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# CHERN-SIMONS TERMS AS AN EXAMPLE OF THE RELATIONS BETWEEN MATHEMATICS AND PHYSICS

*by* STANLEY DESER

*The inevitability of Chern-Simons terms in constructing a variety of physical models, and the mathematical advances they in turn generate, illustrate the unexpected but profound interactions between the two disciplines.*

I begin with warmest greetings to the Institut des Hautes Études Scientifiques (IHÉS) on the occasion of its $40^{th}$ birthday, and look forward to its successes in the years to come. My own association dates back to its early days in 1966-67 and it has continued fruitfully ever since.

The IHÉS represents a unique synthesis between Mathematics and Physics, as emphasized by this volume's title. I propose to illustrate this synthesis through a particular set of examples, Chern-Simons "effects" in physics. This should both reflect the interplay of the two disciplines, as well as the uncanny way mathematical constructs become incorporated into physics (and sometimes even require the physicist to be a little precise). I must of necessity be succinct here, and refer (also compactly) to the literature for details. I shall not have the space here to illustrate the "backreaction", how such borrowing by physics in turn stimulates new mathematics; the application of CS to knot theory is a salient recent example.

To do justice to the full web of interconnections involving Chern-Simons (CS) terms [1] in physics would require one of those complicated tree (or loop) diagrams. I will have to omit entirely any discussion of some of the principal ones, for example the relation of CS to: 1) conformal field theory [2] (descending from 3 to 2 dimension in particular), 2) anomalies [3], via its Pontryagin $F \wedge F$ ancestor (ascending from 3 to 4) and 3) to integrable systems [4]. Instead, I will stick to some more concrete applications in which I have been involved.

The first sighting of CS in physics may have been in 1978, when $D = 11$ supergravity (now back, after two decades, in a central role) was constructed. It arose there as a strange but unavoidable term needed for consistency of the theory, by preserving its local supersymmetry, then rapidly invaded lower dimensional, $4 < D < 11$, models [3]. That a metric-independent, "topological", term (as physicists sometimes call them) should come to the rescue of a gravitational model is the first example of its uncanny properties! The theory necessarily contains a 3-form potential A, and it was found that there has to be an addition ${}_{11}I_{CS}[A] = \kappa \int A \wedge F \wedge F$, $F \equiv dA$ to the usual $F^2$ kinetic term in the action. The Einstein gravitational constant $\kappa$ appears here, but not (of course) the metric. A smaller paradox is that despite appearances, $I_{CS}$ is both parity and T even. From a physical point of view, this term generates a cubic self-interaction of the form field that is in fact essential in constructing its supersymmetry-preserving invariants [4]. These invariants are important as they can serve both as a check of M-theories currently thought to incorporate D = 11 supergravity as a limiting case and as counter terms in higher loop corrections to it [5]. This first physics appearance of CS passed relatively unnoticed for several reasons, not least the cubic nature of ${}_{11}I_{CS}[A]$, so that it did not directly affect the kinematics. Soon afterwards, and with no apparent connection to the above, the possibility and interest of introducing the 1-form CS term in spacetime dimensions D = 3 was suggested by several authors [6, 7]. This time the context was more auspicious both because D = 3 is closer to D = 4 and because physics in this planar world may even have observable consequences, in condensed matter settings as well as in high temperature limits of our D = 4 world. Most of all, the interest was due to the fact that CS is here quadratic, ${}_3I_{CS}[A] = \mu \int A \wedge F$ and hence can affect free-field (Maxwell) electromagnetism, and indeed lead to a finite-range but still gauge-invariant model. In its nonabelian incarnation, where A is a Lie-algebra valued 1-form, the term has the remarkable property that its numerical coefficient must be quantized for the quantum theory to be well-defined [7]. This idea, coming entirely from homotopy analysis, was of course a revelation to physicists on how *a priori* arbitrary parameters could in fact be restricted in their possible values (and hence had better also be renormalized by integer amounts only).

Before we consider some of the novel consequences of CS in this D = 3 context, we first mention a quite different direction that gave rise to an enormous literature on so-called topological quantum field theories, including D = 3 gravity, as we shall see. For the moment, we take the geometry to be Minkowskian $R^1 \times R^2$, to avoid global and topological complications. Then the Euler-Lagrange equations of purely CS actions simply become ${}^*F^\mu = 0$, in the absence of sources or ${}^*F^\mu = j^\mu$ when charged currents are present (${}^*F$ is the dual field strength, a 1-form). Thus the field is locally pure gauge wherever there are no sources; to find the general global solution with the properties that the field strength is equal to the current and vanishes elsewhere is then an interesting exercise. This is even more so in the nonabelian case where the abelian part is supplemented by the famous $\frac{1}{3} tr \int (A \wedge A \wedge A)$ addition to yield the same (but now nonabelian) Euler-Lagrange equation ${}^*F = 0$. Now in D = 3, general relativity has a very similar property: spacetime is flat in the absence of source, since Einstein (G) and Riemann (R) tensors are equivalent, obeying the double-duality identity $G \equiv {}^*R^*$. Hence the Einstein equations $G = 0$ imply local flatness. Classification of such Minkowski signature locally flat (or more generally locally constant curvature if there is also a cosmological constant so that $G + \Lambda g = 0$), spacetimes [8] has also become a physical industry of its own. Here the physics involves global matching of flat patches at particle trajectories where the sources $T_{\mu\nu}$ (and therefore curvature) do not vanish. This zero "field strength" field equation in source-free regions in gravity is of course very reminiscent of the above Yang-Mills CS story and indeed there is a CS form of Einstein gravity [9]. This insight has led to another large topic of its own ever since, namely the uses of the "antigeometrical" CS as geometry! There is also a direct mathematical connection, namely that between the Riemann-Hilbert problem and that of $D = 3$ gravity coupled to several moving particles [12].

