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YAKOV VARSHAVSKY

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# $p$-ADIC UNIFORMIZATION OF UNITARY SHIMURA VARIETIES

*by* YAKOV VARSHAVSKY

## Introduction

Let $\Gamma \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ be a torsion-free cocompact lattice. Then $\Gamma$ acts on the unit ball $\mathrm{B}^{d-1} \subset \mathbf{C}^{d-1}$ by holomorphic automorphisms. The quotient $\Gamma \backslash \mathrm{B}^{d-1}$ is a complex manifold, which has a unique structure of a complex projective variety $\mathrm{X}_\Gamma$ (see [Sha, Ch. IX, § 3]).

Shimura had proved that when $\Gamma$ is an arithmetic congruence subgroup, $\mathrm{X}_\Gamma$ has a canonical structure of a projective variety over some number field K (see [Del] or [Mil]). For certain arithmetic problems it is desirable to know a description of the reduction of $\mathrm{X}_\Gamma$ modulo $w$, where $w$ is some prime of K. In some cases it happens that the projective variety $\mathrm{X}_\Gamma$ has a $p$-adic uniformization. By this we mean that the $\mathrm{K}_w$-analytic space $(\mathrm{X}_\Gamma \otimes_{\mathrm{K}} \mathrm{K}_w)^{\mathrm{an}}$ is isomorphic to $\Delta \backslash \Omega$ for some $p$-adic analytic symmetric space $\Omega$ and some group $\Delta$, acting on $\Omega$ discretely. Then a formal scheme structure on $\Delta \backslash \Omega$ gives us an $\mathcal{O}_{\mathrm{K}_w}$-integral model for $\mathrm{X}_\Gamma \otimes_{\mathrm{K}} \mathrm{K}_w$.

Cherednik was the first who obtained a result in this direction. Let F be a totally real number field, and let B/F be a quaternion algebra, which is definite at all infinite places, except one, and ramified at a finite prime $v$ of F. Then Cherednik proved in [Ch2] that the Shimura curve corresponding to B has a $p$-adic uniformization by the $p$-adic upper half-plane $\Omega^2_{\mathrm{F}_v}$, constructed by Mumford (see [Mum1]), when the subgroup defining the curve is maximal at $v$. Cherednik's proof is based on the method of elliptic elements, developed by Ihara in [Ih].

The next significant step was done by Drinfel'd in [Dr2]. First he constructed certain covers of $\Omega^2_{\mathrm{F}_v}$ (see below). Then, when $\mathrm{F} = \mathbf{Q}$, he proved the existence of a $p$-adic uniformization by some of his covers for all Shimura curves, described in the previous paragraph, without the assumption of maximality at $v$. The basic idea of Drinfel'd's proof was to invent some moduli problem, whose solution is the Shimura curve as well as a certain $p$-adically uniformized curve, showing, therefore, that they are isomorphic.

Developing Drinfel'd's method, Rapoport and Zink (see [RZ1, Ra]) obtained some higher-dimensional generalizations of the above results.

In this paper we generalize Cherednik's method and prove that certain unitary Shimura varieties and automorphic vector bundles over them have a $p$-adic uniformization. Our results include all previously known results as particular cases.

We now describe our work in more detail. Let $\mathrm{F}$ be a totally real number field of degree $g$ over $\mathbf{Q}$, and let $\mathrm{K}$ be a totally imaginary quadratic extension of $\mathrm{F}$. Let $\mathrm{D}$ and $\mathrm{D}^{\mathrm{int}}$ be central simple algebras of dimension $d^2$ over $\mathrm{K}$ with involutions of the second kind $\alpha$ and $\alpha^{\mathrm{int}}$ respectively over $\mathrm{F}$. Let $\mathrm{G} := \mathrm{GU}(\mathrm{D}, \alpha)$ and $\mathrm{G}^{\mathrm{int}} := \mathrm{GU}(\mathrm{D}^{\mathrm{int}}, \alpha^{\mathrm{int}})$ be the corresponding algebraic groups of unitary similitudes (see Definition 2.1.1 and Notation 2.1.2 for the notation).

Let $v$ be a non-archimedean prime of $\mathrm{F}$ that splits in $\mathrm{K}$, let $w$ and $\bar{w}$ be the primes of $\mathrm{K}$ that lie over $v$, and let $\infty_1$ be an archimedean prime of $\mathrm{F}$. Suppose that $\mathrm{D}^{\mathrm{int}} \otimes_{\mathrm{K}} \mathrm{K}_w$ has Brauer invariant $1/d$, that $\mathrm{D} \otimes_{\mathrm{K}} \mathrm{K}_w \cong \mathrm{Mat}_d(\mathrm{K}_w)$, and that the pairs $(\mathrm{D}, \alpha) \otimes_{\mathrm{F}} \mathrm{F}_u$ and $(\mathrm{D}^{\mathrm{int}}, \alpha^{\mathrm{int}}) \otimes_{\mathrm{F}} \mathrm{F}_u$ are isomorphic for all primes $u$ of $\mathrm{F}$, except $v$ and $\infty_1$. Assume also that $\alpha$ is positive definite at all archimedean places $\mathrm{F}_{\infty_i} \cong \mathbf{R}$ of $\mathrm{F}$, that is that $\mathrm{G}(\mathrm{F}_{\infty_i}) \cong \mathrm{GU}_d(\mathbf{R})$ for all $i = 1, \ldots, g$, and that the signature of $\alpha^{\mathrm{int}}$ at $\infty_1$ is $(d-1, 1)$, so that $\mathrm{G}^{\mathrm{int}}(\mathrm{F}_{\infty_1}) \cong \mathrm{GU}_{d-1,1}(\mathbf{R})$.

Let $\mathbf{A}_{\mathrm{F}}^{f}$ and $\mathbf{A}_{\mathrm{F}}^{f;v}$ be the ring of finite adeles of $\mathrm{F}$ and the ring of finite adeles of $\mathrm{F}$ without the $v$-th component respectively. Set $\mathrm{E}' := \mathrm{F}_v^{\times} \times \mathrm{G}(\mathbf{A}_{\mathrm{F}}^{f;v})$, and fix a central simple algebra $\widetilde{\mathrm{D}}_w$ over $\mathrm{K}_w$ of dimension $d^2$ with Brauer invariant $1/d$. Then $\mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f}) \cong \widetilde{\mathrm{D}}_w^{\times} \times \mathrm{E}'$ and $\mathrm{G}(\mathbf{A}_{\mathrm{F}}^{f}) \cong \mathrm{GL}_d(\mathrm{K}_w) \times \mathrm{E}'$. In particular, the group $\mathrm{GL}_d(\mathrm{K}_w)$ acts naturally on $\mathrm{G}(\mathbf{A}_{\mathrm{F}}^{f})$ by left multiplication.

Let $\Omega_{\mathrm{K}_w}^{d}$ be the Drinfel'd's $(d-1)$-dimensional upper half-space over $\mathrm{K}_w$ constructed in [Dr1], and let $\{ \Sigma_{\mathrm{K}_w}^{d,n} \}_{n \in \mathbf{N} \cup \{0\}}$ be the projective system of étale coverings of $\Omega_{\mathrm{K}_w}^{d}$ constructed in [Dr2]. This system is equipped with an equivariant action of the group $\mathrm{GL}_d(\mathrm{K}_w) \times \widetilde{\mathrm{D}}_w^{\times}$ such that if $\mathrm{T}_n$ denotes the $n$-th congruence subgroup of $\mathcal{O}_{\widetilde{\mathrm{D}}_w}^{\times}$, then we have $\mathrm{T}_n \backslash \Sigma_{\mathrm{K}_w}^{d,m} \cong \Sigma_{\mathrm{K}_w}^{d,n}$ for all $m \geqslant n$ (see 1.3.1 and 1.4.1 for our notation and conventions, which differ from those of Drinfel'd).

Denote by $\mathrm{G}^{\mathrm{int}}(\mathrm{F})_{+}$ the set of all $d \in (\mathrm{D}^{\mathrm{int}})^{\times}$ such that $d \cdot \alpha^{\mathrm{int}}(d)$ is a totally positive element of $\mathrm{F}$. Choose an embedding $\mathrm{K} \hookrightarrow \mathbf{C}$, extending $\infty_1 : \mathrm{F} \hookrightarrow \mathbf{R}$. It defines us an embedding $\mathrm{G}^{\mathrm{int}}(\mathrm{F})_{+} \hookrightarrow \mathrm{GU}_{d-1,1}(\mathbf{R})^0 = \mathrm{Aut}(\mathrm{B}^{d-1})$. Choose finally an embedding of $\mathrm{K}_w$ into $\mathbf{C}$, extending that of $\mathrm{K}$.

For each compact and open subgroup $\mathrm{S}$ of $\mathrm{E}'$ and each non-negative integer $n$ let $\mathrm{X}_{\mathrm{S},n}$ be the weakly-canonical model over $\mathrm{K}_w$ of the Shimura variety corresponding to the complex analytic space $(\mathrm{T}_n \times \mathrm{S}) \backslash [\mathrm{B}^{d-1} \times \mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f})] / \mathrm{G}^{\mathrm{int}}(\mathrm{F})_{+}$ and to the morphism $h : \mathrm{S} \to \mathrm{G}^{\mathrm{int}} \otimes_{\mathbf{Q}} \mathbf{R}$, described in 3.1.1 (see Definition 3.1.12 and Remark 3.1.13 for the definitions). The experts might notice that our $h$ is not the one usually used in moduli problems of abelian varieties.

Let $\mathrm{V}_{\mathrm{S},n}$ be the canonical model of the automorphic vector bundle on $\mathrm{X}_{\mathrm{S},n}$ (see [Mil, III] or the last paragraph of the proof of Proposition 4.3.1 for the definitions), corresponding to the complex analytic space

$$(\mathrm{T}_n \times \mathrm{S}) \backslash [\beta_{\mathbf{R}}^{*}(\mathrm{W}^{\mathrm{tw}} \otimes_{\mathrm{K}_w} \mathbf{C})_{1}^{\mathrm{an}} \times \mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f})] / \mathrm{G}^{\mathrm{int}}(\mathrm{F})_{+}$$

(see 4.1.1 for the necessary notation).

Let $\mathrm{P}_{\mathrm{S},n}$ be the canonical model of the standard principal bundle over $\mathrm{X}_{\mathrm{S},n}$ (see [Mil, III] or Corollary 4.7.2 for the definitions), corresponding to the complex analytic space $(\mathrm{T}_n \times \mathrm{S})\backslash[\mathrm{B}^{d-1} \times (\mathrm{PG}^{\mathrm{int}} \otimes_{\mathbf{Q}} \mathbf{C})^{\mathrm{an}} \times \mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f})]/\mathrm{G}^{\mathrm{int}}(\mathrm{F})_{+}$ (see 4.1.1 for the necessary notation).

*Main Theorem. — For each compact and open subgroup* S *of* E′ *and each* $n \in \mathbf{N} \cup \{0\}$ *we have isomorphisms of* $\mathrm{K}_w$*-analytic spaces:*

*a)* $(\mathrm{X}_{\mathrm{S},n})^{\mathrm{an}} \overset{\sim}{\to} \mathrm{GL}_d(\mathrm{K}_w)\backslash[\Sigma_{\mathrm{K}_w}^{d,n} \times (\mathrm{S}\backslash\mathrm{G}(\mathbf{A}_{\mathrm{F}}^{f})/\mathrm{G}(\mathrm{F}))]$;

*b)* $(\mathrm{V}_{\mathrm{S},n})^{\mathrm{an}} \overset{\sim}{\to} \mathrm{GL}_d(\mathrm{K}_w)\backslash[\beta_{w,n}^{*}(\mathrm{W}^{\mathrm{an}}) \times (\mathrm{S}\backslash\mathrm{G}(\mathbf{A}_{\mathrm{F}}^{f})/\mathrm{G}(\mathrm{F}))]$ *(see* 4.1.1 *for the necessary notation), where the group* $\mathrm{GL}_d(\mathrm{K}_w)$ *acts on* $\beta_{w,n}^{*}(\mathrm{W}^{\mathrm{an}})$ *as the direct factor of* $(\mathrm{G} \otimes_{\mathbf{Q}} \mathrm{K}_w)(\mathrm{K}_w)$, *corresponding to the natural embedding* $\mathrm{K} \hookrightarrow \mathrm{K}_w$;

*c)* $(\mathrm{P}_{\mathrm{S},n})^{\mathrm{an}} \overset{\sim}{\to} [\mathrm{GL}_d(\mathrm{K}_w)\backslash[\Sigma_{\mathrm{K}_w}^{d,n} \times ((\mathrm{PG} \otimes_{\mathbf{Q}} \mathrm{K}_w)^{\mathrm{an}} \times (\mathrm{S}\backslash\mathrm{G}(\mathbf{A}_{\mathrm{F}}^{f})))/\mathrm{G}(\mathrm{F})]]^{\mathrm{tw}}$ *(see* 4.1.1 *for the definition of the twisting* $(\ )^{\mathrm{tw}}$*), where the group* $\mathrm{GL}_d(\mathrm{K}_w)$ *acts trivially on* $(\mathrm{PG} \otimes_{\mathbf{Q}} \mathrm{K}_w)^{\mathrm{an}}$.

*These isomorphisms commute with the natural projections for* $\mathrm{S}_1 \subset \mathrm{S}_2$, $n_1 \geqslant n_2$ *and with the action of* $\mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f}) \cong \widetilde{\mathrm{D}}_w^{\times} \times \mathrm{F}_v^{\times} \times \mathrm{G}(\mathbf{A}_{\mathrm{F}}^{f;v})$.

The idea of the proof is the following. Consider the $p$-adic analytic varieties $\widetilde{\mathrm{Y}}_{\mathrm{S},n}$ of the right hand side of *a)* of the Main Theorem. They form a projective system and each of them has a natural structure $\mathrm{Y}_{\mathrm{S},n}$ of a projective variety over $\mathrm{K}_w$. Kurihara proved in [Ku] that for every torsion-free cocompact lattice $\Gamma \subset \mathrm{PGL}_d(\mathrm{K}_w)$ the Chern numbers of $\Gamma\backslash\Omega_{\mathrm{K}_w}^{d}$ are proportional to those of the $(d-1)$-dimensional projective space and that the canonical class of $\Gamma\backslash\Omega_{\mathrm{K}_w}^{d}$ is ample. The result of Yau (see [Ya]) then implies that $\mathrm{B}^{d-1}$ is the universal covering of each connected component of the complex analytic space $(\mathrm{Y}_{\mathrm{S},n} \otimes_{\mathrm{K}_w} \mathbf{C})^{\mathrm{an}}$ for all sufficiently small $\mathrm{S} \in \mathscr{F}(\mathrm{E})$ and all embeddings $\mathrm{K}_w \hookrightarrow \mathbf{C}$.

It is technically better to work with the inverse limit of the $\mathrm{Y}_{\mathrm{S},n}$'s equipped with the action of the group $\mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f}) \cong \widetilde{\mathrm{D}}_w^{\times} \times \mathrm{E}'$ on it rather then to work with each $\mathrm{Y}_{\mathrm{S},n}$ separately. Generalizing the ideas of Cherednik [Ch2] we prove that there exists a subgroup $\Delta \subset \mathrm{GU}_{d-1,1}(\mathbf{R})^{0} \times \mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f})$ such that

$$(\mathrm{Y}_{\mathrm{S},n} \otimes_{\mathrm{K}_w} \mathbf{C})^{\mathrm{an}} \cong (\mathrm{T}_n \times \mathrm{S})\backslash(\mathrm{B}^{d-1} \times \mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f}))/\Delta$$

for all compact open subgroups $\mathrm{S} \subset \mathrm{E}'$ and all $n \in \mathbf{N} \cup \{0\}$.

