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ISOTROPY OF QUADRATIC FORMS OVER FUNCTION FIELDS OF p-ADIC CURVES

by R. PARIMALA *and* V. SURESH

INTRODUCTION

Let k be a field of characteristic not equal to 2. We recall the notion of the u-invariant $u(k)$ of k:

$$u(k) = \sup\{ \text{ dimension of } q \mid q \text{ an anisotropic quadratic form over } k \}$$

It is a longstanding question whether the finiteness of $u(k)$ implies the finiteness of $u(k(t))$. This was open even in the case k is a p-adic field. Recently, by using a theorem of Saltman ([S], 3.4, [S1], [HV], 2.5) on bounding the index of central simple algebras over the function field $k(\mathrm{X})$ in one variable over a non-dyadic p-adic field by the square of the exponent, Hoffmann - Van Geel ([HV], 3.7) and independently Merkurjev ([M2]) proved the finiteness of the u-invariant of $k(\mathrm{X})$. Hoffmann and Van Geel ([HV], 3.7) proved that $u(k(\mathrm{X})) \leqslant 22$. In this paper, we follow the techniques of Saltman to prove that the u-invariant of $k(\mathrm{X})$ is bounded by 10. We remark that conjecturally $u(k(\mathrm{X})) = 8$. Recall that if F is a finite field, $k = \mathrm{F}((t))$ is C_2 and if X is an irreducible curve over k, then $k(\mathrm{X})$ is a C_3 field ([Gre], p 36, p 22) and hence $u(k(\mathrm{X})) = 8$.

The main step of the proof is to kill any element in $\mathrm{H}^3(k(\mathrm{X}), \mathbf{Z}/2)$ in a quadratic extension of $k(\mathrm{X})$ (3.8). This is done by killing the ramification of any element of $\mathrm{H}^3(k(\mathrm{X}), \mathbf{Z}/2)$ on a regular proper model $\mathscr{X}$ of a quadratic extension L of $k(\mathrm{X})$ and using a theorem of Kato ([K], 5.2) that the unramified cohomology group $\mathrm{H}^3_{\mathrm{nr}}(\mathrm{L}/\mathscr{X}, \mathbf{Z}/2) = 0$. This shows that every element α in $\mathrm{H}^3(k(\mathrm{X}), \mathbf{Z}/2)$ is of the form $(f) \cup \beta$, with $(f) \in \mathrm{H}^1(k(\mathrm{X}), \mathbf{Z}/2) = k(\mathrm{X})^*/k(\mathrm{X})^{*2}$ and $\beta \in \mathrm{H}^2(k(\mathrm{X}), \mathbf{Z}/2)$. In view of a theorem of Saltman (cf. 2.2), β and hence α, is a sum of two symbols. A subtler choice of a biquadratic extension (2.1) which splits $\beta \in \mathrm{H}^2(k(\mathrm{X}), \mathbf{Z}/2)$ leads to the fact that every element in $\mathrm{H}^3(k(\mathrm{X}), \mathbf{Z}/2)$ is a symbol $(f) \cup (g) \cup (h)$. In fact we also prove (3.9) that given $\alpha_i \in \mathrm{H}^3(k(\mathrm{X}), \mathbf{Z}/2)$, $1 \leqslant i \leqslant n$, there exist $f, g, h_i \in k(\mathrm{X})^*$ such that $\alpha_i = (f) \cup (g) \cup (h_i)$. This is a local two-dimensional analogue of a result of Tate for number fields ([T], 5.2).

Using methods of Hoffmann and Van Geel ([HV]) and the fact that every element in $\mathrm{H}^3(k(\mathrm{X}), \mathbf{Z}/2)$ is a symbol, one can deduce that $u(k(\mathrm{X})) \leqslant 12$ (4.2). One shows further that given $\alpha \in \mathrm{H}^3(k(\mathrm{X}), \mathbf{Z}/2)$, a suitable choice of a quadratic extension $\mathrm{L} = k(\mathrm{X})(\sqrt{f}\,)$ which splits α can be made so that f is a value of a given binary quadratic form (4.4). This leads to $u(k(\mathrm{X})) \leqslant 10$ (4.5).

Let k be a p-adic field and C a smooth, projective, geometrically integral curve over k. Let $\pi : X \to C$ be an admissible quadric fibration (cf. [CSk]) and $CH_0(X/C)$ the kernel of the induced homomorphism $\pi_* : CH_0(X) \to CH_0(C)$, where CH_0 denotes the group of zero-cycles modulo rational equivalence. In ([CSk]), Colliot-Thelene and Skorobogatov posed the question whether $CH_0(X/C)$ is zero if $\dim(X) \geqslant 4$. In ([HV], 4.2), Hoffmann and Van Geel showed that if k is non-dyadic and $\dim X \geqslant 6$, then, $CH_0(X/C) = 0$. They further proved that if every element in $H^3(k(X), \mathbf{Z}/2)$ is a symbol and $\dim(X) \geqslant 4$, then $CH_0(X/C) = 0$ ([HV], 4.4). Thus, as a consequence of our result, it follows that if $\dim(X) \geqslant 4$, then $CH_0(X/C) = 0$ (5.2), answering the above question of Colliot-Thélène and Skorobogatov in the affirmative.

In ([Se], §8.3), Serre raised the question whether for a p-adic field k, every element in $H^3(k(t), \mathbf{Z}/2)$ is a symbol. In this were true, he has the following explicit description of the set of isomorphism classes of Cayley algebras over $k(t)$ as the set

$$C(P) = \{ f: P \to \mathbf{Z}/2 \mid \text{Supp}(f) \text{ finite and } \sum_{x \in P} f(x) = 0 \},$$

where P denotes the set of closed points of $\mathbf{P}^1_k$. Using our theorem and a result of Kato ([K]), we give a description (6.3), following Serre's method, of the set of isomorphism classes of Cayley algebras over $k(X)$, where X is a smooth, irreducible curve over a non-dyadic p-adic field, which reduces to that of Serre in the case $X = \mathbf{P}^1_k$.

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1. Some Preliminaries

We recall (cf. [Sc]) some basic definitions and facts about quadratic forms and (cf. [C]) some results on Galois cohomology and unramified cohomology. Let F be a field of characteristic not equal to 2. By a *quadratic form* over F we mean a pair (V, q), where V is a finite dimensional vector space, $q : V \to F$ is a map such that $q(\lambda v) = \lambda^2 q(v)$, for $\lambda \in F$, $v \in V$ and the map $b_q : V \times V \to F$ given by $b_q(v, w) = q(v + w) - q(v) - q(w)$ is a non-singular bilinear form. We shall abbreviate (V, q) by q. Let q be a quadratic form over F. The rank of q, denoted by $\mathrm{rk}(q)$, is defined as the dimension of V over F. We say that a quadratic form q over F is *isotropic* if there exists $v \in V$, $v \neq 0$, such that $q(v) = 0$; otherwise q is called *anisotropic*. The *u-invariant* of F, denoted by $u(F)$, is defined as

$$u(\mathrm{F}) = \sup\{ \mathrm{rk}(q) \mid q \text{ an anisotropic quadratic form over F} \}.$$

Let q be a quadratic form over F. Since char(F) $\neq 2$, q is isometric to a diagonal form $< a_1, \cdots, a_n >$, for some $a_i \in \mathrm{F}^*$. A quadratic form is isotropic if and only if $q \simeq < 1, -1 > \perp q'$ for some quadratic form q' over F. A quadratic form q is said to be *hyperbolic* if $q \simeq < 1, -1 > \perp \cdots \perp < 1, -1 >$. Let W(F) be the Witt group of quadratic forms over F. Note that every element in W(F) is represented by an anisotropic quadratic form over F. A quadratic form q represents 0 in W(F) if and only if q is hyperbolic. Tensor product of quadratic forms makes W(F) into a ring. Let I(F) be the ideal of W(F) consisting of even rank forms. For $n \geqslant 1$, let $\mathrm{I}^n(\mathrm{F})$ denote the n^{th} power of I(F). The abelian group $\mathrm{I}^n(\mathrm{F})$ is generated by quadratic forms of the type $< 1, a_1 > \otimes \cdots \otimes < 1, a_n >$, with $a_i \in \mathrm{F}^*$. A quadratic form of the type $< 1, a_1 > \otimes \cdots \otimes < 1, a_n >$ is called an n-fold Pfister form. Let $\mathrm{P}_n(\mathrm{F})$ denote the set of n-fold Pfister forms over F.

The rank induces an isomorphism $\mathrm{rk} : \mathrm{W(F)}/\mathrm{I(F)} \simeq \mathbf{Z}/2$. For a quadratic form over F, let $d(q)$ be the discriminant of q and $c(q)$ the Clifford invariant of q. Then the discriminant induces an isomorphism $d : \mathrm{I(F)}/\mathrm{I}^2(\mathrm{F}) \to \mathrm{F}^*/\mathrm{F}^{*2}$. A celebrated theorem of Merkurjev ([M1]) asserts that c induces an isomorphism

$$\frac{\mathrm{I}^2(\mathrm{F})}{\mathrm{I}^3(\mathrm{F})} \xrightarrow{\sim} \mathrm{H}^2(\mathrm{F}, \mathbf{Z}/2),$$

where for any $n \geqslant 0$, $\mathrm{H}^n(\mathrm{F}, \mathbf{Z}/2)$ denotes the n^{th} Galois cohomology group $\mathrm{H}^n(\mathrm{Gal}(\mathrm{F}_s/\mathrm{F}), \mathbf{Z}/2)$, F_s denoting the separable closure of F. For $a \in \mathrm{F}^*$, let (a) denote the class in $\mathrm{H}^1(\mathrm{F}, \mathbf{Z}/2) = \mathrm{F}^*/\mathrm{F}^{*2}$. For $a_1, \cdots, a_n \in \mathrm{F}^*$, let $(a_1) \cdot \cdots \cdot (a_n)$ denote the element $(a_1) \cup \cdots \cup (a_n) \in \mathrm{H}^n(\mathrm{F}, \mathbf{Z}/2)$. Let $n \geqslant 1$. For $a_1, \cdots, a_n \in \mathrm{F}^*$, let

$$e_n : \mathrm{P}_n(\mathrm{F}) \to \mathrm{H}^n(\mathrm{F}, \mathbf{Z}/2)$$

be defined by $e_n(< 1, -a_1 > \otimes \cdots \otimes < 1, -a_n >) = (a_1) \cdot \cdots \cdot (a_n) \in \mathrm{H}^n(\mathrm{F}, \mathbf{Z}/2)$. Then e_1 is the discriminant and e_2 is the Clifford invariant. Suppose that the 2-cohomological dimension $\mathrm{cd}_2(\mathrm{F})$ of F is at most 3. Then by a theorem of Arason, Elman and Jacob ([AEJ], Corollary 4 and Theorem 2), $\mathrm{I}^4(\mathrm{F}) = 0$ and

$$e_3 : \mathrm{I}^3(\mathrm{F}) \to \mathrm{H}^3(\mathrm{F}, \mathbf{Z}/2)$$

is an isomorphism.

