PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Publications mathématiques de l'I.H.É.S., tome S88 (1998), p. 73-79

<http://www.numdam.org/item?id=PMIHES_1998__S88__73_0>

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A MATHEMATICIAN'S VIEW OF THE DEVELOPMENT OF PHYSICS*

by LUDVIG FADDEEV

Mathematics in its clean form is the product of the free human mind.

Physics is a natural science with just a single goal – uncovering the structure of matter.

In their quest physicists naturally use mathematical tools to correlate data, to express the laws found by means of formulas and to make relevant calculations. To a greater or smaller extent this is done in all sciences. And there is no *a priori* reason for such a distinguished role of mathematics in physics which we are witnessing nowadays – namely that of an imminent language of physical theory.

I shall not elaborate on the examples to prove this role. Everybody in his profession can choose his favorite. It is enough to recall that such purely mathematical structures as Riemann geometry or Lie groups theory are indispensable in modern theory of gravity, formulated by Einstein, or in the description of kinematical and dynamical symmetries of any physical system.

This role of mathematics as the language of physics is taken by physicists with mixed feelings of admiration and irritation. Take for example the title of the famous essay by Wigner: "On the unreasonable effectiveness of mathematics in natural sciences". The complex formed by these feelings is sometimes resolved in malicious jokes on mathematics which some great men allowed themselves to tell. I shall not comment more on this.

Instead, I shall take seriously the stated role of mathematics as a fact and try to present in this spirit the analysis of modern trends in physics. To do this I shall need some formalized framework and I proceed now to its description.

In the description of the physical system we use two main notions: those of observables and those of states. The set of observables $\mathfrak{U}$ comprises all physical entities A, B, C, ... constituting the system. The set of states Ω, with elements ω, μ, ..., describes the possible results of the measurements of observables. More formally, each state ω gives to each

* The original source of this article is: *Proceedings of the 25th Anniversary Conference – Frontiers in Physics, High Technology & Mathematics*, eds. H.A. Cerdeira & S.O. Lundqvist (1990) p. 238-246. This article first appeared in *Miscellanea mathematica*, Berlin Heidelberg Springer-Verlag (1991), p. 119–127.

observable A its probability distribution – a nonnegative, monotone increasing function $\omega_A(\lambda)$ of a real variable λ, $-\infty < \lambda\ \infty$, normalized by the conditions $\omega_A(-\infty) = 0$, $\omega_A(\infty) = 1$. In particular, the mean value of an observable A in a state ω is given by

$$\langle\omega|A\rangle = \int_{-\infty}^{\infty} \lambda\, d\,\omega_A(\lambda)\,.$$

The completeness of such a description is expressed in the requirement that states separate observables. Namely, if two observables A and B have the same mean value in all states, then they coincide. This is a formal expression of the main epistemological principle of the ability of cognition of the universe.

Mathematically this principle introduces some structure in the set of observables:

1. $\mathfrak{U}$ is a real linear space.

Indeed, observables A + B and kA for real k are defined as having the mean values

$$\langle\omega|A+B\rangle = \langle\omega|A\rangle + \langle\omega|B\rangle$$

and

$$\langle\omega|kA\rangle = k\langle\omega|A\rangle$$

in all states ω.

2. For each real-valued function $\varphi(\lambda)$ of the real-variable λ and each observable A we can construct the observable $\varphi(A)$ by means of the formula

$$\langle\omega|\varphi(A)\rangle = \int_{-\infty}^{\infty} \varphi(\lambda)\, d\omega_A(\lambda)\,,$$

valid for any state ω. Alternatively, we can say that the probability distribution of $\varphi(A)$ is given by

$$\omega_{\varphi(A)}(\lambda) = \omega_A(\varphi^{-1}(\lambda))\,.$$

To introduce the dynamics of the system we are to describe the notion of motions or one-parameter automorphisms $A \to A(s)$ in the set of observables. This is done by means of a binary operation (bracket) $\{A, B\}$ which allows one to associate a particular motion $A \to A(s)$ with every observable B by means of the differential equation

$$\frac{dA(s)}{ds} = \{B, A(s)\}, \qquad A(0) = A\,.$$

It is natural to require that the generating observable B does not change, so that $\{B, B\} = 0$. The compatibility with the linear structure implies that $\{\,,\}$ must be a Lie bracket, namely it must be linear

$$\{kA + lB, C\} = k\{A, C\} + l\{B, C\}$$

and satisfy the Jacobi identity

$$\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0\,.$$

Moreover, the notion of function is to commute with motion

$$\varphi(A(s)) = \varphi(A)(s)\,.$$

This is essentially all that we need to provide a general framework for the description of a physical system. I admit that the dynamical principle is less intuitive than the kinematical one. However, I do not see any other way of formalizing the experience we have until now.