Using Margulis' theorem on arithmeticity we show that the groups $\Delta$ and $\mathrm{G}^{\mathrm{int}}(\mathrm{F})_{+}$ are almost isomorphic modulo centers. More precisely, we show that $(\mathrm{Y}_{\mathrm{S},n} \otimes_{\mathrm{K}_w} \mathbf{C})^{\mathrm{an}}$ is isomorphic to a finite covering of $(\mathrm{X}_{\mathrm{S},n} \otimes_{\mathrm{K}_w} \mathbf{C})^{\mathrm{an}}$. Using Kottwitz' results [Ko] on local Tamagawa measures we find that the volumes of $(\mathrm{Y}_{\mathrm{S},n} \otimes_{\mathrm{K}_w} \mathbf{C})^{\mathrm{an}}$ and $(\mathrm{X}_{\mathrm{S},n} \otimes_{\mathrm{K}_w} \mathbf{C})^{\mathrm{an}}$ are equal. It follows that the varieties $\mathrm{Y}_{\mathrm{S},n} \otimes_{\mathrm{K}_w} \mathbf{C}$ and $\mathrm{X}_{\mathrm{S},n} \otimes_{\mathrm{K}_w} \mathbf{C}$ are isomorphic over $\mathbf{C}$. Comparing the action of the Galois group on the set of special points on both sides we conclude that $\mathrm{Y}_{\mathrm{S},n}$ and $\mathrm{X}_{\mathrm{S},n}$ are actually isomorphic over $\mathrm{K}_w$.

Notice that if one considers only Shimura varieties corresponding to subgroups which are maximal at $w$, then the use of Drinfel'd's covers in the proof of the $p$-adic uniformization is very minor. (We use them only for showing that the $p$-adically uniformized Shimura varieties have Brauer invariant $1/d$ at $w$; that probably can be done directly.) In this case the proof would be technically much easier but contain all the essential ideas.

The proof of the $p$-adic uniformization of standard principal bundles is similar. In addition to the above considerations it uses the connection on principal bundles. Using the ideas from [Mil, III] we show that the $p$-adic uniformization of standard principal bundles implies the $p$-adic uniformization of automorphic vector bundles. In fact Tannakian arguments show (see [DM]) that these statements are equivalent.

This paper is organized as follows. In the first section we introduce certain constructions of projective systems of projective algebraic varieties, give their basic properties and do other technical preliminaries.

In the second section we give two basic examples of such systems. Then we formulate and prove the complex version of our Main Theorem for Shimura varieties.

The third and the forth sections are devoted to the proof of the theorem on the $p$-adic uniformization of Shimura varieties and of automorphic vector bundles respectively.

Our proof appears to be very general. That is starting from any reasonable $p$-adic symmetric space, whose quotient by an arithmetic cocompact subgroup is algebraizable, we find Shimura varieties uniformized by it. For example, in another work ([Va]) we extend our results to Shimura varieties uniformized by the product of Drinfel'd's upper half-spaces. Hence it would be interesting to have more examples of such $p$-adic symmetric spaces.

Our result on the $p$-adic uniformization of automorphic vector bundles is not complete, because we prove the $p$-adic uniformization only under the assumption that the center acts trivially. In fact our proof of the complex version of the theorem works also in the general case, but to get an isomorphism over $K_w$ one should understand better the action of the Galois group on the set of special points.

After this work was completed, it was pointed out to the author that Rapoport and Zink have recently obtained similar results concerning the uniformization of Shimura varieties by completely different methods (see [RZ2]).

## Notation and conventions

1) For a group G let $Z(G)$ be the center of G, let $PG := G/Z(G)$ be the adjoint group of G, and let $G^{der}$ be the derived group of G.

2) For a Lie group or an algebraic group G let $G^0$ be its connected component of the identity.

3) For a totally disconnected topological group E let $\mathscr{F}(E)$ be the set of all compact and open subgroups of E, and let $E^{disc}$ be the group E with the discrete topology.

4) For a subgroup $\Gamma$ of a group G let $\mathrm{Comm}_G(\Gamma)$ be the commensurator of $\Gamma$ in G.

5) For a subgroup $\Gamma$ of a topological group G let $\overline{\Gamma}$ be the closure of $\Gamma$ in G.

6) For a set X and a group G acting on X let $X^G$ be the set of all elements of X fixed by all $g \in G$.

7) For a set X, a subset Y of X and a group G acting on X let $\mathrm{Stab}_G(Y)$ be the set of all elements of G mapping Y into itself.

8) For an analytic space or a scheme X let T(X) be the tangent bundle on X.

9) For a vector bundle V on X and a point $x \in X$ let $V_x$ be the fiber of V over $x$.

10) For an algebra D let $D^{\mathrm{opp}}$ be the opposite algebra of D.

11) For a finite dimensional central simple algebra D over a field let $SD^\times$ be the subgroup of $D^\times$ consisting of elements with reduced norm 1.

12) For a number field F and a finite set N of finite primes of F let $\mathbf{A}_F^f$ be the ring of finite adeles of F, and let $\mathbf{A}_F^{f;N}$ be the ring of finite adeles of F without the components from N.

13) For a field extension K/F let $R_{K/F}$ be the functor of the restriction of scalars from K to F.

14) For a natural number $n$ let $I_n$ be the $n \times n$ identity matrix and let $B^n \subset \mathbf{C}^n$ be the $n$-dimensional complex unit ball.

15) For a scheme X over a field K and a field extension L of K write $X_L$ or $X \otimes_K L$ instead of $X \times_{\mathrm{Spec}\,K} \mathrm{Spec}\, L$.

16) For an analytic space X over a complete non-archimedean field K and $a$ for a complete non-archimedean field extension L of K let $X \hat{\otimes}_K L$ be a field extension from K to L. (A completion sign will be omitted in the case of a finite extension.)

17) By a $p$-adic field we mean a finite field extension of $\mathbf{Q}_p$ for some prime number $p$. Let $\mathbf{C}_p$ be the completion of the algebraic closure of $\mathbf{Q}_p$.

18) By a $p$-adic analytic space we mean an analytic space over a $p$-adic field in the sense of Berkovich [Be1].

19) For an affinoid algebra A let $\mathscr{M}(A)$ be the affinoid space associated to it.

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## CONTENTS

## 1. BASIC DEFINITIONS AND CONSTRUCTIONS

### 1.1. General preparations

*Definition* **1.1.1.** — A *locally profinite group* is a locally compact totally disconnected topological group. In such a group E, the set $\mathscr{F}(\mathrm{E})$ forms a fundamental system of neighbourhoods of the identity element, and $\bigcap_{\mathrm{S}\in\mathscr{F}(\mathrm{E})} \mathrm{S} = \{1\}$.

*Lemma* **1.1.2.** — *Let* E *be a locally profinite group, and let* X *be a separated topological space with a continuous action* $\mathrm{E}\times\mathrm{X}\to\mathrm{X}$ *of* E. *For each* $\mathrm{S}\in\mathscr{F}(\mathrm{E})$, *set* $\mathrm{X}_\mathrm{S} := \mathrm{S}\backslash\mathrm{X}$. *Then* $\{\mathrm{X}_\mathrm{S}\}_\mathrm{S}$ *is a projective system and* $\mathrm{X}\cong\varprojlim_{\mathrm{S}}\mathrm{X}_\mathrm{S}$.

*Proof.* — [Mil, Ch. II, Lem. 10.1]. □

This lemma motivates the following definition.

*Definition* **1.1.3.** — Let X be a separated scheme over a field L, let E be a locally profinite group, acting L-rationally on X. We call X an (E, L)-*scheme* (or simply an E-*scheme* if L is clear or is not important) if for each $S \in \mathscr{F}(E)$ there exists a quotient $X_S := S\backslash X$, which is a projective scheme over L, and $X \cong \varprojlim_S X_S$.

The following remarks show that E-schemes are closely related to projective systems of projective schemes, indexed by $\mathscr{F}(E)$.

*Remark* **1.1.4.** — If X is an E-scheme or merely a topological space with a continuous action of E, then for each $g \in E$ and each $S, T \in \mathscr{F}(E)$ with $S \supset gTg^{-1}$ we have a morphism $\rho_{S,T}(g) : X_T \to X_S$, induced by the action of $g$ on X and satisfying the following conditions:

*a)* $\rho_{S,S}(g) = \mathrm{Id}$ if $g \in S$;

*b)* $\rho_{S,T}(g) \circ \rho_{T,R}(h) = \rho_{S,R}(gh)$;

*c)* if T is normal in S, then $\rho_{T,T}$ defines the action of the finite group S/T on $X_T$, and $X_S$ is isomorphic to the quotient of $X_T$ by the action of S/T.

*Remark* **1.1.5.** — Conversely, suppose that for each $S \in \mathscr{F}(E)$ there is given a scheme $X_S$, and for each $g \in E$ and each $S, T \in \mathscr{F}(E)$ with $S \supset gTg^{-1}$, there is given a morphism $\rho_{S,T}(g) : X_T \to X_S$, satisfying the conditions *a)-c)* of 1.1.4. Then for each $T \subset S$ there is a map $\rho_{S,T}(1) : X_T \to X_S$, which is finite, by condition *c)*. In this way we get a projective system of schemes and we can form an inverse limit scheme $X := \varprojlim_S X_S$. Then there is a unique action of E on X such that for each $g \in E$ and each $S \in \mathscr{F}(E)$ the action of $g$ on X induces an isomorphism $\rho_{gSg^{-1},S} : X_S \xrightarrow{\sim} X_{gSg^{-1}}$. It follows from *c)* that $X_S \xrightarrow{\sim} S\backslash X$ for each $S \in \mathscr{F}(E)$.

*Definition* **1.1.6.** — Let $\widetilde{E}$ be a topological group, which is isomorphic to E under an isomorphism $\Phi : E \xrightarrow{\sim} \widetilde{E}$. We say that an (E, L)-scheme X is $\Phi$-*equivariantly isomorphic* to an $(\widetilde{E}, L)$-scheme $\widetilde{X}$ if there exists an isomorphism $\varphi : X \xrightarrow{\sim} \widetilde{X}$ of schemes over L such that for each $g \in E$ we have $\varphi \circ g = \Phi(g) \circ \varphi$. If in addition $E = \widetilde{E}$ and $\varphi$ is the identity, then we say that $\varphi$ is an *isomorphism of* (E, L)-*schemes.*

*Definition* **1.1.7.** — Let $L_2/L_1$ be a field extension. We say that an $(E, L_1)$-scheme X is *an* $L_2/L_1$-*descent* of an $(E, L_2)$-scheme Y if the $(E, L_2)$-schemes $X_{L_2}$ and Y are isomorphic.

Suppose from now on that E is a noncompact locally profinite group.

*Notation* **1.1.8.** — For a topological group G and a subgroup $\Gamma \subset G \times E$ let $\mathrm{pr}_G$ and $\mathrm{pr}_E$ be the projection maps from $\Gamma$ to G and E respectively. Set $\Gamma_G := \mathrm{pr}_G(\Gamma)$, $\Gamma_E := \mathrm{pr}_E(\Gamma)$ and $\Gamma_S := \mathrm{pr}_G(\Gamma \cap (G \times S))$ for each $S \in \mathscr{F}(E)$. For each $\gamma \in \Gamma$ set $\gamma_G := \mathrm{pr}_G(\gamma)$ and $\gamma_E := \mathrm{pr}_E(\gamma)$.

*Lemma* **1.1.9.** — *Let* $\Gamma \subset G \times E$ *be a cocompact lattice. Suppose that* $pr_G$ *is injective. Then for each* $S \in \mathcal{F}(E)$ *we have the following*:

*a)* $| S\backslash E/\Gamma_E | < \infty$;
*b)* $[\Gamma_G : \Gamma_S] = \infty$;
*c)* $\Gamma_S$ *is a cocompact lattice of* $G$;
*d)* $\Gamma_G \subset \mathrm{Comm}_G(\Gamma_S)$.

*Proof.* — *a)* Since the double quotient $(G \times S)\backslash(G \times E)/\Gamma \cong S\backslash E/\Gamma_E$ is compact and discrete, it is finite.

*b)* The group E is noncompact, therefore $| S\backslash E | = \infty$. Hence, by *a)*,

$$[\Gamma_E : S \cap \Gamma_E] = | S\backslash S\Gamma_E | = \infty.$$

But $\Gamma_G = pr_G(\Gamma) = pr_G(pr_E^{-1}(\Gamma_E))$, and likewise $\Gamma_S = pr_G(pr_E^{-1}(\Gamma_E \cap S))$. Since $pr_G$ is injective, we are done.

*c)* The group $\Gamma$ is a cocompact lattice in $G \times E$, hence $\Gamma \cap (G \times S)$ is a cocompact lattice in $G \times S$, and the statement follows by projecting to G (see [Shi, Prop. 1.10]).

*d)* Let $\gamma \in \Gamma$, and set $S' = \gamma_E S \gamma_E^{-1} \in \mathcal{F}(E)$. Then

$$\gamma(\Gamma \cap (G \times S))\,\gamma^{-1} = \Gamma \cap (G \times S').$$

But $S \cap S' \in \mathcal{F}(E)$ is a subgroup of finite index in both S and S', hence $\gamma_G \Gamma_S \gamma_G^{-1} \cap \Gamma_S = \Gamma_{S \cap S'}$ is a subgroup of finite index in both $\Gamma_S$ and $\gamma_G \Gamma_S \gamma_G^{-1}$. □

Suppose that $d \geqslant 2$ and take G equal to $PGL_d(K_w)$ for some $p$-adic field $K_w$ or to $PGU_{d-1,1}(\mathbf{R})^0$. We shall call these the $p$-adic and the real (or the complex) cases respectively.

*Proposition* **1.1.10.** — *Under the assumptions of Lemma* 1.1.9 *we have*:

*a)* $\overline{\Gamma_G} \supset G^{der}$;
*b)* $pr_E$ *is injective*;
*c) for each* $S \in \mathcal{F}(E)$, *the group* $\Gamma_S$ *is an arithmetic subgroup of* G *in the sense of Margulis (see* [Ma, p. 292]*)*;
*d) if* $S \in \mathcal{F}(E)$ *is sufficiently small, then the subgroup* $\Gamma_{aSa^{-1}}$ *is torsion-free for each* $a \in E$.

*Proof.* — *a)* For each $S \in \mathcal{F}(E)$, $\Gamma_S$ is cocompact in G and $[\Gamma_G : \Gamma_S] = \infty$. It follows that $\overline{\Gamma_G}$ is a closed non-discrete cocompact subgroup of G. Therefore its inverse image $\pi^{-1}(\overline{\Gamma_G})$ in $SU_{d-1,1}(\mathbf{R})$ (resp. $SL_d(K_w)$) is also closed, non-discrete and cocompact, hence by [Ma, Ch. II, Thm. 5.1] it is all of $SU_{d-1,1}(\mathbf{R})$ (resp. $SL_d(K_w)$). This completes the proof.

*b)* Set $\Gamma_0 := pr_G(\mathrm{Ker}\, pr_E)$. This is a discrete (hence a closed) subgroup of G, which is normal in $\Gamma_G$. Therefore it is normal in $\overline{\Gamma_G} \supset G^{der}$. It follows that each $\gamma \in \Gamma_0$ must commute with some open neighborhood of the identity in $G^{der}$, hence $\Gamma_0$ is trivial.

*c)* is a direct corollary of [Ma, Ch. IX, Thm. 1.14] by *b)*-*d)* of Lemma 1.1.9.

*d)* (compare the proof of [Ch. 1, Lem. 1.3]). Choose an $S \in \mathscr{F}(E)$, then $\Gamma_S \subset G$ is a cocompact lattice.

*Lemma* **1.1.11.** — *The torsion elements of* $\Gamma_S$ *comprise a finite number of conjugacy classes in* $\Gamma_S$.