Let R be a discrete valuation ring, F its quotient field and κ its residue field. Assume that the characteristic of κ is not equal to 2. For $q \geqslant 1$, let

$$\partial_{\mathrm{R}} : \mathrm{H}^q(\mathrm{F}, \mathbf{Z}/2) \to \mathrm{H}^{q-1}(\kappa, \mathbf{Z}/2)$$

be the residue homomorphism defined with respect to R. If P is the maximal ideal of R, then sometimes we denote ∂_R by ∂_P. For u_i units in R, $1 \leqslant i \leqslant q-1$ and π a parameter in R, we have $\partial_R((u_1)\cdot \cdots (u_{q-1})\cdot(\pi)) = (\bar{u}_1)\cdot \cdots (\bar{u}_{q-1})$, where bar denotes the image in κ.

Let $\mathscr{X}$ be a regular integral scheme of dimension n and F its function field. For $i \geqslant 0$, let $\mathscr{X}^i$ denote the set of points of $\mathscr{X}$ of codimension i. For any $x \in \mathscr{X}$, let $\kappa(x)$ denote the residue field at x. Assume that the characteristic of $\kappa(x)$ is not equal to 2, for any $x \in \mathscr{X}$. For $x \in \mathscr{X}^1$, let $\mathscr{O}_{\mathscr{X},x}$ denote the discrete valuation ring at x and $\partial_x : H^q(F, \mathbf{Z}/2) \to H^{q-1}(\kappa(x), \mathbf{Z}/2)$ the residue homomorphism defined with respect to $\mathscr{O}_{\mathscr{X},x}$. Let

$$H^q_{nr}(F/\mathscr{X}, \mathbf{Z}/2) = \ker(H^q(F, \mathbf{Z}/2) \overset{\partial=(\partial_x)}{\longrightarrow} \bigoplus_{x\in\mathscr{X}^1} H^{q-1}(\kappa(x), \mathbf{Z}/2)).$$

An element $\alpha \in H^q(F, \mathbf{Z}/2)$ is called *unramified at a point* $x \in \mathscr{X}^1$, if $\partial_x(\alpha) = 0$; otherwise it is called ramified at x. We say that $\alpha \in H^q(F, \mathbf{Z}/2)$ is *unramified on* $\mathscr{X}$ if it is unramified at all points of $\mathscr{X}^1$, i.e., $\alpha \in H^q_{nr}(F/\mathscr{X}, \mathbf{Z}/2)$. We define the ramification divisor

$$\mathrm{ram}_{\mathscr{X}}(\alpha) = \sum_{\partial_x(\alpha)\neq 0} x.$$

For $f \in F^*$, we denote by $\mathrm{Supp}_{\mathscr{X}}(f)$ the support of the principal divisor $\mathrm{div}_{\mathscr{X}}(f)$.

Let k be a p-adic field, $p \neq 2$. Let X be a smooth, projective, integral curve over k and $K = k(X)$ the function field of X. Let $\mathscr{O}_k$ be the ring of integers of k. For $\alpha_i \in H^q(K, \mathbf{Z}/2)$ and $f_j \in K^*$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, by a result of Lipman on the resolution of singularities (cf. [S], Proof of 2.1), there exists a regular, projective model $\mathscr{X}$ of X over $\mathscr{O}_k$ and two regular curves C and E on $\mathscr{X}$ with only normal crossings (i.e., for every $x \in C \cap E$, the maximal ideal of the local ring $\mathscr{O}_{\mathscr{X},x}$ is generated by local equations of C and E at x), such that

$$\cup_{1\leqslant i\leqslant n} \mathrm{Supp}\,(\mathrm{ram}_{\mathscr{X}}(\alpha_i)) \cup \cup_{1\leqslant j\leqslant m} \mathrm{Supp}_{\mathscr{X}}(f_j) \subset \mathrm{Supp}(C + E).$$

We use this result throughout this paper without further reference.

Let F be a field of characteristic not equal to 2 and L a field extension of F. For any $\alpha \in H^q(F, \mathbf{Z}/2)$, the image of α in $H^q(L, \mathbf{Z}/2)$ under the restriction map is denoted by α_L. Let $\mathscr{X}$ be a scheme and $x \in \mathscr{X}$. Let $\mathscr{O}_{\mathscr{X},x}$ be the local ring at x. For any $f \in \mathscr{O}_{\mathscr{X},x}$, the image of f in $\kappa(x)$ is denoted by $f(x)$. For any ring A, let A^* denote the group of units in A. Let $A \subset B$ be local rings with maximal ideals m_A and m_B respectively. We say that B *dominates* A if $m_A \subset m_B$. In the rest of the paper, we assume that 2 is invertible in all the rings concerned.

2. Cohomology in degree 2

Let k be a non-dyadic p-adic field and $\mathcal{O}_k$ the ring of integers in k. Let X be a smooth, projective, irreducible curve over k and $K = k(X)$ the function field of X over k.

Proposition **2.1.** — *Let k, X and K be as above. Let $\alpha_i \in H^2(K, \mathbf{Z}/2)$, $1 \leqslant i \leqslant n$. Let $\mathscr{X}$ be a regular, projective model of* X *over $\mathcal{O}_k$ such that*

$$\bigcup_{i=1}^{n} \mathrm{Supp}\,(\mathrm{ram}_{\mathscr{X}}(\alpha_i)) \subset \mathrm{Supp}\,(C + E),$$

where C *and* E *are regular curves on $\mathscr{X}$ having only normal crossings. Suppose there exists $f \in K^*$ such that*

$$\mathrm{div}_{\mathscr{X}}(f) = C + E + F,$$

where F *is a divisor on $\mathscr{X}$ whose support does not contain any point of* $C \cap E$ *and no component of* C *or* E *is contained in* F. *Let* T *be the finite set of closed points consisting of* $C \cap E$, $C \cap F$, $E \cap F$. *Let* B *be the semi-local ring at* T. *Let $h \in B$, $h \neq 0$, be such that $\mathrm{Supp}_{\mathrm{Spec(B)}}(h) \subset \mathrm{Supp}(C + E)$ and h is square free in B. Suppose $x \in C \cap E$ is a closed point. Let π_x and δ_x be local equations at x for C and E respectively. We write $h = \pi_x^{\varepsilon_1}\delta_x^{\varepsilon_2} w_x$ and $f = \pi_x \delta_x w'_x$, where w_x, w'_x are units at x and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$. Suppose there exists an element $h_1 \in B^*$ such that for $x \in T$,*

(i) *if $h(x) \neq 0$, then $(hh_1)(x)$ is not a square in $\kappa(x)$.*

(ii) *if $h(x) = 0$ and either $x \in C \cap F$ or $x \in E \cap F$, then h_1 is a unit at x.*

(iii) *if $h(x) = 0$ and $x \in C \cap E$, then $(w_x w'_x h_1)(x)$ is not a square in $\kappa(x)$.*

Then the image of α_i in $H^2(K(\sqrt{f}, \sqrt{hh_1}), \mu_2)$ is zero, for $1 \leqslant i \leqslant n$.

Proof. — Let $L = K(\sqrt{f}, \sqrt{hh_1})$ and S be a discrete valuation ring, containing $\mathcal{O}_k$, with quotient field L. Since $\mathscr{X}$ is projective over $\mathcal{O}_k$, there exists a point $x \in \mathscr{X}$ of codimension 1 or 2 such that S dominates the local ring $A = \mathcal{O}_{\mathscr{X},x}$. We show that, for $1 \leqslant i \leqslant n$, $(\alpha_i)_L$ is unramified at S. Fix i, $1 \leqslant i \leqslant n$ and let $\alpha = \alpha_i$.

Suppose that $x \notin C \cup E$. Then α is unramified on A and hence unramified over S ([S], 1.4). Assume that $x \in C \cup E$.

Suppose that dim(A) = 1. Then f is a parameter at x and hence S is ramified over A. Therefore α is unramified on S.

Suppose that dim(A) = 2. Let m_S be the maximal ideal of S and ν_S the valuation of S. We show that $\partial_S(\alpha_L) = 0$.

Suppose that $x \in C \setminus (E \cup f)$ (resp. $x \in E \setminus (C \cup f)$). Then f is a local equation for C (resp. E) at x and α can be ramified only at (f) in A. By ([S], 1.2), we have

$\alpha = \alpha' + (u)\cdot(f)$, where α' is unramified on A and $u \in A^*$. Since $(u)\cdot(f)_L = (u)\cdot(1) = 0$, $\alpha_L = \alpha'_L$ is unramified at S.

Suppose that $x \in C \cap F$. Then $x \notin E$ and hence, by ([S], 1.2), $\alpha = \alpha' + (u)\cdot(\pi_x)$, where α' is unramified on A, $u \in A^*$. Suppose further that $h(x) \neq 0$. Then $(hh_1)(x)$ is not a square in $\kappa(x)$. We have $\partial_S((u)\cdot(\pi_x)) = \bar{u}^{v_S(x)}$, bar denoting the image modulo m_S. Since $(hh_1(x))$ is not a square in the finite field $\kappa(x)$, $u(x)$ is a square in $\kappa(x)(\sqrt{hh_1(x)})$. Since $\kappa(x)(\sqrt{(hh_1)(x)}) \subset S/m_S$, $\bar{u}$ is a square in S/m_S and hence $(u)\cdot(\pi_x)$ is unramified on S. Suppose that $h(x) = 0$. Since $h_1(x)$ is a unit at x and $\mathrm{Supp}_{\mathrm{Spec(B)}}(h) \subset \mathrm{Supp}(C + E)$, hh_1 is a local equation for C at x, $\pi_x = hh_1 v$, $v \in A^*$ and $\alpha = \alpha' + (u)\cdot(hh_1 v)$. Since $(u)\cdot(hh_1 v)_L = (u)\cdot(v)_L$, α is unramified at S. Similarly, one proves that α is unramified at S, if $x \in E \cap F$.

