w, \mathcal{T}_n(1))\cdot \#L_w(\mathcal{T}_n)}{\#H^1(D_w, \mathcal{T}_n)}.$$

We now estimate these factors. For two positive quantities B and C (possibly depending on n and Σ), we write $B \ll C$ to mean that the ratio B/C is bounded independently of n and the places in Σ (it may, however, depend on ρ and $\#\Sigma$). Similarly, we write $B \asymp C$ to mean that $\max(B/C, C/B) \ll 1$. A simple computation using our hypotheses on ρ shows that

$$\textbf{(6.4)} \qquad h_\infty(\mathcal{T}_n) \asymp \#(A/\lambda^n)^{2[F:\mathbf{Q}]}.$$

Almost by definition,

$$\textbf{(6.5)} \qquad h_w(\mathcal{T}_n) = \#H^0(D_w, \mathcal{T}_n(1)) \text{ if } \rho\otimes\overline{K} \text{ is unramified at } w.$$

Suppose that $\rho\otimes\overline{K}$ is type A or type B at w. From the definition of $L_w(\mathcal{T}_n)$, it is clear that $H^1(\mathbf{F}_w, \mathcal{T}_n^{I_w}) \hookrightarrow L_w(\mathcal{T}_n)$. The order of the quotient $L_w(\mathcal{T}_n)/H^1(\mathbf{F}_w, \mathcal{T}_n^{I_w})$ is bounded by the order of $K_n^{D_w}$, where

$$K_n = \ker\{\, H^1(I_w, \mathcal{T}_n) \longrightarrow H^1(I_w, \mathcal{T}_n/\mathcal{T}_n^w)\,\}.$$

The exact sequence

$$0 \longrightarrow \mathscr{T}_n^w \longrightarrow \mathscr{T}_n \longrightarrow \mathscr{T}_n/\mathscr{T}_n^w \longrightarrow 0$$

gives rise to an exact sequence

$$\begin{aligned} 0 \longrightarrow \mathscr{T}_n^{\mathrm{I}_w}/\mathscr{T}_n^w \longrightarrow (\mathscr{T}_n/\mathscr{T}_n^w)^{\mathrm{I}_w} \cong \mathrm{A}/\lambda^n \longrightarrow \quad & \mathrm{H}^1(\mathrm{I}_w, \mathscr{T}_n^w) \cong \mathscr{T}_n^w \\ & \cong \mathrm{A}/\lambda^n \longrightarrow \mathrm{K}_n \longrightarrow 0 \end{aligned}$$

if $\rho \otimes \overline{\mathrm{K}}$ is type A at w, and

$$0 \longrightarrow \mathscr{T}_n^{\mathrm{I}_w} \cong (\mathscr{T}_n/\mathscr{T}_n^w)^{\mathrm{I}_w} \longrightarrow \mathrm{H}^1(\mathrm{I}_w, \mathscr{T}_n^w) \longrightarrow \mathrm{K}_n \longrightarrow 0$$

if $\rho \otimes \overline{\mathrm{K}}$ is type B at w. In the former case it follows that $\#\mathrm{K}_n = \#(\mathscr{T}_n^{\mathrm{I}_w}/\mathscr{T}_n^w) \asymp 1$, and in the latter case it follows that $\#\mathrm{K}_n^{\mathrm{D}_w} = \#\mathrm{H}^1(\mathrm{I}_w, \mathscr{T}_n^w)^{\mathrm{D}_w} = \#\mathrm{H}^0(\mathrm{D}_w, \mathscr{T}_n^w(1)) \asymp 1$. It now follows from local duality that

(6.6) $\qquad h_w(\mathscr{T}_n) \asymp 1$ if $\rho \otimes \overline{\mathrm{K}}$ is type A or type B at w.

It remains to estimate $h_v(\mathscr{T}_n)$ for $v|p$. To do so, consider the diagram

$$\begin{array}{ccccc} & & \mathrm{H}^1(\mathrm{D}_v, \mathscr{T}_n) & & \\ & & \varphi \big\downarrow & \searrow \psi & \\ 0 \longrightarrow \mathrm{H}^1(\mathbf{F}_v, (\mathscr{T}_n/\mathscr{T}_{n,v}^{\mathrm{ord}})^{\mathrm{I}_v}) & \longrightarrow & \mathrm{H}^1(\mathrm{D}_v, \mathscr{T}_n/\mathscr{T}_{n,v}^{\mathrm{ord}}) & \longrightarrow & \mathrm{H}^1(\mathrm{I}_v, \mathscr{T}_n/\mathscr{T}_{n,v}^{\mathrm{ord}}). \end{array}$$

Our hypothesis (vi) of (6.2) implies that $\#\mathrm{cok}(\varphi) \asymp 1$. It follows that

(6.7)
$$\begin{aligned} \#\mathrm{L}_v(\mathscr{T}_n) &= \#\ker(\psi) = \frac{\#\mathrm{H}^1(\mathrm{D}_v, \mathscr{T}_n)}{\#\mathrm{im}(\psi)} \\ &\asymp \frac{\#\mathrm{H}^1(\mathrm{D}_v, \mathscr{T}_n) \cdot \#\mathrm{H}^1(\mathbf{F}_v, (\mathscr{T}_n/\mathscr{T}_{n,v}^{\mathrm{ord}})^{\mathrm{I}_v})}{\#\mathrm{im}(\varphi)}. \end{aligned}$$

From the long exact cohomology sequence for $0 \longrightarrow \mathscr{T}_{n,v}^{\mathrm{ord}} \longrightarrow \mathscr{T}_n \longrightarrow \mathscr{T}_n/\mathscr{T}_{n,v}^{\mathrm{ord}} \longrightarrow 0$ and the fact that D_v has cohomological dimension two, one finds

$$\#\mathrm{im}(\varphi) = \frac{\#\mathrm{H}^1(\mathrm{D}_v, \mathscr{T}_n/\mathscr{T}_{n,v}^{\mathrm{ord}}) \cdot \#\mathrm{H}^2(\mathrm{D}_v, \mathscr{T}_n)}{\#\mathrm{H}^2(\mathrm{D}_v, \mathscr{T}_{n,v}^{\mathrm{ord}}) \cdot \#\mathrm{H}^2(\mathrm{D}_v, \mathscr{T}_n/\mathscr{T}_{n,v}^{\mathrm{ord}})}.$$

Substituting this into (6.7) yields

(6.8)
$$h_v(\mathscr{T}_n) \asymp \frac{\#\mathrm{H}^2(\mathrm{D}_v, \mathscr{T}_{n,v}^{\mathrm{ord}})}{\#(\mathrm{A}/\lambda^n)^{2[\mathrm{F}_v:\mathbf{Q}_p]}} \asymp \cdot\#(\mathrm{A}/\lambda^n)^{-2[\mathrm{F}_v:\mathbf{Q}_p]}.$$

Here, we have again used hypothesis (vi) of (6.2) (really its implication that $\chi_1\chi_2^{-1} \neq \varepsilon$).

Now writing $\Sigma = \Sigma_0 \cup \Sigma'$ with $\{v|p\} \subseteq \Sigma_0$, ρ ramified at each prime in $\Sigma_0 \backslash \{v|p\}$, and unramified at each prime in Σ' and combining (6.3)-(6.6) and (6.8) yields

$$\textbf{(6.9)} \qquad \#\mathrm{H}_\Sigma(\mathscr{T}_n) \asymp \#\mathrm{H}^*_\Sigma(\mathscr{T}_n) \cdot \prod_{w \in \Sigma'} \#\mathrm{H}^0(\mathrm{D}_w, \mathscr{T}_n(1)).$$

The exact sequence

$$0 \longrightarrow \mathscr{T}_n \longrightarrow \mathscr{T}_m \xrightarrow{\lambda^n} \mathscr{T}_{m-n} \longrightarrow 0$$

gives rise to the commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{H}_\Sigma(\mathscr{T}_n) & \longrightarrow & \mathrm{H}_\Sigma(\mathscr{T}_m) & \xrightarrow{\lambda^n} & \mathrm{H}_\Sigma(\mathscr{T}_{m-n}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{H}^1(\mathrm{F}_\Sigma/\mathrm{F}, \mathscr{T}_n) & \longrightarrow & \mathrm{H}^1(\mathrm{F}_\Sigma/\mathrm{F}, \mathscr{T}_m) & \xrightarrow{\lambda^n} & \mathrm{H}^1(\mathrm{F}_\Sigma/\mathrm{F}, \mathscr{T}_{m-n}) \\
 & & \downarrow & & \downarrow & & \downarrow \\
\oplus_{w \in \Sigma_0} \mathrm{M}_{w,n} & \hookrightarrow & \oplus_{w \in \Sigma_0} \mathrm{H}_w(\mathscr{T}_n) & \longrightarrow & \oplus_{w \in \Sigma_0} \mathrm{H}_w(\mathscr{T}_m) & \xrightarrow{\lambda^n} & \oplus_{w \in \Sigma_0} \mathrm{H}_w(\mathscr{T}_{m-n})
\end{array}$$

whose last two rows are exact. Each $\mathrm{M}_{w,n}$ is a finite group such that $\#\mathrm{M}_{w,n} \asymp 1$. It is apparent from the diagram that

$$\textbf{(6.10)} \qquad \mathrm{H}_\Sigma(\mathscr{T}_n) \hookrightarrow \mathrm{H}_\Sigma(\mathscr{T}_m) \text{ and } \#\mathrm{H}_\Sigma(\mathscr{T}_n) \asymp \#\mathrm{H}_\Sigma(\mathscr{T}_m)[\lambda^n].$$

Similar considerations show that

$$\textbf{(6.11)} \qquad \#\mathrm{H}^*_{\Sigma_0}(\mathscr{T}_n) \asymp \#\mathrm{H}^*_{\Sigma_0}(\mathscr{T}_m)[\lambda^n].$$

We will combine the above computations with the following lemma to deduce some results about "divisible ranks" of various Selmer groups.

Lemma **6.7.** — *The groups* $\mathrm{H}_\Sigma(\mathscr{T}_n)$ *and* $\mathrm{H}^*_\Sigma(\mathscr{T}_n)$ *are finite* A*-modules whose minimal number of generators is bounded in terms of* $\#\Sigma$ *but independently of* n*.*

Proof. — This follows from (6.10) and (6.11). Note that as $\mathrm{H}^*_\Sigma(\mathscr{T}_n)$ is a submodule of $\mathrm{H}^*_{\Sigma_0}(\mathscr{T}_n)$ it suffices to prove that the number of generators of $\mathrm{H}^*_{\Sigma_0}(\mathscr{T}_n)$ is bounded independently of n. □

A refinement of this lemma using also (6.9) is the following result.

Lemma **6.8.** — *Forming limits with respect to the obvious maps*

$$\varinjlim_n \mathrm{H}_{\Sigma_0}(\mathscr{T}_n) \cong (\mathrm{K}/\mathrm{A})^r \oplus \mathrm{X} \text{ and } \varinjlim_n \mathrm{H}^*_{\Sigma_0}(\mathscr{T}_n) \cong (\mathrm{K}/\mathrm{A})^r \oplus \mathrm{X}^*,$$

with $r < \infty$ *and* X *and* X^* *finite groups.*

The following lemma is an analogue of [W1, Proposition 1.11] and it occupies a similar place in the proof of the main result of this section.

Lemma **6.9.** — *Let* E *be the splitting field of* ρ, *and let* E_∞ *be the extension of* E *obtained by adjoining all p-th power roots of unity. There exists an integer* $N_1 > 0$ *such that for each* n, $H^1(E_\infty/F, \mathcal{T}_n(1))$ *is annihilated by* λ^{N_1}.

Proof. — Let F^+ and F' be as defined prior to Lemma 6.2. There is an exact sequence

$$0 \longrightarrow H^1(F^+/F, \mathcal{T}_n(1)^{\mathrm{Gal}(E_\infty/F^+)}) \longrightarrow H^1(E_\infty/F, \mathcal{T}_n(1)) \longrightarrow H^1(E_\infty/F^+, \mathcal{T}_n(1)). \tag{6.12}$$

The first term in this exact sequence is bounded independent of n by Remark 6.6.

Now consider the last term of the sequence (6.12). Let $\Delta = \mathrm{Gal}(F'/F^+) \cong \mathbf{Z}/2$. There are isomorphisms

$$H^1(E_\infty/F^+, \mathcal{T}_n(1)) \cong H^1(E_\infty/F', \mathcal{T}_n(1))^{\Delta=1} \cong H^1(E_\infty/F', \mathcal{T}_n)^{\Delta=-1}, \tag{6.13}$$

the first by restriction and the second by the fact that $\mathcal{T}_n(1)$ and $\mathcal{T}_n$ are isomorphic as $\mathrm{Gal}(E_\infty/F')$-modules. Note that $\mathrm{Gal}(E_\infty/F^+)$ and $\mathrm{Gal}(E_\infty/F')$ project isomorphically onto subgroups H^+ and H' of $\mathrm{Gal}(E/F)$, respectively. In particular, $H^1(E_\infty/F', \mathcal{T}_n)^{\Delta=-1} = H^1(H', \mathcal{T}_n)^{\Delta=-1}$, and an element of the latter corresponds to an equivalence class of representations into $GL_2(A \oplus \varepsilon A/\lambda^n)$ having trivial determinant and reducing to ρ modulo ε. Here $A\oplus\varepsilon A/\lambda^n$ is given a ring structure by setting $\varepsilon^2 = 0$. This correspondence is given by

$$\alpha \in H^1(H', \mathcal{T}_n) \longleftrightarrow \rho_\alpha : H' \longrightarrow GL_2(A \oplus \varepsilon A/\lambda^n), \ \ \rho_\alpha(\sigma) = \rho(\sigma)(1 + \varepsilon\alpha(\sigma)).$$

Put $H = H^+\backslash H'$, and define a map

$$\varphi : H \longrightarrow H', \quad \varphi(\sigma) = \lim_{n\to\infty} (\sigma^{1+p^n}).$$

Consider the open set $H_M = \{\sigma \in H : \mathrm{trace}(\rho(\sigma)) \not\equiv 0 \bmod \lambda^M\}$. By Lemma 6.3(ii), H_M is non-empty if M is large. Fix such an M. We will show that the closure of $\varphi(H_M)$ has positive measure (with respect to the Haar measure of H'). For each n, write ρ_n for $\rho \bmod \lambda^n$, and write H_n^+, H_n', and $H_{M,n}$ for the respective images of H^+, H', and H_M under ρ_n. Being open, H_M contains a translate of an open subgroup of H', so there exists a constant $C > 0$ such that

$$\frac{\#H_{M,n}}{\#H_n'} \geqslant C \text{ for all } n. \tag{6.14}$$

Suppose $h, h' \in \mathrm{H_M}$ are such that $\rho_n(h) \neq \rho_n(h')$ but $\varphi(h) = \varphi(h') \bmod \lambda^n$. It is not hard to deduce that $\operatorname{trace}\rho_n(h) = \operatorname{trace}\rho_n(h')$. With respect to a basis for ρ_n such that $\rho_n(h) = \begin{pmatrix} \beta & \\ & -\beta^{-1} \end{pmatrix}$, $\rho_n(h') = g^{-1}\begin{pmatrix} \beta & \\ & -\beta^{-1} \end{pmatrix} g$ for some $g \in \mathrm{GL}_2(\mathrm{A}/\lambda^n)$ commuting with $\rho_n(\varphi(h)) = \begin{pmatrix} \beta & \\ & \beta^{-1} \end{pmatrix}$ but not with $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Therefore, given $x \in \mathrm{H}'_n$, we find that

(6.15) $$\#\{ h \in \mathrm{H}_{\mathrm{M},n} : \varphi(h) = x \} \leqslant \#(\mathrm{A}/\lambda^{2\mathrm{M}}).$$

Let $\mathrm{S_M} = \varphi(\mathrm{H_M})$, and let $\mathrm{S}_{\mathrm{M},n}$ be the image of $\mathrm{S_M}$ under ρ_n. Combining (6.14) and (6.15) shows that

$$\frac{\#\mathrm{S}_{\mathrm{M},n}}{\#\mathrm{H}'_n} \geqslant \mathrm{C} \cdot \#(\mathrm{A}/\lambda^{2\mathrm{M}})^{-1} > 0,$$

from which it follows that

$$\overline{\mathrm{S}}_{\mathrm{M}} = \varprojlim_n \mathrm{S}_{\mathrm{M},n},$$

the closure of $\varphi(\mathrm{H_M})$, has positive measure.

Fix $\alpha \in \mathrm{H}^1(\mathrm{H}', \mathscr{T}_n)^{\Delta=-1}$. For $\sigma \in \mathrm{H_M}$, $\operatorname{trace}\rho_\alpha(\varphi(\sigma)) = \operatorname{trace}\rho(\varphi(\sigma))$. As $\operatorname{trace}(\cdot)$ is continuous, this equality holds for all $s \in \overline{\mathrm{S}}_{\mathrm{M}}$. Fix a $\sigma_0 \in \mathrm{S_M}$ having infinite order (this is possible provided M is large enough). Choose a basis for ρ such that $\rho(\sigma_0) = \begin{pmatrix} \beta & \\ & \beta^{-1} \end{pmatrix}$. Then $\rho_\alpha(\sigma_0) = \begin{pmatrix} \beta & \\ & \beta^{-1} \end{pmatrix} \begin{pmatrix} 1+\varepsilon x & \varepsilon z \\ \varepsilon w & 1-\varepsilon x \end{pmatrix}$ with $\lambda^{\mathrm{M}} x = 0$. Put $\alpha_1 = \lambda^{\mathrm{M}}\alpha$. It follows that $\rho_{\alpha_1}(\sigma_0)$ is diagonalizable with eigenvalues β and β^{-1}. Pick a basis for ρ_{α_1} such that $\rho_{\alpha_1}(\sigma_0) = \begin{pmatrix} \beta & \\ & \beta^{-1} \end{pmatrix}$. As $\overline{\mathrm{S}}_{\mathrm{M}}$ has positive measure, there must be some $r > 0$ such that $\mathrm{X}_r = \overline{\mathrm{S}}_{\mathrm{M}} \cap \sigma_0^{-r}\overline{\mathrm{S}}_{\mathrm{M}}$ also has positive measure. For any $\sigma \in \mathrm{X}_r$,

$$\operatorname{trace}\rho_{\alpha_1}(\sigma) = \operatorname{trace}\rho(\sigma) \quad \text{and} \quad \operatorname{trace}\rho_{\alpha_1}(\sigma_0^r\sigma) = \operatorname{trace}\rho(\sigma_0^r\sigma).$$

Write $\rho(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$ and $\rho_{\alpha_1}(\sigma) = \begin{pmatrix} a(\alpha_1)_\sigma & b(\alpha_1)_\sigma \\ c(\alpha_1)_\sigma & d(\alpha_1)_\sigma \end{pmatrix}$. It follows that both $(\beta^{2r} - 1)a(\alpha_1)_\sigma$ and $(\beta^{2r} - 1)d(\alpha_1)_\sigma$ are in A. Thus if $\alpha_2 = \lambda^{2\mathrm{M}r}\alpha_1$, then $a_\sigma = a(\alpha_2)_\sigma$ and $d_\sigma = d(\alpha_2)_\sigma$ for all $\sigma \in \mathrm{X}_r$.

