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A NEW PROOF OF MORLEY'S THEOREM

by ALAIN CONNES

It is now 22 years since IHÉS offered me its hospitality. I have learnt there most of the mathematics I know, mostly thanks to impromptu lunch conversations with visitors or permanent members.

When I arrived, I was engrossed in my own work and had the humbling experience of finding out how little I understood of what was currently discussed. Dennis Sullivan took care of me, and gave me a crash course in geometry which influenced the way I thought for the rest of my life.

It is also in Bures, thanks to the physicists that I understood the truth of the statement of J. Hadamard on the depth of the mathematical concepts coming from physics:

"Not this short lived novelty which can too often only influence the mathematician left to his own devices, but this infinitely fecund novelty which springs from the nature of things."

In order to give some flavor of the atmosphere of friendly competition caracteristic of IHÉS, I have chosen a specific example of a lunch conversation of this last spring which led me to an amusing new result.

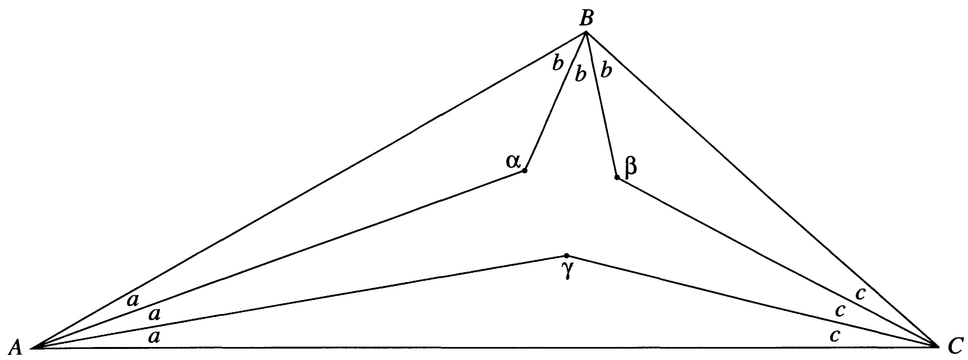

Figure 1

Around 1899, F. Morley proved a remarkable theorem on the elementary geometry of Euclidean triangles:

"Given a triangle A, B, C the pairwise intersections α, β, γ of the trisectors form the vertices of an equilateral triangle" (cf. fig. 1).

One of us mentioned this result at lunch, and (wrongly) attributed it to Napoleon. Bonaparte had indeed studied mathematics at an early age, and, besides learning English, was teaching mathematics to the son of Las Cases during his St Helen's exile in Longwood.

It was the first time I heard about Morley's result and when I came back home, following one of the advices of Littlewood, I began to look for a proof, not in books but in my head. My only motivation besides curiosity was the obvious challenge "This is one of the rare achievements of Bonaparte I should be able to compete with". After a few unsuccessful attempts I quickly realized that the intersections of consecutive trisectors are the fixed points of pairwise products of rotations g_i around the vertices of the triangle (with angles two thirds of the corresponding angles of the triangle). It was thus natural to look for the three fold symmetry g of the equilateral triangle as an element of the group Γ generated by the three rotations g_i. Now, it is easy to construct an example (in spherical geometry) showing that Morley's theorem does not hold in NonEuclidean geometry, so that the proof should make use of special Euclidean properties of the group of isometries.

I thus spent some time trying to find a formula for g in terms of g_i, using the easy construction (any isometry with angle $2\pi/n$, $n \geqslant 2$ is automatically of order n), of plenty of elements of order 3 in the group Γ, such as $g_1 g_2 g_3$. After much effort I realized that this was in vain (cf. remark 2 below) and that the relevant group is the affine group of the line, instead of the isometry group of the plane.

The purpose of this short note is to give a conceptual proof of Morley's theorem as a group theoretic property of the action of the affine group on the line. It will be valid for any (commutative) field k (with arbitrary characteristic, though in characteristic 3 the hypothesis of the theorem cannot be fulfilled). Thus we let k be such a field and G be the affine group over k, in other words the group of 2×2 matrices $g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ where $a \in k$, $a \neq 0$, $b \in k$. For $g \in$ G we let,

$$\delta(g) = a \in k^* \, . \tag{1}$$

By construction δ is a morphism from G to the multiplicative group k^* of non zero elements of k, and the subgroup T = Ker δ is the group of translations, i.e. the additive group of k.

Each $g \in$ G defines a transformation,

$$g(x) = ax + b \qquad \forall\, x \in k \, , \tag{2}$$

and if $a \neq 1$, it admits one and only one fixed point,

$$\operatorname{fix}(g) = \frac{b}{1-a} \, . \tag{3}$$

Let us prove the following simple fact:

Theorem. *Let* $g_1, g_2, g_3 \in G$ *be such that* $g_1 g_2, g_2 g_3, g_3 g_1$ *and* $g_1 g_2 g_3$ *are* not *translations and let* $j = \delta(g_1 g_2 g_3)$. *The following two conditions are equivalent,*

a) $g_1^3 g_2^3 g_3^3 = 1$.

b) $j^3 = 1$ *and* $\alpha + j\beta + j^2\gamma = 0$ *where* $\alpha = \operatorname{fix}(g_1 g_2)$, $\beta = \operatorname{fix}(g_2 g_3)$, $\gamma = \operatorname{fix}(g_3 g_1)$.