Suppose that $x \in C \cap E$. Let π_x and δ_x be local equations for C and E at x given in the statement of the proposition. Then we have $f = \pi_x\delta_x w'_x$ with $w'_x \in A^*$. We have ([S], 1.2) $\alpha = \alpha' + \alpha''$, where α' is unramified on A and α'' is a sum of symbols of the type $(u)\cdot(\pi_x)$, $(v)\cdot(\delta_x)$ and $(\pi_x)\cdot(\delta_x)$, $u, v \in A^*$. For $u \in A^*$, we have

(*) $(u)\cdot(\delta_x)_L = (u)\cdot(\delta_x f)_L = (u)\cdot(\pi_x w'_x)_L$,

(**) $(u)\cdot(\pi_x)_L = (u)\cdot(\pi_x f)_L = (u)\cdot(\delta_x w'_x)_L$,

(***) $(\pi_x)\cdot(\delta_x)_L = (\pi_x f)\cdot(\delta_x)_L = (\delta_x w'_x)\cdot(\delta_x)_L = (-w'_x)\cdot(\delta_x)_L$.

Suppose further that $h(x) \neq 0$. Then $hh_1(x)$ is not a square in $\kappa(x)$. As before, $(v)\cdot(\pi_x)_L$ and $(v)\cdot(\delta_x)_L$ are unramified at S for any $v \in A^*$. Therefore α_L is unramified at S. Suppose that $h(x) = 0$. Then either $h = \pi_x w_x$ or $h = \delta_x w_x$ or $h = \pi_x \delta_x w_x$, where $w_x \in A^*$. If $h = \pi_x w_x$ or $\delta_x w_x$, then, by (*), (**), (***) it follows that $\partial_S(\alpha'') = 0$ and hence α is unramified at S. Suppose $h = \pi_x \delta_x w_x$. Since $\sqrt{f}$, $\sqrt{hh_1} \in L^*$, $\sqrt{w'_x w_x h_1} \in L^*$. Since $(w'_x w_x h_1)(x)$ is not a square in $\kappa(x)$, once again using (***) and arguing as above, it follows that α'' and hence α is unramified at S.

Let k' be the field of constants in L. Let X′ be the smooth, projective, irreducible curve over k' with L as its function field. Let $\mathscr{X}'$ be a regular, projective model of X′ over $\mathcal{O}_{k'}$. For every $x' \in \mathscr{X}'$ of codimension 1, $\mathcal{O}_{\mathscr{X}',x'}$ dominates $\mathcal{O}_{\mathscr{X},x}$, where $x \in \mathscr{X}$ is a point of codimension 1 or 2. The element α_L is unramified at x' for every $x' \in \mathscr{X}'^1$. Since the Brauer group of $\mathscr{X}'$ is trivial (cf. [L], Theorem 4 or [Gr], 2.15 and 3.1), it follows that $\alpha_L = 0$. This completes the proof of the proposition. □

Corollary **2.2** ([S], 3.4). — *Let* D *be a central division algebra over* K *of exponent 2 in the Brauer group of* K. *Then the degree of* D *is at most 4. In particular, every element in* $H^2(K, \mathbf{Z}/2)$ *is a sum of two symbols.*

Proof. — Let $\alpha \in H^2(K, \mathbf{Z}/2)$ denote the class of D. Let $\mathscr{X}$, C and E be as in (2.1) defined with respect to α. By a semi-local argument, due to Colliot-Thélène (cf. [HV], Lemma 2.4), we choose $f \in K^*$ such that

$$\operatorname{div}_{\mathscr{X}}(f) = C + E + F,$$

where F is a divisor on $\mathscr{X}$ whose support does not contain any point of $C \cap E$ not any component of C or E. Let T and B be as in (2.1). Let $h \in B^*$ be such that for every $x \in T$, $h(x)$ is not a square in $\kappa(x)$. We set $h_1 = 1$. Then h and h_1 satisfy the hypotheses of (2.1). Therefore by (2.1), the image of α in $H^2(K(\sqrt{f}, \sqrt{h}), \mathbf{Z}/2)$ is zero. Hence $D \otimes K(\sqrt{f}, \sqrt{h})$ is a split algebra. In particular, the degree of D is at most 4 and D is a tensor product of two quaternion algebras ([A]). Hence α is a sum of two symbols. □

3. Cohomology in degree 3

Lemma **3.1.** — *Let* F *be a finite field of characteristic not equal to 2 and* Y *a smooth, projective curve over* F. *Let* $\beta \in H^2(F(Y), \mathbf{Z}/2)$ *and* $P_1, \ldots, P_n$ *be the closed points of* Y *where* β *is ramified. Let* $f \in F(Y)^*$ *be such that at each* P_i *either* f *has odd valuation or* f *is a unit at* P_i *and* $f(P_i)$ *is not a square in* $\kappa(P_i)$. *Then* $\beta \otimes F(Y)(\sqrt{f}) = 0$.

Proof. — By class field theory, it is enough to prove that $\beta \otimes F(Y)(\sqrt{f})$ is unramified at each discrete valuation ring of $F(Y)(\sqrt{f})$. Let S be a discrete valuation ring with $F(Y)(\sqrt{f})$ as its quotient field. Let R be the discrete valuation ring of F(Y) such that $R \subset S$. If β is unramified at R, then β is unramified at S. Suppose that β is ramified at R and $R = \mathscr{O}_{Y,P_i}$ for some i. If f has odd valuation at P_i, then S over R is ramified and hence β is unramified at S. If f has even valuation at P_i, then by the choice of f, f is a unit at P_i and not a square in $\kappa(P_i)$. Therefore the residue field $\overline{S}$ of S is a quadratic extension of the residue field $\kappa(P_i)$ at R. Since S over R is unramified, $\partial_S(\alpha_L) = \partial_R(\alpha) \otimes_{\kappa(P_i)} \overline{S}$ (cf. [S]. 1.3). Since $\overline{S}$ is a quadratic extension of $\kappa(P_i)$ and $\kappa(P_i)$ is a finite field, every element of $\kappa(P_i)$ is a square in $\overline{S}$. Therefore β is unramified at S. □.

Lemma **3.2.** — *Let* R *be a discrete valuation ring,* K *its quotient field and* κ *its residue field, with char* $\kappa \neq 2$. *Let* δ *be a parameter in* R *and* $u \in R^*$. *If* $(u) \cdot (\delta)$ *is unramified at* R, *then* $(u) \cdot (\delta) = (u) \cdot (u')$ *for some* $u' \in R^*$.

Proof. — Suppose that $(u) \cdot (\delta)$ is unramified at R. Since $\partial_R((u) \cdot (\delta)) = (\overline{u})$, where bar denotes the image in κ, $\overline{u}$ is a square in κ. Let $a \in R$ be such that $\overline{a}^2 = \overline{u}$. We write $a^2 - u = v\delta^r$ for some $r \geqslant 1$ and $v \in R^*$. Suppose that $r \geqslant 2$. We have $(a+\delta)^2 - u = v\delta^r + \delta^2 + 2a\delta = \delta(v\delta^{r-1} + \delta + 2a)$. Since $r \geqslant 2$ and a is a unit in R, $v\delta^{r-1} + \delta + 2a$ is a unit in R. Replacing a by $a + \delta$ we assume that $r = 1$. Therefore we have, $(u) \cdot (\delta) = (a^2 - v\delta) \cdot (\delta) = (1 - a^{-2}v\delta) \cdot (\delta) =$ (since $(x) \cdot (1-x)$ is trivial) $(1 - a^{-2}v\delta) \cdot (a^{-2}v\delta^2) = (1 - a^{-2}v\delta) \cdot (v) = (u) \cdot (v)$. □

Proposition **3.3.** — *Let* A *be a regular local ring of dimension 2,* K *its quotient field and* κ *its residue field, with char* $\kappa \neq 2$. *For every regular parameter* π *of* A *(i.e.,* $A/(\pi)$ *is regular) with residue field* $\kappa(\pi)$, *suppose that every element of* $H^2(\kappa(\pi), \mathbf{Z}/2)$ *is represented by a symbol* $(a)\cdot(b)$ *for some* $a, b \in \kappa(\pi)^*$. *Let* $\alpha \in H^3(K, \mathbf{Z}/2)$.

(i) *Suppose* α *is ramified only at* π *among the prime elements of* A. *Assume that* π *is a regular parameter in* A. *Then*

$$\alpha = \alpha' + (u)\cdot(v)\cdot(\pi)$$

for some $\alpha' \in H^3_{nr}(K/\mathrm{Spec}(A), \mathbf{Z}/2)$ *and* $u, v \in A^*$.

(ii) *Suppose* α *is ramified only at* π *and* δ *among the prime elements of* A. *Further assume that* π *and* δ *generate the maximal ideal* m *of* A. *Then*

$$\alpha = \alpha_1 + \alpha_2,$$

where $\alpha_1 \in H^3_{nr}(K/\mathrm{Spec}(A), \mathbf{Z}/2)$ *and* α_2 *is a sum of symbols of the type*

$$(u)\cdot(v)\cdot(\pi), \quad (u)\cdot(v)\cdot(\delta), \quad (u)\cdot(\delta)\cdot(\pi),$$

u, v *running over the units of* A.

Proof. — Let α and π be as in (i). Since π is a regular parameter of A, there exists a prime element δ in A such that the maximal ideal m of A is generated by π and δ. We have a complex ([K], Prop. 1.7)

$$H^3(K, \mathbf{Z}/2) \xrightarrow{\partial} \bigoplus_{x\in \mathrm{Spec}(A)^1} H^2(\kappa(x), \mathbf{Z}/2) \xrightarrow{\partial} H^1(\kappa, \mathbf{Z}/2):$$

By the assumption on $\kappa(\pi)$, there exist $a, b \in A$ such that $\partial_\pi(\alpha) = (\bar{a})\cdot(\bar{b})$, bar denoting the image in $A/(\pi)$. Since m is generated by π and δ, $A/(\pi)$ is a discrete valuation ring with $\bar{\delta}$ as a parameter. Without loss of generality we assume that $\partial_\pi(\alpha)$ is equal to either $(\bar{u})\cdot(\bar{v})$ or $(\bar{u})\cdot(\bar{v}\,\bar{\delta})$ for some $u, v \in A^*$. Suppose $\partial_\pi(\alpha) = (\bar{u})\cdot(\bar{v}\,\bar{\delta})$. Since α has residue only at π, $\partial\partial(\alpha) = \partial((\bar{u})\cdot(\bar{v}\,\bar{\delta}))$ is the square class of the image of u in κ^*. Since $\partial\partial = 0$, u is a square modulo m. Thus $(\bar{u})\cdot(\bar{v}\,\bar{\delta})$ over $\kappa(\pi)$ is unramified at $\bar{\delta}$ and by (3.2) $(\bar{u})\cdot(\bar{v}\,\bar{\delta}) = (\bar{u})\cdot(\bar{v}')$ for some $v' \in A^*$. Thus we assume that $\partial_\pi(\alpha) = (\bar{u})\cdot(\bar{v})$ for some $u, v \in A^*$. Let $\alpha' = \alpha - (u)\cdot(v)\cdot(\pi)$. Since $\partial_\pi(\alpha) = \partial_\pi((\bar{u})\cdot(\bar{v})\cdot(\pi))$ and $\partial_{\pi'}((u)\cdot(v)\cdot(\pi)) = \partial_{\pi'}(\alpha) = 0$ for any prime element π' of A not equal to π, we have $\partial(\alpha') = 0$. Hence $\alpha' \in H^3_{nr}(K/\mathrm{Spec}(A), \mathbf{Z}/2)$ and $\alpha = \alpha' + (u)\cdot(v)\cdot(\pi)$.