There exists $\tau \in \mathrm{X}_r$ such that $b_\tau c_\tau \neq 0$. If this were not so, then the image of $\rho|_{\mathrm{H}^+}$ would be triangular on a set of positive measure and also on the group generated by the set. This is ruled out by Lemma 6.4. Let $n_1 = \operatorname{ord}_\lambda(b_\tau)$. If $\alpha_3 = \lambda^{n_1}\alpha_2$, then $b(\alpha_3)_\tau = b_\tau(1 + \varepsilon t)$ for some t. Rescaling the basis for ρ_{α_3}, we can assume that $b(\alpha_3)_\tau = b_\tau$. Now put $\alpha_4 = \lambda^{n_1}\alpha_3$. As $b_\tau c(\alpha_3)_\tau = 1 - a(\alpha_3)_\tau d(\alpha_3)_\tau = 1 - a_\tau d_\tau \in \mathrm{A}$, it follows that $c(\alpha_4)_\tau = c_\tau$. In particular, $\rho(\tau) = \rho_{\alpha_4}(\tau)$.

Now pick an integer s such that $\mathrm{Y}_s = \mathrm{X}_r \cap \tau^{-s}\mathrm{X}_r$ has positive measure. As the eigenvalues of $\rho(\tau)$ have infinite order and $\rho(\tau)$ is not triangular, $\rho(\tau^s)$ is not triangular.

In particular, $b_{\tau^s}, c_{\tau^s} \not\equiv 0$. Moreover, if $\sigma \in Y_s$, then by considering $\rho_{\alpha_4}(\tau^s\sigma)$ we see that

$$(6.16) \qquad a_{\tau^s}a_\sigma + b_{\tau^s}c(\alpha_4)_\sigma = a_{\tau^s\sigma} \in A \quad \text{and} \quad c_{\tau^s}b(\alpha_4)_\sigma + d_{\tau^s}d_\sigma = \tau\sigma^s \in A.$$

Let $n_2 = \max(\mathrm{ord}_\lambda(b_{\tau^s}), \mathrm{ord}_\lambda(c_{\tau^s}))$. Put $\alpha_5 = \lambda^{n_2}\alpha_4$. It follows from (6.16) that for $\sigma \in Y_s$, $b(\alpha_5)_\sigma = b_\sigma$ and $c(\alpha_5)_\sigma = c_\sigma$. In other words, $\rho(\sigma) = \rho_{\alpha_5}(\sigma)$ for all $\sigma \in Y_s$. The same holds for all elements σ in the subgroup G generated by Y_s. This subgroup has positive measure and hence has finite index. Choose a subgroup $G' \subseteq G$ of finite index that is normal in H^+. Consider the exact sequence

$$0 \longrightarrow H^1(H'/G', (\mathcal{T}_n)^{G'}) \longrightarrow H^1(H', \mathcal{T}_n) \xrightarrow{\text{res}} H^1(G', \mathcal{T}_n).$$

We have shown that if $\alpha \in H^1(H', \mathcal{T}_n)^{\Delta=-1}$, then $\mathrm{res}(\alpha) \in H^1(G', \mathcal{T}_n)$ is annihilated by $\lambda^{3Mr+2n_1+n_2}$. By Remark 6.6 there is an integer N_2 (depending on G') such that λ^{N_2} annihilates $(\mathcal{T}_n)^{G'}$. Therefore,

$$\lambda^{3Mr+N_2+2n_1+n_2}H^1(H', \mathcal{T}_n)^{\Delta=-1} = 0.$$

Combining this with (6.12) and (6.13) yields the lemma. □

The next result, the principal result of this section, will enable us to control the ranks of various "tangent spaces" in the auxiliary deformation rings and Hecke rings that appear in the proof of the fundamental isomorphism (see section §7).

Proposition **6.10.** — *Let* $\sigma \in \mathrm{Gal}(F_\Sigma/F')$ *be an element such that the eigenvalues of* $\rho(\sigma)$ *are in* A *and have infinite order, as in Lemma* 6.3(i). *Then there exists an integer* $r = r(\rho)$ *such that for each* $m > 0$ *there are infinitely many sets* $Q = \{w_1, ..., w_r\}$ *such that*

(i) $\mathrm{Nm}(w_i) \equiv 1 \bmod p^m$ *for each i.*

(ii) $\rho_\mathfrak{p}(\mathrm{Frob}_{w_i}) \equiv \rho_\mathfrak{p}(\sigma) \bmod \lambda^m$ *for each i.*

(iii) $\varinjlim_n H_{\Sigma_Q}(\mathcal{T}_n) \simeq (K/A)^r \oplus X_{\Sigma_Q}$

with $\Sigma_Q = \Sigma_0 \cup Q$, $\#X_{\Sigma_Q} < C(\sigma, r) < \infty$ *for some constant* $C(\sigma, r)$ *depending only on* σ *and r. Moreover r is given by Lemma* 6.8.

Proof. — If the eigenvalues of σ are α and β then let λ^e be the highest power of λ dividing $(\alpha/\beta - 1)$. We fix a free, rank one quotient $\mathcal{M}$ of $\mathcal{T}(1)$ on which σ acts trivially. We denote by π_n the projection of $\mathcal{T}_n(1)$ onto $\mathcal{M}_n = \mathcal{M}/\lambda^n$.

Write

$$\varinjlim_n H^*_{\Sigma_0}(\mathcal{T}_n) \simeq (K/A)^r \oplus X^*_{\Sigma_0}$$

as in Lemma 6.8. Let f be the smallest integer such that λ^f annihilates $X^*_{\Sigma_0}$. Let N_1 be as in Lemma 6.9 and let N_2 be as in Lemma 6.5 (for the group $\mathrm{Gal}(F_\Sigma/F^+)$).

As the natural map $H^*_{\Sigma_0}(\mathcal{T}_n) \longrightarrow H^*_{\Sigma_0}(\mathcal{T}_{n'})$ for $n' > n$ has kernel of order bounded independent of n we can choose $n \geqslant m$ sufficiently large that

$$H^*_{\Sigma_0}(\mathcal{T}_n) \simeq \bigoplus_{i=1}^{r}(A/\lambda^{r_i}) \oplus X^*_{\Sigma_0}$$

with $r_i > N_1 + N_2 + e + f$. Let $[c_1] \in H^*_{\Sigma_0}(\mathcal{T}_n)$ be a cohomology class of exact order λ^{r_1} (where the r_i are indexed arbitrarily) where c_1 is a cocycle representing $[c_1]$. Let E_n denote the field generated by the splitting field of $\rho \bmod \lambda^n$ together with a primitive p^n-th root of unity. Suppose the annihilator of $\mathrm{res}([c_1]) \in H^1(F_\Sigma/E_n, \mathcal{T}_n(1))$ is λ^{s_1}. Then by Lemma 6.9 $s_1 \geqslant r_1 - N_1$.

The restriction $\mathrm{res}([c_1])$ determines a homomorphism

$$f_1 \in \mathrm{Hom}(\mathrm{Gal}(F_\Sigma/E_n), \mathcal{T}_n(1))^{\mathrm{Gal}(F_\Sigma/F)}.$$

Since f_1 has order λ^{s_1}, $\mathrm{im} f_1 \supseteq \lambda^{a_1}\mathcal{T}_n(1)$ with $n - a_1 = s_1 - N_2$ by Lemma 6.5. Let M_1 be the fixed field of $\ker f_1$ and pick $\tau_1 \in \mathrm{Gal}(M_1/E_n)$ such that $\pi_n(f_1(\tau_1))$ has order at least $\lambda^{s_1 - N_2}$. Let $t_1 = s_1 - N_2$. Put

$$g_1 = \begin{cases} \sigma|_{M_1} & \text{if } \lambda^{t_1 - 1}\pi_n(c_1(\sigma)) \neq 0 \\ \sigma|_{M_1} \cdot \tau_1 & \text{otherwise.} \end{cases}$$

Then $\pi_n(c_1(g_1))$ has order $\geqslant \lambda^{t_1}$. We choose a prime w_1 of F such that ρ is unramified at w_1 and $\mathrm{Frob}_{w_1} = g_1$ in $\mathrm{Gal}(M_1/F)$. By the choice of g_1, $\mathrm{Nm}(w_1) \equiv 1 \bmod p^n$. Moreover the image of $[c_1]$ in $H^1(F_{w_1}, \mathcal{M}_n(1))$ has order $\geqslant \lambda^{t_1}$.

Now suppose $n' > n$. Write $H^*_{\Sigma_0}(\mathcal{T}_{n'}) \simeq \bigoplus_{i=1}^{r}(A/\lambda^{r'_i}) \oplus X^*_{\Sigma_0}$. We may assume that there is a cohomology class $[c'_1] \in H^*_{\Sigma_0}(\mathcal{T}_{n'})$ of exact order $\lambda^{r'_1}$ such that $\lambda^{m_1}[c'_1]$ is the image of $[c_1]$ in $H^*_{\Sigma_0}(\mathcal{T}_{n'})$ for some $m_1 \geqslant r'_1 - r_1$. It follows that the order of $\mathrm{res}(\lambda^{m_1}[c'_1]) \in H^1(D_{w_1}, \mathcal{T}_{n'}(1))$ is at least $\lambda^{t_1 - e}$. As $t_1 - e = r_1 - N_1 - N_2 - e > 0$ the order of $\mathrm{res}([c'_1])$ is at least $\lambda^{r'_1 - N_1 - N_2 - e}$. Let $\Sigma_1 = \Sigma_0 \cup \{w_1\}$. It then follows that

$$\varinjlim_{n'} H^*_{\Sigma_1}(\mathcal{T}_{n'}) \simeq (K/A)^{r-1} \oplus X^*_{\Sigma_1}$$

with $\# X^*_{\Sigma_1} \leqslant \# X^*_{\Sigma_0} \cdot \#(A/\lambda^{N_1+N_2+e+f})$.

Suppose that inductively we have picked primes $w_1, \ldots, w_j$ such that for $\Sigma_j = \Sigma_0 \cup \{w_1, \ldots, w_j\}$

$$\varinjlim_{n} H^*_{\Sigma_j}(\mathcal{T}_n) \simeq (K/A)^{r-j} \oplus X^*_{\Sigma_j}$$

with

$$\#X^*_{\Sigma_j} \leqslant \#X^*_{\Sigma_{j-1}} \cdot \#(A/\lambda^{N_1+N_2+e+f})^j.$$

We repeat the construction given for w_1 and obtain a new prime w_{j+1}. When we reach $j = r$ we have

$$\varinjlim_n H^*_{\Sigma_r}(\mathcal{T}_n) \simeq X^*_{\Sigma_r}, \quad \#X^*_{\Sigma_r} \leqslant \#X^*_{\Sigma_0} \cdot \#(A/\lambda^{N_1+N_2+e+f})^{r^2}.$$

From this it follows that $\#H^*_{\Sigma_r}(\mathcal{T}_n) \asymp \#X^*_{\Sigma_r}$, and by (6.9) $\#H_{\Sigma_r}(\mathcal{T}_n) \asymp \#(A/\lambda^n)^r \cdot \#X^*_{\Sigma_r}$. The proposition now follows from this together with (6.10). □

7. Nice primes at minimum level

In this section we assume that F is a totally real field of even degree. Associated to a cohomology class c as in (2.1) is a deformation datum $\mathcal{D} = (\mathcal{O}, \Sigma, c, \mathcal{M})$. We suppose that we are given a prime $\mathfrak{p} \subseteq \mathbf{T}_{\mathcal{D}}$ which is nice for $\mathcal{D}$ in the sense of §4.2. Note that since $p \in \mathfrak{p}$, $\mathfrak{p}$ must come from a prime of $\mathbf{T}^{\min}_{\mathcal{D}}$, and we also use $\mathfrak{p}$ to denote this prime.

Now $\mathbf{T}^{\min}_{\mathcal{D}}$ acts on the module $M_{\mathcal{D}} = M_\infty(U^{\min}_{\mathcal{D}})_{\mathfrak{m}}$ defined at the end of §3.5. Furthermore, $\mathbf{T}^{\min}_{\mathcal{D}}$ is a finite, torsion-free $\Lambda_{\mathcal{O}}$-algebra. On the other hand, associated to $\mathcal{D}$ we have a deformation ring $R^{\min}_{\mathcal{D}}$ defined in §2.3 which is also a $\Lambda_{\mathcal{O}}$-algebra. There is no natural map $R^{\min}_{\mathcal{D}} \longrightarrow \mathbf{T}^{\min}_{\mathcal{D}}$ since we have no natural representation with coefficients in $\mathbf{T}^{\min}_{\mathcal{D}}$. However there is a pseudo-representation with coefficients in $\mathbf{T}^{\min}_{\mathcal{D}}$ inducing the horizontal map in

$$\text{(7.1)} \qquad \begin{array}{ccc} R_{\mathcal{D}\,\mathrm{ps}} & \xrightarrow{\pi^{\min}_{\mathcal{D}}} & \mathbf{T}^{\min}_{\mathcal{D}} \\ & {\scriptstyle r^{\min}_{\mathcal{D}}} \searrow & \\ & & R^{\min}_{\mathcal{D}} \end{array}$$

(cf. (3.4)). The map $r^{\min}_{\mathcal{D}}$ is the one which induces the pseudo-representation associated to $\rho^{\min}_{\mathcal{D}}$ (see §2.4).

Since $\mathfrak{p}$ is nice, $\mathbf{T}^{\min}_{\mathcal{D}}/\mathfrak{p}$ is of dimension one and $p \in \mathfrak{p}$. So the integral closure A of $\mathbf{T}^{\min}_{\mathcal{D}}/\mathfrak{p}$ is isomorphic to $k'[[\lambda]]$ for some finite extension k' of k and some λ. Furthermore the assumption that $\mathfrak{p}$ is nice ensures that under the composite map

$$\Lambda_{\mathcal{O}} \longrightarrow \mathbf{T}^{\min}_{\mathcal{D}}/\mathfrak{p} \hookrightarrow A$$

the ring A is finite over $\Lambda_{\mathcal{O}}$. (This is because the associated representation $\rho_{\mathfrak{p}}|_{D_i}$ has at least one character of infinite order on the diagonal.) Writing $\Lambda_{\mathcal{O}} = \mathcal{O}[[z_1, \ldots, z_m]]$

let us suppose that $z_i \longmapsto \lambda^{r_i} u_i \in A$ with u_i a unit or zero for each i. Then we may take $r_i > 0$ for each i (as follows from the definition of the $\Lambda_{\mathscr{O}}$-action on $\mathbf{T}_{\mathscr{D}}$ in §3.3) and we may assume, after possibly renumbering, that u_1 is a unit. Set

$$\widetilde{\Lambda}_{\mathscr{O}} = \mathscr{O}'[[W_1, \ldots, W_m]], \tag{7.2}$$

where $\mathscr{O}' = \mathscr{O} \otimes_{W(k)} W(k')$. There is a map $\widetilde{\Lambda}_{\mathscr{O}} \twoheadrightarrow A$ defined by $W_1 \longmapsto \lambda$ and $W_i \longmapsto 0$ for $2 \leqslant i \leqslant m$. Define a homomorphism $\Lambda_{\mathscr{O}} \longrightarrow \widetilde{\Lambda}_{\mathscr{O}}$ by

$$z_1 \longmapsto W_1^{r_1} \widetilde{u}_1, \quad z_i \longmapsto -W_i + W_1^{r_i} \widetilde{u}_i \quad \text{for} \quad 2 \leqslant i \leqslant m.$$

Here $\widetilde{u}_i$ denotes any fixed choice of lift of u_i to $\widetilde{\Lambda}_{\mathscr{O}}$. Then $\widetilde{\Lambda}_{\mathscr{O}}$ is finite and free over Λ and we have a commutative diagram of rings

$$\begin{array}{ccccc} & & \widetilde{\Lambda}_{\mathscr{O}} & & \\ & \nearrow & & \searrow & \\ \Lambda_{\mathscr{O}} & & & & A. \\ & \searrow & & \nearrow & \\ & & \mathbf{T}_{\mathscr{D}}^{\min} & & \end{array}$$

From this diagram we deduce the existence of a prime $\widetilde{\mathfrak{p}}$ of $\mathbf{T}_{\mathscr{D}}^{\min} \otimes_{\Lambda_{\mathscr{O}}} \widetilde{\Lambda}_{\mathscr{O}}$ extending $\mathfrak{p}$. Similarly we deduce the existence of a prime $\widetilde{\mathfrak{p}}_{\mathscr{D}}$ of $R_{\mathscr{D}}^{\min} \otimes_{\Lambda_{\mathscr{O}}} \widetilde{\Lambda}_{\mathscr{O}}$ extending $\mathfrak{p}_{\mathscr{D}}$, where $\mathfrak{p}_{\mathscr{D}}$ is the prime of $R_{\mathscr{D}}^{\min}$ associated to $\mathfrak{p}$ as defined at the end of §4.2. For simplicity of notation, we may write $\widetilde{\mathfrak{p}}$ to denote $\widetilde{\mathfrak{p}}_{\mathscr{D}}$ if the context makes this usage clear. It will also be convenient to write $\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min}$ for $\mathbf{T}_{\mathscr{D}}^{\min} \otimes_{\Lambda_{\mathscr{O}}} \widetilde{\Lambda}_{\mathscr{O}}$ and $\widetilde{R}_{\mathscr{D}}^{\min}$ for $R_{\mathscr{D}}^{\min} \otimes_{\Lambda_{\mathscr{O}}} \widetilde{\Lambda}_{\mathscr{O}}$ from now on. Let $(\widetilde{R}_{\mathscr{D}}^{\min})_{\widetilde{\mathfrak{p}}}$ and $(\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\widetilde{\mathfrak{p}}}$ denote the localization *and* completions of $\widetilde{R}_{\mathscr{D}}^{\min}$ and $\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min}$, respectively at $\widetilde{\mathfrak{p}}_{\mathscr{D}}$ and $\widetilde{\mathfrak{p}}$. Define $\widetilde{R}_{\mathscr{D}^{\mathrm{ps}}}$, $\widetilde{\mathfrak{p}}^{\mathrm{ps}}$, and $(\widetilde{R}_{\mathscr{D}^{\mathrm{ps}}})_{\widetilde{\mathfrak{p}}^{\mathrm{ps}}}$ similarly, where $\mathfrak{p}^{\mathrm{ps}}$ is the inverse image of $\mathfrak{p}$ under $r_{\mathscr{D}}^{\min} : R_{\mathscr{D}^{\mathrm{ps}}} \to R_{\mathscr{D}}^{\min}$.