We first complete the proof of the proposition assuming the lemma. Let $a_1, \ldots, a_n \in E$ be representatives of double classes $\Gamma_E \backslash E/S$ (use Lemma 1.1.9, *a)*). For each $i = 1, \ldots, n$ let $M_i \subset \Gamma_{a_i S a_i^{-1}}$ be a finite set of representatives of conjugacy classes of torsion non-trivial elements of $\Gamma_{a_i S a_i^{-1}}$. Then the image of all non-trivial torsion elements of $\Gamma_{a_i S a_i^{-1}}$ under the natural injection $j_i : \Gamma_{a_i S a_i^{-1}} \xrightarrow{\sim} \Gamma \cap (G \times a_i S a_i^{-1}) \hookrightarrow a_i S a_i^{-1} \xrightarrow{\sim} S$ is contained in the set $X_i = \{ s \cdot j_i(M_i) \cdot s^{-1} \mid s \in S \}$, which is compact and does not contain 1. Hence there exists $T \in \mathscr{F}(E)$ not intersecting any of the $X_i$'s. By taking a smaller subgroup we may suppose that T is a normal subgroup of S. Since all the $j_i$'s are injective, the subgroup $\Gamma_{a_i T a_i^{-1}} = j_i^{-1}(T)$ is torsion-free for each $i = 1, \ldots, n$. For each $a \in E$ there exist $i \in \{ 1, \ldots, n \}$, $s \in S$ and $\gamma \in \Gamma$ such that $a = \gamma_E a_i s$. Hence the subgroup

$$\begin{aligned} \Gamma_{aTa^{-1}} &\cong \Gamma \cap (G \times aTa^{-1}) = \Gamma \cap (\gamma_G G \gamma_G^{-1} \times \gamma_E a_i T a_i^{-1} \gamma_E^{-1}) \\ &\cong \gamma(\Gamma \cap (G \times a_i T a_i^{-1}))\, \gamma^{-1} \cong \Gamma_{a_i T a_i^{-1}} \end{aligned}$$

is torsion-free. □

*Proof* (of the lemma). — The group G acts continuously and isometrically on some complete negatively curved metric space Y. Indeed, in the real case $Y = B^{d-1}$ with the hyperbolic metric. In the $p$-adic case Y is a geometric realization (see [Br, Ch. I, appendix]) of the Bruhat-Tits building $\Delta$ of $SL_d(K_w)$. This is a locally finite simplicial complex of dimension $d - 1$ which can be described as follows. Its vertices are the equivalence classes of free $\mathcal{O}_{K_w}$-submodules of rank $d$ of the vector space $K_w^n$, where M and N are said to be equivalent when there exists $a \in K_w^\times$, such that $M = aN$. The distinct vertices $\Delta_1, \Delta_2, \ldots, \Delta_k$ form a simplex when there exist for them representative lattices $M_1, M_2, \ldots, M_k$, such that $M_1 \supset M_2 \supset \ldots \supset M_k \supset \pi M_1$. For more information see [Mus, § 1] or [Br, Ch. V, § 8].

The geometric realization Y of $\Delta$ has a canonical metric, that makes Y a complete metric space with negative curvature (see [Br, Ch. VI, § 3]). Moreover, the natural action of $PGL_d(K_w)$ on the set of vertices of $\Delta$ can be (uniquely) extended to the simplicial, continuous and isometric action on Y.

Now the Bruhat-Tits fixed point theorem (see [Br, Ch. VI, § 4, Thm. 1]) implies that any compact subgroup of G has a fixed point on Y. In particular, any torsion element of G has a fixed point on Y. Notice that in the $p$-adic case it then stabilizes the minimal simplex, containing the fixed point.

Conversely, the stabilizer in G of each point of Y is compact. In the real case this is true, since the group $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ acts transitively on $\mathrm{B}^{d-1}$ and the group $\mathrm{K} = \mathrm{Stab}_{\mathrm{B}^{d-1}}(0) \cong \mathrm{U}_{d-1}(\mathbf{R})$ is compact. In the $p$-adic case the group $\mathrm{PGL}_d(\mathrm{K}_w)$ acts transitively on the set of vertices, and the stabilizer of the equivalence class of $\mathcal{O}^d_{\mathrm{K}_w} \subset \mathrm{K}^d_w$ is $\mathrm{PGL}_d(\mathcal{O}_{\mathrm{K}_w})$, hence it is compact. Since the stabilizer in G of any point $y \in \mathrm{Y}$ must stabilize the minimal simplex $\sigma$ containing $y$, it must permute the finitely many vertices of $\sigma$, so that it is also compact. It follows that the stabilizer of any point of Y in $\Gamma_{\mathrm{S}}$ is compact and discrete, hence it is finite.

To finish the proof of the lemma in the real case we note that for each $x \in \mathrm{B}^{d-1}$ there exists an open neighbourhood $\mathrm{U}_x$ of $x$ such that

$$\Gamma_x := \{ g \in \Gamma_{\mathrm{S}} \mid g(\mathrm{U}_x) \cap \mathrm{U}_x \neq \emptyset \} = \{ g \in \Gamma_{\mathrm{S}} \mid g(x) = x \}$$

is finite (see [Shi, Prop. 1.6 and 1.7]). The space $\Gamma_{\mathrm{S}}\backslash\mathrm{B}^{d-1}$ is compact, hence there exist a finite number of points $x_1, x_2, \dots, x_m$ of $\mathrm{B}^{d-1}$, such that $\Gamma_{\mathrm{S}}(\bigcup_{i=1}^m \mathrm{U}_{x_i}) = \mathrm{B}^{d-1}$. If $\gamma$ is a torsion element of $\Gamma_{\mathrm{S}}$, then it fixes some point of $\mathrm{B}^{d-1}$. By conjugation we may assume that it fixes a point in some $\mathrm{U}_{x_i}$, therefore $\gamma$ is conjugate to an element of the finite set $\bigcup_{i=1}^m \Gamma_{x_i}$.

In the $p$-adic case we first assert that $\Delta$ has only a finite number of equivalence classes of simplexes under the action of $\Gamma_{\mathrm{S}}$. Since $\Delta$ is locally finite, it is enough to prove this assertion for vertices. The group G acts transitively on the set of vertices, and $\mathrm{G} = \Gamma_{\mathrm{S}} \cdot \mathrm{K}$ for some compact set $\mathrm{K} \subset \mathrm{G}$. Hence if $v$ is a vertex of $\Delta$, then $\mathrm{K} \cdot v$ is a compact and discrete (because the set of all vertices of $\Delta$ is a discrete set in Y) subset of Y, and our assertion follows. Now the same considerations as in the real case complete the proof. □

## 1.2. GAGA results

In what follows we will need some GAGA results. Let L be equal to $\mathrm{K}_w$ in the $p$-adic case and to **C** in the complex case. We will call both the complex and the $p$-adic (L-)analytic spaces simply (L-)*analytic spaces*. Recall that for each scheme X of locally finite type over L and each coherent sheaf F on X a certain L-analytic space $\mathrm{X}^{\mathrm{an}}$ and a coherent analytic sheaf $\mathrm{F}^{\mathrm{an}}$ on $\mathrm{X}^{\mathrm{an}}$ can be associated (see [Bel, Thm. 3.4.1] in the $p$-adic case and [SGA1, Exp. XII] in the complex one).

*Theorem* **1.2.1.** — *Let* X *be a projective* L*-scheme. The functor* $\mathrm{F} \mapsto \mathrm{F}^{\mathrm{an}}$ *from the category of coherent sheaves on* X *to the category of coherent analytic sheaves on* $\mathrm{X}^{\mathrm{an}}$ *is an equivalence of categories.*

*Proof.* — In the complex case the theorem is proved in [Sel, § 12, Thm. 2 and 3], in the $p$-adic one the proof is the same. One first shows by a direct computation that the $p$-adic analytic and the algebraic cohomology groups of $\mathbf{P}^n$ coincide. Next, one concludes from Kiehl's theorem (see [Bel, Prop. 3.3.5]) that the cohomology group of an analytic coherent sheaf on $\mathbf{P}^n$ is a finite-dimensional vector space. Now the arguments of Serre's proof in the complex case hold in the $p$-adic case as well. See [Bel, 3.4] for the relevant definitions and basic properties. □

*Corollary* **1.2.2.** — *a) If* X *is an algebraic variety over* L *and* X′ *is a compact* L-*analytic subvariety of* X, *then* X′ *is a proper* L-*algebraic subvariety of* X.

*b) The functor which associates to a proper* L-*scheme* X *the analytic space* $X^{an}$ *is fully faithful.*

*Proof.* — Serre's arguments (see [Sel, § 19, Prop. 14 and 15]) hold in both the complex and the $p$-adic cases. □

*Corollary* **1.2.3.** — *Let* X *be a projective* L-*scheme. The functor* $X' \mapsto (X')^{an}$ *induces an equivalence between:*

*a) the category of vector bundles of finite rank on* X *and the category of analytic vector bundles of finite rank on* $X^{an}$;

*b) the category of finite schemes over* X *and the category of finite* L-*analytic spaces over* $X^{an}$, *if* L *is a* $p$-*adic field.*

*Proof.* — *a)* To prove the statement we first notice that the category of vector bundles of finite rank is equivalent to the category of locally free sheaves of finite rank. In the algebraic case this is proved in [Ha, II, Ex. 5.18]. In the analytic case the proof is similar. Now the corollary would follow from the theorem if we show that locally free analytic sheaves of finite rank correspond to locally free algebraic ones. The analytic structure sheaf is faithfully flat over the algebraic one (see [Sel, § 2, Prop. 3] and [Bel, Thm. 3.4.1]). Therefore the statement follows from the fact that an algebraic flat coherent sheaf is locally free (see [Mi2, Thm. 2.9]).

*b)* We first show that the correspondence $(\varphi : Y \to X) \mapsto \varphi_*(\mathcal{O}_Y)$ (resp. $(\varphi : \tilde{Y} \to X^{an}) \mapsto \varphi_*(\mathcal{O}_{\tilde{Y}})$) gives an equivalence between the category of finite schemes (resp. analytic spaces) over X (resp. $X^{an}$) and the category of coherent $\mathcal{O}_X$— (resp. $\mathcal{O}_{X^{an}}$—)algebras. In the algebraic case this is proved in [Ha, II, Ex. 5.17]. In the analytic case the proof is exactly the same, because a finite algebra over an affinoid algebra has a canonical structure of an affinoid algebra (see [Bel, Prop. 2.1.12]). □

*Remark* **1.2.4.** — If X′ is finite over X, then it is projective over X, therefore if, in addition, X is projective over $K_w$, then X′ is also projective over $K_w$.

*Corollary* **1.2.5.** — *Let* X *and* Y *be projective* L-*schemes, and let* W *and* V *be algebraic vector bundles of finite ranks on* X *and* Y *respectively. Then for each analytic map of vector bundles* $\tilde{f} : W^{an} \to V^{an}$ *covering some map* $f : X \to Y$ *there exists a unique algebraic morphism* $g : W \to V$ *such that* $g^{an} = \tilde{f}$.

*Proof.* — By definition, $\tilde{f}$ factors uniquely as

$$W^{an} \xrightarrow{\tilde{f}'} V^{an} \times_{Y^{an}} X^{an} \cong (V \times_Y X)^{an} \xrightarrow{\text{proj}} V^{an}.$$

Corollary 1.2.3 implies that there exists a unique $g' : W \to V \times_Y X$ such that $(g')^{an} = \tilde{g}'$. Set $g := \text{proj} \circ g'$.

For the uniqueness observe that if $h : W \to V$ satisfies $h^{an} = \tilde{g}$, then it covers $f$. Hence $h$ factors as $W \xrightarrow{h'} V \times_Y X \xrightarrow{\text{proj}} V$. Since $\tilde{f}'$ and $g'$ are unique, we have $h' = g'$ and $h = g$. □

*Remark* **1.2.6.** — Using the results and ideas of [SGA1, Exp. XII] one can replace in the above results the assumption of projectivity by properness.

We now introduce two constructions of E-schemes which are basic for this work.

## 1.3. First construction

**1.3.1.** Let $\Omega^d_{K_w}$ be an open $K_w$-analytic subset of $(\mathbf{P}^{d-1}_{K_w})^{an}$, obtained by removing from $(\mathbf{P}^{d-1}_{K_w})^{an}$ the union of all the $K_w$-rational hyperplanes (see [Be1] and [Be3] for the definition and basic properties of analytic spaces). It is called the $(d-1)$-dimensional Drinfel'd upper half-space over $K_w$ (see also [Dr1, § 6]). Then $\Omega^d_{K_w}$ is the generic fiber of a certain formal scheme $\hat{\Omega}^d_{K_w}$ over $\mathcal{O}_{K_w}$, constructed in [Mus, Ku], generalizing [Mum1].

The group $PGL_d(K_w)$ acts naturally on $\Omega^d_{K_w}$. (It will be convenient for us to consider $\mathbf{P}^{d-1}$ as the set of lines in $\mathbf{A}^d$ and not as the set of hyperplanes, as Drinfel'd does. Therefore our action differs by transpose inverse from that of Drinfel'd.) Moreover, this action naturally extends to the $\mathcal{O}_{K_w}$-linear action of $PGL_d(K_w)$ on $\hat{\Omega}^d_{K_w}$. Furthermore, $PGL_d(K_w)$ is the group of all formal scheme automorphisms of $\hat{\Omega}^d_{K_w}$ over $\mathcal{O}_{K_w}$ (see [Mus, Prop. 4.2]) and of all analytic automorphisms of $\Omega^d_{K_w}$ over $K_w$ (see [Be2]). Though the action of $PGL_d(K_w)$ on $\Omega^d_{K_w}$ is far from being transitive, we have the following

*Lemma* **1.3.2.** — *There is no non-trivial closed analytic subspace of* $\Omega^d_{K_w} \hat{\otimes}_{K_w} \mathbf{C}_p$, *invariant under the subgroup*

$$U = \left\{ U_{\bar{x}} := \left( \begin{array}{c|c} I_{d-1} & \bar{x} \\ \hline 0 & 1 \end{array} \right) \middle| \; \bar{x} \in K_w^{d-1} \right\} \subset PGL_d(K_w).$$

*Proof.* — Suppose that our lemma is false. Let Y be a non-trivial U-invariant closed analytic subset of $\Omega^d_{K_w} \hat{\otimes}_{K_w} \mathbf{C}_p$. Then $\dim Y < \dim \Omega^d_{K_w} \hat{\otimes}_{K_w} \mathbf{C}_p = d - 1$. Choose a regular point $y \in Y(\mathbf{C}_p)$ (the set of regular points is open and non-empty). Then $\dim T_y(Y) = \dim Y < d - 1$. Next we identify $\Omega^d_{K_w} \hat{\otimes}_{K_w} \mathbf{C}_p$ with an open analytic subset of $(\mathbf{A}^{d-1}_{\mathbf{C}_p})^{an}$ by the map $(z_1 : \ldots : z_d) \mapsto \left( \frac{z_1}{z_d}, \ldots, \frac{z_{d-1}}{z_d} \right)$. Then

$U_{\bar{x}}(z) = z + \bar{x}$ for every $z \in \mathbf{A}^{d-1}_{\mathbf{C}_p}$ and every $\bar{x} \in \mathbf{A}^{d-1}(\mathrm{K}_w)$. In particular, $y + \bar{x} \in \mathrm{Y}$ for every $\bar{x} \in \mathbf{A}^{d-1}(\mathrm{K}_w)$, contradicting the assumption that $\dim \mathrm{T}_y(\mathrm{Y}) < d - 1$. □

Recall also that the group $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ acts transitively on $\mathrm{B}^{d-1}$ and that it is the group of all analytic (holomorphic) automorphisms of $\mathrm{B}^{d-1}$ (see [Ru, Thm. 2.1.3 and 2.2.2]).

In what follows we will need the notion of a pro-analytic space.

*Definition* **1.3.3.** — A *pro-analytic space* is a projective system $\{ \mathrm{X}_\alpha \}_{\alpha \in \mathrm{I}}$ of analytic spaces such that for some $\alpha_0 \in \mathrm{I}$ all transition maps $\varphi_{\beta\alpha} : \mathrm{X}_\beta \to \mathrm{X}_\alpha$; $\beta \geqslant \alpha \geqslant \alpha_0$ are étale and surjective.