Now let α, π and δ be as in (ii). Since every element in $H^2(\kappa(\pi), \mathbf{Z}/2)$ is represented by a symbol, there exist $u, v \in A^*$, such that $\partial_\pi(\alpha)$ is equal to $(\bar{u})\cdot(\bar{v})$ or $(\bar{u})\cdot(\bar{v}\,\bar{\delta})$. Set $\alpha_1 = \alpha - (u)\cdot(v)\cdot(\pi)$ if $\partial_\pi(\alpha) = (\bar{u})\cdot(\bar{v})$ and $\alpha_1 = \alpha - (u)\cdot(v\delta)\cdot(\pi)$ if $\partial_\pi(\alpha) = (\bar{u})\cdot(\bar{v}\,\bar{\delta})$. Since α is ramified only at π and δ, α_1 is unramified except possibly at δ. Now we can apply (i) to describe α_1. This completes the proof of the proposition. □

Remark **3.4.** — *Suppose that in the above proposition,* K *is a function field in one variable over a non-dyadic local field k, $\mathscr{X}$ a regular 2-dimensional scheme over the integers $\mathscr{O}_k$ and* A *the local ring at a codimension 2 point of $\mathscr{X}$. Then for every prime $\pi \in$ A, the residue field $\kappa(\pi)$ at π is either a local field or a function field in one variable over a finite field. Therefore every element in* $H^2(\kappa(\pi), \mathbf{Z}/2)$ *is represented by a symbol. Thus* A *satisfies the hypothesis of (3.3).*

Let k be a non-dyadic p-adic field and $\mathscr{O}_k$ the ring of integers in k. Let X be a smooth, projective, irreducible curve over k and $K = k(X)$ the function field of X over k.

Let $\alpha \in H^3(K, \mathbf{Z}/2)$. Let $\mathscr{X}$ be a regular, projective model of X over $\mathscr{O}_k$ such that

$$\mathrm{ram}_{\mathscr{X}}(\alpha) \subset C + E,$$

where C and E are regular curves on $\mathscr{X}$ having only normal crossings.

Lemma **3.5.** — *Let k,* K *and $\mathscr{X}$ be as above. Let x be a codimension 2 point of $\mathscr{X}$ and* $A = \mathscr{O}_{\mathscr{X},x}$. *Let* S *be a discrete valuation ring which dominates* A. *Then every symbol of the type* $(u)\cdot(v)\cdot(\pi)$, *with* $u, v \in A^*$ *and* $\pi \in K^*$, *is unramified at* S.

Proof. — Let $u, v \in A^*$. We have $\partial_S((u)\cdot(v)\cdot(\pi)) = ((\bar{u})\cdot(\bar{v}))^{v_S(\pi)}$, bar denoting the image in the residue field of S and v_S denoting the valuation of S. Since $u, v \in A^*$ and $\kappa(x)$ is a finite field, it follows that $(\bar{u})\cdot(\bar{v}) = 0$. Hence $(u)\cdot(v)\cdot(\pi)$ is unramified at S. □

Lemma **3.6.** — *Let k,* K, $\alpha \in H^3(K, \mathbf{Z}/2)$, $\mathscr{X}$, C *and* E *be as above. Let* L *be an extension of* K *and* S *a discrete valuation ring with quotient field* L. *Suppose that there exists* $x \in C \cap E$ *such that* S *dominates* $\mathscr{O}_{\mathscr{X},x}$. *Suppose one of the following conditions holds.*

(i) *The residue field of* S *contains a quadratic extension of* $\kappa(x)$.

(ii) *There exist local equations* π_x, δ_x *for* C *and* E *respectively at x such that either* π_x *or* δ_x *or* $\pi_x\delta_x$ *is of the form* $w\theta^2$, $\theta \in S$, $w \in S^*$, *with the image of w in the residue field of* S *having its square class coming from* $\kappa(x)^*$.

Then α_L *is unramified at S.*

Proof. — Let $A = \mathscr{O}_{\mathscr{X},x}$. By (3.3), $\alpha = \alpha' + \alpha''$, where α' is unramified on A and α'' is a sum of the symbols of the type $(u)\cdot(v)\cdot(\pi_x)$, $(u)\cdot(v)\cdot(\delta_x)$, $(u)\cdot(\pi_x)\cdot(\delta_x)$, with $u, v \in A^*$. Let v_S denote the discrete valuation of S, ∂_S denote the residue homomorphism at S and m_S denote the maximal ideal of S. By (3.5), $(u)\cdot(v)\cdot(\pi_x)$, $(u)\cdot(v)\cdot(\delta_x)$ are unramified at S.

Suppose that the residue field of S contains a quadratic extension of $\kappa(x)$. We have

$$\partial_S((u)\cdot(\pi_x)\cdot(\delta_x)) = (\bar{u}) \cup \partial_S((\pi_x)\cdot(\delta_x)).$$

Since the unique quadratic extension of $\kappa(x)$ is contained in the residue field of S, $\bar{u}$ is a square in the residue field of S. Therefore $\partial_S(\alpha_L) = 0$.

Suppose that $\pi_x = w\theta^2$ for some $w \in S^*$ such that $\bar{w} = \lambda\lambda_1^2$ with $\lambda \in \kappa(x)^*$, and $\theta \in S$. Then, we have $((u)\cdot(\pi_x)\cdot(\delta_x))_L = ((u)\cdot(w)\cdot(\delta_x))_L$. We have $\partial_S((u)\cdot(\pi_x)\cdot(\delta_x)) = ((\bar{u})\cdot(\bar{w}))^{v_S(\delta_x)} = ((\bar{u})\cdot(\lambda))^{v_S(\delta_x)}$. Since $\bar{u}, \lambda \in \kappa(x)^*$, as before, it follows that $(\bar{u})\cdot(\lambda) = 0$. Similarly, one can prove that if $\delta_x = w\theta^2$, with w, θ as above, then $\partial_S((u)\cdot(\pi_x)\cdot(\delta_x)) = 0$. Suppose that $\pi_x\delta_x = w\theta^2$, with w, θ as above. Since $(u)\cdot(\pi_x)\cdot(\delta_x) = (u)\cdot(-\pi_x\delta_x)\cdot(\delta_x)$, we have $((u)\cdot(\pi_x)\cdot(\delta_x))_L = ((u)\cdot(-w)\cdot(\delta_x))_L$ and $\partial_S(((u)\cdot(\pi_x)\cdot(\delta_x))_L) = ((\bar{u})\cdot(-\bar{w}))^{v_S(\pi_x)} = 0$. Therefore α is unramified at S. □

Lemma **3.7.** — *Let k and* K *be as in (3.6). Let* A *be a regular local ring of dimension 2 with* K *as its quotient field. and* S *a discrete valuation ring containing* A. *Then the map* $H^3(K, \mathbf{Z}/2) \to H^3(L, \mathbf{Z}/2)$ *restricts to a map*

$$H^3_{nr}(K/\mathrm{Spec}(A), \mathbf{Z}/2) \to H^3_{nr}(L/\mathrm{Spec}(S), \mathbf{Z}/2).$$

Proof. — The lemma follows from the absolute purity theorem of Gabber for two dimensional regular local rings. We give a proof here for the sake of completeness.

Let W(A) denote the Witt group of A. Since A is a two-dimensional regular local ring, one has the following exact sequence ([O], [CS])

$$0 \to W(A) \to W(K) \to \bigoplus_{x \in \mathrm{Spec}(A)^1} W(\kappa(x)).$$

For $n \geqslant 0$, let $I_n(A) := I^n(K) \cap W(A)$. Since cd (K) $\leqslant 3$ and cd ($\kappa(x)$) $\leqslant 2$, in view of ([AEJ], Theorem 2), the homomorphisms $e_n : I^n(F) \to H^n(F, \mathbf{Z}/2)$ exist and are surjective with kernel $I^{n+1}(F)$, for $F = K$ or $\kappa(x)$. Since the following diagram is commutative (cf. [P]),

$$\begin{array}{ccc} I^3(K) & \xrightarrow{\partial} & \bigoplus_{x \in \mathrm{Spec}(A)^1} I^2(\kappa(x)) \\ \downarrow e_3 & & \downarrow e_2 \\ H^3(K, \mathbf{Z}/2) & \xrightarrow{\partial} & \bigoplus_{x \in \mathrm{Spec}(A)^1} H^2(\kappa(x), \mathbf{Z}/2) \end{array}$$

with e_3 and e_2 isomorphisms, e_3 induces an isomorphism

$$e_3 : I_3(A) \to H^3_{nr}(K/\mathrm{Spec}(A), \mathbf{Z}/2)$$

Let $\alpha \in H^3_{nr}(K/\mathrm{Spec}(A), \mathbf{Z}/2)$ and $q \in I_3(A)$ with $e_3(q) = \alpha$. Then $q_L \in I_3(S)$ and $\alpha_L = e_3(q_L)$ in $H^3(L, \mathbf{Z}/2)$. In view of the following commutative diagram

$$\begin{array}{ccc} I^3(L) & \xrightarrow{\partial_S} & I^2(S/m_S) \\ \downarrow e_3 & & \downarrow e_2 \\ H^3(L, \mathbf{Z}/2) & \xrightarrow{\partial_S} & H^2(S/m_S, \mathbf{Z}/2), \end{array}$$

we have $\partial_S(\alpha_L) = \partial_S(e_3(q_L)) = e_2\partial_S(q_L) = 0$.