Lemma **7.1.** — *There is a natural local surjective map*

$$\psi(\mathscr{D}, \mathfrak{p}) : (\widetilde{R}_{\mathscr{D}}^{\min})_{\widetilde{\mathfrak{p}}} \twoheadrightarrow (\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\widetilde{\mathfrak{p}}} \tag{7.3}$$

under which $\operatorname{trace} \rho_{\mathscr{D}}^{\min}(\mathrm{Frob}_\ell) \otimes 1 \mapsto T(\ell) \otimes 1$ *and* $\det \rho_{\mathscr{D}}^{\min}(\mathrm{Frob}_\ell) \otimes 1 \mapsto S(\ell)\mathrm{Nm}(\ell) \otimes 1$ *for all* $\ell \notin \Sigma$.

Proof. — Let π be a uniformizer of $\mathcal{O}$, and let $P = (\pi, W_2, \ldots, W_m) \subseteq \widetilde{\Lambda}_{\mathcal{O}}$. Write $\widetilde{\Lambda}_{\mathcal{O}, P}$ for the localization *and* completion of $\widetilde{\Lambda}_{\mathcal{O}}$ at P. Note that $P = \widetilde{\mathfrak{p}} \cap \widetilde{\Lambda}_{\mathcal{O}}$. Let $\mathcal{Q}_0$ be the set of primes $\mathfrak{q}$ of $\widetilde{\Lambda}_{\mathcal{O}}$ of dimension two contained in P and such that each quotient $\widetilde{\Lambda}_{\mathcal{O}}/\mathfrak{q}$ is again a regular ring. It is easy to see that if $\mathfrak{q} \in \mathcal{Q}_0$, then $\mathfrak{q}\widetilde{\Lambda}_{\mathcal{O}, P}$ is also a prime ideal and that the set $\mathcal{Q}_0$ is Zariski-dense in spec $(\widetilde{\Lambda}_{\mathcal{O}, P})$.

Let Q be a minimal prime of $\widetilde{\mathbf{T}}_{\mathcal{D}}^{\min}$ contained in $\widetilde{\mathfrak{p}}$. Let B be the integral closure of $\widetilde{\mathbf{T}}_{\mathcal{D}}^{\min}/Q$ in its field of fractions. Observe that B is a finite, torsion-free $\widetilde{\Lambda}_{\mathcal{O}}$-algebra (cf. [G, 7.8.3]). Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the (finitely many) primes of B extending $\widetilde{\mathfrak{p}}$. For each $i = 1, \ldots, s$ let $\mathcal{Q}_i$ be the set of primes of B of dimension two contained in $\mathfrak{p}_i$ and extending some prime in $\mathcal{Q}_0$. Note that given any $\mathfrak{q} \in \mathcal{Q}_0$ there exists a prime $\mathfrak{Q} \in \mathcal{Q}_i$ extending $\mathfrak{q}$ (cf. [Mat, Theorem 9.4(ii)]). For each $i = 1, \ldots, s$ we have a commutative diagram

(7.4)
$$\begin{array}{ccc} \widetilde{\Lambda}_{\mathcal{O}, P} & \hookrightarrow & \prod_{\mathfrak{q} \in \mathcal{Q}_0} \widetilde{\Lambda}_{\mathcal{O}, P}/\mathfrak{q} \\ \downarrow & & \downarrow \\ \widehat{B}_{\mathfrak{p}_i} & \hookrightarrow & \prod_{\mathfrak{Q} \in \mathcal{Q}_i} \widehat{B}_{\mathfrak{p}_i}/\mathfrak{Q}. \end{array}$$

The arrows are the obvious maps. That the top arrow is an injection follows from the observation in the preceding paragraph. That the rightmost vertical arrow is also an injection follows from the fact that $\widehat{B}_{\mathfrak{p}_i}$ is a finite $\widetilde{\Lambda}_{\mathcal{O}, P}$-algebra and that each $\widetilde{\Lambda}_{\mathcal{O}, P}/\mathfrak{q}$ is a one-dimensional domain. If the bottom arrow were not an injection, then neither would be its composition with the leftmost vertical arrow, since $\widehat{B}_{\mathfrak{p}_i}$ is a domain and an integral extension of $\widetilde{\Lambda}_{\mathcal{O}, P}$ (cf. [G, 7.8.3]).

Choose $\mathfrak{Q} \in \mathcal{Q}_i$. Let $\mathbf{T} = B/\mathfrak{Q}$ and let R be the integral closure of $\mathbf{T}$ in its field of fractions. Let $\mathfrak{q} = \mathfrak{Q} \cap \widetilde{\Lambda}_{\mathcal{O}}$ and let $\Lambda = \widetilde{\Lambda}_{\mathcal{O}}/\mathfrak{q}$. Note that Λ is regular. Since $\mathbf{T}$ is a finite, torsion-free Λ-algebra so is R. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_{t_i}$ be the (finitely many) primes of R extending $\mathfrak{p}_i$. By Proposition 2.15, for each $j = 1, \ldots, t_i$ there exists an extension R_j^+ of R, R_j^+ a domain, an extension $\mathcal{O}_j$ of $\mathcal{O}$, and a prime $\mathfrak{P}_j^+$ of R_j^+ extending $\mathfrak{P}_j$ such that there exists a deformation ρ_j into $GL_2(R_j^+)$ that is $(\mathcal{O}_j, \Sigma, c, \mathcal{M})$-minimal and whose associated pseudo-deformation is just that obtained from the one into $\mathbf{T}_{\mathcal{D}}^{\min}$ described in §3.3 via the obvious map $\mathbf{T}_{\mathcal{D}}^{\min} \to \mathbf{T}$. We have natural injections

$$\mathbf{T}_{\mathfrak{p}_i} \hookrightarrow \prod_{j=1}^{t_i} R_{\mathfrak{P}_j}$$

and

$$\prod_{j=1}^{t_i} R_{\mathfrak{P}_j} \hookrightarrow \prod_{j=1}^{t_i} R^+_{j, \mathfrak{P}_j^+}.$$

We claim that both maps remain injective upon passing to completions of the rings in question. For the first map this is an easy consequence of the fact that both $\mathbf{T}$ and R are finite Λ-algebras. For the second map we note that each $R_{\mathfrak{P}_j}$ is an integrally closed one-dimensional domain and hence so is each $\widehat{R}_{\mathfrak{P}_i}$. On the other hand, each $\widehat{R}^+_{j,\mathfrak{P}_i^+}$ is reduced, so the kernel of the induced map $\widehat{R}_{\mathfrak{P}_j} \to \widehat{R}^+_{j,\mathfrak{P}_j^+}$ must be either (0) or $\mathfrak{P}_j$. It cannot be the latter, as we have $R_{\mathfrak{P}_j} \hookrightarrow R^+_{j,\mathfrak{P}_j^+} \hookrightarrow \widehat{R}^+_{i,\mathfrak{P}_i^+}$. This proves the desired injectivity. We have an induced injection

$$\text{(7.5a)} \qquad \widehat{\mathbf{T}}_{\mathfrak{p}_i} \hookrightarrow \prod_{j=1}^{t_i} \widehat{R}^+_{j,\mathfrak{P}_j^+}.$$

We also have a map $R^{\min}_{\mathscr{D}} \otimes_{\mathscr{O}} \mathscr{O}_j \to R^+_j$ inducing ρ^+_j. It is easy to see that the inverse image of $\mathfrak{P}^+_j$ in $R^{\min}_{\mathscr{D}}$ is just $\mathfrak{p}_{\mathscr{D}}$. We thus have a map

$$\text{(7.5b)} \qquad (\widetilde{R}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}_{\mathscr{D}}} \to \prod_{j=1}^{t_i} \widehat{R}^+_{j,\mathfrak{P}_j^+}.$$

Composing (7.5a) with the map $(\widetilde{R}_{\mathscr{D}\,\mathrm{ps}})_{\widetilde{\mathfrak{p}}^{\mathrm{ps}}} \to \widehat{\mathbf{T}}_{\mathfrak{p}_i}$ coming from $\pi^{\min}_{\mathscr{D}}$ and composing (7.5b) with the map $(\widetilde{R}_{\mathscr{D}\,\mathrm{ps}})_{\widetilde{\mathfrak{p}}^{\mathrm{ps}}} \twoheadrightarrow (\widetilde{R}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}_{\mathscr{D}}}$ (see Proposition 2.11 for surjectivity) yields the same map. Therefore we have a map

$$(\widetilde{R}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}_{\mathscr{D}}} \to \widehat{\mathbf{T}}_{\mathfrak{p}_i}$$

through which $(\widetilde{R}_{\mathscr{D}\,\mathrm{ps}})_{\widetilde{\mathfrak{p}}^{\mathrm{ps}}} \to \widehat{\mathbf{T}}_{\mathfrak{p}_i}$ factors. Combining this with the injectivity of the bottom row of (7.4) yields a map

$$(\widetilde{R}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}_{\mathscr{D}}} \to \prod_{i=1}^{s} \widehat{B}_{\mathfrak{p}_i}$$

through which the map $(\widetilde{R}_{\mathscr{D}\,\mathrm{ps}})_{\widetilde{\mathfrak{p}}^{\mathrm{ps}}} \to \prod_{i=1}^{s} \widehat{B}_{\mathfrak{p}_i}$ coming from $\pi^{\min}_{\mathscr{D}}$ factors. The image of this last map is just that of the natural map from $(\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}/Q$. As the latter map is injective, we obtain a surjection

$$(\widetilde{R}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}_{\mathscr{D}}} \twoheadrightarrow (\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}/Q$$

through which the map $(\widetilde{R}_{\mathscr{D}\,\mathrm{ps}})_{\widetilde{\mathfrak{p}}^{\mathrm{ps}}} \twoheadrightarrow (\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}/Q$ coming from $\pi^{\min}_{\mathscr{D}}$ factors. From this we obtain a map

$$\text{(7.6)} \qquad (\widetilde{R}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}_{\mathscr{D}}} \to \prod (\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}/Q$$

where the product is over the minimal primes of $\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}}$ contained in $\widetilde{\mathfrak{p}}$. Since the maps $(\widetilde{\mathrm{R}}_{\mathscr{D}^{\mathrm{ps}}})_{\widetilde{\mathfrak{p}}^{\mathrm{ps}}} \twoheadrightarrow (\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}/\mathrm{Q}$ factor through the surjection $(\widetilde{\mathrm{R}}_{\mathscr{D}^{\mathrm{ps}}})_{\widetilde{\mathfrak{p}}^{\mathrm{ps}}} \twoheadrightarrow (\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}$ coming from $\pi^{\min}_{\mathscr{D}}$, the image in (7.6) is the same as that of the natural map from $(\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}$. This latter map is injective, and we therefore have a surjection

$$(\widetilde{\mathrm{R}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}_{\mathscr{D}}} \twoheadrightarrow (\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}$$

through which the map $(\widetilde{\mathrm{R}}_{\mathscr{D}^{\mathrm{ps}}})_{\widetilde{\mathfrak{p}}^{\mathrm{ps}}} \twoheadrightarrow (\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}$ factors. □

Remark **7.2.** — The same result holds for $\mathscr{D}$ replaced by any of the auxiliary deformation data $\mathscr{D}_{\mathrm{Q}}$, with the same proof.

Since $\mathfrak{p}$ is nice for $\mathscr{D}$, there is some choice of basis for $\rho_{\mathfrak{p}}$ such that $\rho_{\mathfrak{p}}$ has image in $\mathrm{GL}_2(\mathrm{A})$ and the corresponding representation into $\mathrm{GL}_2(\mathrm{A})$ is a deformation of type-$(\mathscr{O}', \Sigma, c, \mathscr{M})$. From here on we write $\rho_{\mathfrak{p}}$ for this deformation.

We will assume for the rest of this section that $\mathscr{D} = \mathscr{D}_c$ (i.e., that $\Sigma = \Sigma_c$ is the set of primes at which ρ_c ramifies together with $\mathscr{P} = \{v_i : v_i | p\}$, and that $\mathscr{M} = \mathscr{M}_c = \Sigma_c \backslash \mathscr{P}$).

Next we define the rings and modules which will be used for patching. Let $\widetilde{\mathrm{R}}_{\mathscr{D}^{\mathrm{ps}}}$ denote the ring $\mathrm{R}_{\mathscr{D}^{\mathrm{ps}}} \otimes_{\Lambda_{\mathscr{O}}} \widetilde{\Lambda}_{\mathscr{O}}$ and let $\widetilde{\mathrm{M}_{\mathscr{D}}} = \mathrm{M}_{\mathscr{D}} \otimes_{\Lambda_{\mathscr{O}}} \widetilde{\Lambda}_{\mathscr{O}}$. Then we define

$$\mathrm{N}^{(0)} = \mathrm{im}\{\widetilde{\mathrm{M}_{\mathscr{D}}} \longrightarrow (\widetilde{\mathrm{M}_{\mathscr{D}}})_{\widetilde{\mathfrak{p}}}/\mathrm{P}\} \tag{7.7}$$

where $(\widetilde{\mathrm{M}_{\mathscr{D}}})_{\widetilde{\mathfrak{p}}}$ is the localization and completion of $\widetilde{\mathrm{M}_{\mathscr{D}}}$ with respect to $\widetilde{\mathfrak{p}} \subset \widetilde{\mathbf{T}}^{\min}_{\mathscr{D}}$ and $\mathrm{P} \subseteq \widetilde{\Lambda}_{\mathscr{O}}$ is the prime defined in the proof of Lemma 7.1. There is a commutative triangle

(7.8)

$$\begin{array}{ccc} & \widetilde{\mathrm{R}}_{\mathscr{D}^{\mathrm{ps}}} & \\ \swarrow & & \searrow \varphi_1 \\ \widetilde{\mathrm{R}}^{\min}_{\mathscr{D}} & \xrightarrow[\varphi_2]{} & (\widetilde{\mathrm{R}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}/(\widetilde{\mathfrak{p}} \cdot \mathrm{F}_0, \mathrm{P}) \end{array}$$

where $\mathrm{F}_0 = \mathrm{Fitt}((\widetilde{\mathrm{M}_{\mathscr{D}}})_{\widetilde{\mathfrak{p}}}) \subset (\widetilde{\mathrm{R}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}$ is the Fitting ideal of $(\widetilde{\mathrm{M}_{\mathscr{D}}})_{\widetilde{\mathfrak{p}}}$ as an $(\widetilde{\mathrm{R}}^{\min}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}$-module. The maps φ_1 and φ_2 are the obvious ones. We define rings

$$\mathrm{R}^{(0)} = \mathrm{im}(\varphi_2), \qquad \mathrm{R}^{\mathrm{tr}(0)} = \mathrm{im}(\varphi_1). \tag{7.9}$$

Now we introduce auxiliary levels. First we fix a $\sigma \in \mathrm{Gal}(\mathrm{F}_{\Sigma}/\mathrm{F}')$ as in Lemma 6.3(i). Then there exists an integer $r = r(\rho_{\mathfrak{p}})$ as in Proposition 6.10 with the following property. For each odd integer N there is a set of primes $Q_N = \{w_1^{(N)}, \dots, w_r^{(N)}\}$ of F satisfying $\mathrm{Nm}(w_i^{(N)}) \equiv 1 (\mathrm{mod}\, p^N)$ as well as property (ii) and (iii) of Proposition 6.10. We can and do choose the sets Q_N to be disjoint from each other as well as from Σ. For such a set $Q = Q_N$, we earlier introduced a deformation problem $\mathcal{D}_Q$ as well as associated Hecke and deformation rings $\mathbf{T}_{\mathcal{D}_Q}$ and $R_{\mathcal{D}_Q}$. In particular at the end of §3.5 we associated a $\mathbf{T}_{\mathcal{D}_Q}$-module $M_{\mathcal{D}_Q}$ to $\mathcal{D}_Q$. We now set

$$\widetilde{M}_{\mathcal{D}_Q} = M_{\mathcal{D}_Q} \underset{\Lambda_{\mathcal{O}}}{\otimes} \widetilde{\Lambda}_{\mathcal{O}}$$

as before and note that this is a $\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_Q}$-module.

For each $w_i = w_i^{(N)} \in Q = Q_N$ there is an associated element $\delta_{w_i} \in \mathrm{End}_{\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_Q}}(\widetilde{M}_{\mathcal{D}_Q})$ as in Lemma 3.21. We let $s_i = \delta_{w_i} - 1$. We can then define, for each odd integer $1 \leqslant a \leqslant N$,

$$\textbf{(7.10)} \qquad M_a^{(N)} = \mathrm{im}\{\widetilde{M}_{\mathcal{D}_Q} \longrightarrow (\widetilde{M}_{\mathcal{D}_Q})_{\tilde{\mathfrak{p}}}/(P, s_1^{a+1}, \dots, s_r^{a+1})\}$$

where the completion is as a $\widetilde{\mathbf{T}}^{\min}_{\mathcal{D}_Q}$-module (with respect to $\tilde{\mathfrak{p}}$) and $Q = Q_N$. Then $M_a^{(N)}$ is a module over the ring $A_a = A[[s_1, \dots, s_r]]/(s_1^{a+1}, \dots, s_r^{a+1})$ by construction.