*Definition* **1.3.4.** — By a *point* of $\mathrm{X} := \{ \mathrm{X}_\alpha \}_{\alpha \in \mathrm{I}}$ we mean a system $\{ x_\alpha \}_{\alpha \in \mathrm{I}}$, where $x_\alpha$ is a point of $\mathrm{X}_\alpha$ for all $\alpha \in \mathrm{I}$ and $\varphi_{\beta\alpha}(x_\beta) = x_\alpha$ for all $\beta \geqslant \alpha$ in I. For a point $x = \{ x_\alpha \}_{\alpha \in \mathrm{I}}$ of $\mathrm{X} = \{ \mathrm{X}_\alpha \}_{\alpha \in \mathrm{I}}$ let

$$\mathrm{T}_x(\mathrm{X}) := \{ v = \{ v_\alpha \}_{\alpha \in \mathrm{I}} \mid v_\alpha \in \mathrm{T}_{x_\alpha}(\mathrm{X}_\alpha),\ d\varphi_{\alpha\beta}(v_\beta) = v_\alpha \text{ for all } \beta \geqslant \alpha \text{ in I} \}$$

be the *tangent space* of $x$ in X.

*Definition* **1.3.5.** — Let $\mathrm{X} = \{ \mathrm{X}_\alpha \}_{\alpha \in \mathrm{I}}$ and $\mathrm{Y} = \{ \mathrm{Y}_\beta \}_{\beta \in \mathrm{J}}$ be two pro-analytic spaces. To give a *pro-analytic morphism* $f : \mathrm{X} \to \mathrm{Y}$ is to give an order-preserving map $\sigma : \mathrm{I} \to \mathrm{J}$, whose image is cofinal in J, and a projective system of analytic morphisms $f_\alpha : \mathrm{X}_\alpha \to \mathrm{Y}_{\sigma(\alpha)}$. A morphism $f$ is called *étale* if there exists $\alpha_0 \in \mathrm{I}$ such that for each $\alpha \geqslant \alpha_0$ the morphism $f_\alpha$ is étale.

*Construction* **1.3.6.** — Suppose that $\Gamma \subset \mathrm{G} \times \mathrm{E}$ satisfies the conditions of Lemma 1.1.9. We are going to associate to Γ a certain (E, L)-scheme.

Let $\mathrm{X}^0$ be $\mathrm{B}^{d-1}$ in the real case and $\Omega^d_{\mathrm{K}_w}$ in the $p$-adic one. Consider the L-analytic space $\widetilde{\mathrm{X}} := (\mathrm{X}^0 \times \mathrm{E}^{\mathrm{disc}})/\Gamma$, where Γ acts on $\mathrm{X}^0 \times \mathrm{E}^{\mathrm{disc}}$ by the natural right action: $(x, g)\,\gamma := (\gamma_{\mathrm{G}}^{-1} x, g\gamma_{\mathrm{E}})$. Then E acts analytically on $\widetilde{\mathrm{X}}$ by left multiplication.

*Proposition* **1.3.7.** — *For each* $\mathrm{S} \in \mathscr{F}(\mathrm{E})$ *the quotient* $\mathrm{S}\backslash\widetilde{\mathrm{X}} = \mathrm{S}\backslash(\mathrm{X}^0 \times \mathrm{E})/\Gamma$ *exists and has a natural structure of a projective scheme* $\mathrm{X}_{\mathrm{S}}$ *over* L.

*Proof.* — First take $\mathrm{S} \in \mathscr{F}(\mathrm{E})$ satisfying part *d)* of Proposition 1.1.10. Then $\mathrm{S}\backslash\widetilde{\mathrm{X}}$ has $| \mathrm{S}\backslash\mathrm{E}/\Gamma_{\mathrm{E}} | < \infty$ connected components, each of them is isomorphic to $\Gamma_{a\mathrm{S}a^{-1}}\backslash\mathrm{X}^0$ for some $a \in \mathrm{E}$. By *c), d)* of Proposition 1.1.10, each $\Gamma_{a\mathrm{S}a^{-1}}$ is a torsion-free arithmetic cocompact lattice of G.

By [Shi, Prop. 1.6 and 1.7], [Sha, Ch. IX, 3.2] in the real case and by [Mus] or [Ku] in the $p$-adic one, each quotient $\Gamma_{a\mathrm{S}a^{-1}}\backslash\mathrm{X}^0$ exists and has a unique structure of a projective algebraic variety over L. Therefore there exists a projective scheme $\mathrm{X}_{\mathrm{S}}$ over L such that $\mathrm{X}_{\mathrm{S}}^{\mathrm{an}} \cong \mathrm{S}\backslash\widetilde{\mathrm{X}}$.

Take now an arbitrary $S \in \mathscr{F}(E)$. It has a normal subgroup $T \in \mathscr{F}(E)$ which satisfies part *d)* of Proposition 1.1.10. The finite group $S/T$ acts on $T\backslash\widetilde{X} \cong X_T^{an}$ by analytic automorphisms and $S\backslash\widetilde{X} \cong (S/T)\backslash X_T^{an}$. Corollary 1.2.2 implies that the analytic action of $S/T$ on $X_T^{an}$ defines an algebraic action on $X_T$ and that the projective scheme $X_S := (S/T)\backslash X_T$ (the quotient exists by [Mum2, § 7]) satisfies $(X_S)^{an} \cong S\backslash\widetilde{X}$. Moreover, the same corollary implies also that the algebraic structure on $S\backslash\widetilde{X}$ is unique. □

For all $g \in E$ and all $S, T \in \mathscr{F}(E)$ with $S \supset gTg^{-1}$ we obtain by Remark 1.1.4 analytic morphisms $\rho_{S,T}(g) : X_T^{an} \rightarrow X_S^{an}$. They give us by Corollary 1.2.2 uniquely determined algebraic morphisms $\rho_{S,T}(g) : X_T \rightarrow X_S$, which provide us by Remark 1.1.5 an $(E, L)$-scheme $X := \varprojlim_{S} X_S$.

*Proposition* **1.3.8.** — *a) There exists the inverse limit* $X^{an}$ *of the* $X_S^{an}$*'s in the category of* L*-analytic spaces, which is isomorphic to* $\widetilde{X}$.

*b)* $\mathrm{Stab}_E(X^0 \times \{1\}) = \Gamma_E$.

*c) Let* $X_0$ *be the connected component of* $X$ *such that* $X_0^{an} \supset X^0 \times \{1\}$ *(note that* $X^0 \times \{1\}$ *is a connected component of* $X^{an}$*, and that the analytic topology is stronger then the Zariski topology). Then* $\mathrm{Stab}_E(X_0) = \overline{\Gamma_E}$.

*d) The group* E *acts faithfully on* X.

*e) For each* $x \in X$ *the orbit* $E \cdot x$ *is (geometrically) Zariski dense. In particular,* E *acts transitively on the set of geometrically connected components of* X.

*f) For each* $S \in \mathscr{F}(E)$ *satisfying part d) of* 1.1.10, *the map* $X \rightarrow X_S$ *is étale;*

*g) For each embedding* $K_w \hookrightarrow \mathbf{C}$ *and each* $S \in \mathscr{F}(E)$ *as in c),* $B^{d-1}$ *is the universal covering of each connected component of* $(X_{S,\mathbf{C}})^{an}$ *in the p-adic case and of* $X_S^{an}$ *in the complex one.*

*Proof.* — *a)* We start from the following

*Lemma* **1.3.9.** — *a) Let* $\Pi$ *be a torsion-free discrete subgroup of* G. *Then the natural projection* $X^0 \rightarrow \Pi\backslash X^0$ *is an analytic (topological) covering.*

*b) For each* $x \in X^0$ *the stabilizer of* $x$ *in* G *is compact.*

*Proof.* — *a)* follows from [Shi, Prop. 1.6 and 1.7] in the real case and from [Be2, Lem. 4 and 6] in the $p$-adic one.

*b)* By [Dr2, § 6] there exists a $PGL_d(K_w)$-equivariant map from $\Omega_{K_w}^d$ to the Bruhat-Tits building $\Delta$ of $SL_d(K_w)$, thus it suffice to show the required property for stabilizers of points in $\Delta$ and $B^{d-1}$. This was done in the proof of Lemma 1.1.11. □

The lemma implies that for each sufficiently small $S \in \mathscr{F}(E)$ the analytic space $X_S^{an}$ admits a covering by open analytic subsets $U_i$ satisfying the following condition: for each $i$ and each subgroup $S \supset T \in \mathscr{F}(E)$ the inverse image $\rho_T^{-1}(U_i)$ of $U_i$ under the natural projection $\rho_T : X_T^{an} \rightarrow X_S^{an}$ splits as a disjoint union of analytic spaces, each of them isomorphic to $U_i$ under $\rho_T$.

Now we will define a certain L-analytic space $X^{an}$ associated to X. As a set it is the inverse limit of the underlying sets of the $X_S^{an}$'s. To define an analytic structure on $X^{an}$ consider subsets $V_\alpha \subset X^{an}$ such that for some (hence for every) sufficiently small $S \in \mathscr{F}(E)$, the natural projection $\pi_S : X^{an} \to X_S^{an}$ induces a bijection of $V_\alpha$ with an open analytic subset $\pi_S(V_\alpha)$ of $X_S^{an}$, described in the previous paragraph. Provide then such a $V_\alpha$ with an analytic structure by requiring that $\pi_S : V_\alpha \to \pi_S(V_\alpha)$ is an analytic isomorphism. Then the analytic structure of the $V_\alpha$'s does not depend on the choice of the S's, and there exists a unique L-analytic structure on $X^{an}$ such that each $V_\alpha$ is an open analytic subset of $X^{an}$.

By the construction, $X^{an}$ is the inverse limit of the $X_S^{an}$'s in the category of L-analytic spaces. Hence there exists a unique E-equivariant analytic map $\pi : \widetilde{X} \to X^{an}$ such that for each $S \in \mathscr{F}(E)$ the natural projection $\widetilde{X} \to X_S^{an}$ factors as $\widetilde{X} \xrightarrow{\pi} X^{an} \xrightarrow{\pi_S} X_S^{an}$, where by $\pi_S$ we denote the natural projection. It remains to show that $\pi$ is an isomorphism.

For each $S \in \mathscr{F}(E)$ satisfying part *d)* of Proposition 1.1.10, the natural projection $X^0 \to \Gamma_S \backslash X^0$ is a local isomorphism, hence the projections $\widetilde{X} \to X_S$ and $\pi$ are local isomorphisms as well.

The map $\pi_S \circ \pi$ is surjective, hence for each $x \in X^{an}$ there exists a point $y \in \widetilde{X}$ such that $\pi_S(x) = \pi_S \circ \pi(y)$. Therefore, $\pi(y) = sx$ for some $s \in S$. Since $\pi$ is E-equivariant, we conclude that $\pi(s^{-1}(y)) = x$. Hence $\pi$ is surjective.

Suppose that $\pi(y_1) = \pi(y_2)$ for some $y_1, y_2 \in \widetilde{X}$. Let $(x_1, g_1)$ and $(x_2, g_2)$ be their representatives in $X^0 \times E$. Then for each $S \in \mathscr{F}(E)$ there exist $s \in S$ and $\gamma \in \Gamma$ such that $x_1 = \gamma_G^{-1}(x_2)$ and $g_1 = sg_2\gamma_E$. Such $\gamma_G$'s belong to the set $\{ g \in G \mid g(x_1) = x_2 \} \cap \Gamma_{\sigma_2^{-1} S \sigma_1}$, which is compact (by the lemma) and discrete, hence finite. Therefore we can choose sufficiently small $S \in \mathscr{F}(E)$ such that $g_1 \gamma_E^{-1} g_2^{-1} = s \in S$ must be equal to 1. This means that $y_1 = y_2$. Thus $\pi$ is a surjective, one-to-one local isomorphism, hence it is an isomorphism.

*b)* is clear.

*c)* For each $S \in \mathscr{F}(E)$ let $Y_S$ be the connected component of $X_S$ such that $Y_S^{an}$ is the image of $X^0 \times \{1\} \subset X^{an}$ under the natural projection $\pi_S : X^{an} \to X_S^{an}$. Then $X_0 = \varprojlim_S Y_S$. It follows that $g \in E$ satisfies $g(X_0) = X_0$ if and only if $g(Y_S) = Y_S$ for each $S \in \mathscr{F}(E)$ if and only if $X^0 \times \{g\} \subset S(X^0 \times 1)\,\Gamma$ for each $S \in \mathscr{F}(E)$ if and only if $g \in S\Gamma_E$ for each $S \in \mathscr{F}(E)$ if and only if $g \in \bigcap_{S \in \mathscr{F}(E)} S\Gamma_E = \overline{\Gamma_E}$.

*d)* If $g \in E$ acts trivially on X, then it acts trivially on $X^{an} \cong (X^0 \times E^{disc})/\Gamma$. By *b)*, $g = \gamma_E$ for some $\gamma \in \Gamma$, and $\gamma_G$ acts trivially on $X^0$. Since $pr_G$ is injective, $\gamma = g = 1$.

*e)* Let Y be the Zariski closure of $E.x$. Then Y is E-invariant and, therefore, $Y^{an} \cap (X^0 \times \{1\})$ is a closed $\Gamma$-invariant analytic subspace of $X^0 \times \{1\} \cong X^0$. By Proposition 1.1.10 *a)*, it is $G^{der}$-invariant. Since $G^{der}$ acts transitively on $X^0$ in the real case and by Lemma 1.3.2 in the $p$-adic one, $Y^{an} \cap (X^0 \times \{1\})$ has to be all of $X^0 \times \{1\}$. It follows that $Y = X$.

*f)* holds, since the projection $\pi_S : X^{an} \to X_S^{an}$ is a local isomorphism (see the proof of *a)*).

*g)* The real case is clear, the *p*-adic case is deep. It uses Yau's theorem (see [Ku, Rem. 2.2.13]). □

*Remark* **1.3.10.** — The functorial property of projective limits implies that $X^{an}$ satisfies the functorial properties of analytic spaces associated to schemes (see [Bel, Thm. 3.4.1] or [SGA1, Exp. XII, Thm. 1.1]).

*Lemma* **1.3.11.** — *Let* $\Gamma \subset G \times E$ *and* X *be as above, let* $E'$ *be a compact normal subgroup of* E, *and let* $\Gamma' \subset G \times (E' \backslash E)$ *be the image of* $\Gamma$ *under the natural projection. Then we have the following*:

*a) the map* $\varphi : \Gamma \to \Gamma'$ *is an isomorphism*;

*b)* $\Gamma'$ *satisfies the conditions of Lemma* 1.1.9;

*c) the quotient* $E' \backslash X$ *exists and is isomorphic to the* $(E' \backslash E, L)$*-scheme corresponding to* $\Gamma'$.

*Proof.* — *a)* The composition map $\Gamma \xrightarrow{\varphi} \Gamma' \xrightarrow{pr_G} G$ is injective, therefore $\varphi$ is an isomorphism and $pr_G : \Gamma' \to G$ is injective.

*b)* $\Gamma'$ is clearly cocompact. Let $U \times S \subset G \times (E' \backslash E)$ be an open neighbourhood of the identity with a compact closure. Then $\varphi^{-1}(U \times S)$ is an open neighbourhood of the identity of $G \times E$ with a compact closure. It follows that $\varphi^{-1}(U \times S) \cap \Gamma$ is finite, thus $(U \times S) \cap \Gamma'$ is also finite. Hence $\Gamma'$ is discrete.