Thus $\alpha_L \in H^3_{nr}(L/\mathrm{Spec}(S), \mathbf{Z}/2)$. □

Theorem **3.8.** — *Let k be a non-dyadic p-adic field, X a smooth, projective, irreducible curve over k. Let $K = k(X)$ and $\alpha_i \in H^3(K, \mathbf{Z}/2)$, $1 \leqslant i \leqslant n$. Then there exists $f \in K^*$ such that $\alpha_i \otimes K(\sqrt{f}) = 0$ for $1 \leqslant i \leqslant n$.*

Proof. — Let $\mathscr{X}$ be a regular, projective model of X over $\mathscr{O}_k$ with

$$\cup_{i=1}^n \mathrm{Supp}(\mathrm{ram}_{\mathscr{X}}(\alpha_i)) \subset \mathrm{Supp}(C + E),$$

where C and E are regular curves on $\mathscr{X}$ with only normal crossings. Let $f \in K^*$ be such that

$$\mathrm{div}_{\mathscr{X}}(f) = C + E + F,$$

where F is a divisor on $\mathscr{X}$ whose support does not contain any point of $C \cap E$, nor any component of C or E. Let $L = K(\sqrt{f})$. Let k' be the field of constants in L. Let X′ be the smooth, projective, irreducible curve over k' with function field L. Let $\mathscr{X}'$ be a regular, projective model for X′ over $\mathscr{O}_{k'}$. Fix i, $1 \leqslant i \leqslant n$ and let $\alpha = \alpha_i$. We show that $\alpha_L \in H^3_{nr}(L/\mathscr{X}', \mathbf{Z}/2)$. Let $y \in \mathscr{X}'$ be a point of codimension 1 and $S = \mathscr{O}_{\mathscr{X}',y}$ be the discrete valuation ring at y. Since $\mathscr{X}$ is proper over $\mathscr{O}_k$, there exists a point $x \in \mathscr{X}$ of codimension 1 or 2, such that S dominates the local ring $A = \mathscr{O}_{\mathscr{X},x}$.

Suppose $\dim(A) = 1$. Then A is a discrete valuation ring. If x corresponds to a component of C or E, then f is a parameter at x and S over A is ramified. Hence, α_L is unramified at S. Suppose that x does not correspond to a component of C or E. Since $\mathrm{ram}_{\mathscr{X}}(\alpha) \subset C + E$, α is unramified at R and hence α_L is unramified at S.

Suppose $\dim(A) = 2$. Suppose first that x does not belong to $\mathrm{Supp}(C) \cup \mathrm{Supp}(E)$. Then α is unramified on A and hence unramified at S (3.7). Suppose $x \in \mathrm{Supp}(C) \setminus \mathrm{Supp}(E)$ or $x \in \mathrm{Supp}(E) \setminus \mathrm{Supp}(C)$, then by (3.3) and (3.5), α is unramified on A and hence by (3.7), α_L is unramified at S. Suppose that $x \in \mathrm{Supp}(C) \cap \mathrm{Supp}(E)$. Let π_x and δ_x be local equations for C and E at x respectively. Then we have $f = \pi_x\delta_x w$ for some $w \in A^*$. Since f is a square in L, it follows from (3.6) that α_L is unramified at S. Therefore $\alpha_L \in H^3_{nr}(L/\mathscr{X}', \mathbf{Z}/2)$. Since $H^3_{nr}(L/\mathscr{X}', \mathbf{Z}/2) = 0$ ([K], 5.2), we have $\alpha_L = 0$. □

Theorem **3.9.** — *Let k be a non-dyadic p-adic field and* K *a function field in one variable over k. Let $\alpha_i \in H^3(K, \mathbf{Z}/2)$, $1 \leqslant i \leqslant n$. Then there exist $f, g, h_i \in K^*$ such that $\alpha_i = (f) \cdot (g) \cdot (h_i)$. In particular, every element in $H^3(K, \mathbf{Z}/2)$ is a symbol.*

Proof. — By (3.8), there exists $h \in K^*$ such that $\alpha_i \otimes K(\sqrt{h}\,) = 0$, for $1 \leqslant i \leqslant n$. Therefore, there exist ([Ar], 4.6) $\beta_i \in H^2(K, \mathbf{Z}/2)$, such that $\alpha_i = (h) \cup \beta_i$, for $1 \leqslant i \leqslant n$. Let X be a smooth, projective, irreducible curve over k with $k(X) = K$. Let $\mathscr{X}$ be a regular, projective model of X over $\mathscr{O}_k$ such that

$$\cup_{i=1}^{n} \mathrm{Supp}(\mathrm{ram}_{\mathscr{X}}(\beta_i)) \cup \mathrm{Supp}_{\mathscr{X}}(h) \subset \mathrm{Supp}(C + E)$$

where C and E are as before. Let $f \in K^*$ be such that

$$\mathrm{div}_{\mathscr{X}}(f) = C + E + F,$$

where F is a divisor on $\mathscr{X}$ whose support does not contain any point of $C \cap E$, nor any component of C or E. Let T be the finite set of codimension 2 points of $\mathscr{X}$ consisting of $C \cap E$, $C \cap F$ and $E \cap F$. Let B be the semi local ring at T. Since $\mathscr{X}$ is regular, B is a regular ring and hence a unique factorisation domain with quotient field K. Hence, without loss of generality, we assume that $h \in B$ and is square free with $\mathrm{Supp}_{\mathrm{Spec(B)}}(h) \subset \mathrm{Supp}(C + E)$. Let $x \in C \cap E$. Let π_x and δ_x be local equations at x for C and E respectively. Then $h = \pi_x^{\varepsilon_1} \delta_x^{\varepsilon_2} w_x$ and $f = \pi_x \delta_x w'_x$, where $w_x, w'_x \in B$ are units at x and $\varepsilon_1, \varepsilon_2 \in \{ 0, 1 \}$. Choose $w \in B^*$ such that w is a unit at one closed point of each component of C and E and $-w(x)w_x(x)w'_x(x)$ is not a square in $\kappa(x)$. Replacing f by wf, we assume that $-w_x(x)w'_x(x)$ is not a square in $\kappa(x)$ for all $x \in C \cap E$ and $div_{\mathscr{X}}(f) = C + E + F'$, with C, E as above and F′ is a divisor on $\mathscr{X}$ whose support does not contain any point of $C \cap E$ and any component of C or E. We claim that there exist $a_i \in K^*$ such that $\alpha_i = (h) \cdot (f) \cdot (a_i)$, $1 \leqslant i \leqslant n$. For $x \in T$,

(i) if $h(x) \neq 0$, let $a_x, b_x \in \kappa(x)$ be such that $h(x)(h(x)a_x^2 - b_x^2)$ is not a square.

(ii) if $h(x) = 0$, let $a_x = 0$ and $b_x = 1$ in $\kappa(x)$.

Let $a, b \in B$ be such that $a(x) = a_x$ and $b(x) = b_x$ for all $x \in T$. Let $h_1 = ha^2 - b^2$. Since $-w_x(x)w'_x(x)$ is not a square in $\kappa(x)$ for any $x \in C \cap E$, it is easy to see that f, h, h_1 satisfy the conditions in (2.1). Therefore, by (2.1), $\beta_i \otimes K(\sqrt{f}, \sqrt{hh_1}) = 0$, for $1 \leqslant i \leqslant n$. Hence there exist $a_i, b_i \in K^*$ such that $\beta_i = (f) \cdot (a_i) + (hh_1) \cdot (b_i)$, for $1 \leqslant i \leqslant n$ (cf. [HV], 3.1). Since $hh_1 = (ha)^2 - hb^2$, hh_1 is norm from $K(\sqrt{h})$ and hence $(h) \cdot (hh_1) = 0$. For $1 \leqslant i \leqslant n$, we have

$$\begin{aligned} \alpha_i &= (h) \cup \beta_i \\ &= (h) \cdot (f) \cdot (a_i) + (h) \cdot (hh_1) \cdot (b_i) \\ &= (h) \cdot (f) \cdot (a_i): \end{aligned}$$

This completes the proof of the theorem. □

4. u-invariant

Theorem **4.1.** — *Let k be a non-dyadic p-adic field and* K *a function field in one variable over k. Then every element of* $I^3(K)$ *is represented by a 3-fold Pfister form.*

Proof. — Let q be an anisotropic quadratic form over K representing an element of $I^3(K)$. Let $\alpha = e_3(q)$. Then by (3.9), $\alpha = (f)\cdot(g)\cdot(h)$. Since $e_3 : I^3(K) \to H^3(K, \mathbf{Z}/2)$ is an isomorphism ([AEJ], Theorem 2), $q = <1, -f><1, -g><1, -h>$ in $I^3(K)$. Since q is anisotropic, $q \simeq <1, -f><1, -g><1, -h>$. □

Corollary **4.2.** — *Let* K *be as in (4.1). Then every quadratic form over* K *of rank at least 13 is isotropic.*

Proof. — Let q be a quadratic form over q of rank 13. By the theorem of Saltman (cf. 2.2), $c(q)$ is a biquaternion algebra over K. Let q_0 be a quadratic form over K such that $\mathrm{rk}\,(q_0) = 5$, $d(q+q_0) = 1$ and $c(q+q_0) = 0$ (cf. [HV], 3.2). Then $q + q_0 \in I^3(K)$ ([M1]). By (4.1), we have $q + q_0 = <1, f><1, g><1, h>$ for some $f, g, h \in K^*$. Since $\mathrm{rk}\,(q) = 13$, $q \simeq <1, f><1, g><1, h> \perp -q_0$. Since $I^4(K) = 0$, every element in $I^3(K)$ represents every element of K^*. In particular $<1, f><1, g><1, h>$ represents a value of q_0. Therefore q is isotropic. □

To prove that every quadratic form over K of rank at least 11 is isotropic, we need a subtler choice of a quadratic extension which splits the given element in $H^3(K, \mathbf{Z}/2)$.

Let k be a non-dyadic p-adic field, X a smooth, projective, integral curve over k and $K = k(X)$. Let $\alpha \in H^3(K, \mathbf{Z}/2)$ and $\mathscr{X}$ be a regular, projective model of X over the ring $\mathscr{O}_k$ of integers in k, such that

$$\mathrm{ram}_{\mathscr{X}}(\alpha) \subset C + E,$$

where C and E are regular curves on $\mathscr{X}$ such that C and E have only normal crossings. Let $T = C \cap E$ and B be the semi-local ring at T. Since $\mathscr{X}$ is regular, B is a regular semi-local ring and hence a unique factorisation domain.

Lemma **4.3.** — *With the notation as above, let* L *be a quadratic extension of* K. *Let* S *be a discrete valuation ring with* L *as its quotient field. Assume that* $S \cap K = B_{(\pi)}$, *where* π *is a prime element in* B *giving a local equation for a component* C_1 *of* C. *If* $C_1 \cap E \neq \emptyset$, *let* $C_1 \cap E = \{x_1, \cdots, x_r\}$ *and* δ_{x_i} *be a local equation of* E *at* x_i, $1 \leqslant i \leqslant r$. *Suppose that either* $C_1 \cap E = \emptyset$ *or* $L = K(\sqrt{f})$ *with* $f \in B$ *satisfying one of the following conditions:*

(i) *f is a parameter in* $B_{(\pi)}$,

(ii) *f is a unit in* $B_{(\pi)}$ *such that either* $v_{\overline{\delta}_{x_i}}(\overline{f}) = 1$ *or* $f(x_i)$ *is not a square in* $\kappa(x_i)$, $1 \leqslant i \leqslant r$,

where bar denotes the image modulo (π) *and* $v_{\overline{\delta}_{x_i}}$ *denotes the discrete valuation of* $B/(\pi)$ *at* $\overline{\delta}_{x_i}$.