Let $\rho_{\mathcal{D}_Q^{\mathrm{ps}}}$ denote the universal pseudo-deformation associated to $\mathcal{D}_Q$. Let $\sigma_{w_i} \in \mathrm{Gal}(\overline{F}/F)$ be such that $\sigma_{w_i} \in I_{w_i}$, the inertia group of w_i, and σ_{w_i} is a generator of the p-part of tame inertia as in Lemma 3.21. Then there is a map of rings

$$\textbf{(7.11)} \qquad \widetilde{\Lambda}_{\mathcal{O}}[[t_1, \dots, t_r]] \longrightarrow \widetilde{R}_{\mathcal{D}_Q^{\mathrm{ps}}} = R_{\mathcal{D}_Q^{\mathrm{ps}}} \underset{\Lambda_{\mathcal{O}}}{\otimes} \widetilde{\Lambda}_{\mathcal{O}}$$

given by $t_i \longmapsto \mathrm{trace}\,(\rho_{\mathcal{D}_Q^{\mathrm{ps}}}(\sigma_{w_i}) - 2) \otimes 1$, where here $Q = Q_N = \{w_1, \dots, w_r\}$. We have a commutative diagram

(7.12)

$$\begin{array}{ccc} & \widetilde{R}_{\mathcal{D}_Q^{\mathrm{ps}}} & \\ \swarrow & & \searrow{\scriptstyle \varphi_{1,a}^{(N)}} \\ \widetilde{R}_{\mathcal{D}_Q^{\min}} & \xrightarrow[\varphi_{2,a}^{(N)}]{} & (\widetilde{R}^{\min}_{\mathcal{D}_Q})_{\tilde{\mathfrak{p}}}/(P, t_1^{\frac{a+1}{2}}, \dots, t_r^{\frac{a+1}{2}}, \tilde{\mathfrak{p}}F_N) \end{array}$$

where $Q = Q_N$ and $F_N = \mathrm{Fitt}((\widetilde{M}_{\mathscr{D}_Q})_{\tilde{\mathfrak{p}}})$ is the Fitting ideal of $(\widetilde{M}_{\mathscr{D}_Q})_{\tilde{\mathfrak{p}}}$ with respect to the ring $(\widetilde{R}^{\min}_{\mathscr{D}_Q})_{\tilde{\mathfrak{p}}}$. The maps are the obvious ones. We define rings $R_a^{(N)}$ and $R_a^{\mathrm{tr}(N)}$ by

(7.13) $$R_a^{(N)} = \mathrm{im}(\varphi_{2,a}^{(N)}), \quad R_a^{\mathrm{tr}(N)} = \mathrm{im}(\varphi_{1,a}^{(N)}).$$

These rings are both algebras over $B_a = A[[t_1, \dots, t_r]]/(t_1^{\frac{a+1}{2}}, \dots, t_r^{\frac{a+1}{2}})$ by construction.

Let $\tilde{\rho}_{\mathscr{D}_Q} : \mathrm{Gal}(\overline{F}/F) \to GL_2((\widetilde{R}^{\min}_{\mathscr{D}_Q})_{\tilde{\mathfrak{p}}})$ be the representation obtained from $\rho_{\mathscr{D}_Q}$. It is easy to see that for each $w_i \in Q$, $\tilde{\rho}_{\mathscr{D}_Q}|_{I_{w_i}} \simeq \begin{pmatrix} \psi_i & \\ & \psi_i^{-1} \end{pmatrix}$ for some character ψ_i. Define a map $A[[s_1, \dots, s_n]] \to R_a^{(N)} \otimes_A K$ by $s_i \mapsto \psi_i(\sigma_{w_i}) - 1$. This makes $R_a^{(N)} \otimes_A K$ into an $A_a \otimes_A K$-algebra compatibly with its structure as a B_a-algebra. Moreover, it is obvious that each s_i maps to 0 under the canonical map $R_a^{(N)} \otimes_A K \to R^{(0)} \otimes_A K$.

The action of $R_a^{\mathrm{tr}(N)}$ on $M_a^{(N)}$ is obtained as follows. First $\widetilde{T}^{\min}_{\mathscr{D}_Q}$ acts on $\widetilde{M}_{\mathscr{D}_Q}$ whence it acts on $M_a^{(N)}$ through the image of $\widetilde{T}^{\min}_{\mathscr{D}_Q}$ in $(\widetilde{T}^{\min}_{\mathscr{D}_Q})_{\tilde{\mathfrak{p}}}$. Now we have homomorphisms

(7.14)
$$\begin{array}{ccccc} R^{ps}_{\mathscr{D}_Q} & \longrightarrow & \widetilde{T}^{\min}_{\mathscr{D}_Q} & \longrightarrow & (\widetilde{T}^{\min}_{\mathscr{D}_Q})_{\tilde{\mathfrak{p}}}/(P, t_1^{\frac{a+1}{2}}, \dots, t_r^{\frac{a+1}{2}}, \tilde{\mathfrak{p}} F_N) \\ & \searrow & & \nearrow {\scriptstyle \psi(\mathscr{D}_Q, \mathfrak{p})} & \\ & & R_a^{\mathrm{tr}(N)} \subseteq (\widetilde{R}^{\min}_{\mathscr{D}_Q})_{\tilde{\mathfrak{p}}}/(P, t_1^{\frac{a+1}{2}}, \dots, t_r^{\frac{a+1}{2}}, \tilde{\mathfrak{p}} F_N) & & \end{array}$$

and the diagram commutes by Lemma 7.1 and the remark following it. So by Lemma 3.21 we get an induced action of $R_a^{\mathrm{tr}(N)}$ on $M_a^{(N)}$ which is compatible with the A_a-action of the subring B_a. Now put

$$M^{(0)} = \bigoplus_{i=1}^{2^r} N^{(0)}$$

where $N^{(0)}$ is as in (7.7). The same reasoning shows that $R^{\mathrm{tr}(0)}$ acts on $M^{(0)}$, and a diagram as in (7.14) holds for $N = 0$. We define $R_0^{(N)}$ and $R_0^{\mathrm{tr}(N)}$ by

$$R_0^{(N)} = R^{(0)} \quad \text{and} \quad R_0^{\mathrm{tr}(N)} = R^{\mathrm{tr}(0)}.$$

To define $M_0^{(N)}$ we first define a natural map $\widetilde{M}_{\mathscr{D}_Q} \longrightarrow \widetilde{M}_{\mathscr{D}}^{2^r}$ $(Q = Q_N)$. Let $\mathscr{D}_0 = \mathscr{D}$, $\mathscr{D}_i = \mathscr{D}_{\pi_i}$ where $\pi_i = \{w_1^{(N)}, \dots, w_i^{(N)}\}$. For each $0 \leqslant i < r$ define a map $H^+_\infty(U^{\min}_{\mathscr{D}_i})^2 \longrightarrow H^+_\infty(U^{\min}_{\mathscr{D}_{i+1}})$ by $(f_1(g), f_2(g)) \longmapsto f_1(g) + f_2\left(g\begin{pmatrix} 1 & \\ & \lambda_{i+1} \end{pmatrix}\right)$ where

$\lambda_{i+1} \in \mathscr{O}_F \otimes \widehat{\mathbf{Z}}$ is the element chosen in the definition of the Hecke operator $T(\ell_{w_{i+1}})$. Repeated composition of these maps gives a map $H^+_\infty(U_{\mathscr{D}})^{2^r} \longrightarrow H^+_\infty(U_{\mathscr{D}_Q})$ and taking Pontryagin duals and tensoring with $\widetilde{\Lambda}_{\mathscr{O}}$ gives the desired map $\widetilde{M}_{\mathscr{D}_Q} \longrightarrow \widetilde{M}^{2^r}_{\mathscr{D}}$. We define $M_0^{(N)}$ to be the image of $\widetilde{M}_{\mathscr{D}_Q} \longrightarrow (\widetilde{M}_{\mathscr{D}})^{2^r}_{\tilde{\mathfrak{p}}}/P$. Clearly $M_0^{(N)} \subseteq M^{(0)}$.

We now verify that these constructions satisfy the properties in §5 needed for formal patching. A bound for the number of generators of $R^{(N)} = R_N^{(N)}$ is given by $\dim_k(H^1(F_{\Sigma\cup Q}/F, \mathrm{ad}^0\rho_c))$, which is easily bounded independent of N. A bound for the number of generators of $R^{\mathrm{tr}(N)} = R_N^{\mathrm{tr}(N)}$ is more subtle. When $N = 0$ this follows from Lemma 2.10, which bounds the number of generators of $R_{\mathscr{D}^{\mathrm{ps}}}$. In general a similar argument applies using $R_{\mathscr{D}^{\mathrm{ps}}_Q}$, and a uniform bound independent of N can be given for example by applying Lemma 2.10 with Σ replaced by $\Sigma \cup Q$. We can choose the generators in each case so that (5.2ii) and (5.6ii) hold by subtracting suitable elements of $\mathscr{O}[[W_1]]$. The other properties in (5.2) and (5.4) follow from the definitions.

Next we consider the properties (5.10) of $M_a^{(N)}$. Properties (iii)-(v) are straightforward, as are the first two assertions of (ii), but we need to check (i), (vi), and the last assertion of (ii). Property (vi) follows immediately from Lemma 3.19 provided the hypothesis that $U'/F^\times \cap U'$ acts without fixed points on $D^\times \backslash G^D(\mathbf{A}_f)$ holds. Here we need to take U successively as $U_{\mathscr{D}_0}, U_{\mathscr{D}_1}, \dots, U_{\mathscr{D}_r} = U_{\mathscr{D}_Q}$ where $U_{\mathscr{D}_0} = U_{\mathscr{D}}$, $\mathscr{D}_i = \mathscr{D}_{\pi_i}$, and $\pi_i = \{w_1^{(N)}, \dots, w_i^{(N)}\}$ and check the conditions of Lemma 3.19 with $v = w_{i+1}^{(N)}$ for $U = U_{\mathscr{D}_i}$. However these conditions need not hold, and instead we consider modules with an auxiliary level structure at primes $\ell_1, \dots, \ell_s \notin \Sigma \cup Q$, chosen so that $M_\infty(U''_{\mathscr{D}_i})$, with $U'' = U_{\mathscr{D}_i} \cap U_1(\ell_1 \cdots \ell_s)$, is related to $M_\infty(U_{\mathscr{D}_i})$ in a simple way.

To achieve this, pick primes $\ell_1, \dots, \ell_s$ satisfying the hypotheses of Corollary 3.6 as well as satisfying the conditions

(7.15) (i) $\mathrm{Nm}(\ell_i) \not\equiv 1 (\mathrm{mod}\, p)$

(ii) $\rho_{\mathfrak{p}}(\mathrm{Frob}_{\ell_i}) \equiv \rho_{\mathfrak{p}}(\sigma_0) \bmod \lambda^e$ for e sufficiently large, where σ_0 is chosen as in Lemma 6.3(ii).

(In order that condition (ii) make sense, we identify $\rho_{\mathfrak{p}}$ with $\rho_{\mathscr{D}} \bmod \mathfrak{p}_{\mathscr{D}}$, which in turn we view as taking values in $GL_2(A)$.) Condition (ii), for a sufficiently large, ensures that

$$\textbf{(7.16)} \qquad \mathrm{trace}\, \rho_{\mathfrak{p}}(\mathrm{Frob}_{\ell_i})^2 \not\equiv \det \rho_{\mathfrak{p}}(\mathrm{Frob}_{\ell_i})(1 + \mathrm{Nm}(\ell_i))^2 \mathrm{Nm}(\ell_i)^{-1}.$$

This, together with (i), ensures that

$$M_\infty(U''_{\mathscr{D}_i})_{\mathfrak{p}} \simeq M_\infty(U_{\mathscr{D}_i})^{2^s}_{\mathfrak{p}}$$

by Lemma 3.29, where the isomorphism is of $\Lambda_{\mathscr{O}, \mathfrak{p}_0}[\Delta_{w_i}]$-modules. Now Lemma 3.19 and Corollary 3.6 can be used to check that $M_\infty(U''_{\mathscr{D}_i})_{\mathfrak{p}}$ is free over $\Lambda_{\mathscr{O}, \mathfrak{p}_0}[\Delta_{w_i}]$ for each $w_i = w_i^{(N)} \in Q_N$. We then deduce the same result for $M_\infty(U_{\mathscr{D}_i})_{\mathfrak{p}}$ and so also for $(\widetilde{M}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}$ and $M_a^{(N)} \underset{\mathscr{O}}{\otimes} K$ over $\widetilde{\Lambda}_{\mathscr{O},P}[\Delta_{w_i}]$ and $A_a \otimes_A K$ respectively. The second property of (5.10vi) follows similarly from Lemma 3.19(ii).

Property (5.10i) follows from property (5.10vi) which was just established. It remains to show that the last assumption of (5.10ii) holds. For this we pick some $\sigma_1 \in \mathrm{Gal}(\overline{F}/F^{ab})$ such that $\rho_{\mathfrak{p}}(\sigma_1)$ has infinite order. That such a σ_1 exists is an easy consequence of Lemma 6.3. Let $\rho_{\mathscr{D}}^{\min}$ be the pseudo-deformation associated to $\mathbf{T}_{\mathscr{D}}^{\min}$. It follows from Lemma 3.26 that $z = \mathrm{trace}\, \rho_{\mathscr{D}}^{\min}(\sigma_i) - 2$ annihilates the cokernel of the map $\widetilde{M}_{\mathscr{D}_Q} \longrightarrow \widetilde{M}_{\mathscr{D}}^{2^r}$ in the definition of $M_0^{(N)}$. Therefore z also annihilates the cokernel of $\widetilde{M}_{\mathscr{D}_Q} \longrightarrow M^{(0)}$ and hence also that of $M_0^{(N)} \longrightarrow M^{(0)}$. By our choice of σ_1, $z \notin \widetilde{\mathfrak{p}}$. Note that z is independent of N.

The properties in (5.9) are consequences of those in (5.10) as well as of the definitions of the $M^{(N)}$'s.

Next we verify properties (5.5i) and (5.3i). Let $d_1(a) = \dim_K(M_a^{(N)} \otimes_A K)$. This is independent of N by (5.10vi). Again using (5.10vi)

$$d_1(1) \geqslant \mu^{(N)}_{R_a \otimes_A K}(M_a^{(N)} \otimes_A K)$$

where as before $\mu_S(X)$ denotes the minimal number of generators of the S-module X. Now $\widetilde{\mathfrak{p}}^{d_1(a)}$ annihilates $M_a^{(N)} \otimes_A K$ and hence

$$\widetilde{\mathfrak{p}}^{d_1(a)d_1(1)} \subseteq \mathrm{Fitt}_{R_a^{(N)} \otimes_A K}(M_a^{(N)} \otimes_A K).$$

From the definition of $R_a^{(N)} \otimes_A K$ it follows that $\widetilde{\mathfrak{p}}^{d_1(a)d_1(1)+1} = 0$ in this ring so we may take $d(a) = d_1(a)d_1(1) + 1$.

Now we check (5.5ii) and (5.3ii). Recall that we are given a set $Q_N = \{w_1^{(N)}, \ldots, w_r^{(N)}\}$ of primes satisfying the hypotheses of Proposition 6.10, and that by the same proposition

(7.17) $$\varinjlim_n H_{\Sigma_Q}(\mathscr{T}_n) \simeq (K/A)^r \oplus X_{\Sigma_Q}$$

with $\Sigma_Q = \Sigma \cup Q_N$ and $\# X_{\Sigma_Q}$ bounded independent of N. Now let $Q = Q_N$ and suppose that $\widetilde{\mathfrak{p}}_{\mathscr{D}_Q}$, $\widetilde{\mathfrak{p}}^{\min}_{\mathscr{D}_Q}$ are the primes corresponding to $\mathfrak{p}$ in $\widetilde{R}_{\mathscr{D}_Q}$ and $\widetilde{R}^{\min}_{\mathscr{D}_Q}$. Then we also have the usual isomorphism in the style of [W1, Proposition 1.2]

(7.18) $$\mathfrak{X}_Q = \mathrm{Hom}_A\left((\widetilde{\mathfrak{p}}^{\min}_{\mathscr{D}_Q}/(\widetilde{\mathfrak{p}}^{\min}_{\mathscr{D}_Q})^2, P), \lambda^{-n}A/A\right) \xrightarrow{\sim} H_{\Sigma_Q}(\mathscr{T}_n).$$

The isomorphism is obtained as follows. To an element $\varphi \in \mathfrak{X}_Q$ we associate the representation

(7.19) $$\rho_\varphi : \mathrm{Gal}(\overline{F}/F) \longrightarrow GL_2(\widetilde{R}^{\min}_{\mathscr{D}_Q}/((\widetilde{\mathfrak{p}}^{\min}_{\mathscr{D}_Q})^2, P, \ker \varphi).$$

This is a deformation of $\rho_{\mathfrak{p}}$ with values in $A[\varepsilon]/(\lambda^n\varepsilon, \varepsilon^2)$ and its associated cohomology class lies in $H_{\Sigma_Q}(\mathcal{T}_n)$. Conversely to such a cohomology class, we obtain a deformation with values in $A[\varepsilon]/(\lambda^n\varepsilon, \varepsilon^2)$, and hence by universality a homomorphism $R_{\mathcal{D}_Q} \otimes_{\mathcal{O}} \mathcal{O}' \longrightarrow A[\varepsilon]/(\lambda^n\varepsilon, \varepsilon^2)$. Extending scalars we get an A-algebra homomorphism $\widetilde{R}_{\mathcal{D}_Q}/P \longrightarrow A[\varepsilon]/(\lambda^n\varepsilon, \varepsilon^2)$ which factors through $\widetilde{R}^{\min}_{\mathcal{D}_Q}/P$. Restricting to $\widetilde{\mathfrak{p}}^{\min}_{\mathcal{D}}$ we recover an element of $\mathfrak{X}_Q$. The restrictions on classes in $H_{\Sigma_Q}(\mathcal{T}_n)$ correspond to the restrictions required for ρ to be of type $\mathcal{D}^{\min}_Q$. In particular, the "min" condition, together with reduction mod P, ensure that ρ_φ does not deform the determinant. This is why the cohomology class we obtain is associated to $\mathcal{T}_n = \mathrm{ad}^0\rho/\lambda^n$ rather than $\mathrm{ad}\,\rho/\lambda^n$. We omit the details and refer to [W1] for a more detailed argument in a similar situation. Of course (7.18) also holds with $\mathcal{D}_Q$ replaced by $\mathcal{D}$ and Σ_Q by Σ.