The existing physical theories give us concrete realizations of this general scheme. Take classical mechanics. The basic notion there is that of the phase space Γ consisting of the generalized coordinates q and momenta p. Observables are real-valued functions $f(p, q)$ on Γ. The linear structure is evident, $\varphi(f)$ is understood as the superposition of the functions f and φ. The remarkable mathematical theorem of Markov then defines the set of states Ω in a unique manner as that of normalized measures ω on Γ. The distribution function $\omega_f(\lambda)$ is given by

$$\omega_f(\lambda) = \int\limits_{f(p,q)\leqslant\lambda} d\omega = \int\limits_{\Gamma} \theta(f-\lambda)\,d\omega\,,$$

where $\theta(\lambda)$ is the Heaviside function.

The dynamical Lie operation is given by the Poisson bracket which looks in the canonical variables p, q as follows

$$\{f, g\} = \Sigma\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\right)\,.$$

A particular motion corresponding to evolution or time development is given by the Hamilton equation

$$\frac{df}{dt} = \{H, f\}\,,$$

where the observable H is called the energy.

Quantum mechanics is just another realization of our general scheme. In the usual description of quantum mechanics the role of observables is played by selfadjoint operators A, B, ... in some (auxillary) Hilbert space $\mathfrak{H}$. The states are given by positive operators M with trace equal to 1. The distribution of A in the state M is given by

$$\omega_A(\lambda) = \mathrm{tr}(MP_A(\lambda))\,,$$

where $P_A(\lambda)$ is the spectral function of A (which can be formally written as $P_A(\lambda) = \Theta(\lambda - A)$). The definition of the function $\varphi(A)$ of A is given by

$$\varphi(A) = \int_{-\infty}^{\infty} \varphi(\lambda)\,dP_A(\lambda)\,,$$

and the number of degrees of freedom is equal to the number of functionally independent commuting observables. The Lie bracket is given by

$$\{A, B\} = \frac{i}{\hbar}(AB - BA) ,$$

where $i = \sqrt{-1}$ and $\hbar$ is a fixed parameter of dimension

$$[\hbar] = [p][q] ,$$

the famous Planck constant.

Let us mention that in both realizations the notion of function could be introduced purely algebraically by means of the associative product existing in the set of observables. It is not clear if there exists a more general realization where such a product does not appear at all.

Another mathematical observation, already vaguely mentioned, is that the structure of the set of observables is sufficient to define the set of states as the dual object – the convex set of positive functionals.

I am now ready to proceed to speculations on the development of physics. However, it is worth making some general comments on what was already said.

1. The scheme was formulated already after the advent of quantum mechanics. The main role here belongs to Dirac who in particular introduced the term "observables". In the particular way of arranging the formulas and notions I am influenced by my mathematical colleagues, I. Segal and G. Mackey.

2. The fact that classical mechanics bears all the essential features of the scheme makes it plausible that it could already be formulated in the previous century for example by Hamilton or Gibbs. Also, the mathematics available at that time allowed for the search for other realizations, leading to quantum mechanics. However, history did not take that path.

3. It is often said that quantum mechanics is "indeterministic" because the notion of probability is used in its formulation. This is a misleading statement. We have seen that distribution functions in the role of states appeared already in classical mechanics. The specific feature of the classical case is that all observables are exact in pure states, whereas in the quantum case they are exact only in eigenstates, which is, however, enough to describe them completely. So it is just an undeserved luxury what we have in classical mechanics and it is only justified that it is eliminated during the passage to quantum mechanics.