Then α_L *is unramified at* S.

Proof. — Let $A = B_{(\pi)}$. Then the residue field $\kappa(\pi)$ of A is the quotient field of $B/(\pi)$. Since $\mathrm{ram}_{\mathscr{X}}\alpha \subset C + E$ and C_1 is a regular curve on $\mathscr{X}$, it follows from the complex ([K], 1.7)

$$H^3(K, \mathbf{Z}/2) \xrightarrow{\partial} \bigoplus_{\eta\in\mathscr{X}^1} H^2(\kappa(\eta), \mathbf{Z}/2) \xrightarrow{\partial} \bigoplus_{y\in\mathscr{X}^2} H^1(\kappa(y), \mathbf{Z}/2)$$

that $\partial_{(\pi)}(\alpha)$ is possibly ramified only at the discrete valuations of $\kappa(\pi)$ corresponding to $C_1 \cap E$. Suppose that $C_1 \cap E = \emptyset$. Then it follows that $\partial_{(\pi)}(\alpha)$ is unramified at every discrete valuation ring of $\kappa(\pi)$. Since $\kappa(\pi)$ is either a global field of positive characteristic (so that there are no archimedean primes) or a local field, by class field theory, we have $\partial_{(\pi)}(\alpha) = 0$ and hence α_L is unramified at S.

Suppose that $C_1 \cap E \neq \emptyset$. Suppose that f is a parameter in A. Then S over A is ramified and hence α_L is unramified at S. Suppose that f is as in (ii). Since $v_{\overline{\delta}_{x_i}}(\overline{f}) = 1$ or $f(x_i)$ is not a square in $\kappa(x_i)$, for $1 \leqslant i \leqslant r$, it follows that $\overline{f}$ is not a square in $B/(\pi)$. Since C and E have only normal crossings, $B/(\pi)$ is a regular semi local ring and is integrally closed. Hence $\overline{f}$ is not a square in the residue field $\kappa(\pi)$ of A. Since $H^3(K, \mathbf{Z}/2)$ is generated by symbols and the ramification map is natural on unramified extensions, one sees easily that if S over A is unramified, then $\partial_S(\alpha_L) = \partial_A(\alpha)\otimes\kappa(\pi)(\sqrt{\overline{f}})$. Suppose that $\kappa(\pi)$ is a p-adic field. Since the residue field of S is the quadratic extension $\kappa(\pi)(\sqrt{\overline{f}})$, it follows that $\partial_A(\alpha)$ is split over $\kappa(\pi)(\sqrt{\overline{f}})$. Since f is a unit in A, S over A is unramified and hence $\partial_S(\alpha_L) = \partial_A(\alpha) \otimes \kappa(\pi)(\sqrt{\overline{f}}) = 0$ and α_L is unramified at S. Suppose that $\kappa(\pi)$ is a function field in one variable over a finite field. As above, it follows that $\partial_A(\alpha)$ can be ramified only at the discrete valuation rings of $\kappa(\pi)$ given by the prime elements $\overline{\delta}_{x_i}$ in $B/(\pi)$, $1 \leqslant i \leqslant r$. By the assumption on $\overline{f}$, in view of (3.1), $\partial_A(\alpha) \otimes \kappa(\pi)(\sqrt{\overline{f}}) = 0$ and the lemma follows. □

Proposition **4.4.** — *Let* k, K *be as above. Let* $\alpha \in H^3(K, \mathbf{Z}/2)$ *and* $a, b \in K^*$. *Then there exists* $f \in K^*$ *which is a value of the quadratic form* $< a, b >$ *such that* $\alpha \otimes K(\sqrt{f}) = 0$.

Proof. — Let $\mathscr{X}$ be a regular, projective model of X over $\mathscr{O}_k$ such that

$$\mathrm{Supp}(a) \cup \mathrm{Supp}(b) \cup \mathrm{Supp}(\mathrm{ram}_{\mathscr{X}}(\alpha)) \subset \mathrm{Supp}(C + E),$$

where C and E are regular curves on $\mathscr{X}$ with only normal crossings. Let $T = C \cap E$. Let B be the semi-local ring at T. For $x \in T$, let $\pi_x, \delta_x \in B$ be local equations for C and E at x, respectively. Since B is a unique factorisation domain with quotient field K, without loss of generality, we assume that a, b are square free in B and $\mathrm{Supp}_{\mathrm{Spec(B)}}(ab) \subset \mathrm{Supp}(C + E)$. Let $c \in B$ be the greatest common divisor of a and b, so that $a = ca'$, $b = cb'$, with $a', b' \in B$. Since a and b are square free, c, a', b' are pairwise coprime. For $x \in T$, choose $u_x, v_x \in \kappa(x)$ as follows:

(i) Suppose $c(x) = 0$. Let m_x denote the maximal ideal of B at x. Since c, a', b' are pairwise coprime and the only prime elements of B_{m_x} which divide $ca'b'$ are π_x, δ_x, at least one of a' and b' is coprime with π_x and δ_x, and hence is a unit at x. Thus $a'(x) \neq 0$ or $b'(x) \neq 0$. Let $u_x, v_x \in \kappa(x)$ be such that $a'(x)u_x^2 + b'(x)v_x^2 \neq 0$.

(ii) Suppose that $c(x) \neq 0$ and $a'b'(x) = 0$. Let $u_x = v_x = 1$.

(iii) Suppose that $c(x)a'(x)b'(x) \neq 0$. Since $\kappa(x)$ is a finite field of characteristic not equal to 2, every element of $\kappa(x)$ is represented by the quadratic form $< a'(x), b'(x) >$. Let $u_x, v_x \in \kappa(x)$ be such that $c(x)a'(x)b'(x)\,(a'(x)u_x^2 + b'(x)v_x^2) \notin \kappa(x)^{*2}$.

Let $u, v \in$ B be such that $u(x) = u_x$ and $v(x) = v_x$ for all $x \in$ T. Let $f = ca'b'(a'u^2 + b'v^2)$. Clearly f is a value of $c < a', b' >=< a, b >$. We now show that $\alpha \otimes K(\sqrt{f}\,) = 0$. Let $L = K(\sqrt{f}\,)$ and k' be the field of constants of L. Let X$'$ be a smooth, projective, irreducible curve over k' with $k'(X') = L$. Let $\mathscr{X}'$ be a regular proper model of X$'$ over $\mathcal{O}_{k'}$ and $y \in \mathscr{X}'$ be a point of codimension one. Let $S = \mathcal{O}_{\mathscr{X}',y}$ be the discrete valuation ring at y. As in the proof of (3.9), it is enough to show that α_L is unramified at S. Since $\mathscr{X}$ is projective over $\mathcal{O}_k$, there exists a point $z \in \mathscr{X}$ of codimension 1 or 2, such that S dominates the local ring $A = \mathcal{O}_{\mathscr{X},z}$.

Suppose dim(A) = 1. Then A is a discrete valuation ring. Suppose that z does not correspond to a component of C or E. Then α is unramified at A and hence α_L is unramified at S. Let z correspond to a component C_1 of C. The case where z corresponds to a component of E is similar.

Suppose that $C_1 \cap E = \emptyset$. Then by (4.3), α_L is unramified at S.

Suppose that $C_1 \cap E \neq \emptyset$. Let π be a prime element of B corresponding to the component C_1. Since c, a', b' are pairwise coprime in B, it follows that at most one of c, a', b' is divisible by π.

Suppose π divides c. Let $x \in C_1 \cap E$. Then by (i), $a'u^2 + b'v^2$ is a unit in $\mathcal{O}_{\mathscr{X},x}$. Since A is a localisation of $\mathcal{O}_{\mathscr{X},x}$, $a'u^2 + b'v^2$ is a unit in A. Further, since π divides c, both a' and b' are units in A. Therefore f is a parameter in A and hence by (4.3), α_L is unramified at S.

Suppose π does not divide c and divides a' or b'. Let $x \in C_1 \cap E$. If $c(x) = 0$, then by (i), $a'u^2 + b'v^2$ is a unit at x and hence it is a unit in A. If $c(x) \neq 0$, then by (ii), u and v are units at x and hence units in A. Since only one of the a', b' is divisible by π, $a'u^2 + b'v^2$ is a unit in A. Therefore, as above, f is a parameter in A and α_L is unramified at S.

Suppose that π does not divide $ca'b'$. Let $x \in C_1 \cap E$. If $c(x) = 0$, then by (i), $a'u^2 + b'v^2$ is a unit at x and hence a unit in A. Suppose that $c(x) \neq 0$. Since π does not divide $a'b'$, the only prime elements of B_{m_x} which divide $a'b'$ being π and δ_x, either $a'(x) \neq 0$ or $b'(x) \neq 0$. Therefore if $a'b'(x) = 0$, then by (ii), $a'u^2 + b'v^2$ is a unit at x and if $a'b'(x) \neq 0$, then by iii), $a'u^2 + b'v^2$ is a unit at x. Therefore $a'u^2 + b'v^2$ is a unit in A and hence $v_{\overline{\delta}_x}(\overline{f}) = v_{\overline{\delta}_x}(\overline{ca'b'})$, which is equal to 0 or 1. Further, if $v_{\overline{\delta}_x}(\overline{f}) = 0$, by (iii), $f(x)$ is not a square in $\kappa(x)$. Therefore, by (4.3), α_L is unramified at S.

Suppose $\dim(A) = 2$. Then z is a closed point of $\mathscr{X}$. If $z \notin C \cup E$, then α is unramified on A and hence unramified at S (3.7). Assume that $z \in C \cup E$. If $z \notin C \cap E$, then by (3.3 and 3.5), α_L is unramified at S. Suppose that $z \in C \cap E$. Then $A = B_{m_z}$, where m_z is the maximal ideal of B at z.

Suppose that $c(z) = 0$. Then, by the choice of u, v, $a'u^2 + b'v^2$ is a unit at z. Since the only prime elements of A which divide $ca'b'$ are π_z, δ_z and c, a', b' are pairwise coprime, $f = ca'b'(a'u^2 + b'v^2)$ is of the form $w\pi_z$ or $w\delta_z$ or $w\pi_z\delta_z$, with $w \in A^*$. Since $f \in L^{*2}$, by (3.6), α_L is unramified at S.