To apply this we use the sequence of homomorphisms

$$\widetilde{\mathfrak{p}}^{\min}_{\mathcal{D}_Q}/((\widetilde{\mathfrak{p}}^{\min}_{\mathcal{D}_Q})^2, P) \longrightarrow \mathfrak{p}^{(N)}/(\mathfrak{p}^{(N)})^2 \longrightarrow \mathfrak{p}^{(N)}_a/(\mathfrak{p}^{(N)}_a)^2 \longrightarrow \mathfrak{p}^{(0)}/(\mathfrak{p}^{(0)})^2$$

$$\widetilde{\mathfrak{p}}^{\min}_{\mathcal{D}_Q}/((\widetilde{\mathfrak{p}}^{\min}_{\mathcal{D}_Q})^2, P) \xrightarrow{\ \beta_Q\ } \widetilde{\mathfrak{p}}^{\min}_{\mathcal{D}}/((\widetilde{\mathfrak{p}}^{\min}_{\mathcal{D}})^2, P) \xrightarrow{\ \beta_0\ } \mathfrak{p}^{(0)}/(\mathfrak{p}^{(0)})^2$$

Here $Q = Q_N$. The horizontal maps are surjections arising from the definitions in (7.9) and (7.13). The maps β_Q and β_0 are surjective and β_0 is an isomorphism after tensoring with K. That β_Q is also an isomorphism upon tensoring with K follows from Proposition 6.10 and (7.18). Properties (5.5ii) and (5.3ii) then follow from (7.17) and (7.18).

Next we verify property (5.7). Using (7.18) the condition in (5.7) translates into the requirement that $\#G_N$, where

$$G_N = \{[\rho_\varphi] : \mathrm{trace}\ \rho_\varphi = \mathrm{trace}\ \rho_{\mathfrak{p}}, \quad \varphi \in \mathfrak{X}_Q\},$$

is bounded independent of N and n. (Here as before $Q = Q_N$). Fix a basis for $\rho_{\mathfrak{p}}$ such that $\rho_{\mathfrak{p}}(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $\rho_{\mathfrak{p}}(\sigma') = \begin{pmatrix} * & u \\ * & * \end{pmatrix}$ for some unit $u \in A^\times$, where z_1 is a complex conjugation and σ' is some element of $\mathrm{Gal}(\overline{F}/F)$. With respect to this basis write $\rho_{\mathfrak{p}}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$. Fix a τ such that $c_\tau \neq 0$. Now suppose that $[\rho_\varphi]$ is a class in G_N. The class $[\rho_\varphi]$ has a representative ρ_φ such that

$$\rho_\varphi(\sigma) = \begin{pmatrix} a_\sigma & b'_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$$

and $b'_\sigma c_\tau \in A$ for all $\sigma \in \mathrm{Gal}(\overline{F}/F)$. Hence c_τ annihilates $[\rho_\varphi]$. Since the number of generators of G_N over A is bounded independent of N and n (as follows, for example, from (7.17) and (7.18)) we obtain the desired bound on $\#G_N$.

It remains to prove the existence of an element $x^{(N)}$ as in (5.11). Let us write M for $\widetilde{M}_{\mathscr{D}}$, which is a $\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}}$-module.

Let $J = \ker\{(M/P)^{2^r} \twoheadrightarrow M^{(0)}\}$. Since $M^{(0)} \hookrightarrow (\widetilde{M}_{\mathscr{D}})^{2^r}_{\tilde{\mathfrak{p}}}/P$ by definition, J is just the kernel of $M^{2^r}/P \longrightarrow (M/P)^{2^r}_{\tilde{\mathfrak{p}}}$. Let $m_1, \ldots, m_s$ be generators of J as a $\widetilde{\mathbf{T}}_{\mathscr{D}}$-module. For each i, choose $x_i \in \widetilde{\mathbf{T}}^{\min}_{\mathscr{D}}$, $x_i \notin \tilde{\mathfrak{p}}$, such that $x_i \cdot m_i = 0$. Put $x = x_1 \cdots x_s$. Clearly, x annihilates J. Also, $x \notin \tilde{\mathfrak{p}}$ since each $x_i \notin \tilde{\mathfrak{p}}$.

Now set $M_N = \widetilde{M}_{\mathscr{D}_Q}$ where $Q = Q_N$. Then suppose that we have an element $y^{(N)} \in \widetilde{\mathbf{T}}^{\min}_{\mathscr{D}_Q}$ with the property that

$$y^{(N)} \cdot \ker\{M_N/(s_1, \ldots, s_r) \longrightarrow (M/P)^{2^r}\} = 0. \tag{7.20}$$

It would follow that $\tilde{x}y^{(N)}$ would annihilate $\ker\{M_N/(s_1, \ldots, s_r) \longrightarrow M^{(0)}\}$ where $\tilde{x}$ is a lift of x to $\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}_Q}$, and so $\tilde{x}y^{(N)}$ would also annihilate $\ker\{M_a^{(N)}/(s_1, \ldots, s_r) \longrightarrow M^{(0)}\}$. Thus it would satisfy condition (i) of (5.11) except of course that it is not in the desired ring.

Our construction of such a $y^{(N)}$ is an involved procedure. We begin by introducing auxiliary level structures much as we did in the proof of property (5.10vi). Let $\ell_1, \ldots, \ell_s \notin \Sigma \cup Q$, where $Q = Q_N$, be primes satisfying the hypotheses of Corollary 3.6. as well as (7.15) and (7.16). We can and do choose the ℓ_i to be independent of N. Now let $U'_{\mathscr{D}_Q} = U^{\min}_{\mathscr{D}_Q} \cap U_1(\ell_1 \cdots \ell_s)$ and put $M'_N = M_\infty(U'_{\mathscr{D}_Q})_{\mathfrak{m}} \otimes_{\Lambda_{\mathscr{O}}} \tilde{\Lambda}_{\mathscr{O}}$. Let also $\mathscr{D}' = (\mathscr{O}, \Sigma', c, \mathscr{M})$ with $\Sigma' = \Sigma \cup \{\ell_1, \ldots, \ell_s\}$. Let $\mathbf{T}'_N$ denote $\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}'_Q}$. It is clear that M'_N is a $\mathbf{T}'_N$-module.

There is a natural map $M'_N \longrightarrow M_N^{2^s}$ defined analogously to the map $M_N \longrightarrow M^{2^r}$ used in the definition of $M_0^{(N)}$. Composing these maps we obtain a similar map $M'_N \longrightarrow M^{2^{r+s}}$. Arguing just as in the verification of (5.10ii) (see the first full paragraph following (7.16)) we find that there is some $y_1^{(N)} \in \widetilde{\mathbf{T}}^{\min}_{\mathscr{D}_Q}$ such that

$$y_1^{(N)} \cdot \operatorname{coker}\{M'_N \longrightarrow M_N^{2^s}\} = 0. \tag{7.21a}$$

and

$$0 \neq \operatorname{ord}_\lambda(y_1^{(N)} \bmod \tilde{\mathfrak{p}}) \quad \text{is independent of N.} \tag{7.21b}$$

We next construct $y_2^{(N)} \in \mathbf{T}'_N$ such that

$$y_2^{(N)} \cdot \ker\{M'_N/(s_1, \ldots, s_r) \longrightarrow (M/P)^{2^{r+s}}\} = 0 \tag{7.22}$$

for then $y^{(N)} = y_1^{(N)} \cdot y_2^{(N)}$ will satisfy (7.20). (We view s_i as an element of $\operatorname{End}(M'_N)$ just as we did for M_N. These actions are compatible under the map $M'_N \longrightarrow M_N^{2^r}$.)

Let $U''_{\mathscr{D}_Q} = U^{\min}_{\mathscr{D}} \cap U_0(w_1, \dots, w_s) \cap U_1(\ell_1, \dots, \ell_r)$ and write M''_N for the module $M_\infty(U''_{\mathscr{D}_Q})_{\mathfrak{m}} \otimes_{\Lambda_{\mathscr{O}}} \widetilde{\Lambda}_{\mathscr{O}}$. By our choice of $\ell_1, \dots, \ell_r$, $U''_{\mathscr{D}_Q}$ satisfies the hypotheses of Lemma 3.19, from which we deduce that

$$M'_N/(s_1, \dots, s_r) = M''_N.$$

Let I_N be the set of minimal primes $\mathfrak{q}$ of $\mathbf{T}'_N$ such that $M''_{N,\mathfrak{q}} \neq 0$. Let $I \subseteq I_N$ be the subset consisting of the inverse images of the minimal primes of $\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}}$. It is relatively straightforward to see that for $\mathfrak{q} \in I_N \backslash I$ the representation $\rho_{\mathfrak{q}}$ is type A at each of the primes ℓ_i and $w_j^{(N)}$. If $\mathscr{L}$ is the field of fractions of $\widetilde{\Lambda}_{\mathscr{O}}$ then we have

$$M''_N \hookrightarrow M''_N \otimes_{\widetilde{\Lambda}_{\mathscr{O}}} \mathscr{L} = \prod_{\mathfrak{q} \in I_N} M''_{N,\mathfrak{q}}$$
$$M \hookrightarrow M \otimes_{\widetilde{\Lambda}_{\mathscr{O}}} \mathscr{L} = \prod_{q \in I} M_{\mathfrak{q}}.$$

The kernel of the map $M''_N \otimes_{\widetilde{\Lambda}_{\mathscr{O}}} \mathscr{L} \longrightarrow (M \otimes_{\widetilde{\Lambda}_{\mathscr{O}}} \mathscr{L})^{2^{r+s}}$ is just $\prod_{q \in I_N \backslash I} M''_{N,\mathfrak{q}}$. Therefore, if $y_3^{(N)} \in \bigcap_{q \in I_N \backslash I} \mathfrak{q}$, then

$$\textbf{(7.23)} \qquad y_3^{(N)} \cdot \ker(M''_N \longrightarrow M^{2^{r+s}}) = 0.$$

We choose $y_3^{(N)}$ as follows. Let ρ'_N be the pseudo representation associated to $\mathbf{T}'_N$. Let $\tau_i \in \mathrm{Gal}(\overline{F}/F)$ be a lift of $\mathrm{Frob}_{w_i(N)}$ and let δ_i be a lift of Frob_{ℓ_i}. Put

$$y_3^{(N)} = \prod_{i=1}^{r} \left(T^2_{w_i} - d_{w_i}(q_i^{(N)} + 1)^2\right) \cdot \prod_{i=1}^{s} \left(T^2_{\ell_i} - d_{\ell_i}(\mathrm{Nm}(\ell_i) + 1)^2\right)$$

where $T_{w_i} = \mathrm{trace}\, \rho'_N(\tau_i)$, $q_i(N) = \mathrm{Nm}(w_i^{(N)})$, $d_{w_i} = \det \rho'_N(\tau_i) \cdot q_i(N)^{-1}$ and similarly for T_{ℓ_i} and d_{ℓ_i}. Then $y_3^{(N)} \in \mathfrak{q}$ for every $\mathfrak{q} \in I_N \backslash I$ as can be seen by examining the actions of D_{w_i} and D_{ℓ_i} (decomposition groups at $w_i = w_i^{(N)}$ and ℓ_i, respectively) on the Galois representations associated to such primes $\mathfrak{q}$. From our choice of w_i (and our choice of σ as in Proposition 6.10)

$$T^2_{w_i} - d_{w_i}(q_i^{(N)} + 1)^2 \equiv \mathrm{trace}\, \rho_{\mathfrak{p}}(\sigma)^2 - 4 \not\equiv 0 \bmod (\widetilde{\mathfrak{p}}, \lambda^e)$$

for some sufficiently large e independent of N. This, together with (7.16) shows that

$$\textbf{(7.24)} \qquad 0 \neq \mathrm{ord}_{\lambda}(y_3^{(N)} \bmod \widetilde{\mathfrak{p}}) \text{ is independent of N.}$$

Finally arguing as we did to establish (5.10ii) as well as (7.21*a*, *b*) shows that there is some $y_4^{(N)} \in \widetilde{\mathbf{T}}_{\mathscr{D}}^{\min}$ such that

(7.25*a*) $$y_4^{(N)} \cdot \operatorname{coker}\{M''_N \longrightarrow M^{2^{r+s}}\} = 0$$

and

(7.25*b*) $$0 \neq \operatorname{ord}_\lambda(y_4^{(N)} \bmod \widetilde{\mathfrak{p}}) \text{ is independent of N.}$$

Let $\widetilde{y}_3^{(N)}$ and $\widetilde{y}_4^{(N)}$ be lifts of $y_3^{(N)}$ and $y_4^{(N)}$ to $\mathbf{T}'_N$. Combining (7.23) and (7.25*a*) shows that $y_2^{(N)} = \widetilde{y}_3^{(N)} \cdot \widetilde{y}_4^{(N)}$ satisfies (7.22). Moreover by (7.24) and (7.25*b*)

$$0 \neq \operatorname{ord}_\lambda(y_2^{(N)} \bmod \widetilde{\mathfrak{p}}) \quad \text{is independent of N.}$$

We may then take $x^{(N)}$ to be any lift of $\widetilde{x} \cdot y^{(N)}$ to $\widetilde{R}_{\mathscr{D}_Q^{ps}}$. Its image in $R^{tr(N)}$ satisfies (5.11i, ii).

We have now verified all the hypotheses in §5 and are thus in a position to prove the main result of this section.

Proposition **7.3.** — *Suppose that* F *is a totally real field of even degree. Suppose that* $\mathscr{D}$ *is a deformation datum and that* $\mathscr{D} = \mathscr{D}_c$. *Suppose finally that* $\mathfrak{p} \subseteq \mathbf{T}_{\mathscr{D}}$ *is a prime which is nice for* $\mathscr{D}$ *in the sense of* §4.2. *Then*

(i) $\psi(\mathscr{D}, \mathfrak{p}) : (\widetilde{R}_{\mathscr{D}}^{\min})_{\widetilde{\mathfrak{p}}} \longrightarrow (\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\widetilde{\mathfrak{p}}}$ *is an isomorphism and* $(\widetilde{R}_{\mathscr{D}}^{\min})_{\widetilde{\mathfrak{p}}}$ *is a complete intersection over* $\widetilde{\Lambda}_{\mathscr{O},P}$ *and reduced;*

(ii) $\widetilde{M}_{\mathscr{D},\widetilde{\mathfrak{p}}}$ *is a free* $(\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\widetilde{\mathfrak{p}}}$*-module.*

Proof. — Our constructions give the following identifications:

$$R^{(0)} \underset{A}{\otimes} K = (\widetilde{R}_{\mathscr{D}}^{\min})_{\widetilde{\mathfrak{p}}}/(\widetilde{\mathfrak{p}}F_0, P)$$

$$R^{tr(0)} \underset{A}{\otimes} K \twoheadrightarrow (\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\widetilde{\mathfrak{p}}}/P$$

$$N^{(0)} \underset{A}{\otimes} K = (\widetilde{M}_{\mathscr{D}})_{\widetilde{\mathfrak{p}}}/P, \quad M^{(0)} \underset{A}{\otimes} K = \bigoplus_{i=1}^{2^r}(N^{(0)} \underset{A}{\otimes} K).$$

By Lemma 5.2 the natural map

(7.26) $$R^{tr(0)} \otimes_A K \longrightarrow R^{(0)} \otimes_A K$$

is an isomorphism. By Proposition 5.8, $M^{(0)} \otimes_A K$ is free over $R^{(0)} \otimes_A K$. As the action of $R^{(0)} \otimes_A K$ on $M^{(0)} \otimes_A K$ factors through the composite map $R^{(0)} \otimes_A K \simeq R^{tr(0)} \otimes_A K \twoheadrightarrow (\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\widetilde{\mathfrak{p}}}/P$ we conclude that $M^{(0)} \otimes_A K$ is a free $(\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\widetilde{\mathfrak{p}}}/P$-module and that $\psi(\mathscr{D}, \mathfrak{p})$

induces an isomorphism $(\widetilde{\mathrm{R}}_{\mathscr{D}}^{\min})_{\tilde{\mathfrak{p}}}/(\tilde{\mathfrak{p}}\mathrm{F}_0, \mathrm{P}) \simeq (\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\tilde{\mathfrak{p}}}/\mathrm{P}$. Picking generators of $\mathrm{M}^{(0)} \otimes_{\mathrm{A}} \mathrm{K}$ as an $\mathrm{R}^{\mathrm{tr}(0)} \otimes_{\mathrm{A}} \mathrm{K}$-module and lifting them to $(\widetilde{\mathrm{M}}_{\mathscr{D}})_{\tilde{\mathfrak{p}}}$ we get a map (for some minimal s)

$$(\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\tilde{\mathfrak{p}}}^{s} \twoheadrightarrow (\widetilde{\mathrm{M}}_{\mathscr{D}})_{\tilde{\mathfrak{p}}} \tag{7.27}$$

which is an isomorphism modulo P. Since $(\widetilde{\mathrm{M}}_{\mathscr{D}})_{\tilde{\mathfrak{p}}}$ is free over $\widetilde{\Lambda}_{\mathscr{O},\mathrm{P}}$ it follows that (7.27) is an isomorphism. In particular $(\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\tilde{\mathfrak{p}}}$ is free over $\widetilde{\Lambda}_{\mathscr{O},\mathrm{P}}$.

As observed, the reduction mod P of the map

$$(\widetilde{\mathrm{R}}_{\mathscr{D}}^{\min})_{\tilde{\mathfrak{p}}}/(\tilde{\mathfrak{p}}\mathrm{F}_0) \longrightarrow (\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\tilde{\mathfrak{p}}} \tag{7.28}$$

induced by $\psi(\mathscr{D}, \mathfrak{p})$ is an isomorphism. Using that $(\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\tilde{\mathfrak{p}}}$ is free over $\widetilde{\Lambda}_{\mathscr{O}}$ we now deduce that (7.28) is an isomorphism. Under (7.28) F_0 maps to zero as $\widetilde{\mathrm{M}}_{\mathscr{D},\tilde{\mathfrak{p}}}$ is a free $(\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\tilde{\mathfrak{p}}}$-module. So $\mathrm{F}_0/\tilde{\mathfrak{p}}\mathrm{F}_0 = 0$ whence $\mathrm{F}_0 = 0$. Finally $(\widetilde{\mathrm{R}}_{\mathscr{D}}^{\min})_{\tilde{\mathfrak{p}}}$ is a complete intersection since $(\widetilde{\mathrm{R}}_{\mathscr{D}}^{\min})_{\tilde{\mathfrak{p}}}/\mathrm{P}$ is by Proposition 5.9. (Note that $(\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\tilde{\mathfrak{p}}}$ is reduced as $\mathbf{T}_{\mathscr{D}}^{\min}$ is reduced.) This completes the proof of the proposition. □

8. Raising the level for nice primes

8.1. Preliminaries

In this section we complete the proof that property (P1) holds for a deformation datum $\mathscr{D}$. However, before doing so we need some auxiliary results. We begin by imposing a partial ordering on the deformation data. If $\mathscr{D}_1 = (\mathscr{O}_1, \Sigma_1, c_1, \mathscr{M}_1)$ and $\mathscr{D}_2 = (\mathscr{O}_2, \Sigma_2, c_2, \mathscr{M}_2)$ are data, then we write $\mathscr{D}_1 \geqslant \mathscr{D}_2$ to mean $\mathscr{O}_1 = \mathscr{O}_2$, $c_1 = c_2$, $\Sigma_1 \supseteq \Sigma_2$, and $\mathscr{M}_1 \subseteq \mathscr{M}_2$. If $\#(\Sigma_1\backslash\Sigma_2) + \#(\mathscr{M}_2\backslash\mathscr{M}_1) = 1$, then we say that the inequality $\mathscr{D}_1 \geqslant \mathscr{D}_2$ is *strict*.