*c)* Since $E'$ is compact and normal, we have $E' S = SE' \in \mathscr{F}(E)$ for each $S \in \mathscr{F}(E)$. Hence $E' \backslash X := \varprojlim_S X_{E' S}$ is the required quotient. Next we notice that for each $S \in \mathscr{F}(E)$ the subgroup $\overline{S} := S \backslash E' S$ belongs to $\mathscr{F}(E' \backslash E)$ and that each $T \in \mathscr{F}(E' \backslash E)$ is of this form. Since $X_{E' S}^{an} \cong E' S \backslash [X^0 \times E]/\Gamma \cong \overline{S} \backslash [X^0 \times (E' \backslash E)]/\Gamma'$, we are done. □

### 1.4. Drinfel'd's covers

**1.4.1.** Now we need to recall some Drinfel'd's results [Dr2] concerning covers of $\Omega_{K_w}^d$. (A detailed treatment is given in [BC] for $d = 2$ and in [RZ2] for the general case.)

Let $K_w$ be as before and let $D_w$ be a central skew field over $K_w$ with invariant $1/d$. Let $\mathcal{O}_{D_w} \subset D_w$ be the ring of integers. Fix a maximal commutative subfield $K_w^{(d)}$ of $D_w$ unramified over $K_w$. Let $\pi \in K_w$ be a uniformizer and let $Fr_w$ be the Frobenius automorphism of $K_w^{(d)}$ over $K_w$. Then $D_w$ is generated by $K_w$ and an element $\Pi$ with the following defining relations: $\Pi^d = \pi$, $\Pi \cdot a = Fr_w(a) \cdot \Pi$ for each $a \in K_w^{(d)}$.

Denote by $\widehat{\mathcal{O}}_w^{nr}$ the ring of integers of the completion of the maximal unramified extension $\widehat{K}_w^{nr}$ of $K_w$. Drinfel'd had constructed a commutative formal group Y over $\widehat{\Omega}_{K_w}^d \widehat{\otimes}_{\mathcal{O}_w} \widehat{\mathcal{O}}_w^{nr}$ with an action of $\mathcal{O}_{D_w}$ on it. For a natural number *n* denote by $\Gamma_n$ the kernel of the homomorphism $Y \xrightarrow{\pi^n} Y$. Let $\mathscr{X}_n := \Gamma_n \widehat{\otimes}_{\mathcal{O}_w} K_w$ be the generic fiber of $\Gamma_n$ and let $\mathscr{X}_{n-1/d} \subset \mathscr{X}_n$ be the kernel of $\Pi^{nd-1} (= \pi^{n-1/d})$.

Put $\Sigma^{d,n}_{\mathrm{K}_w} := \mathscr{X}_n - \mathscr{X}_{n-1/d}$, and set $\mathrm{T}_n := 1 + \pi^n \mathcal{O}_{\mathrm{D}_w} \in \mathscr{F}(\mathrm{D}^\times_w)$. Then $\Sigma^{d,n}_{\mathrm{K}_w}$ is an étale Galois covering of $\Sigma^{d,0}_{\mathrm{K}_w} := \Omega^d_{\mathrm{K}_w} \hat{\otimes}_{\mathrm{K}_w} \hat{\mathrm{K}}^{\mathrm{nr}}_w$ with Galois group $(\mathcal{O}_{\mathrm{D}_w}/\pi^n)^\times \cong \mathcal{O}^\times_{\mathrm{D}_w}/\mathrm{T}_n$. We also denote $\mathcal{O}^\times_{\mathrm{D}_w}$ by $\mathrm{T}_0$. The action of $\pi$ induces étale covering maps $\pi_n : \Sigma^{d,n}_{\mathrm{K}_w} \to \Sigma^{d,n-1}_{\mathrm{K}_w}$, giving a $\hat{\mathrm{K}}^{\mathrm{nr}}_w$-pro-analytic space $\Sigma^d_{\mathrm{K}_w} := \{ \Sigma^{d,n}_{\mathrm{K}_w} \}_n$. The group $\mathcal{O}^\times_{\mathrm{D}_w}$ acts naturally on $\Sigma^d_{\mathrm{K}_w}$, and we have $\Sigma^{d,n}_{\mathrm{K}_w} \cong \mathrm{T}_n \backslash \Sigma^d_{\mathrm{K}_w}$ for each $n \in \mathbf{N} \cup \{ 0 \}$. Moreover, Drinfel'd had also constructed an action of the group $\mathrm{GL}_d(\mathrm{K}_w) \times \mathrm{D}^\times_w$ on $\Sigma^d_{\mathrm{K}_w}$, viewed as a pro-analytic space over $\mathrm{K}_w$, which extends the action of $\mathcal{O}^\times_{\mathrm{D}_w}$ and satisfies the following properties (notice that our convention 1.3.1 differ from those of Drinfel'd):

*a)* the diagonal subgroup $\{ (k, k) \in \mathrm{GL}_d(\mathrm{K}_w) \times \mathrm{D}^\times_w \mid k \in \mathrm{K}^\times_w \}$ acts trivially;

*b)* $\mathrm{GL}_d(\mathrm{K}_w)$ (resp. $\mathrm{D}^\times_w$) acts on $\Sigma^{d,0}_{\mathrm{K}_w} = \Omega^d_{\mathrm{K}_w} \hat{\otimes}_{\mathrm{K}_w} \hat{\mathrm{K}}^{\mathrm{nr}}_w$ by the product of the natural action of $\mathrm{PGL}_d(\mathrm{K}_w)$ on $\Omega^d_{\mathrm{K}_w}$ (resp. the trivial action on $\Omega^d_{\mathrm{K}_w}$) and the Galois action $g \mapsto \mathrm{Fr}_w^{\mathrm{val}_w(\det(g))}$ (resp. $g \mapsto \mathrm{Fr}_w^{-\mathrm{val}_w(\det(g))}$) on $\hat{\mathrm{K}}^{\mathrm{nr}}_w$.

**1.4.2.** In the case $d = 1$ Drinfel'd's coverings can be described explicitly. Let L be a $p$-adic field. Then, by property *a)* above, the action of $\mathrm{L}^\times \times \mathrm{L}^\times$ on $\Sigma^1_{\mathrm{L}}$ is determined uniquely by its restriction to the second factor. Denote by $\theta_{\mathrm{L}} : \mathrm{L}^\times \to \mathrm{Gal}(\mathrm{L}^{\mathrm{ab}}/\mathrm{L})$ the Artin homomorphism (sending the uniformizer to the arithmetic Frobenius automorphism).

*Lemma* **1.4.3.** — *One has* $\Sigma^1_{\mathrm{L}} \cong \mathscr{M}(\hat{\mathrm{L}}^{\mathrm{ab}})$, *and the action of* $(1, l) \in \{ 1 \} \times \mathrm{L}^\times$ *on* $\Sigma^1_{\mathrm{L}}$ *is given by the action of* $\theta_{\mathrm{L}}(l)^{-1} \in \mathrm{Gal}(\mathrm{L}^{\mathrm{ab}}/\mathrm{L})$ *on* $\mathrm{L}^{\mathrm{ab}}$.

*Proof.* — This follows from the fact that Drinfel'd's construction for $d = 1$ is equivalent to the construction of Lubin-Tate of the maximal abelian extension of L (see, for example, [CF, Ch. VI, § 3]). □

**1.4.4.** Let L be an extension of $\mathrm{K}_w$ of degree $d$ and of ramification index $e$. For every embeddings $\mathrm{L} \hookrightarrow \mathrm{Mat}_d(\mathrm{K}_w)$, $\mathrm{L} \hookrightarrow \mathrm{D}_w$ (such exist by [CF, Ch. VI, § 1, App.]) and $\mathrm{K}^{\mathrm{nr}}_w \hookrightarrow \mathrm{L}^{\mathrm{nr}}$ and for every $n \in \mathbf{N} \cup \{ 0 \}$ there exists a closed L-rational embedding $i_n : \Sigma^{1,en}_{\mathrm{L}} \hookrightarrow \Sigma^{d,n}_{\mathrm{K}_w}$, which is $(\mathrm{L}^\times \times \mathrm{L}^\times)$-equivariant and commutes with the projections $\pi_n$. Moreover, $i_0 : \Omega^1_{\mathrm{L}} \hat{\otimes}_{\mathrm{L}} \hat{\mathrm{L}}^{\mathrm{nr}} \hookrightarrow \Omega^d_{\mathrm{K}_w} \hat{\otimes}_{\mathrm{K}_w} \hat{\mathrm{K}}^{\mathrm{nr}}_w$ is the product of our embedding $\hat{\mathrm{K}}^{\mathrm{nr}}_w \hookrightarrow \hat{\mathrm{L}}^{\mathrm{nr}}$ and a closed embedding $i : \Omega^1_{\mathrm{L}} \hookrightarrow \Omega^d_{\mathrm{K}_w}$, with image $(\Omega^d_{\mathrm{K}_w})^{\mathrm{L}^\times}$ (see [Dr2, Prop. 3.1]). Taking an inverse limit we obtain an embedding $\tilde{\imath} : \Sigma^1_{\mathrm{L}} \hookrightarrow \Sigma^d_{\mathrm{K}_w}$.

*Lemma* **1.4.5.** — *Let* H *be a subgroup of* $\mathrm{R}_{\mathrm{L}/\mathrm{K}_w}(\mathbf{G}_m)(\mathrm{K}_w) \cong \mathrm{L}^\times$, *Zariski dense in* $\mathrm{R}_{\mathrm{L}/\mathrm{K}_w}(\mathbf{G}_m)$. *Then* $\mathrm{Im}\, \tilde{\imath} = \{ x \in \Sigma^d_{\mathrm{K}_w} \mid (l, l)\, x = x$ *for every* $l \in \mathrm{H} \}$.

*Proof.* — Since for each $l \in \mathrm{H} \subset \mathrm{L}^\times$ the action of $(l, l)$ on $\Sigma^1_{\mathrm{L}}$ is trivial, and since $\tilde{\imath}$ is $(\mathrm{L}^\times \times \mathrm{L}^\times)$-equivariant, $\mathrm{Im}\, \tilde{\imath}$ is contained in the set of fixed points of $(l, l)$, $l \in \mathrm{H}$.

Conversely, if $x \in \Sigma^d_{\mathrm{K}_w}$ is fixed by all $(l, l)$, $l \in \mathrm{H}$, then its image $\bar{x} \in \Omega^d_{\mathrm{K}_w}$ under the natural projection $\rho : \Sigma^d_{\mathrm{K}_w} \to \Omega^d_{\mathrm{K}_w}$ belongs to $(\Omega^d_{\mathrm{K}_w})^{\mathrm{H}} = (\Omega^d_{\mathrm{K}_w})^{\mathrm{L}^\times} = i(\Omega^1_{\mathrm{L}})$. Since $\rho(\mathrm{Im}\, \tilde{\imath}) = \mathrm{Im}\, i$, there exists $y \in \mathrm{Im}\, \tilde{\imath}$ such that $\rho(y) = \bar{x} (= \rho(x))$. Recall that $\Omega^d_{\mathrm{K}_w} = \mathrm{D}^\times_w \backslash \Sigma^d_{\mathrm{K}_w}$. Therefore $y = \delta x$ for some $\delta \in \mathrm{D}^\times_w$. It follows that $(l, \delta l \delta^{-1})\, y = y$

for each $l \in \mathrm{H}$, hence also $(1, \delta l \delta^{-1} l^{-1}) y = y$. Since the covering $\Sigma^{d}_{\mathrm{K}_w} \to \mathcal{O}^{\times}_{\mathrm{D}_w} \backslash \Sigma^{d}_{\mathrm{K}_w}$ is étale, the group $\mathcal{O}^{\times}_{\mathrm{D}_w}$ acts freely on $\Sigma^{d}_{\mathrm{K}_w}$. Therefore $\delta l \delta^{-1} l^{-1} = 1$ for each $l \in \mathrm{H}$. Hence $\delta$ belongs to the centralizer of H in $\mathrm{D}^{\times}_{w}$, so that to $\mathrm{L}^{\times}$. It follows that $x = \delta^{-1} y \in \mathrm{L}^{\times} \cdot \mathrm{Im}\, \tilde{i} = \mathrm{Im}\, \tilde{i}$. □

*Proposition* **1.4.6.** — *For each $n \in \mathbf{N} \cup \{0\}$ the group $\mathrm{SD}^{\times}_{w} \cap \mathrm{T}_1$ acts trivially on the set $\pi_0$ of connected components of $\Sigma^{d,n}_{\mathrm{K}_w} \hat{\otimes}_{\mathrm{K}_w} \mathbf{C}_p$.*

*Proof.* — Recall (see 1.4.4) that each maximal commutative subfield $\mathrm{L} \subset \mathrm{D}_w$ gives us (after some choices) a closed L-rational $\mathrm{L}^{\times}$-equivariant embedding $i_n : \Sigma^{1, en}_{\mathrm{L}} \to \Sigma^{d,n}_{\mathrm{K}_w}$. Let $\mathcal{Y}$ be a connected component of $\Sigma^{1, en}_{\mathrm{L}} \hat{\otimes}_{\mathrm{L}} \mathbf{C}_p$. Take $\mathcal{X} \in \pi_0$, which contains $i_n(\mathcal{Y})$. Then, by Lemma 1.4.3, $\mathcal{X}$ is defined over $\hat{\mathrm{L}}^{\mathrm{ab}}$ and

$$l(\mathcal{X}) = (\theta_{\mathrm{L}}(l))^{-1}(\mathcal{X}) \text{ for each } l \in \mathrm{L}^{\times}. \tag{1.1}$$

Fix a $\mathcal{X}_0 \in \pi_0$, and let M be the field of definition of $\mathcal{X}_0$. Then $\mathrm{M} \supset \hat{\mathrm{K}}^{\mathrm{nr}}_{w}$. Since the quotient $\mathrm{D}^{\times}_{w} \backslash \Sigma^{d,n}_{\mathrm{K}_w} \cong \Omega^{d}_{\mathrm{K}_w}$ is geometrically connected, $\mathrm{D}^{\times}_{w}$ acts transitively on $\pi_0$. Since the action of $\mathrm{D}^{\times}_{w}$ on $\pi_0$ is $\mathrm{K}_w$-rational, M is the field of definition of every $\mathcal{X} \in \pi_0$. In particular, M is the closure of a Galois extension of $\mathrm{K}_w$, and $\mathrm{M} \subset \hat{\mathrm{L}}^{\mathrm{ab}}$ for every extension L of $\mathrm{K}_w$ of degree $d$. Taking L be unramified we see that the group $\mathrm{Aut}^{\mathrm{cont}}_{\mathrm{K}_w}(\mathrm{M})$ of continuous automorphisms of M over $\mathrm{K}_w$ is meta-abelian (= extension of two abelian groups). Set $\mathrm{H} := \{ \delta \in \mathrm{D}^{\times}_{w} \mid \text{there exists a } \sigma(\delta) \in \mathrm{Aut}^{\mathrm{cont}}_{\mathrm{K}_w}(\mathrm{M}) \text{ such that } \delta(\mathcal{X}_0) = \sigma(\delta)^{-1}(\mathcal{X}_0) \}$. Then H is a group and $\sigma : \mathrm{H} \to \mathrm{Aut}^{\mathrm{cont}}_{\mathrm{K}_w}(\mathrm{M})$ is a well-defined homomorphism.