Suppose that $c(z) \neq 0$ and $a'(z)b'(z) = 0$. If $a'(z)$ or $b'(z)$ is not zero, then, as above, one shows that either π_z or δ_z or $\pi_z\delta_z$ is as in (3.6, ii) and hence, by (3.6), α_L is unramified at S. Suppose that $a'(z) = b'(z) = 0$. Since the only prime elements of A which divide a', b' are π_z, δ_z and a', b' are coprime and non units at z, we have $a' = w\pi_z$ and $b' = w'\delta_z$ or $a' = w\delta_z$ and $b' = w'\pi_z$ for some $w, w' \in A^*$. Consider the case where $a' = w\pi_z$ and $b' = w'\delta_z$, with $w, w' \in A^*$ (the other case being similar). Let ν_S denote the valuation at S. Since S dominates A, we have $\nu_S(a') \geqslant 1$ and $\nu_S(b') \geqslant 1$. We assume without loss of generality that $\nu_S(a') \leqslant \nu_S(b')$. Then, $b'/a' \in S$ and

$$f = cb'(u^2 + \left(\frac{b'}{a'}\right) v^2)(a')^2.$$

Suppose that $\nu_S(a') < \nu_S(b')$. Then $u^2 + \frac{b'}{a'}v^2 \in S^*$. Since $b' = w'\delta_z$, $w' \in A^*$, $c \in A^*$ and $f \in L^{*2}$, it follows that δ_z is as in (3.6, ii) and α_L is unramified at S. Suppose that $\nu_S(a') = \nu_S(b')$. If $u^2 + \frac{b'}{a'}v^2 \in S^*$, then $\nu_S(f) = \nu_S(b') + 2\nu_S(a') = 3\nu_S(b')$. Since $\nu_S(f)$ is even, it follows that $\nu_S(a') = \nu_S(b')$ is even. In particular $\nu_S(\pi_z) = \nu_S(\delta_z)$ is even. By (3.3, (ii), we have $\alpha = \alpha' + \alpha''$, where α' is unramified on A and α'' is a sum of symbols of the type $(\mu)\cdot(\mu')\cdot(\pi_z)$, $(\mu)\cdot(\mu')\cdot(\delta_z)$, $(\mu)\cdot(\pi_z)\cdot(\delta_z)$, with μ, μ' running over A^*. Since π_z and δ_z have even valuations at S, clearly α''_L is unramified at S. By (3.7), α'_L, and hence α_L, is unramified at S. Assume that $u^2 + \frac{b'}{a'}v^2$ is not a unit in S. Let $n = \nu_S(a') = \nu_S(b')$. Let θ be a parameter in S and write $a' = w_1\theta^n$, $b' = w_2\theta^n$, with $w_1, w_2 \in S^*$. By (ii), $u, v \in A^*$. Since $u^2 + \frac{b'}{a'}v^2 = u^2 + \frac{w_2}{w_1}v^2$ is not a unit in S, we have

$$\frac{\overline{u}^2}{\overline{v}^2} = -\frac{\overline{w_2}}{\overline{w_1}}.$$

By 3.3, (ii), we have $\alpha = \alpha' + \alpha''$, where α' is unramified on A and α'' is a sum of symbols of the type $(\mu)\cdot(\mu')\cdot(\pi_z)$, $(\mu)\cdot(\mu')\cdot(\delta_z)$, $(\mu)\cdot(\pi_z)\cdot(\delta_z)$, with μ, μ' running over

A^*. By (3.5), $(\mu)\cdot(\mu')\cdot(\pi_z)$ and $(\mu)\cdot(\mu')\cdot(\delta_z)$ are unramified at S. Since $a' = w\pi_z = w_1\theta^n$, $b' = w'\delta_z = w_2\theta^n$, we have

$$(\mu)\cdot(\pi_z)\cdot(\delta_z) = (\mu)\cdot(ww_1\theta^n)\cdot(w'w_2\theta^n).$$

If n is even, then clearly $(\mu)\cdot(\pi_z)\cdot(\delta_z)$ is unramified at S. Assume that n is odd. Then, we have

$$(\mu)\cdot(\pi_z)\cdot(\delta_z) = (\mu)\cdot(ww_1\theta)\cdot(w'w_2\theta) = (\mu)\cdot(ww_1\theta)\cdot(-ww_1w'w_2)$$

and

$$\partial_S((\mu)\cdot(\pi_z)\cdot(\delta_z))_L) = (\overline{\mu})\cdot(-\overline{ww_1w'w_2}).$$

Since $-\overline{w_2}/\overline{w_1}$ is a square in the residue field of S, we have $\partial_L((\mu)\cdot(\pi_z)\cdot(\delta_z)) = (\overline{\mu})\cdot(\overline{ww'})$. Since $\mu, w, w' \in A^*$ and $\kappa(z)$ is a finite field, it follows that $(\overline{\mu})\cdot(\overline{ww'}) = 0$. Hence α_L is unramified at S.

Suppose that $c(z)a'(z)b'(z) \neq 0$. Then by the choice of u, v it follows that $f(z) \notin \kappa(z)^{*2}$. Since f is a square in S, it follows from (3.6) that α_L is unramified at S. This completes the proof of the proposition. □

Theorem **4.5.** — *Let k be a non-dyadic p-adic field and* K *a function field in one variable over k. Then every quadratic form over* K *of rank at least 11 is isotropic.*

Proof. — Let q be a quadratic form over K of rank 11. Then by a theorem of Saltman (cf. 2.2) $c(q)$ is a biquaternion algebra. Let q_0 be a quadratic form over K with $\mathrm{rk}\,(q_0) = 5$, $d(q+q_0) = 1$ and $c(q+q_0) = 0$ (cf. [HV], 3.2). Then $q + q_0 \in I^3(K)$ ([M]). Therefore, by (4.1), there exists a 3-fold Pfister form q_1 over K such that $q = q_1 - q_0$. Since for any $\lambda \in K^*$, q is isotropic if and only if λq is isotropic, we assume that $q_0 = <1, a, b, c, d>$ for some $a, b, c, d \in K^*$. Let $\alpha = e_3(q_1)$. Then by (4.4), there exists $f \in K^*$ which is a value of $<-a, -b>$ such that $\alpha \otimes K(\sqrt{f}) = 0$. Since e_3 is an isomorphism, $q_1 \otimes K(\sqrt{f})$ is hyperbolic. Therefore there exist $g, h \in K^*$ such that $q_1 = <1, -f><1, g><1, h>$. Since $-f$ is a value of $<a, b>$, there exists $f' \in K^*$ such that $<a, b> \simeq <-f, f'>$. We have

$$\begin{aligned} q &= q_1 - q_0 \\ &= <1, -f><1, g><1, h> - <1, -f, f', c, d> \\ &= <1, -f><g, h, gh> - <f', c, d>. \end{aligned}$$

Since $\mathrm{rk}\,(q) = 11$ and the rank of $<1, f><g, h, gh> - <f', c, d>$ is 9, it follows that q is isotropic over K. □

Theorem **4.6.** — *Let k be a non-dyadic p-adic field and* K *a function field in one variable over k. Let q be a quadratic form over* K *of rank at least 9. Suppose that c(q) is of index at most 2. Then q is isotropic.*

Proof. — By (4.5), if the rank of q is at least 11, then q is isotropic. Assume that rank of q is 9 or 10. Since $c(q)$ is of index at most 2, there exist $a, b \in K^*$ such that $c(q) = (-a)\cdot(-b)$ in $H^2(K, \mathbf{Z}/2)$. Suppose that the rank of q is 9. By scaling, we can assume that $d(q) = 1$. Let $q_0 =< a, b, ab >$. Then $d(q - q_0) = 1$ and $c(q - q_0) = 0$. Therefore $q = q_0 + q_1$ for some $q_1 \in I^3(K)$. As in the proof of (4.5), there exists $f \in K^*$ which is a value of $< a, b >$ and $q_1 \otimes K(\sqrt{-f}\,)$ is hyperbolic. Therefore we have $< a, b >=< f, f' >$ and $q_1 =< 1, f >< 1, g >< 1, h >$ for some $f', g, h \in K^*$. Since $I^4(K) = 0$ and $q_1 \in I^3(K)$, we have $\lambda q_1 = q_1$ for every $\lambda \in K^*$. Thus, we have

$$\begin{aligned}(-ab)q &= (-ab)q_0 + (-ab)q_1\\ &= (-ab)q_0 + q_1\\ &=< -b, -a, -1 > + q_1\\ &=< -f, -f', -1 > + < 1, f > + < 1, f >< g, h, gh >\\ &=< -f' > + < 1, f >< g, h, gh > .\end{aligned}$$

Therefore q is isotropic. Suppose that the rank of q is 10. Let $q' = q \perp < 1 >$. Then $c(q) = c(q')$. Since the rank of q' is 11, it is isotropic by (4.5). Write $q' =< 1, -1 > \perp q''$. Then the rank of q'' is 9 and $c(q'') = c(q') = c(q)$. Therefore q'' is isotropic. Since $q = q'' \perp < -1 >$, q is isotropic. □

5. Zero-cycles on quadric fibrations

Let k be a p-adic field and C a smooth, projective, geometrically integral curve over k. Let $\pi : X \to C$ be an admissible quadric fibration over C (cf. [CSk], §3). For a variety Y, let $CH_0(Y)$ denote the Chow group of zero-cycles on Y. Let $\pi_* : CH_0(X) \to CH_0(C)$ be the induced homomorphism and $CH_0(X/C) = \ker(\pi_*)$. If $\dim(X) = 2$, then it was proved in ([G]) that the group $CH_0(X/C)$ is finite. In ([CSk]), Colliot-Thélène and Skorobogatov proved that if $\dim(X) = 3$, then $CH_0(X/C)$ is finite and they raised the following question:

If $\dim(X) \geqslant 4$, is the group $CH_0(X/C)$ zero or at least finite?

In ([PS], 4.8), it was shown that the group $CH_0(X/C)$ is finite, answering the latter part of the above question. Recently Hoffmann and Van Geel ([HV], 4.2) proved that if k is non-dyadic and $\dim(X) \geqslant 6$, then $CH_0(X/C) = 0$. Using results proved in §4, we show that $CH_0(X/C) = 0$ if $\dim(X) \geqslant 4$ and k is a non-dyadic p-adic field.