Let $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$ be a deformation datum and suppose that $\mathfrak{p} \subseteq \mathbf{T}_{\mathscr{D}}$ is nice for $\mathscr{D}$. As $p \in \mathfrak{p}$, $\mathfrak{p}$ is the inverse image of a prime of $\mathbf{T}_{\mathscr{D}}^{\min}$ which we also denote by $\mathfrak{p}$. We adopt the notation and conventions from the beginning of §7. The primary goal of this section is to prove the following proposition.

Proposition **8.1.** — *If* $\mathfrak{p} \subseteq \mathbf{T}_{\mathscr{D}}$ *is nice for* $\mathscr{D}$, *then the map* $\psi(\mathscr{D}, \mathfrak{p}) : (\widetilde{\mathrm{R}}_{\mathscr{D}}^{\min})_{\tilde{\mathfrak{p}}} \longrightarrow (\widetilde{\mathbf{T}}_{\mathscr{D}}^{\min})_{\mathfrak{p}}$ *in Proposition* 7.1 *is an isomorphism.*

For $\mathscr{D} = \mathscr{D}_c$ this was proven in §5 - §7 (see Proposition 7.3). We will deduce the general result from this case by a generalization of the arguments in [W1, Chapter 2].

8.2. Congruence maps

A key ingredient in our proof that $\psi(\mathscr{D}, \mathfrak{p})$ is an isomorphism will be a lower bound for the length of a certain "congruence module". In this subsection we construct maps between various $\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}}$-modules that will be instrumental in obtaining this lower bound.

We first fix a sequence of deformation data $\mathscr{D}_c = \mathscr{D}_0 \leqslant \mathscr{D}_1 \leqslant \cdots \leqslant \mathscr{D}_n = \mathscr{D}$ such that $\mathscr{D}_i \geqslant \mathscr{D}_{i-1}$ is a strict inequality for $1 \leqslant i \leqslant n$. Put

$$\mathrm{R}_i = (\widetilde{\mathrm{R}}^{\min}_{\mathscr{D}_i})_{\widetilde{\mathfrak{p}}}, \quad \mathbf{T}_i = (\widetilde{\mathbf{T}}^{\min}_{\mathscr{D}_i})_{\widetilde{\mathfrak{p}}},$$
$$\mathrm{M}_i = (\widetilde{\mathrm{M}}_{\mathscr{D}_i})_{\widetilde{\mathfrak{p}}}, \text{ and } \mathrm{M}_i^+ = (\widetilde{\mathrm{M}}^+_{\mathscr{D}_i})_{\widetilde{\mathfrak{p}}},$$

where $\widetilde{\mathrm{M}}^+_{\mathscr{D}_i}$ is defined as is $\widetilde{\mathrm{M}}_{\mathscr{D}_i}$ but using $\mathrm{M}^+_{\mathscr{D}_i}$ instead. Let $\mathrm{P} = \widetilde{\mathfrak{p}} \cap \widetilde{\Lambda}_{\mathscr{O}}$.

Lemma **8.2.** — *Each* M_i *and* M_i^+ *is a free* $\widetilde{\Lambda}_{\mathscr{O},\mathrm{P}}$*-module. Also, there exists an integer* s *such that* $\mathrm{M}_i^{2^s} \simeq \mathrm{Hom}_{\widetilde{\Lambda}_{\mathscr{O},\mathrm{P}}}(\mathrm{M}_i^+, \widetilde{\Lambda}_{\mathscr{O},\mathrm{P}})^{2^s}$ *and* $\mathrm{M}_i^{+,2^s} \simeq \mathrm{Hom}_{\widetilde{\Lambda}_{\mathscr{O},\mathrm{P}}}(\mathrm{M}_i, \widetilde{\Lambda}_{\mathscr{O},\mathrm{P}})^{2^s}$ *as* $\mathbf{T}_i$*-modules.*

Proof. — Choose a set of places $\{r_1, \ldots, r_s\}$, distinct from those in Σ, satisfying the hypotheses of Corollary 3.6 and such that $\omega(\mathrm{Frob}_{r_i}) \not\equiv 1$ and $\rho_{\mathfrak{p}}(\mathrm{Frob}_{r_i})$ has eigenvalues of infinite order for each i. Put $\mathrm{U}_i = \mathrm{U}_{\mathscr{D}_i} \cap \mathrm{U}_1(r_1, \ldots, r_s)$ and $\mathrm{U}_i^{\min} = \mathrm{U}^{\min}_{\mathscr{D}_i} \cap \mathrm{U}_1(r_1, \ldots, r_s)$. By Lemma 3.29, $\mathbf{T}_\infty(\mathrm{U}_i, \mathscr{O})_{\mathfrak{p}} \simeq \mathbf{T}_{\mathscr{D}_i,\mathfrak{p}}$ and

$$\mathrm{M}_\infty(\mathrm{U}_i^{\min})_{\mathfrak{p}} \simeq \mathrm{M}^{2^s}_{\mathscr{D}_i,\mathfrak{p}} \text{ and } \mathrm{M}_\infty(\mathrm{U}_i^{\min})^+_{\mathfrak{p}} \simeq \mathrm{M}^{+,2^s}_{\mathscr{D}_i,\mathfrak{p}}.$$

The lemma now follows from Proposition 3.3 and (3.17). □

Fix an s as in Lemma 8.2. Now let

$$\{w_i\} = \Sigma_i \backslash \Sigma_{i-1} \cup \mathscr{M}_{i-1} \backslash \mathscr{M}_i, \quad r_i = \begin{cases} 2 & \text{if } w_i \in \mathscr{M}_{i-1} \\ 3 & \text{if } w_i \in \Sigma_i \backslash \Sigma_{i-1}, \end{cases}$$

and

$$\eta_i = \begin{cases} (q_i - 1)\,(\mathrm{T}(\ell_i)^2 - \mathrm{S}(\ell_i)\,(q_i + 1)^2) & \text{if } w_i \in \Sigma_i \backslash \Sigma_{i-1} \\ (q_i - 1)\,(q_i + 1) & \text{if } w_i \in \mathscr{M}_{i-1} \text{ and } \chi|_{\mathrm{I}_{w_i}} = 1 \\ (q_i - 1) & \text{if } w_i \in \mathscr{M}_{i-1} \text{ and } \chi|_{\mathrm{I}_{w_i}} \not\equiv 1. \end{cases}$$

Here $\ell_i = \ell_{w_i}$ and $q_i = \mathrm{Nm}(\ell_i)$.

Next we define maps of $\mathbf{T}_i$-modules

$$\Phi_i : \mathrm{M}_{i-1}^{2^s r_i} \longrightarrow \mathrm{M}_i^{2^s}, \quad \widehat{\Phi}_i : \mathrm{M}_i^{2^s} \longrightarrow \mathrm{M}_{i-1}^{2^s r_i}$$
$$\Psi_i, \Theta_i : \mathrm{M}_{i-1}^{2^s r_i} \simeq \mathrm{M}_{i-1}^{2^s r_i}$$

such that

(8.1) *a)* Φ_i is injective with $\widetilde{\Lambda}_{\mathcal{O},\mathrm{P}}$-free cokernel and $\widehat{\Phi}_i$ is surjective.

b) $\Theta_i \circ \widehat{\Phi}_i \circ \Phi_i \circ \Psi_i = \left(\begin{array}{c|c} (\mathrm{unit})\times\eta_i & * \\ \hline & \mathrm{A}_i \end{array}\right) \in \mathrm{M}_{2^s r_i}(\mathbf{T}_{i-1})$ with the image of $\det(\mathrm{A}_i)$ not a zero-divisor in $\mathbf{T}_0$.

c) $\mathrm{im}(\Psi_i \circ (\Phi_{i-1}, \ldots, \Phi_{i-1})) = \mathrm{im}(\Phi_{i-1}, \ldots, \Phi_{i-1})$.

Let $\lambda_i = \lambda^{(\ell_i)}$ be as in the definition of $\mathrm{T}(\ell_i)$. For any $f\colon \mathrm{G^D}(\mathrm{A}^f) \longrightarrow \mathrm{R}$ (R an $\mathcal{O}$-module) let $\alpha_i f\colon \mathrm{G^D}(\mathbf{A}^f) \longrightarrow \mathrm{R}$ be given by $(\alpha_i f)(x) = f\left(x\begin{pmatrix}1 & \\ & \lambda_i\end{pmatrix}\right)$. We define Φ_i to be the direct sum of 2^s copies of the localization of the map $(\varprojlim \Phi_i^a)\otimes 1 : \mathrm{M}_{\mathscr{D}_{i-1}}^{r_i} \otimes_{\Lambda_\mathcal{O}} \widetilde{\Lambda}_\mathcal{O} \longrightarrow \mathrm{M}_{\mathscr{D}_i} \otimes_{\Lambda_\mathcal{O}} \widetilde{\Lambda}_\mathcal{O} = \widetilde{\mathrm{M}}_{\mathscr{D}_i}$, where $\Phi_i^a : e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_{i-1,a}), \mathcal{O})^{r_i} \longrightarrow e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_{i,a}), \mathcal{O})$ is given by $(f_1, f_2) \longmapsto f_1 + \alpha_i f_2$ if $r_i = 2$ and $(f_1, f_2, f_3) \longmapsto f_1 + \alpha_i f_2 + \alpha_i^2 f_3$ if $r_i = 3$. We define $\Phi_i^+ : \mathrm{M}_{i-1}^{+,r_i} \longrightarrow \mathrm{M}_i^+$ similarly and take for $\widehat{\Phi}_i$ the dual map obtained from Φ_i^+ by applying $\mathrm{Hom}_{\widetilde{\Lambda}_{\mathcal{O},\mathrm{P}}}(\cdot, \widetilde{\Lambda}_{\mathcal{O},\mathrm{P}})$. Similarly, let $\widehat{\Phi}_i^+$ be the dual of Φ_i.

We now verify (8.1*a*). Choose $\mathfrak{n}$ to be an ideal such that $\mathrm{U}_0(\mathfrak{n}) \supseteq \mathrm{U}_n \supseteq \mathrm{U}(\mathfrak{n})$. If $w_i \in \Sigma_i \backslash \Sigma_{i-1}$ (so $r_i = 3$) then by Lemma 3.28 for a sufficiently large both the kernel of Φ_i^a and the cokernel of $\widehat{\Phi}_i^a$ are annihilated by $T(\ell) - 1 - \mathrm{Nm}(\ell)$ for any prime ℓ that splits completely in the ray class field of conductor $p^a \cdot \mathfrak{n} \cdot \infty$. Here $\widehat{\Phi}_i^a$ is the adjoint of Φ_i^a with respect to the pairings defined in §3.2. Let F_a be the ray class field of conductor $p^a \cdot \mathfrak{n} \cdot \infty$ and let $\mathrm{F}_\infty = \cup \mathrm{F}_a$. Choose $\sigma \in \mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F}_\infty)$. Such a σ is the limit of a sequence of Frobenii $\{\mathrm{Frob}_{\ell_a}\}$ of primes ℓ_a splitting in F_a. In fact such a sequence can be chosen so that $\mathrm{Frob}_{\ell_a} = \mathrm{Frob}_{\ell_b}$ $(b \geqslant a)$ in $\mathrm{Gal}(\mathrm{F}_a/\mathrm{F})$. As trace $\rho_{\mathscr{D}_i}^{\mathrm{mod}}(\sigma)$ is the limit of $\{\mathrm{T}(\ell_a)\}$, it follows that trace $\rho_{\mathscr{D}_i}^{\mathrm{mod}}(\sigma) - 1 - \varepsilon(\sigma)$ annihilates $\ker(\Phi_i^a)$ and $\mathrm{coker}(\widehat{\Phi}_i^a)$. As $\rho_\mathfrak{p}$ is neither reducible nor dihedral it must be that $\mathrm{trace}\,\rho_{\mathscr{D}_i}^{\mathrm{mod}}(\sigma) - 1 - \varepsilon(\sigma) \notin \mathfrak{p}$ for some $\sigma \in \mathrm{Gal}(\overline{\mathrm{F}}/\mathrm{F}_\infty)$. It follows that both $\ker(\varprojlim \Phi_i^a)$ and $\mathrm{coker}\,(\varprojlim \widehat{\Phi}_i^a)$ vanish when localized at $\mathfrak{p}$. Thus Φ_i is injective and $\widehat{\Phi}_i$ is surjective. A similar argument shows that Φ_i^+ is injective and $\widehat{\Phi}_i^+$ is surjective. This proves (8.1*a*).

Now if $w_i \in \mathscr{M}_{i-1}$, then it follows from Lemma 3.27 that $\ker(\Phi_i)$ and $\mathrm{coker}(\widehat{\Phi}_i)$ are isomorphic to submodules of $(\widetilde{\mathrm{M}_\infty(\mathrm{U}')})_\mathfrak{p}^{2^s}$ where

$$\mathrm{U}' = \mathrm{U}_{\mathscr{D}_{i-1}}^{\min} \cdot \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_{\mathrm{F},w_i}) : a - 1, c \in \ell_i^{\max(1, r(\ell_i)) - 1}\right\}$$

with $r(\ell_i)$ as in the definition of $\mathrm{U}_{\mathscr{D}_i}$. By considering $\rho_\mathfrak{p}|_{I_{w_i}}$ one sees that $\mathfrak{p}$ is not a prime in $\mathbf{T}_\infty(\mathrm{U}', \mathcal{O})$ (so $\mathbf{T}_\infty(\mathrm{U}', \mathcal{O})_\mathfrak{p} = 0$) whence $\widetilde{\mathrm{M}_\infty(\mathrm{U}')}_\mathfrak{p} = 0$. This proves that

Φ_i and $\widehat{\Phi}_i$ are, respectively, injective and surjective in this case. The same argument applies to Φ_i^+ and $\widehat{\Phi}_i^+$, thereby establishing (8.1*a*) in this case as well.

Next we define Ψ_i and Θ_i. If $w_i \in \Sigma_i \backslash \Sigma_{i-1}$, then we put $\Psi_i = \oplus_{r=1}^{2^s} \Psi_i'$ and $\Theta_i = \oplus_{r=1}^{2^s} \Theta_i'$ where

$$\Psi_i' = \begin{pmatrix} -S(\ell_i) & 0 & 0 \\ T(\ell_i) & -S(\ell_i)^{-1} & 0 \\ -q_i & 0 & -1 \end{pmatrix}, \quad \Theta_i' = \begin{pmatrix} 0 & 0 & S(\ell_i) \\ 0 & S(\ell_i)^{-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

If $w_i \in \mathscr{M}_{i-1}$, then we take

$$\Psi_i' = \begin{pmatrix} T(\ell_i) & 0 \\ -q_i & T(\ell_i)^{-1} \end{pmatrix}, \quad \Theta_i' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We note that while $T(\ell_i)$ is not included in the definition of $\mathbf{T}_{i-1}$ if $w_i \in \mathscr{M}_{i-1}$, it is in fact in $\mathbf{T}_{i-1}$ and is a unit, so the definition of Ψ_i makes sense in this case. To see this, let Q be any minimal prime of $\mathbf{T}_{\mathscr{D}_{i-1}}$. Then $T(\ell_i)$ is the eigenvalue of the action of Frob_{ℓ_i} on the maximal unramified quotient of $\rho_Q|_{D_{\ell_i}}$. (This can be checked on the representations associated to algebraic primes containing Q.) As $w_i \in \mathscr{M}_{i-1}$, the representation $\rho^{\min}_{\mathscr{D}_{i-1}}|_{D_{\ell_i}}$ has a non-trivial maximal unramified quotient, and it follows that the image in $\mathbf{T}_{i-1}$ under $\psi(\mathscr{D}, \mathfrak{p})$ of the eigenvalue of Frob_{ℓ_i} on this quotient must be $T(\ell_i)$.

As (8.1*c*) is obvious from the definition of Ψ_i, it remains to verify (8.1*b*). Suppose first that $w_i \in \Sigma_i \backslash \Sigma_{i-1}$. A straightforward calculation shows that $\widehat{\Phi}_i \circ \Phi_i$ is a direct sum of 2^s copies of

$$\begin{pmatrix} q_i(q_i+1) & T(\ell_i)q_i & T(\ell_i)^2 - S(\ell_i)(q_i+1) \\ S(\ell_i)^{-1}T(\ell_i)q_i & q_i(q_i+1) & T(\ell_i)q_i \\ T(\ell_i)^2 S(\ell_i)^{-2} - S(\ell_i)^{-1}(1+q_i) & T(\ell_i)\,S(\ell_i)^{-1}q_i & q_i(q_i+1) \end{pmatrix}.$$

Thus

$$\Theta_i \circ \widehat{\Phi}_i \circ \Phi_i \circ \Psi_i = \left(\begin{array}{c|c} \eta_i & * \\ \hline & A_i \end{array}\right), \quad \det(A_i) = (q_i S(\ell_i)^{-2})^{2^s} \eta_i^{2^s-1}.$$

That the determinant of A_i is not a zero-divisor in $\mathbf{T}_0$ is easily checked. As $\mathbf{T}_0$ is reduced, we need only verify that $\det(A_i) \notin Q$ for all minimal primes Q of $\mathbf{T}_{\mathscr{D}_0}$. Suppose that $\det(A_i)$ is contained in such a Q. Let $P \subseteq \mathbf{T}_{\mathscr{D}_0}$ be an algebraic prime containing Q (and hence $\det(A_i)$). Let $t = T(\ell_i) \bmod P$ and $s = S(\ell_i) \bmod P$. We will show that $t^2 - s(1+q_i)^2 \neq 0$. Let α and β be the eigenvalues of $\rho_P(\mathrm{Frob}_{\ell_i})$. Recall that

$t = \alpha + \beta$ and $sq_i = \alpha\beta$. If $t^2 - s(1+q_i)^2 = 0$, then either $\dfrac{\alpha}{\beta} = q_i$ or $\dfrac{\beta}{\alpha} = q_i$. But both possibilities violate (3.3). It follows that $\det(A_i) \notin P$ and hence $\det(A_i) \notin Q$.