We claim that $\mathrm{H} = \mathrm{D}^{\times}_{w}$. Take a $\delta \in \mathrm{D}^{\times}_{w}$, then $\mathrm{K}_w[\delta]$ is a commutative subfield of $\mathrm{D}_w$. Let L be a maximal commutative subfield of $\mathrm{D}_w$ containing $\delta$. Then by (1.1), $\delta(\mathcal{X}) = (\theta_{\mathrm{L}}(\delta))^{-1}(\mathcal{X})$ for some $\mathcal{X} \in \pi_0$. Take $\delta' \in \mathrm{D}^{\times}_{w}$ such that $\mathcal{X} = \delta'(\mathcal{X}_0)$. Then $(\delta')^{-1} \delta \delta'(\mathcal{X}_0) = (\delta')^{-1} \circ (\theta_{\mathrm{L}}(\delta))^{-1} \circ \delta'(\mathcal{X}_0) = (\theta_{\mathrm{L}}(\delta))^{-1}(\mathcal{X}_0)$, so that $(\delta')^{-1} \delta \delta' \in \mathrm{H}$. Thus each element of $\mathrm{D}^{\times}_{w}$ is conjugate to some element of H. In particular, $\mathrm{Z}(\mathrm{D}^{\times}_{w}) \subset \mathrm{H}$. Since $\mathrm{T}_n$ acts trivially on $\Sigma^{d,n}_{\mathrm{K}_w}$, it is also contained in H. Hence $\mathrm{H} \supset \mathrm{T}_n \cdot \mathrm{Z}(\mathrm{D}^{\times}_{w})$ has a finite index in $\mathrm{D}^{\times}_{w}$. Therefore our claim follows from the following

*Lemma* **1.4.7.** — *Let* G *be a group and let* H *be a subgroup of* G *of finite index. Suppose that* $\mathrm{G} = \bigcup_{g \in \mathrm{G/H}} g\mathrm{H}g^{-1}$. *Then* $\mathrm{G} = \mathrm{H}$.

*Proof.* — Set $\mathrm{K} := \bigcap_{g \in \mathrm{G}} g\mathrm{H}g^{-1}$. Then K is a normal subgroup of G of finite index, and $\mathrm{G/K} = \bigcup_{g \in \mathrm{G/H}} g(\mathrm{H/K})\, g^{-1} = \bigcup_{g \in \mathrm{G/H}} (g(\mathrm{H/K})\, g^{-1} - \{1\}) \cup \{1\}$. Hence $| \mathrm{G/K} | \leqslant | \mathrm{G/H} | \, (| \mathrm{H/K} | - 1) + 1 = | \mathrm{G/K} | - | \mathrm{G/H} | + 1$, therefore $\mathrm{G} = \mathrm{H}$. □

Now the proposition follows from the fact that $\mathrm{SD}^{\times}_{w}$ is the derived group of $\mathrm{D}^{\times}_{w}$ (see [PR, 1.4.3]) and that $\mathrm{T}_1 \cap \mathrm{SD}^{\times}_{w}$ is the derived group of $\mathrm{SD}^{\times}_{w}$ (see [PR, 1.4.4, Thm. 1.9]). □

### 1.5. Second construction

*Construction* **1.5.1.** — Suppose that a subgroup $\Gamma \subset \mathrm{GL}_d(\mathrm{K}_w) \times \mathrm{E}$ satisfies the following conditions:

*a)* $\mathrm{Z}(\Gamma) = \mathrm{Z}(\mathrm{GL}_d(\mathrm{K}_w) \times \mathrm{E}) \cap \Gamma$;

*b)* the subgroup $\overline{\mathrm{Z}(\Gamma)} \subset \mathrm{Z}(\mathrm{GL}_d(\mathrm{K}_w) \times \mathrm{E})$ is cocompact;

*c)* $\mathrm{P}\Gamma \subset \mathrm{PGL}_d(\mathrm{K}_w) \times \mathrm{PE}$ satisfies the assumptions of Lemma 1.1.9 (this imply, in particular, that the closure of $\Gamma$ is cocompact in $\mathrm{GL}_d(\mathrm{K}_w) \times \mathrm{E}$);

*d)* the intersection of $\mathrm{Z}(\Gamma)$ with $\mathrm{Z}(\mathrm{GL}_d(\mathrm{K}_w)) \times \{ 1 \}$ is trivial.

We are going to associate to $\Gamma$ a certain $(\mathrm{D}_w^\times \times \mathrm{E}, \mathrm{K}_w)$-scheme.

Consider the quotient $\widetilde{\mathrm{X}} := (\Sigma^d_{\mathrm{K}_w} \times \mathrm{E})/\Gamma$. The group $\mathrm{D}_w^\times \times \mathrm{E}$ acts on $\widetilde{\mathrm{X}}$ by the product of the natural action of $\mathrm{D}_w^\times$ on $\Sigma^d_{\mathrm{K}_w}$ and the left multiplication by E.

*Proposition* **1.5.2.** — *For each* $\mathrm{S} \in \mathscr{F}(\mathrm{D}_w^\times \times \mathrm{E})$ *the quotient* $\mathrm{S}\backslash\widetilde{\mathrm{X}} = \mathrm{S}\backslash(\Sigma^d_{\mathrm{K}_w} \times \mathrm{E})/\Gamma$ *has a natural structure of a* $\mathrm{K}_w$*-analytic space, which has a unique structure* $\mathrm{X}_\mathrm{S}$ *of a projective scheme over* $\mathrm{K}_w$.

*Proof.* — First take $\mathrm{S} = \mathrm{T}_n \times \mathrm{S}'$ for some $n \in \mathbf{N} \cup \{ 0 \}$ and some sufficiently small $\mathrm{S}' \in \mathscr{F}(\mathrm{F})$ (to be specified later). Then $\mathrm{S}\backslash\widetilde{\mathrm{X}} = \mathrm{S}'\backslash(\Sigma^{d,n}_{\mathrm{K}_w} \times \mathrm{E})/\Gamma$ is a disjoint union of $| \mathrm{S}\backslash\mathrm{E}/\Gamma_\mathrm{E} | < \infty$ (as in Lemma 1.1.9) quotients of the form $\Gamma_{a\mathrm{S}'a^{-1}}\backslash\Sigma^{d,n}_{\mathrm{K}_w}$ with $a \in \mathrm{E}$. Thus it remains to prove the statement for quotients $\Gamma_{a\mathrm{S}'a^{-1}}\backslash\Sigma^{d,n}_{\mathrm{K}_w}$. For simplicity of notation we assume that $a = 1$. Set

$$\Gamma_{\mathrm{S}',0} := \Gamma_{\mathrm{S}'} \cap \mathrm{pr}_\mathrm{G}(\mathrm{Z}(\Gamma)) = \mathrm{pr}_\mathrm{G}(\Gamma \cap (\mathrm{Z}(\mathrm{GL}_d(\mathrm{K}_w)) \times (\mathrm{Z}(\mathrm{E}) \cap \mathrm{S}'))).$$

First we construct the quotient $\Gamma_{\mathrm{S}',0}\backslash\Sigma^{d,n}_{\mathrm{K}_w}$. Assumptions *a)* and *b)* of 1.5.1 imply that the closure of $\Gamma_{\mathrm{S}',0}$ is cocompact in $\mathrm{Z}(\mathrm{GL}_d(\mathrm{K}_w)) \cong \mathrm{K}_w^\times$, hence $\mathrm{val}_w(\det(\Gamma_{\mathrm{S}',0})) = dk\,\mathbf{Z}$ for some $k \in \mathbf{N}$. Let $\mathrm{K}_w^{(dk)}$ be the unique unramified extension of $\mathrm{K}_w$ of degree $dk$; then $\Gamma_{\mathrm{S}',0}\backslash\Omega^d_{\mathrm{K}_w} \widehat{\otimes}_{\mathrm{K}_w} \widehat{\mathrm{K}}_w^{\mathrm{nr}} \cong \Omega^d_{\mathrm{K}_w} \otimes_{\mathrm{K}_w} \mathrm{K}_w^{(dk)}$.

Consider the natural étale projection $\pi_n : \Sigma^{d,n}_{\mathrm{K}_w} \to \Sigma^{d,0}_{\mathrm{K}_w} \to \Omega^d_{\mathrm{K}_w}$. Let $\{ \mathscr{M}(\mathrm{A}_i) \}_{i \in \mathrm{I}}$ be an affinoid covering of $\Omega^d_{\mathrm{K}_w}$. Since the projection $\Sigma^{d,n}_{\mathrm{K}_w} \to \Sigma^{d,0}_{\mathrm{K}_w}$ is finite, each $\pi_n^{-1}(\mathscr{M}(\mathrm{A}_i)) \subset \Sigma^{d,n}_{\mathrm{K}_w}$ is finite over the affinoid space $\mathscr{M}(\mathrm{A}_i \widehat{\otimes}_{\mathrm{K}_w} \widehat{\mathrm{K}}_w^{\mathrm{nr}})$. Hence it is isomorphic to an affinoid space $\mathscr{M}(\mathrm{B}_i)$ for a certain $\widehat{\mathrm{K}}_w^{\mathrm{nr}}$-affinoid algebra $\mathrm{B}_i$, finite over $\mathrm{A}_i \widehat{\otimes}_{\mathrm{K}_w} \widehat{\mathrm{K}}_w^{\mathrm{nr}}$. Since $\pi_n$ is $\mathrm{D}_w^\times$-invariant, we have a natural action of $\Gamma_{\mathrm{S}',0}$ on $\mathrm{B}_i$. Set $\mathrm{C}_i := \mathrm{B}_i^{\Gamma_{\mathrm{S}',0}}$. Since an affinoid algebra is noetherian, we see that $\mathrm{C}_i$ is finite over the $\mathrm{K}_w$-affinoid algebra $\mathrm{A}_i$. Hence $\mathrm{C}_i$ has a canonical structure of a $\mathrm{K}_w$-affinoid algebra (see [Bel, Prop. 2.1.12]). Gluing together the $\mathscr{M}(\mathrm{C}_i)$'s, we obtain a $\mathrm{K}_w$-analytic space $\Gamma_{\mathrm{S}',0}\backslash\Sigma^{d,n}_{\mathrm{K}_w}$, finite and étale over $\Omega^d_{\mathrm{K}_w}$.

Put $\overline{\mathrm{S}} := \mathrm{S}\cdot\mathrm{Z}(\mathrm{E})/\mathrm{Z}(\mathrm{E}) \subset \mathrm{PE}$. Then $\overline{\mathrm{S}} \in \mathscr{F}(\mathrm{PE})$. To construct $\Gamma_{\mathrm{S}'}\backslash\Sigma^{d,n}_{\mathrm{K}_w}$ we observe that the action of $\mathrm{P}\Gamma_{\overline{\mathrm{S}}} = \Gamma_{\mathrm{S}',0}\backslash\Gamma_{\mathrm{S}'}$ on $\Gamma_{\mathrm{S}',0}\backslash\Sigma^{d,n}_{\mathrm{K}_w}$ covers its action on $\Omega^d_{\mathrm{K}_w}$. Suppose that $\mathrm{S}'$ is so small that $\overline{\mathrm{S}}$ satisfies part *d)* of Proposition 1.1.10. Recall that by Lemma 1.3.9 each $x \in \Omega^d_{K_w}$ has an open analytic neighbourhood $U_x$ such that $\gamma(U_x) \cap U_x \neq \emptyset$ for all $\gamma \in P\Gamma_{\overline{S}} - \{1\}$ and, as a consequence, $P\Gamma_{\overline{S}} \backslash \Omega^d_{K_w}$ is obtained by gluing the $U_x$'s. Let $\overline{\pi}_n$ be the natural projection from $\Gamma_{S',0} \backslash \Sigma^{d,n}_{K_w}$ to $\Omega^d_{K_w}$. For each $y \in \Gamma_{S',0} \backslash \Sigma^{d,n}_{K_w}$ set $V_y := \overline{\pi}_n^{-1}(U_{\overline{\pi}_n(x)})$. Then the quotient $K_w$-analytic space $P\Gamma_{\overline{S}} \backslash (\Gamma_{S',0} \backslash \Sigma^{d,n}_{K_w}) = \Gamma_{S'} \backslash \Sigma^{d,n}_{K_w}$ is obtained by gluing the $V_y$'s.

Since $\Gamma_{S'} \backslash \Sigma^{d,n}_{K_w}$ is a finite (and étale) covering of $P\Gamma_{\overline{S}} \backslash \Omega^d_{K_w}$, which has a structure of a projective scheme over $K_w$ by [Mus, Ku], $\Gamma_{S'} \backslash \Sigma^{d,n}_{K_w}$ also has such a structure by Corollary 1.2.3 and the remark following it.

Finally consider an arbitrary $S \in \mathscr{F}(D^\times_w \times E)$. It has a normal subgroup $\widetilde{S}$ of the form $\widetilde{S} = T_n \times S'$ with sufficiently small $S' \in \mathscr{F}(E)$, therefore to complete the proof we can use the same considerations as in the end of the proof of Proposition 1.3.7. □

The same argument as in Construction 1.3.6 gives us a $(D^\times_w \times E, K_w)$-scheme $X = \varprojlim_{S} X_S$.

*Proposition* **1.5.3.** — *a) The kernel* $E_0$ *of the action of* $D^\times_w \times E$ *on* $X$ *is the closure of the subgroup* $Z(\Gamma) \subset Z(GL_d(K_w) \times E) = Z(D^\times_w \times E)$ *after the natural identification* $Z(GL_d(K_w)) = K^\times_w = Z(D^\times_w)$.

*b) Let* $\widetilde{E}_0$ *be the closure of* $Z(\Gamma)$ *in* $E$, *and let* $\Gamma' \subset PGL_d(K_w) \times (\widetilde{E}_0 \backslash E)$ *be the image of* $\Gamma$ *under the natural projection. Then* $\Gamma'$ *satisfies the assumptions of Lemma* 1.1.9.

*c) The quotient* $D^\times_w \backslash X$ *exists and is isomorphic to the* $\widetilde{E}_0 \backslash E$*-scheme corresponding to* $\Gamma'$ *by Construction* 1.3.6.

*d) The quotient* $(D^\times_w \times Z(E)) \backslash X$ *exists and is isomorphic to the* $(PE, K_w)$*-scheme* $X'$ *corresponding to* $P\Gamma$ *by Construction* 1.3.6.

*e) For each* $x \in X$ *the orbit* $(D^\times_w \times E)x$ *is Zariski dense in* $X$.

*f) For each sufficiently small* $S \in \mathscr{F}(E)$ *and each* $n \in \mathbf{N} \cup \{0\}$ *the map* $X \to X_{T_n \times S}$ *is étale, and* $B^{d-1}$ *is the universal covering of each connected component of* $(X_{T_n \times S, C})^{an}$ *for each embedding* $K_w \hookrightarrow \mathbf{C}$. *In particular, the projective system* $X^{an} := \{X^{an}_T\}_{T \in \mathscr{F}(D^\times_w \times E)}$, *associated to* $X$, *is a* $K_w$*-pro-analytic space.*

*Proof.* — *a)* Notice that $g \in E_0$ if and only if $g$ acts trivially on $X_S$ (or, equivalently, on $X^{an}_S = S \backslash (\Sigma^d_{K_w} \times E)/\Gamma$) and normalizes $S$ for each $S \in \mathscr{F}(D^\times_w \times E)$. For each $\gamma \in Z(\Gamma) \subset Z(D^\times_w \times E)$ let $\gamma_w$ be the projection of $\gamma$ to the first factor. Since $(\gamma_G \times \gamma_w)$ acts trivially on $\Sigma^d_{K_w}$, we have $\gamma([x, e]) = [\gamma_w(x), \gamma_E e] \sim [(\gamma_G \times \gamma_w)(x), e] = [x, e]$ for each $x \in \Sigma^d_{K_w}$ and $e \in E$, that is $\gamma$ acts trivially on each $X^{an}_S$. Since $\gamma$ is central, it certainly normalizes $S$. This shows that the closure of $Z(\Gamma)$ is contained in $E_0$.