After these comments I return to my main goal and use the general scheme to analyze the relation between classical and quantum mechanics. It will be convenient to describe both theories by means of similar objects. It is enough to stick with the observables, because we know that states are defined by observables. So we shall use the possibility to describe the quantum mechanical observables by means of functions on the phase space. In this description the function $f(p, q)$ is the symbol of the operator A_f. In the simplest case of linear phase space with one degree of freedom we can express A_f as an integral operator in $L_2(\mathbb{R})$ in terms of its kernel $A_f(x, y)$ by the Weyl formula

$$A_f(x,y) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} f\left(p, \frac{x+y}{2}\right) e^{ip(x,y)/\hbar} dp$$

(note the explicit presence of $\hbar$). It is clear that the main structures of quantum observables $\{f, g\}$ and $\varphi(f)$ are different from those of the classical ones. However, the former converge to the latter in the limit $\hbar \to 0$. More exactly, we have the expansion

$$\{f,g\}_\hbar = \{f,g\}_0 + \hbar\{f,g\}^{(1)} + \hbar^2\{f,g\}^{(2)} + \dots$$
$$\varphi(f)_\hbar = \varphi(f)_0 + \hbar\varphi(f)^{(1)} + \hbar^2\varphi(f)^{(2)} + \dots,$$

where we specified the quantum operations by subscript $\hbar$ and the classical ones by 0. Both these expansions are corollaries of the formula for the product of the symbols f and g induced by the operator product of A_f and A_g. If we use the Weyl correspondence between symbols and operators we have for the product $f * g$ the following expansion in powers of $\hbar$

$$f * g = fg + \frac{\hbar}{2i}\{f,g\}_0 + \dots$$

which in turn leads to the corresponding expansions for the Lie bracket

$$\{f,g\}_\hbar = \frac{i}{\hbar}(f * g - g * f) + \dots$$

and the definition of the function $\varphi(f)$ of a given observable f.

Of course, in nature the Planck constant $\hbar$ has a given value, $\hbar \cong 10^{-27}\mathrm{g}\cdot\mathrm{cm}^2\mathrm{s}^{-1}$; the existence of a family of quantum mechanics, labeled by the parameter $\hbar$ is a mathematical play of mind. Mathematicians use the term "deformation" of structure in such cases. Using this term we can say that quantum mechanics is a deformation of the classical one with $\hbar$ playing the role of deformation parameter.

This statement is the shortest and most adequate formulation of the correspondence principle. For a professional mathematician there is nothing to add or to delete in this formulation. Many words used in popular literature to explain the correspondence principle is nothing but "belletristics".

Now we are coming to the most important place in our exposition. The matter is that in the mathematical theory of deformations of structures there exists the notion of stability. We say that the given algebraic structure is stable if all deformations of it lying near to it are equivalent to it. Now, the following fact is true: the structure of quantum mechanics underlined in the general scheme above is stable. On the contrary, classical mechanics is not stable – it has in its vicinity a nonequivalent deformation – quantum mechanics. The degeneracy of classical mechanics is connected with its overdeterministic nature realized in the exactness of pure states for all observables. The passage to quantum mechanics removes this degeneracy and leads to a stable "generic" theory which is nondeformable in the framework of our general scheme.

Thus we are led to an important conclusion: whereas the change of classical mechanics into the quantum one is fully justified, we have no reasons to predict any change of the latter in the future.

The application of the mathematical theory of deformation of structures to analysis of the evolution of the physical picture of nature is not exhausted by this radical statement. We can describe in a similar fashion the passage from the nonrelativistic dynamics to the relativistic one. In fact, it is even easier.

From the mathematical point of view, this passage was connected with the change of the dynamical group – or the group of symmetries of the space-time – from the Galilei transformations to those of Lorentz-Poincaré.