Proof. Let $g_i = \begin{bmatrix} a_i & b_i \\ 0 & 1 \end{bmatrix}$. The equality $g_1^3 g_2^3 g_3^3 = 1$ is equivalent to $\delta(g_1^3 g_2^3 g_3^3) = 1$, and $b = 0$, where b is the translational part of $g_1^3 g_2^3 g_3^3$. The first condition is exactly $j^3 = 1$. Note that $j \neq 1$ by hypothesis. Next one has

$$b = (a_1^2 + a_1 + 1)\, b_1 + a_1^3 (a_2^2 + a_2 + 1)\, b_2 + (a_1 a_2)^3 (a_3^2 + a_3 + 1)\, b_3 . \tag{4}$$

A straightforward computation, using $a_1 a_2 a_3 = j$ gives,

$$b = -j\, a_1^2\, a_2 (a_1 - j)(a_2 - j)(a_3 - j)(\alpha + j\beta + j^2\gamma) , \tag{5}$$

where, α, β, γ are the fixed points

$$\alpha = \frac{a_1 b_2 + b_1}{1 - a_1 a_2} , \ \beta = \frac{a_2 b_3 + b_2}{1 - a_2 a_3} , \ \gamma = \frac{a_3 b_1 + b_3}{1 - a_3 a_1} . \tag{6}$$

Now, $a_k - j \neq 0$ since by hypothesis the pairwise products of g_j's are *not* translations. Thus, and whatever the characteristic of k is, we get that a) $\Leftrightarrow$ b).

Corollary. *Morley's theorem.*

Proof. Take $k = \mathbb{C}$ and let g_1 be the rotation with center A and angle $2a$, where $3a$ is the angle BAC and similarly for g_2, g_3. One has $g_1^3 g_2^3 g_3^3 = 1$ since each g_i^3 can be expressed as the product of the symmetries along the consecutive sides. Moreover for a similar reason $\alpha = \operatorname{fix}(g_1 g_2)$, $\beta = \operatorname{fix}(g_2 g_3)$, $\gamma = \operatorname{fix}(g_3 g_1)$ are the intersections of trisectors. Thus from a) $\Rightarrow$ b) one gets $\alpha + j\beta + j^2\gamma = 0$ which is a classical characterization of equilateral triangles.

Remark 1. Without altering the cubes g_1^3, g_2^3, g_3^3 one can multiply each g_i by a cubic root of 1, one obtains in this way the 18 nondegenerate equilateral triangles of variants of Morley's theorem.

Remark 2. We shall now show that in general the rotation g which permutes cyclically the points α, β, γ does not belong to the subgroup Γ of G generated by g_1, g_2, g_3. Under the hypothesis of the theorem, we can assume that the field k contains a non trivial cubic root of unity, $j \neq 1$, and hence that its characteristic is not equal to 3. The rotation which permutes cyclically the points α, β, γ is thus the element of G given by,

$$g = \begin{bmatrix} j & b \\ 0 & 1 \end{bmatrix} , \ 3b = (1 - j)(\alpha + \beta + \gamma) . \tag{7}$$

Now for any element $g = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$, of the group Γ generated by g_1, g_2, g_3, one has Laurent polynomials P_i, in the variables a_j such that,

$$b = b_1 P_1 + b_2 P_2 + b_3 P_3 . \tag{8}$$

Thus, expressing, with the above notations, b_i in terms of α, β, γ,

$$\begin{aligned} b_1 &= (1+j)^{-1}(a_3^{-1}(a_3 - j)\,\alpha - (a_1 - j)\,\beta + a_1 (a_2 - j)\,\gamma) \\ b_2 &= (1+j)^{-1}(a_2 (a_3 - j)\,\alpha + a_1^{-1}(a_1 - j)\,\beta - (a_2 - j)\,\gamma) \\ b_3 &= (1+j)^{-1}(-(a_3 - j)\,\alpha + a_3 (a_1 - j)\,\beta + a_2^{-1}(a_2 - j)\,\gamma) , \end{aligned} \tag{9}$$

we get Laurent polynomials Q_i such that,

$$b = (a_3 - j)\,\alpha\, Q_1 + (a_1 - j)\,\beta\, Q_2 + (a_2 - j)\,\gamma Q_3 . \tag{10}$$

We can thus assume that we have found Laurent polynomials Q_i such that for any $a_1\, a_2\, a_3 \in k^*$, with $a_1 a_2 a_3 = j$, and any α, β, $\gamma \in k$ with $\alpha + j\beta + j^2\gamma = 0$, the following identity holds,

$$(1 - j)(\alpha + \beta + \gamma) = 3((a_3 - j)\,\alpha\, Q_1 + (a_1 - j)\,\beta\, Q_2 + (a_2 - j)\,\gamma Q_3) . \tag{11}$$

We then choose $a_1 = j$, $a_2 = j$, $a_3 = j^2$, $\alpha = 0$, $\beta = -j$, $\gamma = 1$, and get a contradiction. Passing to a function field over k, this is enough to show that in general, $g \notin \Gamma$.

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