Returning to physical applications of the plain abelian CS term, let me sketch a few of the reasons for their interest. First, if we add the usual Maxwell action to CS, the resulting theory represents a single local degree of freedom, paradoxically endowed with a finite range but still gauge invariant. [This seems a very different way to get a finite mass than the Higgs mechanism but even here things are more interesting! See [10]]. As background, recall that a pure Maxwell excitation in any dimension has (D-2) local excitations, the transverse spatial polarizations, while the gauge-broken (Proca theory) massive version has (D-1) of them. Further, these theories represent excitations of unit intrinsic angular momentum or spin. In $D = 3$, however, it turns out that massless fields, including Maxwell, are (unlike massive ones) entirely devoid of spin [11] (but neutrinos still can have fermi statistics). The Maxwell-CS action inherits from pure Maxwell one local excitation; the CS input is to provide mass and thereby spin to this excitation. Here the CS term does break parity: the two degrees of the normal Proca theory are equivalent to a pair of "mirror" M-CS models. The M-CS field equations $dF + \mu{}^*F = 0$ are readily seen to imply that the field strength obeys Klein-Gordon propagation equations with $\mu$ representing the finite range. This mixing of normal metric and topological terms is what makes these models so different from the usual even-dimensional ones.

Mathematically, we have mentioned the role played by homotopy in CS physics. In fact there are several different roles, as we shall see. One is the cited quantization of the CS coefficient in the nonabelian theory: because the exponential of the action, $e^{iI/\hbar}$ is the basic quantum mechanical object there, actions must be invariant mod $2\pi$ under gauge transformations. Tracing the $\Pi_3/\Pi_1$ properties of CS under large gauge transformations shows it changes by a winding number so that its coefficient is necessarily integer; this is the dimensionless combination $\mu/g^2$ where $g^2$ is the (dimensional in $D = 3$) self-coupling constant. Another effect is that of the topology of planar configuration space – this is related to "anyons" or the loss of the standard spin-statistics relation in planar field theories [12]; it too can be represented in CS language [13].

My final example [14] is the most recent (but definitely not the last!); it deals with the definition and role of – even abelian – CS in nontrivial topologies, which already arises in cases as simple as $S^1 \times \Sigma^2$, finite-temperature ($\beta = \frac{1}{kT}$ is the perimeter of $S^1$) planar electrodynamics with (necessarily quantized) magnetic flux in the closed 2-manifold $\Sigma^2$. It is known that the naive CS term $\int A \wedge F$ now requires corrections to remain well-defined. These corrections to CS, and its behavior under large (not reducible to the identity) gauge transformations can in fact be elucidated in two complementary ways (and different from the known cohomology procedures cited in [14]). The first uses a really "classic" result, the Chern-Weil theorem, which in D = 4 tells us (using the transgression formula) that for two different connections $(A, \tilde{A})$ on a bundle, $F \wedge F - \hat{F} \wedge \hat{F} = d[(A - \hat{A}) \wedge (F + \hat{F})]$. This provides a correct definition of $I_{CS}$ on non-trivial bundles and also tells us that, unlike the simple-minded CS, it changes under large gauge rotations as the product of their "winding number" and the magnetic flux so as to respect the quantum action requirements mentioned above. The second way to reach the correct definition is – surprisingly – to embed the abelian model in a nonabelian SU(2) where all D = 3 bundles are trivial; the fact that the homotopies of U(1) and SU(2) are opposite ($\Pi_1$ of the former and $\Pi_3$ of the latter fail to vanish) is no obstacle. [There is a third, heuristic, way – the one a desperate physicist would use to "guarantee" correctness when all else fails [14, 15], but I do not discuss it here!] I cite this seemingly pedantic formal discussion of CS definition precisely because what is the correct one in topologically nontrivial backgrounds has led to an immense and rather confused physics literature; confused because based on the naive $\int A \wedge F$. "Thermal" quantum electrodynamics in spacetime dimension 3 ($QED_3$) consists of the interaction of charged particles with the electromagnetic field, but in particular replacing the time by temperature through periodic identification of *t*, as we have mentioned. Now the CS miracle here is that, whether or not there is a "primitive" CS term in the original action (or indeed any action at all for the electromagnetic field A), there will arise an effective theory of A if one integrates out the charge particles that (necessarily) couple to it. In particular, if we have massive charged electrons obeying the usual Dirac equation $(\not{D} + m)\psi = 0$, $\not{D} \equiv \gamma(\partial + iA)$, then the (logarithm of the) determinant of the Dirac operator is essentially the functional that defines the effective action $I_{eff}[A]$. Since a fermion mass term is parity (and T) violating in D = 3, there should naturally be CS terms in $I_{eff}$. Now the route to this action involves a careful process of first defining the determinant, *e.g.*, by $\zeta$-function regularization. This enables one to expand in Seeley-deWitt coefficients, and find the correct, automatically gauge-invariant $I_{eff}[A]$. In particular the CS term always enters in a way that preserves invariance namely as part of deeper $\eta$-function structures. It was neglecting or omitting this necessary complication that gave rise to paradoxes involving large gauge transformations, that of course do *not* leave CS invariant. Indeed, to a physicist CS is basically the reminder already in the abelian but globally non-trivial space context that there is a further, discrete, gauge invariant variable besides the field strengths, namely the so-called flat connection. I must refer you for details and earlier literature to a long paper that will appear soon [15].

In this "hommage" to IHÉS, I have tried to give one short glimpse of how a theoretical physicist is often forced to use – and greatly benefits from – *a priori* far-removed mathematical constructs, a process that ultimately leads to further advances in mathematics as well. This symbiosis is one of the hallmarks of IHÉS!

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