Conversely, suppose that some $(g_1, g_2) \in D^\times_w \times E$ with $g_1 \in D^\times_w$ and $g_2 \in E$ belongs to $E_0$. Choose $S' \in \mathscr{F}(E)$ and $n \in \mathbf{N} \cup \{0\}$. It suffice to show that $(g_1, g_2) \in (T_n \times S')\, Z(\Gamma)$. Since $(g_1, g_2)$ acts trivially on $S' \backslash (\Sigma^{d,n}_{K_w} \times E)/\Gamma$, we have $[g_1(x), g_2] \sim [x, 1]$ for each $x \in \Sigma^{d,n}_{K_w}$. This means that there exists an element $\gamma = \gamma_x \in \Gamma$ such that $g_1(x) = \gamma_G^{-1}(x)$ and $g_2 \in S' \gamma_E$. Let $x'$ be the projection of $x$ to $\Sigma^{d,0}_{K_w}$, and let $x''$

be its projection to $\Omega^d_{K_w}$. The group $D^\times_w$ acts trivially on $\Omega^d_{K_w}$, therefore $\gamma_G^{-1}(x'') = x''$. Choose $x$ so that no non-trivial element of $PGL_d(K_w)$ fixes $x''$, then $\gamma_G$ belongs to $Z(GL_d(K_w)) \cong K^\times_w$. Assumption *c)* of 1.5.1 implies that $\gamma \in Z(\Gamma)$. Since

$$g_1(x) = \gamma_G^{-1}(x) = \gamma_w(x),$$

we conclude that $g_1^{-1}\,\gamma_w(x) = x$. Hence $x' = (g_1^{-1}\,\gamma_w)\,(x') = Fr_w^{-\mathrm{val}(\det(g_1^{-1}\gamma_w))}(x')$, so that $g_1^{-1}\,\gamma_w \in \mathcal{O}^\times_{D_w}$. Since $\Sigma^{d,n}_{K_w}$ is an étale Galois covering of $\Sigma^{d,0}_{K_w}$ with Galois group $\mathcal{O}^\times_{D_w}/T_n$, the equality $(g_1^{-1}\,\gamma_w)\,(x) = x$ implies that $g_1^{-1}\,\gamma_w \in T_n$. It follows that $(g_1, g_2) \in (T_n \times S')\,(\gamma_w, \gamma_E) \subset (T_n \times S')\,Z(\Gamma)$, as claimed.

*b)* The natural projection $PGL_d(K_w) \times (\widetilde{E}_0\backslash E) \to PGL_d(K_w) \times PE$ induces an isomorphism $\Gamma' \xrightarrow{\sim} P\Gamma$. Hence $\Gamma'$ is discrete and has injective projection to $PGL_d(K_w)$. It is cocompact, because so is $\Gamma \subset PGL_d(K_w) \times E$.

*c)* Notice first that for each open subgroup $E_0 \subset S \subset D^\times_w \times E$, compact modulo $E_0$, the quotient $S\backslash X$ exists and is projective. Assumption *b)* of 1.5.1 implies that $E_0$ is cocompact in $D^\times_w \times Z(E)$. Therefore for each $S \in \mathcal{F}(D^\times_w \times E)$ the quotient $D^\times_w\, S\backslash X = (D^\times_w\, E_0\, S)\backslash X = (D^\times_w \times \widetilde{E}_0)\, S\backslash X$ exists. Set

$$\overline{S} := (D^\times_w \times \widetilde{E}_0)\backslash(D^\times_w \times \widetilde{E}_0)\, S \in \mathcal{F}(\widetilde{E}_0\backslash E).$$

Then $(D^\times_w\, S\backslash X)^{\mathrm{an}} \cong (D^\times_w \times \widetilde{E}_0)\, S\backslash[\Sigma^d_{K_w} \times E]/\Gamma \cong \overline{S}\backslash[\Omega^d_{K_w} \times (\widetilde{E}_0\backslash E)]/\Gamma'$, and the statement follows as in the proof of Lemma 1.3.11 *c)*.

*d)* follows from *c)* and Lemma 1.3.11 *c)*.

*e)* follows from *c)* and Proposition 1.3.8 *e)*.

*f)* Take $T \in \mathcal{F}(PE)$ satisfying part *d)* of Proposition 1.1.10. Then there exists $S \in \mathcal{F}(E)$ such that $Z(E)\backslash S\cdot Z(E) = T$. Since we have shown in the proof of Proposition 1.5.2 that $X_{T_n \times S}$ is étale over $T\backslash X'$ for each $n \in \mathbf{N} \cup \{\,0\,\}$, the statement follows immediately from Proposition 1.3.8 *f)*, *g)*. □

*Corollary* **1.5.4.** — *For each $a \in E$ the composition map*

$$\rho_a : \Sigma^d_{K_w} \xrightarrow{\sim} \Sigma^d_{K_w} \times \{\,a\,\} \hookrightarrow (\Sigma^d_{K_w} \times E^{\mathrm{disc}})/\Gamma \to X^{\mathrm{an}}$$

*of pro-analytic spaces over $K_w$ is étale and one-to-one.*

*Proof.* — The étaleness is clear. Let $x_1$ and $x_2$ be points of $\Sigma^d_{K_w}$ such that $\rho_a(x_1) = \rho_a(x_2)$. Let $\bar{a} \in PE'$ be projection of $a$, and let $\rho_{\bar{a}}$ be the injection

$$\Omega^d_{K_w} \xrightarrow{\sim} \Omega^d_{K_w} \times \{\,\bar{a}\,\} \hookrightarrow (\Omega^d_{K_w} \times (PE')^{\mathrm{disc}})/P\Gamma \xrightarrow{\sim} (X')^{\mathrm{an}}.$$

Then we conclude from the commutative diagram

$$\begin{tikzcd}
\Sigma^{d,n}_{K_w} \arrow[r, "\rho_a"] \arrow[d, "\mathrm{proj}"'] & X^{\mathrm{an}} \arrow[d, "\mathrm{proj}"] \\
\Omega^d_{K_w} \arrow[r, "\rho_{\bar{a}}"'] & (X')^{\mathrm{an}}
\end{tikzcd}$$

that $x_1$ and $x_2$ have the same projection $y \in \Omega^d_{K_w}$.

Choose $S \in \mathscr{F}(E)$ so small that the group $P\Gamma_{aSa^{-1}}$ is torsion-free (use Proposition 1.1.10 *d)*). Then no non-central element of $\Gamma_{aSa^{-1}}$ fixes $y$. For each $n \in \mathbf{N}$ let $\pi_{n,S}$ be the projection $X^{an} \to (X_{T_n \times S})^{an}$. Then the image of $\pi_{n,S} \circ \rho_a$ is isomorphic to $\Gamma_{aSa^{-1}} \backslash \Sigma_{K_w}^{d,n}$. Hence there exists $\gamma_n \in \Gamma_{aSa^{-1}}$ such that the projections of $\gamma_n(x_1)$ and $x_2$ to $\Sigma_{K_w}^{d,n}$ coincide. Therefore $\gamma_n(y) = y$, so that $\gamma_n \in Z(GL_d(K_w)) = K_w^{\times}$. It follows that the sequence $\{\gamma_n\}_n$ converges to some $\gamma \in K_w^{\times}$, which satisfies $\gamma(x_1) = x_2$. Then $(\gamma, 1) \in Z(D_w^{\times} \times E)$ fixes $z := \rho_a(x_1) = \rho_a(x_2)$. Since $(\gamma, 1)$ is central, it then fixes the whole $(D_w^{\times} \times E)$-orbit of $z$. Hence, by Proposition 1.5.3 *c)*, it acts trivially on X. Therefore by Proposition 1.5.3 *a)*, the element $(\gamma, 1)$ belongs to $\overline{Z(\Gamma)} \subset Z(GL_d(K_w) \times E)$. Assumption *d)* of 1.5.1 implies that $\gamma = 1$, hence $x_1 = x_2$. □

## 1.6. Relation between the *p*-adic and the real constructions

The following proposition (and its proof) is a modification of Ihara's theorem (see [Ch2, Prop. 1.3]). It will allow us to establish the connection between the *p*-adic (1.3.6, 1.5.1) and the real (or complex) (1.3.6) constructions.

*Proposition* **1.6.1.** — *Let* X *be an* (E, **C**)*-scheme. Suppose that*

*a)* E *acts faithfully on* X;

*b)* E *acts transitively on the set of connected components of* X;

*c) there exists* $S \in \mathscr{F}(E)$ *such that the projection* $X \to X_S$ *is étale, and* $B^{d-1}$ *is the universal covering of each connected component of* $X_S^{an}$.

*Then* X *can be obtained from the real case of Construction* 1.3.6.

*Remark* **1.6.2.** — *a)* It follows from Proposition 1.3.8 that all the above conditions are necessary.

*b)* Let X be an (E, **C**)-scheme and let $E_0$ be the kernel of the action of E on X. Then X is an $(E_0\backslash E, \mathbf{C})$-scheme with a faithful action of $E_0\backslash E$. Conversely, any $(E_0\backslash E, \mathbf{C})$-scheme can be viewed as an (E, **C**)-scheme with a trivial action of $E_0$.

*c)* Let X be an (E, **C**)-scheme and let $X_0$ be a connected component of X. Put $X' := \bigcup_{g \in E} g(X_0)$. Then X′ is an (E, **C**)-scheme with a transitive action of E on the set of its connected components, and X is a disjoint union of such (E, **C**)-schemes.

Remarks *b)* and *c)* show that assumptions *a)* and *b)* of the proposition are not so restrictive.

*Proof.* — We start the proof with the following

*Lemma* **1.6.3.** — *Suppose that* $\{X_\alpha\}_{\alpha \in I}$ *is a projective system of complex manifolds such that the transition maps* $X_\beta \to X_\alpha$, *where* $\alpha, \beta \in I$ *with* $\beta \geqslant \alpha$, *are analytic coverings. Then there exists a projective limit* X *of the* $X_\alpha$*'s in the category of complex manifolds.*

*Proof.* — Choose an $\alpha \in \mathrm{I}$. Cover $\mathrm{X}_\alpha$ by open balls $\{ \mathrm{U}_{\alpha\beta} \}_{\beta \in \mathrm{J}}$, and let $\pi : \mathrm{X}' \to \mathrm{X}_\alpha$ be an analytic covering. Then the inverse image $\pi^{-1}(\mathrm{U}_{\alpha\beta})$ of each $\mathrm{U}_{\alpha\beta}$ is a disjoint union of analytic spaces, each of them isomorphic to $\mathrm{U}_{\alpha\beta}$ under $\pi$. Hence the construction of the projective limit from the proof of Proposition 1.3.8, *a)* can be applied. □

Now we return to the proof of the proposition. By assumption *c)*, $\mathrm{X}_\mathrm{S}^{\mathrm{an}}$ is a complex manifold for each sufficiently small $\mathrm{S} \in \mathscr{F}(\mathrm{E})$, and the natural covering $\mathrm{X}_\mathrm{S}^{\mathrm{an}} \to \mathrm{X}_\mathrm{T}^{\mathrm{an}}$ is étale (analytic) for each $\mathrm{T} \subset \mathrm{S}$ in $\mathscr{F}(\mathrm{E})$. Therefore by the lemma there exists an analytic space $\mathrm{X}^{\mathrm{an}} := \varprojlim_{\mathrm{S}} \mathrm{X}_\mathrm{S}^{\mathrm{an}}$.

Since $\mathrm{X}_\mathrm{S}$ is a complex projective scheme for each $\mathrm{S} \in \mathscr{F}(\mathrm{E})$, the set of its connected components coincides with the set of connected components of $\mathrm{X}_\mathrm{S}^{\mathrm{an}}$. Hence assumption *b)* implies that the group E acts transitively on the set of connected components of $\mathrm{X}^{\mathrm{an}}$.

Let M be a connected component of $\mathrm{X}^{\mathrm{an}}$. Denote by $\Gamma_\mathrm{E}$ the stabilizer of M in E. Then $\Gamma_\mathrm{E}$ acts naturally on M, and the transitivity statement above implies that $\mathrm{X}^{\mathrm{an}} \cong (\mathrm{M} \times \mathrm{E}^{\mathrm{disc}})/\Gamma_\mathrm{E}$.

For each $\mathrm{S} \in \mathscr{F}(\mathrm{E})$ the analytic space $\mathrm{X}_\mathrm{S}^{\mathrm{an}} \cong \mathrm{S}\backslash(\mathrm{M} \times \mathrm{E})/\Gamma_\mathrm{E}$ is compact. Therefore, as in the proof of Lemma 1.1.9, $| \mathrm{S}\backslash\mathrm{E}/\Gamma_\mathrm{E} | < \infty$ and $[\Gamma_\mathrm{E} : \Gamma_\mathrm{E} \cap \mathrm{S}] = \infty$. Note that $\mathrm{M}_\mathrm{S} := (\Gamma_\mathrm{E} \cap \mathrm{S})\backslash\mathrm{M}$ is a connected component of $\mathrm{X}_\mathrm{S}^{\mathrm{an}}$. Suppose that S satisfies condition *c)*; then the map $\mathrm{M} \to \mathrm{M}_\mathrm{S}$ is étale and $\mathrm{B}^{d-1}$ is the universal covering of $\mathrm{M}_\mathrm{S}$. Hence it is also the universal covering of M. It follows that $\Gamma_\mathrm{E} \subset \mathrm{Aut}(\mathrm{M})$ can be lifted to $\Gamma_\mathbf{R} \subset \mathrm{Aut}(\mathrm{B}^{d-1}) = \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$.

The kernel $\Delta$ of the natural homomorphism $\pi : \Gamma_\mathbf{R} \to \Gamma_\mathrm{E}$ is the fundamental group of M. Let $\Gamma_\mathrm{S} \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ be the fundamental group of the compact analytic space $\mathrm{M}_\mathrm{S}$, then $\Gamma_\mathrm{S}$ is a cocompact lattice in $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$, satisfying $\Gamma_\mathrm{S} = \pi^{-1}(\Gamma_\mathrm{E} \cap \mathrm{S})$. It follows that $[\Gamma_\mathbf{R} : \Gamma_\mathrm{S}] = [\Gamma_\mathrm{E} : \Gamma_\mathrm{E} \cap \mathrm{S}] = \infty$. Therefore, as in the proof of Proposition 1.1.10 *a)*, we see that $\Gamma_\mathbf{R}$ is dense in $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$. The group $\Delta$ is discrete in $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ and normal in $\Gamma_\mathbf{R}$, thus it is trivial (compare the proof of Proposition 1.1.10 *b)*). In particular, $\mathrm{M} \cong \mathrm{B}^{d-1}$ and $\pi$ is an isomorphism.

Put $\Gamma := \{ (\gamma, \pi(\gamma)) \mid \gamma \in \Gamma_\mathbf{R} \} \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{E}$. Since $\Gamma_\mathrm{S}$ is discrete in $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$, so is $\Gamma$ in $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{E}$. Let $\mathrm{K} \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ be the stabilizer of $0 \in \mathrm{B}^{d-1}$. Then $\mathrm{X}_\mathrm{S}^{\mathrm{an}} \cong \mathrm{S}\backslash(\mathrm{B}^{d-1} \times \mathrm{E})/\Gamma = (\mathrm{K} \times \mathrm{S})\backslash(\mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{E})/\Gamma$. Since K, S and $\mathrm{X}_\mathrm{S}^{\mathrm{an}}$ are compact, $\Gamma$ is cocompact in $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{E}$. Since $\mathrm{Ker}(\mathrm{pr}_\mathrm{G})$ equals the kernel of the action of E on X, the projection $\mathrm{pr}_\mathrm{G}$ is injective. This shows that $\Gamma$ satisfies all the assumptions of Construction 1.3.6. □

*Corollary* **1.6.4.** — *Choose an embedding* $\mathrm{K}_w \hookrightarrow \mathbf{C}$. *Let* X *be an* $(\mathrm{E}, \mathrm{K}_w)$*-scheme obtained by the p-adic case of Construction* 1.3.6 *or an* $(\widetilde{\mathrm{E}}, \mathrm{K}_w)$*-scheme obtained by Construction* 1.5.1. *Then* $\mathrm{X}_\mathbf{C}$ *can be constructed by the real case of Construction* 1.3.6.

*Proof.* — This is an immediate consequence of Propositions 1.6.1, 1.3.8 and 1.5.3. □

## 1.7. Elliptic elements

*Definition* **1.7.1.** — Suppose that a group G acts on a (pro-)analytic space (or a scheme) X. An element $g \in G$ is called *elliptic* if it has a fixed point $x$ such that the linear transformation of the tangent space of $x$, induced by $g$, has no non-zero fixed vectors. In such a situation we call $x$ an *elliptic point* of $g$.