The verification of (8.1*b*) in the case where $w_i \in \mathcal{M}_{i-1}$ is done similarly.

We are now in a position to define our "congruence maps". Put

$$I^{\text{old}} = \ker\{\mathbf{T}_n \longrightarrow \mathbf{T}_0\} \text{ and } I^{\text{new}} = \text{Ann}_{\mathbf{T}_n} I^{\text{old}}.$$

Put also

$$r(i) = \prod_{1 \leqslant j \leqslant i} r_j.$$

Define $\Phi^{(i)} : M_0^{2^s r(i)} \longrightarrow M_i^{2^s}$ and $\tilde{\Phi}^{(i)} : M_0^{2^s r(i)} \longrightarrow M_i^{2^s}$ recursively by

$$\Phi^{(1)} = \Phi_1, \quad \tilde{\Phi}^{(1)} = \Phi_i \circ \Psi_1,$$

and

$$\Phi^{(i)} = \Phi_i \circ (\Phi^{(i-1)} \times \cdots \times \Phi^{(i-1)})$$
$$\tilde{\Phi}^{(i)} = (\Phi_i \circ \Psi_i) \circ (\tilde{\Phi}^{(i-1)} \times \cdots \times \tilde{\Phi}^{(i-1)}).$$

Define $\hat{\Phi}^{(i)} : M_i^{2^s} \longrightarrow M_0^{2^s r(i)}$ and $\tilde{\hat{\Phi}}^{(i)} : M_i^{2^s} \longrightarrow M_0^{2^s r(i)}$ in the same way but using $\hat{\Phi}_i$ and Θ_i and reversing the order of composition as appropriate. Put

$$\Phi = \Phi^{(n)}, \quad \Phi_{\mathscr{D}} = \tilde{\Phi}^{(n)}, \quad \hat{\Phi} = \hat{\Phi}^{(n)}, \text{ and } \hat{\Phi}_{\mathscr{D}} = \tilde{\hat{\Phi}}^{(n)}.$$

Put also

$$r = r(n) \qquad \text{and} \qquad \eta = \prod_{1 \leqslant i \leqslant n} \eta_i.$$

Lemma **8.3.** —

(i) $\text{im}(\Phi_{\mathscr{D}}) = M_n[I^{\text{old}}]^{2^s}$ *and* $\text{coker}(\Phi_{\mathscr{D}})$ *is a free* $\tilde{\Lambda}_{\mathscr{O},P}$*-module.*

(ii) $\hat{\Phi}_{\mathscr{D}}$ *is surjective.*

(iii) $\hat{\Phi}_{\mathscr{D}} \circ \Phi_{\mathscr{D}} = \left(\begin{array}{c|c} (\text{unit}) \times \eta & * \\ \hline & A \end{array}\right) \in M_{2^s r}(\mathbf{T}_0)$ *with* $\det(A)$ *not a zero-divisor.*

(iv) $\ker(\hat{\Phi}_{\mathscr{D}}) = M_n[I^{\text{new}}]^{2^s}$.

Proof. — Part (ii) follows from (8.1*a*). Part (iii) follows from (8.1*b*). We leave the details to the reader noting only that $\det(A)$ is a product of powers of the $\det(A_i)$'s and the η_i's. The freeness over $\tilde{\Lambda}_{\mathscr{O},P}$ of the cokernel of $\Phi_{\mathscr{D}}$ also follows from (8.1). It remains to prove the first assertion of part (i) and part (iv).

Note that by (8.1*c*), $\mathrm{im}(\Phi_{\mathscr{D}}) = \mathrm{im}(\Phi)$, so it suffices to prove that $\mathrm{im}(\Phi) = \mathrm{M}_n[\mathrm{I}^{\mathrm{old}}]$. Next note that Φ is the localization of $(\varprojlim \Phi^a) \otimes 1$ where Φ^a is defined as is Φ but with Φ_i^a replacing Φ_i. Now let

$$\mathrm{I}_a^{\mathrm{old}} = \ker\{\mathbf{T}_2(\mathrm{U}_{\mathscr{D},a}, \mathscr{O}) \longrightarrow \mathbf{T}_2(\mathrm{U}_{\mathscr{D}_c,a}, \mathscr{O})\}.$$

It follows from the theory of "new vectors" that

$$\mathrm{im}(\Phi^a) \otimes \mathrm{K} = e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_{\mathscr{D},a}), \mathrm{K})\,[\mathrm{I}_a^{\mathrm{old}}].$$

(For a more detailed proof in a similar situation see the proof of Lemma 3.29). Now consider the commutative diagram

$$\begin{array}{ccc}
\varprojlim_a e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_{\mathscr{D}_c,a}), \mathscr{O})^r & \xrightarrow{\varprojlim \Phi^a} & \varprojlim_a e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_{\mathscr{D},a}), \mathscr{O})\,[\mathrm{I}_a^{\mathrm{old}}] \twoheadrightarrow C \\
\downarrow & & \downarrow \\
0 \longrightarrow \varprojlim_a \mathrm{im}(\Phi^a) \otimes \mathrm{K} & \longrightarrow & \varprojlim_a e\mathrm{H}^0(\mathrm{X}(\mathrm{U}_{\mathscr{D},a}), \mathrm{K})\,[\mathrm{I}_a^{\mathrm{old}}] \longrightarrow 0
\end{array}$$

having exact rows and with the vertical arrows being the natural maps. Applying the snake lemma we find that C embeds into a quotient of $\mathrm{N} = \varprojlim \ker(\Phi^a \otimes \mathrm{K}/\mathscr{O})$. Arguing as in the proof of Lemma 3.29 shows that $\mathrm{N}_{\mathfrak{p}} = 0$ and hence $\mathrm{C}_{\mathfrak{p}} = 0$. Now let

$$\mathrm{I}_\infty^{\mathrm{old}} = \ker\{\mathbf{T}_\infty(\mathrm{U}_{\mathscr{D}}, \mathscr{O}) \longrightarrow \mathbf{T}_\infty(\mathrm{U}_{\mathscr{D}_c}, \mathscr{O})\}.$$

It follows from the preceding remarks that the quotient $\mathrm{M}_{\mathscr{D}}[\mathrm{I}_\infty^{\mathrm{old}}]/\mathrm{im}(\varprojlim \Phi^a)$ vanishes upon localizing at $\mathfrak{p}$. As $\mathrm{I}^{\mathrm{old}} = \mathrm{I}_\infty^{\mathrm{old}}\mathrm{T}_n$, part (i) follows.

To prove (iv) we first note that $\mathrm{M}_0[\mathrm{I}^{\mathrm{new}}] = 0$ by Lemma 8.2, for $\mathrm{M}_0[\mathrm{I}^{\mathrm{new}}]$ is a $\mathbf{T}_0/\mathrm{I}^{\mathrm{new}}\mathbf{T}_0$-module and hence a torsion $\widetilde{\Lambda}_{\mathscr{O},\mathrm{P}}$-module. Therefore $\mathrm{M}_n[\mathrm{I}^{\mathrm{new}}] \subseteq \ker(\widehat{\Phi}_{\mathscr{D}})$. On the other hand, it follows from (i) and (iii) that $\widehat{\Phi}_{\mathscr{D}}$ maps $\mathrm{M}_n[\mathrm{I}^{\mathrm{old}}] \otimes \mathscr{L}$ isomorphically onto $\mathrm{M}_0^r \otimes \mathscr{L}$, where $\mathscr{L}$ is the field of fractions of $\widetilde{\Lambda}_{\mathscr{O},\mathrm{P}}$. As $\mathrm{M}_n \otimes \mathscr{L} = (\mathrm{M}_n[\mathrm{I}^{\mathrm{old}}] \otimes \mathscr{L}) \oplus (\mathrm{M}_n[\mathrm{I}^{\mathrm{new}}] \otimes \mathscr{L})$ it follows that the quotient $\ker(\widehat{\Phi}_{\mathscr{D}})/\mathrm{M}_n[\mathrm{I}^{\mathrm{new}}]$ is a $\widetilde{\Lambda}_{\mathscr{O},\mathrm{P}}$-torsion module, from which we easily conclude that $\ker(\widehat{\Phi}_{\mathscr{D}}) = \mathrm{M}_n[\mathrm{I}^{\mathrm{new}}]$. (The tensor products are as $\widetilde{\Lambda}_{\mathscr{O}}$-modules.) □

8.3. An auxiliary result

We now state (and prove) a simple result in commutative algebra. This result will be important in the proof of the main result of this section (the proof of Proposition 8.1).

Let $\Lambda = \mathrm{B}[[x_1, \ldots, x_s]]$ be a power series ring over a complete DVR B of characteristic 0. Suppose that $(\mathrm{A}_1, \mathrm{A}_2, \beta, \mathrm{N}_1, \mathrm{N}_2, r, \varphi, \hat{\varphi})$ is an 8-tuple consisting of

- complete local finite Λ-algebras A_1 and A_2 with A_1 reduced,
- a surjection $\beta : A_2 \twoheadrightarrow A_1$ of Λ-algebras,
- for each $i = 1, 2$ an A_i-module N_i with each N_i finite and free over Λ and with N_1 free over A_1,
- an integer $r \geqslant 1$ and maps of A_2-modules $\varphi : N_1^r \hookrightarrow N_2$ and $\widehat{\varphi} : N_2 \twoheadrightarrow N_1^r$ such that $\widehat{\varphi} \circ \varphi \in M_r(A_1) \subseteq \mathrm{End}_\Lambda(N_1^r)$ and $\det(\widehat{\varphi} \circ \varphi)$ is not a zero-divisor in A_1.

We further require that

- $\mathrm{im}(\varphi) = N_2[I]$ and $\ker(\widehat{\varphi}) = N_2[J]$ where $I = \ker(\beta)$ and $J = \mathrm{Ann}_{A_2}(I)$,
- $\mathrm{coker}(\varphi)$ is Λ-free.

Lemma **8.4.** — *For each* $0 \leqslant t \leqslant s$ *there exist* $y_1, \ldots, y_t \in \Lambda$ *such that*

(i) $(y_1, \ldots, y_t)$ *is a prime ideal of* Λ,
(ii) $y_1, \ldots, y_t$ *generate a t-dimensional subspace of* $\mathfrak{m}_\Lambda/(\mathfrak{m}_\Lambda^2, \mathfrak{m}_B)$
(iii) $A_1/(y_1, \ldots, y_t)$ *is reduced,*
(iv) $\Lambda \cap (I + J, y_1, \ldots, y_t) \neq (y_1, \ldots, y_t)$,
(v) $\det(\widehat{\varphi} \circ \varphi) \bmod (y_1, \ldots, y_t)$ *is not a zero-divisor in* $A_1/(y_1, \ldots, y_t)$,
(vi) $\ker(\widehat{\varphi} \bmod (y_1, \ldots, y_t)) = N_2[J]/(y_1, \ldots, y_t)$,
(vii) $\mathrm{im}(\varphi \bmod (y_1, \ldots, y_t)) = (N_2/(y_1, \ldots, y_t))[I]$.

Proof. — Our proof will be by induction on t. Note that if $t = 0$ then all the conclusions are satisfied by the hypotheses on A_i and N_i. Suppose then that we have found $y_1, \ldots, y_t$, $t < s$, satisfying the lemma. We will show how to find y_{t+1}. Let $(y) \subseteq \Lambda' = \Lambda/(y_1, \ldots, y_t)$ be a prime ideal such that

a) $(y_1, \ldots, y_t, y)$ satisfies (i) and (ii),
b) (y) does not contain $\mathrm{Ann}_{\Lambda'}(\Omega_{A_1/(y_1, \ldots, y_t)/\Lambda'})$,
c) $(I + J, y_1, \ldots, y_t, y) \cap \Lambda \neq (y_1, \ldots, y_t, y)$,
d) $(\det(\widehat{\varphi} \circ \varphi), y_1, \ldots, y_t, y) \neq (y_1, \ldots, y_t, y)$.

Clearly all but finitely many (y) satisfy *a*) – *d*), and since there are infinitely many possibilities for (y) some (y) has the desired properties. Note that $A_1/(y_1, \ldots, y_t)$ is a finite and free Λ'-module because N_1 is finite and free over Λ and also free over A_1. Hence the hypothesis that $A_1/(y_1, \ldots, y_t)$ is reduced is equivalent to $\Omega_{A_1/(y_1, \ldots, y_t)/\Lambda'}$ being a torsion Λ'-module (here we use the fact that $\mathrm{char}(B) = 0$).

We now show that one may take for y_{t+1} any y such that (y) satisfies *a*) – *d*). Properties (i) and (ii) follow trivially. Property (iii) is a simple consequence of *b*). Property (iv) follows from *c*) and property (v) from *d*) once we know that $A_1/(y_1, \ldots, y_t, y)$ is reduced. Property (vi) is immediate. It remains to prove property (vii).

It follows from (v) that $\widehat{\varphi}$ maps $\mathrm{im}(\varphi \bmod (y_1, \ldots, y_t, y)) \otimes_{\Lambda''} F_{\Lambda''}$ isomorphically onto $N_1^r/(y_1, \ldots, y_t, y) \otimes_{\Lambda''} F_{\Lambda''}$, where $\Lambda'' = \Lambda'/(y)$ and $F_{\Lambda''}$ is its field of fractions. If we can show that

(8.2) $$(N_2/(y_1, \ldots, y_t, y))[I] \cap \ker(\hat{\varphi} \bmod (y_1, \ldots, y_t, y)) = 0$$

then it will also follow that $\hat{\varphi}$ maps $(N_2/(y_1, \ldots, y_t, y))[I] \otimes_{\Lambda''} F_{\Lambda''}$ isomorphically onto $N_1^r/(y_1, \ldots, y_t, y) \otimes_{\Lambda''} F_{\Lambda''}$ whence

$$\mathrm{im}(\varphi \bmod (y_1, \ldots, y_t, y)) \otimes_{\Lambda''} F_{\Lambda''} = (N_2/(y_1, \ldots, y_t, \ldots, y))[I] \otimes_{\Lambda''} F_{\Lambda''}.$$

The desired equality will follow from this one since $\mathrm{im}(\varphi \bmod (y_1, \ldots, y_t, y))$ is contained in $(N_2/(y_1, \ldots, y_t, y)[I]$ with Λ''-torsion-free cokernel. (Here we are using that coker (φ) is Λ-free.) To prove (8.2) we need merely note that the intersection in question is contained in $(N_2/(y_1, \ldots, y_t, y))[I+J]$ which must be zero as it would be simultaneously a torsion-free Λ''-module if non-zero and annihilated by $0 \neq (I+J) \cap \Lambda''$. (The latter is non-zero by *c*)). □

8.4. $\psi(\mathscr{D}, \mathfrak{p})$ is an isomorphism

We now complete the proof that $\psi(\mathscr{D}, \mathfrak{p})$ is an isomorphism. To do so we return to the notation of §8.2. By Proposition 7.2 M_0 is a free $\mathbf{T}_0$-module and $\psi(\mathscr{D}_c, \mathfrak{p}) : R_0 \simeq \mathbf{T}_0$. Moreover, $\mathbf{T}_0$ is a reduced complete intersection over $\widetilde{\Lambda}_{\mathscr{O}, P}$.

Let $\Lambda = \widetilde{\Lambda}_{\mathscr{O}, P}$. Note that $\Lambda = B[[W_2, \ldots, W_m]]$ where B is the localization and completion of $\mathscr{O}'[[W_1]]$ at the prime ideal (π). Let $\beta : \mathbf{T}_n \twoheadrightarrow \mathbf{T}_0$ be the natural surjection. It follows from the results of §8.2 that the 8-tuple $(\mathbf{T}_n, \mathbf{T}_0, \beta, M_n^{2^s}, M_0^{2^s}, r, \Phi_{\mathscr{D}}, \widehat{\Phi_{\mathscr{D}}})$ satisfies the hypotheses of §8.3. Therefore by Lemma 8.4 there are elements $y_1, \ldots, y_{m-1} \in \Lambda$ such that

(8.3) (i) $\Lambda/(y_1, \ldots, y_{m-1}) \cong B$

(ii) $\mathbf{T}_0/(y_1, \ldots, y_{m-1})$ is reduced.

(iii) $\mathrm{im}(\Phi_{\mathscr{D}} \bmod (y_1, \ldots, y_{m-1})) = (M_n/(y_1, \ldots, y_{m-1}))[I^{\mathrm{old}}]$.

(iv) $$\widehat{\Phi}_{\mathscr{D}} \circ \Phi_{\mathscr{D}} = \left(\begin{array}{c|c} (\text{unit}) \times \eta & * \\ \hline & A \end{array}\right) \in M_r(\mathbf{T}_0).$$

(v) $\det(\widehat{\Phi}_{\mathscr{D}} \circ \Phi_{\mathscr{D}})$ is not a zero-divisor in $\mathbf{T}_0/(y_1, \ldots, y_{m-1})$.

Put $\overline{R}_i = R_i/(y_1, \ldots, y_{m-1})$, $\overline{\mathbf{T}}_i = \mathbf{T}_i/(y_1, \ldots, y_{m-1})$, and $\overline{M}_i = M_i/(y_1, \ldots, y_{m-1})$ for each $0 \leqslant i \leqslant n$. Let Q be a minimal prime of $\overline{\mathbf{T}}_0$. As $\overline{\mathbf{T}}_0$ is reduced and η is not a zero-divisor in $\overline{\mathbf{T}}_0$ (η being a divisor of $\det(\widehat{\Phi}_{\mathscr{D}} \circ \Phi_{\mathscr{D}})$) $\eta \notin Q$. Let C be the integral closure of $\overline{\mathbf{T}}_0/Q$ in its field of fractions. As $\mathbf{T}_0$ is a free Λ-module, $\overline{\mathbf{T}}_0$ is a free B-module. Thus C is a complete DVR and a finite flat extension of B. Put

$$R_i' = \overline{R}_i \otimes_B C, \quad \mathbf{T}_i' = \overline{\mathbf{T}}_i \otimes_B C, \text{ and } M_i' = \overline{M}_i \otimes_B C.$$

Let $\Phi' = (M_0')^{2^s r} \longrightarrow (M_n')^{2^s}$ and $\widehat{\Phi}' : (M_n')^{2^s} \longrightarrow (M_0')^{2^s r}$ be the maps induced from $\Phi_{\mathscr{D}}$ and $\widehat{\Phi}_{\mathscr{D}}$, respectively.