Both these groups contain 10 parameters. Let us compare the changes of coordinates $\mathbf{x}$ and time t, corresponding to the change of the reference frame with the relative velocity $\mathbf{v}$. The Galileo transformation is given by

$$\mathbf{x} \longrightarrow \mathbf{x} + \mathbf{v}t; \qquad t \longrightarrow t$$

whereas in the Lorentz-Poincaré case we have

$$\mathbf{x}^{\|} \longrightarrow \frac{\mathbf{x}^{\|} + \mathbf{v}t}{\sqrt{1 - v^2/c^2}} \qquad \mathbf{x}^{\perp} \longrightarrow \mathbf{x}^{\perp}$$

$$t \longrightarrow \frac{t + (\mathbf{v} \cdot \mathbf{x})/c^2}{\sqrt{1 - v^2/c^2}}$$

$$\mathbf{x}^{\|} = (\mathbf{x} \cdot \mathbf{v})\mathbf{v}/v^2; \qquad \mathbf{x}^{\perp} = \mathbf{x} - \mathbf{x}^{\|} .$$

We see that the Lorentz transformation contains a fixed parameter c the velocity of light with the value $c = 3 \cdot 10^{10}$ cm/s. It is clear that Lorentz transformations go into the Galilei in the limit $c \longrightarrow \infty$. (To do this we are to imagine a mathematical fiction – the existence of the worlds with different values of c in the same fashion as we have done above in the case of Planck constant $\hbar$.) In the meantime both transformations constitute the same mathematical structure – a Lie group. Thus we see another manifestation of the deformation in the development of physics: the group of relativistic dynamics is a deformation of the group of nonrelativistic motion. The role of the deformation parameter is played by $1/c^2$. Now an important statement follows: the passage to relativism is into a stable structure. In other words, the Lorentz group is stable and does not allow any nontrivial deformations.

Let us now turn off the current of mathematical consciousness and look at the results of our analysis. We see that the two main revolutions in physics (and in all modern natural sciences) from the mathematical point of view are deformations from unstable structures into stable ones. The fashionable words on the change of paradigms are not highly relevant from this point of view. But nothing is lost in the realization of this fact. I think that our statement is a short and most adequate description of the evolution of our views on the theory of the structure of matter.

Now a natural question must be asked: does the analysis of the past of physics allow us to say something about its future? A historically minded person would answer no. However, those who believe in the existence of the mathematical structure underlying the world around us could try to make some predictions. So I cannot help but use the opportunity to do so; I realize how speculative such predictions can be.

I have already stressed that the parameters of deformation $\hbar$ and $1/c^2$ have physical dimensions. In terms of basic dimensions – mass M, length L and time T – we have $[\hbar] = M[L]^2[T]^{-1}; [1/c^2] = [L]^{-2}[T]^2$. It is clear that only one dimensional parameter is lacking and one could think that one more deformation of our description of matter is to be sought for with such a parameter playing the role of that of deformation.

In fact, we can admit that such a deformation is already realized. Indeed, the third main achievement of physical theory in our century – the Einstein theory of gravity – can be considered as such a deformation in the stable direction. This theory is based on the use of the curvilinear pseudo-Riemann space as a space-time manifold instead of the linear Minkowski space. It is clear that in the set of Riemann spaces the Minkowski space is a kind of degeneracy, whereas a generic Riemannian space is stable so that in its vicinity all spaces are curvilinear. The measure of the deformation is the gravitation constant γ, entering the Hilbert-Einstein equations of gravity, which was known in physics from the time of Newton. The gravitational constant γ has dimension functionally independent from those of $\hbar$ and c and together with them constitutes the basis for all dimensional parameters.

It is clear what I'm driving at: the unification of relativism, quantum principles and gravity must give us the ultimate physical theory, which is stable and not submittable to changes without really drastically breaking down the established framework, which has a very general character.

I do not see any wrong in the prediction that one more natural science may find a final formulation. Chemistry has already found its fundamental formulation on the basis of the quantum mechanics of many electron systems. So it is not surprising that physics is going to share the same fate.

Unfortunately, the natural synthesis of relativism, quantum principles and gravity is not achieved yet in modern theoretical physics. We have a reasonably complete quantum field theory in Minkowski space ($\hbar$ and c) or classical theory of gravity (c and γ). However, there exists no theory which incorporates all three parameters $\hbar$, c and γ in a natural manner. The main efforts of a numerous army of theoretical and mathematical physicists are directed to the realization of this unification. Not so many participants in this quest would agree that their labour will lead to the end of fundamental physics which will be formulated by means of an adequate mathematical language. But for some of them, including this author, this idea is self-evident, inevitable and, what is most important, a guiding one.

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