We recall the identification of $CH_0(X/C)$ with a certain subquotient of $k(C)^*$ given in ([CSk], 4.2). Let k be a field of characteristic not equal to 2 and C a smooth, projective, geometrically integral curve over k. Let $\pi : X \to C$ be an admissible quadric fibration of relative dimension at least 1. Let q be a quadratic form over $k(C)$ defining the generic fibre of π. Let $N_q(k(C))$ be the subgroup of $k(C)^*$ generated by elements of the type ab with $a, b \in k(C)^*$, which are values of q over $k(C)$. Let $k(C)^*_{\mathrm{dn}}$ be the subgroup of $k(C)^*$ consisting of functions, which, at each closed point P of C, can be written as a product of a unit at P and an element of $N_q(k(C))$. We recall the following result from ([CSk], 4.2).

Proposition **5.1.** — *There is an isomorphism*

$$CH_0(X/C) \xrightarrow{\sim} k(C)^*_{dn}/k^* N_q(k(C)).$$

Theorem **5.2.** — *Let k be a non-dyadic p-adic field and* C *a smooth, projective, geometrically integral curve over k. Let* $\pi : X \to C$ *be an admissible quadric fibration. If dim*(X) $\geqslant 4$, *then* $CH_0(X/C) = 0$.

Proof. — Let q be a quadratic form over $k(C)$ defining the generic fibre of π. Since $\dim(X) \geqslant 4$, the rank of q is at least 5. If q is isotropic, then every element in $k(C)^*$ is represented by q over $k(C)$ and hence $N_q(k(C)) = k(C)^*$. Assume that q is anisotropic over $k(C)$. Let $f \in k(C)^*$. Since $q \otimes <1, -f> \otimes k(C)(\sqrt{f})$ is hyperbolic, $c(q \otimes <1, -f>) \otimes k(C)(\sqrt{f})$ is zero and hence the index of $c(q \otimes <1, -f>)$ is at most 2. Therefore by (4.6), $<1, -f> \otimes q$ is isotropic. That is, there exist v, w in the underlying vector space of q, with at least one of them non-zero such that $q(v) - fq(w) = 0$. Since q is anisotropic, $q(v)q(w) \neq 0$. Therefore $f = q(v)q(w)^{-1} \in N_q(k(C))$ and hence $N_q(k(C)) = k(C)^*$. By (5.1), it follows that $CH_0(X/C) = 0$. □

6. Cayley algebras

In this section, we recall a connection between $H^3_{dec}(K)$ and the set of isomorphism classes of Cayley algebras over a field K of characteristic not 2 ([Se], §8.3). We then give a description of the set of isomorphism classes of Cayley algebras over function fields of non-dyadic p-adic curves in the spirit of Serre, using the fact that $H^3_{dec}(K) = H^3(K)$ and a theorem of Kato.

Theorem **6.1** ([Se], §8, Theorem 9). — *Let* G *be a split algebraic group of type* G_2 *defined over a field* K *of characteristic not equal to 2. There are canonical bijections between the following sets:*

(i) $H^1(K, G)$.

(ii) $H^3_{dec}(K) = \{ \alpha \in H^3(K, \mathbf{Z}/2), \alpha = (a)\cdot(b)\cdot(c), a, b, c \in K^* \}$.

(iii) *The set of isomorphism classes of* K*-forms of G.*

(iv) *The set of isomorphism classes of Cayley algebras over* K.

(v) *The set of isomorphism classes of 3-fold Pfister forms.*

Let k be a p-adic field. Let P be the set of closed points of $\mathbf{P}^1_k$ and

$$C(P) = \{ f : P \to \mathbf{Z}/2 \mid \mathrm{Supp}(f) \text{ finite and } \sum_{x \in P} f(x) = 0 \}:$$

The exact sequence

$$0 \to \mathrm{H}^3(k(t), \mathbf{Z}/2) \to \bigoplus_{x \in \mathrm{P}} \mathrm{H}^2(\kappa(x), \mathbf{Z}/2) \to \mathrm{H}^2(k, \mathbf{Z}/2) \to 0$$

identifies $\mathrm{H}^3(k(t), \mathbf{Z}/2)$ with C(P), noting that $\mathrm{H}^2(\kappa(x), \mathbf{Z}/2) = \mathbf{Z}/2$ for every $x \in \mathrm{P}$ and the map $\oplus_{x \in \mathrm{P}} \mathrm{H}^2(\kappa(x), \mathbf{Z}/2) \to \mathrm{H}^2(k, \mathbf{Z}/2)$ is the addition. In ([Se], §8.3), Serre raises the question whether $\mathrm{H}^1(k(t), \mathrm{G})$ is in bijection with C(P). This is equivalent to the question whether $\mathrm{H}^3_{\mathrm{dec}}(k(t)) = \mathrm{H}^3(k(t), \mathbf{Z}/2)$. In view of (3.9), this is indeed true if k is non-dyadic.

Let k be a non-dyadic p-adic field, and X a smooth, projective, integral curve over k. Using a result of Kato ([K]) and following Serre, we give a description of $\mathrm{H}^1(k(\mathrm{X}), \mathrm{G})$ as follows. Let $\mathscr{X}$ be a regular, proper model of X over $\mathscr{O}_k$. Let $\mathrm{Y} = \mathscr{X} \times_{\mathrm{Spec}(\mathscr{O}_k)} \mathrm{Spec}(\mathbf{F}_q)$ be the special fibre, where $\mathbf{F}_q$ is the residue field of k. Let Y' be the reduced scheme of Y and $\pi : \tilde{\mathrm{Y}} \to \mathrm{Y}'$ be the normalisation of Y'. Let $\mathrm{Y}'_{\mathrm{sing}}$ denote the set of singular points of Y' and $\mathrm{Q} = \pi^{-1}(\mathrm{Y}'_{\mathrm{sing}})$. Let $\tilde{\mathrm{Y}} = \cup_1^r \tilde{\mathrm{Y}}_i$, $\tilde{\mathrm{Y}}_i$ denoting the irreducible components of $\tilde{\mathrm{Y}}$. Let

$$\mathrm{C(Q)} = \{f\colon \mathrm{Q} \to \mathbf{Z}/2 \mid \sum_{x \in \tilde{\mathrm{Y}}_i \cap \mathrm{Q}} f(x) = 0,\ 1 \leqslant i \leqslant r,\ \sum_{x \in \pi^{-1}(y)} f(x) = 0 \text{ for all } y \in \mathrm{Y}'_{\mathrm{sing}}\}.$$

For $y \in \mathrm{Y}^1$, let

$$\partial_i^y : \mathrm{H}^2(\kappa(\tilde{\mathrm{Y}}_i), \mathbf{Z}/2) \to \mathrm{H}^1(\kappa(y), \mathbf{Z}/2)$$

be the homomorphism defined as $\partial_i^y = 0$ if $\pi^{-1}(y) \cap \tilde{\mathrm{Y}}_i = \emptyset$ and otherwise

$$\partial_i^y = \sum_{\tilde{y} \in \pi^{-1}(y) \cap \tilde{\mathrm{Y}}_i} \partial_i^{\tilde{y}},$$

where $\partial_i^{\tilde{y}}$ denotes the residue map at $\tilde{y}$. Let

$$\partial^y = \sum_i \partial_i^y.$$

By a result of Kato ([K], 5.2), we have an isomorphism

$$\mathrm{H}^3_{\mathrm{nr}}(k(\mathrm{X})/\mathrm{X}, \mathbf{Z}/2) \xrightarrow{\sim} \ker\Big(\bigoplus_i \mathrm{H}^2(\kappa(\tilde{\mathrm{Y}}_i), \mathbf{Z}/2) \overset{\partial = (\partial^y)}{\longrightarrow} \bigoplus_{y \in \mathrm{Y}^1} \mathrm{H}^1(\kappa(y), \mathbf{Z}/2)\Big).$$

Lemma **6.2.** — *We have an isomorphism*

$$\ker(\bigoplus_i H^2(\kappa(\tilde{Y}_i), \mathbf{Z}/2) \xrightarrow{\partial} \bigoplus_{y\in Y^1} H^1(\kappa(y), \mathbf{Z}/2)) \simeq C(Q)$$

Proof. — Let $(\alpha_i) \in \oplus_i H^2(\kappa(\tilde{Y}_i), \mathbf{Z}/2)$ be such that $\partial((\alpha_i)) = 0$. Then for a closed point $\tilde{y} \in \tilde{Y}_i \setminus Q$, $\partial_i^{\tilde{y}}(\alpha_i) = 0$. For $\tilde{y} \in Q \cap \tilde{Y}_i$, let $f(\tilde{y}) = \partial_i^{\tilde{y}}(\alpha_i) \in H^1(\kappa(\tilde{y}), \mathbf{Z}/2) = \mathbf{Z}/2$. Then, by class field theory for function fields in one variable over finite fields, it follows that $f \in C(Q)$. Conversely, let $f \in C(Q)$. Then by class field theory, there exist $\alpha_i \in H^2(\kappa(\tilde{Y}_i), \mathbf{Z}/2)$ such that for $\tilde{y} \in Q \cap \tilde{Y}_i$, $\partial_i^{\tilde{y}}(\alpha_i) = f(\tilde{y})$ and if $\tilde{y} \in \cup\tilde{Y}_i \setminus Q$, then $\partial_i^{\tilde{y}}(\alpha_i) = 0$ for all i. Since $f \in C(Q)$, $\partial(\alpha_i) = 0$. This proves the lemma. □

Let P be the set of closed points of X. Let

$$C(P) = \{ f : P \to \mathbf{Z}/2 \mid \mathrm{Supp}(f) \text{ finite and } \sum_{x\in P} f(x) = 0 \}.$$

We have an exact sequence ([K], 5.2)

$$0 \to H^3_{nr}(k(X)/X, \mathbf{Z}/2) \to H^3(k(X), \mathbf{Z}/2) \to \bigoplus_{x\in P} H^2(\kappa(x), \mathbf{Z}/2) \to \mathbf{Z}/2 \to 0.$$

This sequence induces an exact sequence

$$0 \to H^3_{nr}(k(X)/X, \mathbf{Z}/2) \to H^3(k(X), \mathbf{Z}/2) \to C(P) \to 0.$$

By (6.2), we have $H^3_{nr}(k(X)/X, \mathbf{Z}/2) \simeq C(Q)$. In view of (3.9), we have $H^3_{dec}(k(X), \mathbf{Z}/2) = H^3(k(X), \mathbf{Z}/2)$ and we have the following

Theorem **6.3.** — *Let k be a non-dyadic p-adic field and* X *a smooth, projective, irreducible curve over k. The bijection* $H^1(k(X), G) \simeq H^3_{dec}(k(X), \mathbf{Z}/2) = H^3(k(X), \mathbf{Z}/2)$ *makes* $H^1(k(X), G)$ *a* $\mathbf{Z}/2$*-vector space which fits into an exact sequence*

$$0 \to C(Q) \to H^1(k(X), G) \to C(P) \to 0.$$

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