We have maps

$$R_n' \overset{\psi'}{\twoheadrightarrow} \mathbf{T}_n' \overset{\gamma}{\twoheadrightarrow} \mathbf{T}_n'/\mathrm{Ann}_{\mathbf{T}_n'}(M_n') \overset{\alpha}{\twoheadrightarrow} \mathbf{T}_0' \simeq R_0'$$

and $\delta : \mathbf{T}_0' \twoheadrightarrow C$. Put $\beta' = \alpha \circ \gamma$. Here ψ' is the map induced by $\psi(\mathscr{D}, \mathfrak{p})$, β' is the map induced from $\beta : \mathbf{T}_n \twoheadrightarrow \mathbf{T}_0$, and δ is induced from the reduction of $\mathbf{T}_0$ modulo Q. That β' factors through $\mathbf{T}_n'/\mathrm{Ann}_{\mathbf{T}_n'}(M_n')$ is a consequence of the surjectivity of $\widehat{\Phi}'$ and of M_0 being a free $\mathbf{T}_0$-module. Put $\mathbf{T}_n'' = \mathbf{T}_n'/\mathrm{Ann}_{\mathbf{T}_n'}(M_n')$. This is a free C-module. Now put

$$H_0 = \ker(\delta), \qquad G_0 = \mathrm{Ann}_{\mathbf{T}_0'} \ker(\delta),$$
$$H_n = \ker(\delta \circ \beta' \circ \psi'), \quad G_n = \mathrm{Ann}_{\mathbf{T}_n''} \ker(\delta \circ \alpha).$$

As $\mathbf{T}_0'$ is a reduced complete intersection over C, it follows from [DRS, Criterion I] that

$$\textbf{(8.4)} \qquad \ell_C(H_0/H_0^2) = \ell_C(C/\delta(G_0)),$$

where for any C-module X, $\ell_C(X)$ denotes the length of X as C-module. Our goal is to prove a similar equality for $\ell_C(H_n/H_n^2)$ and $\ell_C(C/(\delta \circ \alpha)(G_n))$. First we prove that

$$\textbf{(8.5)} \qquad \ell_C(C/(\delta \circ \alpha)(G_n)) \geqslant \ell_C(C/\delta(G_0)) + \ell_C(C/\delta(\eta)).$$

We prove this as follows. Let $I = \ker(\alpha)$ and $J = \mathrm{Ann}_{\mathbf{T}_n''}(I)$. It follows from the definition of $\mathbf{T}_n''$ that $M_n'[I^{\mathrm{old}}] = M_n'[I]$. Therefore, by Lemma 8.3(i), $(JM_n')^{2^s} \subseteq \mathrm{im}(\Phi')$. In particular, if $j \in J$ and $m \in M_0'$, then there exist $m_1, \dots, m_{2^s r} \in M_0'$ such that

$$\left(\begin{array}{c|c} \eta & * \\ \hline & A \end{array}\right) \cdot \begin{pmatrix} m_1 \\ \cdot \\ \cdot \\ \cdot \\ m_{2^s r} \end{pmatrix} = \begin{pmatrix} jm \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

As $\det(A) m_i = 0$ for $i = 2, \dots, 2^s r$ we have that $m_i = 0$ for $i = 2, \dots, 2^s r$ since $\det(A)$ is not a zero-divisor in $\mathbf{T}_0'$ and M_0' is a free $\mathbf{T}_0'$-module. We conclude that $JM_0' \subseteq \eta M_0'$ and hence

$$\textbf{(8.6)} \qquad J\mathbf{T}_0' \subseteq (\eta).$$

Now suppose that $g \in G_n$. Then $\alpha(g) \in (\eta)$ by (8.6) since g annihilates $I \subseteq \ker(\delta \circ \alpha)$. Write $\alpha(g) = \eta x$. Since ηx annihilates $\alpha(\ker(\delta \circ \alpha)) = \ker(\delta) = H_0$ and since η is a non-zero divisor in $\mathbf{T}'_0$ it must be that $x \in G_0$. We have thus shown that $\alpha(G_n) \subseteq \eta G_0$. It follows that $(\delta \circ \alpha)(G_n) \subseteq \delta(\eta G_0)$. The inequality (8.5) is an immediate consequence of this.

Next we show that

$$\textbf{(8.7)} \qquad \ell_C(H_n/H_n^2) \leqslant \ell_C(H_0/H_0^2) + \ell_C(C/\delta(\eta)).$$

We will prove this by comparing the lengths in question to those of various cohomology groups. First we note that $\rho_{\mathscr{D}} : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(R_{\mathscr{D}})$ determines a representation $\rho : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(C)$ obtained from the composition map

$$R_{\mathscr{D}} \longrightarrow (\widetilde{R}_{\mathscr{D}}^{\min})_{\mathfrak{p}} = R_n \twoheadrightarrow \overline{R}_n \longrightarrow \overline{R}_n \otimes_B C = R'_n \overset{\delta\circ\beta'\circ\psi'}{\twoheadrightarrow} C.$$

Fix a basis for $\rho_{\mathscr{D}}$ such that $\rho_{\mathscr{D}}(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ and $\rho_{\mathscr{D}}(g_0) = \begin{pmatrix} * & u_0 \\ * & * \end{pmatrix}$, $u_0 \in \mathscr{O}^\times$, for some $g_0 \in \mathrm{Gal}(F_\Sigma/F)$ fixed. Let λ be a uniformizer of C. A C-algebra homomorphism $f : R'_n \longrightarrow C \oplus \varepsilon C/\lambda^m$ ($\varepsilon^2 = 0$) determines a representation $\rho_f : \mathrm{Gal}(F_\Sigma/F) \longrightarrow GL_2(C \oplus \varepsilon C/\lambda^m)$ such that $\rho = \rho_f \bmod \varepsilon$. Write $\rho_f(\sigma) = \rho(\sigma)(1 + \varepsilon\gamma_f(\sigma))$, $\gamma_f(\sigma) \in M_2(C/\lambda^m)$. It is readily checked that $\sigma \longmapsto \gamma_f(\sigma)$ is a 1-cocycle of $\mathrm{Gal}(F_\Sigma/F)$ with coefficients in $M_2(C/\lambda^m) \simeq \mathrm{ad}\,\rho/\lambda^m$. We first claim that $f \longmapsto$(cocycle class of γ_f) determines an embedding

$$\textbf{(8.8)} \qquad \mathrm{Hom}_{C\text{-alg}}(R'_n, C \oplus \varepsilon C/\lambda^m) \hookrightarrow H^1(F_\Sigma/F, \mathrm{ad}\,\rho/\lambda^m).$$

Here, and in what follows, all cohomology groups are the usual group cohomology; we do not require the cocycles to be continuous.

To see that (8.8) is an embedding first note that if γ_{f_1} and γ_{f_2} are cohomologous, then ρ_{f_1} and ρ_{f_2} are equivalent. Thus there must be some $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(C \oplus \varepsilon C/\lambda^m)$ such that $A\rho_{f_1}A^{-1} = \rho_{f_2}$. Since $\rho_{f_1}(z_1) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, it follows that $A = \begin{pmatrix} a & \\ & d \end{pmatrix}$. We also have that $\begin{pmatrix} * & (a/d)u_0 \\ * & * \end{pmatrix} = A\rho_{f_1}(g_0)A^{-1} = \rho_{f_2}(g_0) = \begin{pmatrix} *' & u_0 \\ *' & *' \end{pmatrix}$. Thus $a = d$ and A is a scalar, whence $\rho_{f_1} = \rho_{f_2}$. This implies that $f_1 = f_2$ since any map $R_{\mathscr{D}}^{\min} \longrightarrow C \oplus \varepsilon C/\lambda^n$ is completely determined by the images of the elements in the set $\{a_\sigma, b_\sigma, c_\sigma, d_\sigma : \sigma \in \mathrm{Gal}(F_\Sigma/F)\}$ ($\rho_{\mathscr{D}}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$) by Lemma 2.5. This proves injectivity of (8.8).

Recall that there is a decomposition $\mathrm{ad}\,\rho = \mathrm{ad}^0\rho \oplus C$ where $\mathrm{ad}^0\rho$ are those elements in $M_2(C)$ with trace zero. It follows from the definition of $R_{\mathscr{D}}^{\min}$ that if $w \in \Sigma\backslash\mathscr{P}$ then $\chi^{-1}\det\rho_{\mathscr{D}}^{\min}$ is unramified at w. The same is then true of $\chi^{-1}\cdot\det\rho_f$ for every $f \in \mathrm{Hom}_{C\text{-alg}}(R'_n, C \oplus C/\lambda^m\varepsilon)$. Therefore

$$\textbf{(8.9)} \qquad \mathrm{res}_w(\gamma_f) \in H^1(D_w, \mathrm{ad}^0\rho/\lambda^m) \quad \forall w \in \Sigma\backslash\mathscr{P}.$$

Let V be the representation space for ρ. This is a free C-module of rank 2. For each $w \in \mathcal{M}_c \backslash \mathcal{M}$ there is a filtration $0 \subsetneq V_w \subsetneq V$ such that V_w and $V_w^1 = V/V_w$ are free C-modules of rank 1 such that I_w acts via $\tilde{\chi}|_{I_w}$ on V_w^1. For each $w \in \Sigma\backslash\Sigma_c \cup \mathcal{M}_c\backslash\mathcal{M}$ define $U_w \subseteq \mathrm{ad}^0\rho$ by

$$U_w = \begin{cases} \mathrm{Hom}_C(V_w^1, V_w) & \text{if } w \in \mathcal{M}_c\backslash\mathcal{M} \text{ and } \chi|_{I_w} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Using the definition of $R_{\mathcal{D}}^{\min}$ and $R_{\mathcal{D}_c}^{\min}$ one easily checks that if $\mathrm{res}_w(\gamma_f) = 0$ in $H^1(I_w, \mathrm{ad}^0\rho/U_w)^{D_w}$ for all $w \in \Sigma\backslash\Sigma_c \cup \mathcal{M}_c\backslash\mathcal{M}$, then f factors through R_0'. It follows that

(8.10)
$$\begin{aligned} \ell_C(\mathrm{Hom}_{C\text{-alg}}(R_n', C \oplus \varepsilon C/\lambda^m)) &- \ell_C(\mathrm{Hom}_{C\text{-alg}}(R_0', C \oplus \varepsilon C/\lambda^m)) \\ &\leqslant \sum_{w \in \Sigma\backslash\Sigma_c \cup \mathcal{M}_c\backslash\mathcal{M}} \ell_C(H^1(I_w, (\mathrm{ad}^0\rho/U_w)/\lambda^r)^{D_w}) \\ &\leqslant \ell_C(C/\delta(\eta), \lambda^m). \end{aligned}$$

The last inequality follows from an explicit calculation of $\ell_C(H^1(I_w, (\mathrm{ad}^0\rho/U_w)/\lambda^r)^{D_w})$ for each w. As $\delta(\eta) \neq 0$ we see that $\ell_C(C/\delta(\eta), \lambda^m) = \ell_C(C/\delta(\eta))$ for large m. Next we note that there are canonical isomorphisms

$$\mathrm{Hom}_C(H_0/(H_0^2, \lambda^m), C/\lambda^m) \simeq \mathrm{Hom}_{C\text{-alg}}(R_0', C \oplus \varepsilon C/\lambda^m)$$

and

$$\mathrm{Hom}_C(H_n/(H_n^2, \lambda^m), C/\lambda^m) \simeq \mathrm{Hom}_{C\text{-alg}}(R_n', C \oplus \varepsilon C/\lambda^m).$$

It follows from this and from (8.10) that

(8.11) $$\ell_C(H_n/(H_n^2)) - \ell_C(H_0/H_0^2) \leqslant \ell_C(C/\delta(\eta)).$$

Combining (8.11) with (8.4) and (8.5) shows that

$$\ell_C(H_n/H_n^2) \leqslant \ell_C(C/(\delta \circ \alpha)(G_n)).$$

It now follows from [DRS, Criterion I] that $\gamma \circ \psi' : R_n' \twoheadrightarrow T_n''$ is an isomorphism of complete intersections of C-algebras. Therefore ψ' must also be an isomorphism of complete intersections. Since C is faithfully flat over B we conclude that the map $\overline{R}_n \twoheadrightarrow \overline{\mathbf{T}}_n$ induced from $\psi(\mathcal{D}, \mathfrak{p})$ is also an isomorphism of complete intersections, and hence $y_1, ..., y_{m-1}$ is a regular sequence in $\mathbf{T}_n$. It then follows easily that $\psi(\mathcal{D}, \mathfrak{p})$ is itself an isomorphism.

This completes the proof of Proposition 8.1. The following proposition is a simple consequence of that one.

Proposition **8.4.** — *If* $\mathscr{D}$ *is a deformation datum for* F *then property* (P1) *holds for* $\mathscr{D}$.

Proof. — Suppose that $\mathfrak{p} \subseteq \mathbf{T}_{\mathscr{D}}$ is a prime that is nice for $\mathscr{D}$. Let $\mathfrak{p}_{\mathscr{D}} \subseteq \mathrm{R}_{\mathscr{D}}$ be the prime associated to $\mathfrak{p}$ as in §4.2. Let $\mathrm{Q} \subseteq \mathfrak{p}_{\mathscr{D}}$ be any minimal prime and let $\rho = \rho_{\mathscr{D}} \bmod \mathrm{Q}$. Put $\mathrm{R} = \mathrm{R}_{\mathscr{D}}/\mathrm{Q}$. As in §2.3 let $\mathrm{L}_\Sigma/\mathrm{F}$ be the maximal abelian p-extension of F unramified away from Σ and let N_Σ be the torsion subgroup of $\mathrm{Gal}(\mathrm{L}_\Sigma/\mathrm{F})$. Fix a finite character $\psi : \mathrm{Gal}(\mathrm{F}_\Sigma/\mathrm{F}) \longrightarrow \mathrm{R}^\times$ of p-power order such that $\tilde{\chi}^{-1} \cdot \det(\rho \otimes \psi)$ is trivial on N_Σ. Corresponding to the deformation $\rho \otimes \psi$ is a homomorphism $\mathrm{R}_{\mathscr{D}} \longrightarrow \mathrm{R}$ that factors through $\mathrm{R}_{\mathscr{D}}^{\min}$. The kernel of this homomorphism, say Q_1, is contained in $\mathfrak{p}_{\mathscr{D}}$. It then follows easily from Proposition 8.1 that Q_1 is pro-modular.

By Lemma 3.17 there is some map $\mathbf{T}_\infty(\mathrm{U}_{\mathscr{D}} \cap \mathrm{U}_1(\mathrm{cond}^{(p)}(\psi)^2), \mathscr{O}) \longrightarrow \mathrm{R}$ inducing the pseudo-deformation associated to ρ. To show that ρ_1 and hence Q is pro-modular it is enough to show that $\mathrm{U}_{\mathscr{D}} \subseteq \mathrm{U}_1(\mathrm{cond}^{(p)}(\psi)^2)$. To establish this inclusion we first note that since ψ has p-power order $\mathrm{cond}^{(p)}(\psi)$ is square-free. Moreover, if $\mathscr{D} = (\mathscr{O}, \Sigma, c, \mathscr{M})$ and if $\ell | \mathrm{cond}^{(p)}(\psi)$, then $\ell \in \Sigma \backslash \mathscr{M}$. It then follows from the definition of $\mathrm{U}_{\mathscr{D}}$ (see §3.6) that $\mathrm{U}_{\mathscr{D}} \subseteq \mathrm{U}_1(\ell^2)$. Thus $\mathrm{U}_{\mathscr{D}} \subseteq \mathrm{U}_1(\mathrm{cond}^{(p)}(\psi)^2)$.

We have thus shown that any minimal prime of $\mathrm{R}_{\mathscr{D}}$ contained in $\mathfrak{p}_{\mathscr{D}}$ is pro-modular. The same is then true of any prime of $\mathrm{R}_{\mathscr{D}}$ contained in $\mathfrak{p}_{\mathscr{D}}$. □

A. A useful fact from commutative algebra

The following result, in the guise of its corollary stated below, is the linchpin in our proof of the Main Theorem.

Proposition **A.1.** [Ray, Corollaire 4.2] — *If* A *is a local Cohen-Macaulay ring of dimension d, and if* $\mathrm{I} = (f_1, ..., f_r)$ *is an ideal of* A *with* $r \leqslant d - 2$, *then*

$$\mathrm{spec}(\mathrm{A}/\mathrm{I}) \setminus \{\mathfrak{m}_\mathrm{A}\} \quad \textit{is connected.}$$

We are indebted to M. Raynaud for providing us with the reference to a proof of this proposition.

Suppose now that A and I are as in the proposition. Let $\mathscr{C}$ be the set of irreducible components of spec (A/I).

Corollary **A.2.** — *If* $\mathscr{C} = \mathscr{C}_1 \sqcup \mathscr{C}_2$ *is a partition of* $\mathscr{C}$ *with* $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *non-empty, then there exist irreducible components* $\mathrm{C}_1 \in \mathscr{C}_1$ *and* $\mathrm{C}_2 \in \mathscr{C}_2$ *such that* $\mathrm{C}_1 \cap \mathrm{C}_2$ *contains a prime of dimension* $d - r - 1$.

Proof of Corollary. — Our proof is by induction on $d - r$. If $d - r = 2$, then the assertion of the corollary is an immediate consequence of the proposition. Now suppose $d - r > 2$. The conclusion of the proposition implies that there exist $C_1' \in \mathscr{C}_1$ and $C_2' \in \mathscr{C}_2$ such that $C_1' \cap C_2'$ contains a prime $\mathfrak{p}$ of dimension 1, which we may view as a prime of A. Now consider spec($A_\mathfrak{p}/I$). Let $\mathscr{C}'$ be the irreducible components of spec($A_\mathfrak{p}/I$). The embedding spec($A_\mathfrak{p}/I$) $\hookrightarrow$ spec(A/I) of topological spaces induces a decomposition of $\mathscr{C}'$:

$$\mathscr{C}' = \mathscr{C}'_1 \sqcup \mathscr{C}'_2, \quad \mathscr{C}'_i = \{C' = C \cap \mathrm{spec}\,(A_\mathfrak{p}/I) : C \in \mathscr{C}_i\}.$$

By the choice of $\mathfrak{p}$, $C_i' \in \mathscr{C}'_i$, so this is a non-trivial decomposition. As dim $A_\mathfrak{p}$ = dim A − 1, the conclusion of the corollary now follows from the induction hypothesis together with the fact that $A_\mathfrak{p}$ is also Cohen-Macaulay and that the dimension of a prime of $A_\mathfrak{p}/I$ is one less than the dimension of the corresponding prime of A/I. □

Index of selected terminology

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