*Lemma* **1.7.2.** — *Let* $\lambda_1, \lambda_2, \dots, \lambda_d$ *be the eigenvalues of some element* $g \in \mathrm{GL}_d(\mathrm{L})$ *(with multiplicities). Let* $v \in \mathbf{P}^{d-1}(\overline{\mathrm{L}})$ *be one of the fixed points of* $g$ *corresponding to* $\lambda_1$. *Then* $\frac{\lambda_2}{\lambda_1}, \frac{\lambda_3}{\lambda_1}, \dots, \frac{\lambda_d}{\lambda_1}$ *are the eigenvalues of the linear transformation of the tangent space of* $v$, *induced by* $g$.

*Proof.* — Simple verification. □

*Proposition* **1.7.3.** — *The set of elliptic elements of* $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ *with respect to its action on* $\mathrm{B}^{d-1}$ *and of* $\mathrm{PSL}_d(\mathrm{K}_w)$ *with respect to its action on* $\Omega^d_{\mathrm{K}_w}$ *is open and non-empty.*

*Proof.* — In the real case we observe that an element

$$g := \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_d) \in \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$$

fixes $(0, 0, \dots, 0) \in \mathrm{B}^{d-1}$. Therefore by Lemma 1.7.2, $g$ is elliptic if $\lambda_i \neq \lambda_d$ for all $i \neq d$. It follows that the set of elliptic elements is non-empty. It is open, because if $g$ has a fixed point in $\mathrm{B}^{d-1}$ corresponding to an eigenvalue of $g$ appearing with multiplicity 1, then the same is true in some open neighbourhood of $g$.

In the $p$-adic case we start with the following

*Lemma* **1.7.4.** — *An element* $g \in \mathrm{GL}_d(\mathrm{K}_w)$ *is elliptic (acting on* $\Omega^d_{\mathrm{K}_w}$*) if and only if its characteristic polynomial is irreducible over* $\mathrm{K}_w$.

*Proof.* — Suppose that the characteristic polynomial $\chi_g$ of $g$ is irreducible over $\mathrm{K}_w$. Then $g$ has $d$ distinct eigenvalues. Let $\lambda$ be some eigenvalue of $g$, let $v \neq 0$ be the eigenvector of $g$ corresponding to $\lambda$, and let $\bar{v} \in \mathbf{P}^{d-1}(\overline{\mathrm{K}}_w)$ be the fixed point of $g$ corresponding to $v$. By Lemma 1.7.2, the linear transformation of the tangent space of $\bar{v}$, induced by $g$, has no fixed non-zero vector. So it remains to be shown that $\bar{v} \in \Omega^d_{\mathrm{K}_w}$. If $\bar{v} \notin \Omega^d_{\mathrm{K}_w}$, then it lies in a $\mathrm{K}_w$-rational hyperplane. Therefore there exist elements $a_1, \dots, a_d \in \mathrm{K}_w$, not all 0 (say $a_d \neq 0$) such that $(a_1, \dots, a_d) \cdot v = 0$. We also know that $(g - \lambda\mathrm{I})\, v = 0$. Let A be the matrix obtained from $g - \lambda\mathrm{I}$ by replacing the last row by $(a_1, \dots, a_d)$. Then $\mathrm{A}v = 0$, so that $\det \mathrm{A} = 0$. The determinant of A is a polynomial in $\lambda$ of degree $(d-1)$ with coefficients in $\mathrm{K}_w$ with leading coefficient $(-1)^{(d-1)} a_d \neq 0$. This contradicts the fact that the minimal polynomial of $\lambda$ over $\mathrm{K}_w$ has degree $d$.

Suppose now that the characteristic polynomial $\chi_g$ of $g$ equals the product $f_1 \cdot \dots \cdot f_k$

of polynomials irreducible over $K_w$ $(k > 1)$. Consider the matrix $f_1(g)$. If $f_1(g) = 0$, then the minimal polynomial $m_g$ of $g$ divides $f_1$. Hence each root of $\chi_g$, being a root of $m_g$, is a root of $f_1$. Each $f_i$ has only simple roots, therefore $f_i \mid f_1$ for each $i$. Since $f_1$ is irreducible, all the $f_i$'s are equal up to a constant. Hence $\chi_g = cf_1^k$ for some $c \in K_w^\times$. In particular, each root of $\chi_g$ is at least double. Lemma 1.7.2 then implies that $g$ is not elliptic.

Hence we can suppose that $f_i(g) \neq 0$ for all $i = 1, 2, \ldots, k$. Let $\lambda$ be an eigenvalue of $g$, let $v$ be the eigenvector corresponding to $\lambda$, and let $\bar{v} \in \mathbf{P}^{d-1}(\overline{K}_w)$ be the fixed point of $g$ corresponding to $v$. Choose $i \in \{1, \ldots, k\}$ such that $\lambda$ is a root of $f_i$. Then $f_i(g)\, v = f_i(\lambda)\, v = 0$. The matrix $f_i(g) \neq 0$ has all its entries in $K_w$, hence $\bar{v}$ lies in a $K_w$-rational hyperplane. Therefore $g$ is not elliptic. □

Now we return to the proof of the proposition. Embed an extension $L = K_w(\lambda)$ of $K_w$ of degree $d$ in $\mathrm{Mat}_d(K_w)$. Then $\lambda \in L^\times \subset GL_d(K_w)$ has an irreducible characteristic polynomial over $K_w$. Therefore the set of elliptic elements of $PGL_d(K_w)$ is non-empty. It is open because by Krasner's lemma if $g \in GL_d(K_w)$ has a characteristic polynomial irreducible over $K_w$, then any $g' \in GL_d(K_w)$, close enough to $g$, has the same property (see [La, Ch. II, § 3, Prop. 4]). It follows that the set of elliptic elements of $PSL_d(K_w)$ is also open. For showing that it is non-empty observe that if an element $g \in PGL_d(K_w)$ is elliptic but $g^d$ is not elliptic, then by Lemma 1.7.2 the characteristic polynomial of any representative of $g^d$ in $GL_d(K_w)$ has at least two equal roots. Hence such a $g$ belongs to some proper Zariski closed subset of $PGL_d$. It follows that there exists an elliptic element $g \in PGL_d(K_w)$ such that $g^d$ is elliptic as well. Since $g^d$ always belongs to $PSL_d(K_w)$, we are done. □

*Proposition* **1.7.5.** — *a) An element* $(g, \delta) \in GL_d(K_w) \times D_w^\times$ *is elliptic with respect to its action on* $\Sigma^d_{K_w}$ *(viewed as a pro-analytic space over* $K_w$*) if and only if the characteristic polynomials of* $g$ *and* $\delta$ *are* $K_w$*-irreducible and coincide.*

*b) For every element* $g \in GL_d(K_w)$ *elliptic with respect to its action on* $\Omega^d_{K_w}$, *there exists a* $\delta \in D_w^\times$ *such that* $(g, \delta)$ *is elliptic with respect to its action on* $\Sigma^d_{K_w}$.

*Proof.* — *a)* Let $x \in \Sigma^d_{K_w}$ be an elliptic point of $(g, \delta)$, and let $\bar{x} \in \Omega^d_{K_w}$ be its image under the natural projection $\rho : \Sigma^d_{K_w} \to \Omega^d_{K_w}$. Since $\rho$ is étale, it induces an isomorphism of tangent spaces (up to an extension of scalars). Hence $g$ is elliptic with respect to its action on $\Omega^d_{K_w}$. By Lemma 1.7.4, $g$ generates a maximal commutative subfield $L := K_w(g)$ of $\mathrm{Mat}_d(K_w)$.

Choose an embedding $j : K_w(g) \hookrightarrow D_w$ (such exists by [CF, Ch. VI, § 1, App.]). It defines an $L^\times$-equivariant embedding $\tilde{\imath} : \Sigma^1_L \hookrightarrow \Sigma^d_{K_w}$ (see 1.4.1). We know that $\bar{x} \in (\Omega^d_{K_w})^{L^\times} = \rho \circ \tilde{\imath}(\Sigma^1_L)$. In particular, there exists $y \in \tilde{\imath}(\Sigma^1_L)$ such that $\rho(y) = \bar{x}$. Since $\tilde{\imath}$ is $L^\times$-equivariant, the element $(g, j(g)) \in GL_d(K_w) \times D_w^\times$ fixes $y$. Using the fact that $x \in \rho^{-1}(\bar{x})$ and that $D_w^\times \backslash \Sigma^d_{K_w} = \Omega^d_{K_w}$, we have $y = d_0\, x$ for some $d_0 \in D_w^\times$. Hence, the elements $(g, d_0^{-1} j(g)\, d_0) \in GL_d(K_w) \times D_w^\times$ and $\tilde{d} := d_0^{-1} j(g)\, d_0\, \delta^{-1} \in D_w^\times$ fix $x$.

In particular, $\widetilde{d} \in \mathrm{D}_w^\times$ fixes some point (the projection of $x$) on $\Sigma_{\mathrm{K}_w}^{d,0} \widehat{\otimes}_{\mathrm{K}_w} \mathbf{C}_p$, therefore $\widetilde{d} \in \mathcal{O}_{\mathrm{D}_w}^\times$. Since the Galois covering $\Sigma_{\mathrm{K}_w}^d \to \mathcal{O}_{\mathrm{D}_w}^\times \backslash \Sigma_{\mathrm{K}_w}^d$ is étale, $\mathcal{O}_{\mathrm{D}_w}^\times$ acts freely on $\Sigma_{\mathrm{K}_w}^d$. It follows that $\widetilde{d} = 1$, hence $\delta = d_0^{-1} j(g)\, d_0$. This completes the proof of the implication " only if ", because $g \in \mathrm{Mat}_d(\mathrm{K}_w)$ and $j(g) \in \mathrm{D}_w$ have the same characteristic polynomials.

Conversely, suppose that the characteristic polynomials of $g$ and $\delta$ are $\mathrm{K}_w$-irreducible and coincide. Then the subfields $\mathrm{K}_w(g) \subset \mathrm{Mat}_d(\mathrm{K}_w)$ and $\mathrm{K}_w(\delta) \subset \mathrm{D}_w$ have degree $d$ over $\mathrm{K}_w$ and are isomorphic under the $\mathrm{K}_w$-isomorphism sending $g$ to $\delta$. Using this isomorphism we obtain embeddings of the field $\mathrm{L} := \mathrm{K}_w(g)$ into $\mathrm{Mat}_d(\mathrm{K}_w)$ and into $\mathrm{D}_w$. These embeddings define (by 1.4.4) an $(\mathrm{L}^\times \times \mathrm{L}^\times)$-equivariant embedding $\widetilde{\imath} : \Sigma_{\mathrm{L}}^1 \hookrightarrow \Sigma_{\mathrm{K}_w}^d$ such that every point $x \in \widetilde{\imath}(\Sigma_{\mathrm{L}}^1)$ is fixed by all elements of the form $(l, l) \in \mathrm{L}^\times \times \mathrm{L}^\times \subset \mathrm{GL}_d(\mathrm{K}_w) \times \mathrm{D}_w^\times$. In particular, $x$ is a fixed point of $(g, \delta)$. As before, the action of $(g, \delta)$ on the tangent space of $x$ coincides with the action of $g$ on the tangent space of $\bar{x}$. Since $\bar{x}$ is an elliptic point of $g$ (by Lemma 1.7.4), $x$ is an elliptic point of $(g, \delta)$.

*b)* If an element $g \in \mathrm{GL}_d(\mathrm{K}_w)$ is elliptic, then by Lemma 1.7.4 it has an irreducible characteristic polynomial over $\mathrm{K}_w$. Therefore $\mathrm{K}_w(g) \subset \mathrm{Mat}_d(\mathrm{K}_w)$ is a field extension of $\mathrm{K}_w$ of degree $d$. Then for every embedding $j$ of $\mathrm{K}_w(g)$ into $\mathrm{D}_w$ the element $(g, j(g))$ is elliptic by *a)*. □

### 1.8. Euler-Poincaré measures and inner twists

Here we give a brief exposition of Kottwitz' result [Ko, § 1].

**1.8.1.** Let L be a local field of characteristic 0, and let H be a connected reductive group over L. Serre [Se2] proved that there exists a unique invariant measure (called the Euler-Poincaré measure) $\mu_{\mathrm{H}}$ on H(L) such that $\mu_{\mathrm{H}}(\Gamma \backslash \mathrm{H}(\mathrm{L}))$ is equal to the Euler-Poincaré characteristic $\chi_{\mathrm{E}}(\Gamma)$ of $\mathrm{H}^*(\Gamma, \mathbf{Q})$ for every torsion-free cocompact lattice $\Gamma$ in H(L). In particular, $\mu_{\mathrm{H}}(\mathrm{H}(\mathrm{L})) = 1$ if the group H(L) is compact. The Euler-Poincaré measure is either always negative, always positive or identically zero. It is non-zero if and only if H has an anisotropic maximal L-torus. (A result of Kneser shows that in the $p$-adic case this happens if and only if the connected center of H is anisotropic.)

**1.8.2.** Let G be an inner form of H. Choose an inner twisting $\rho : \mathrm{H} \to \mathrm{G}$ over $\overline{\mathrm{L}}$. Choose a non-zero invariant differential form $\omega_{\mathrm{G}}$ of top degree on G. Set $\omega_{\mathrm{H}} := \rho^*(\omega_{\mathrm{G}})$. Using the fact that H is reductive, that the twisting is inner and that $\omega_{\mathrm{G}}$ is invariant, we see that $\omega_{\mathrm{H}}$ is invariant, defined over L, and does not depend on $\rho$. Hence $\omega_{\mathrm{G}}$ and $\omega_{\mathrm{H}}$ define invariant measures $| \omega_{\mathrm{G}} |$ and $| \omega_{\mathrm{H}} |$ on G(L) and H(L) respectively (see [We2, 2.2]).

*Definition* **1.8.3.** — The invariant measures $\mu$ on H(L) and $\mu'$ on G(L) are called *compatible* if there exists some $c \in \mathbf{R}$ such that $\mu = c \, | \omega_{\mathrm{H}} |$ and $\mu' = c \, | \omega_{\mathrm{G}} |$.

**1.8.4.** — Now suppose that H has an anisotropic maximal L-torus T, so that the Euler-Poincaré measure $\mu_{\mathrm{H}}$ on H(L) is non-trivial. (Notice that for semisimple groups of type $A_n$ this assumption is satisfied automatically). Denote by $|\mu_H|$ the absolute value of $\mu_H$. Write $\mathscr{D}(T, H)$ for the finite set $\mathrm{Ker}[H^1(L, T) \to H^1(L, H)]$ and write $|\mathscr{D}(T, H)|$ for its cardinality. It is well known that T transfers to G, thus we can also consider the finite set $\mathscr{D}(T, G)$.

*Proposition* **1.8.5** ([Ko, Thm. 1]). — *The invariant measure* $|\mathscr{D}(T, H)|^{-1} |\mu_H|$ *on* $H(L)$ *is compatible with the invariant measure* $|\mathscr{D}(T, G)|^{-1} |\mu_G|$ *on* $G(L)$.

*Remark* **1.8.6.** — *a)* In the $p$-adic case, the sets $\mathscr{D}(T, H)$ and $\mathscr{D}(T, G)$ always have the same cardinality.

*b)* In the real case, $\mathscr{D}(T, H) = \Omega(H(\mathbf{C}), T(\mathbf{C}))/\Omega(H(\mathbf{R}), T(\mathbf{R}))$, where $\Omega$ stands for the Weyl group. In particular, $|\mathscr{D}(\mathrm{diag}, \mathrm{PGU}_d)| = 1$ and $|\mathscr{D}(\mathrm{diag}, \mathrm{PGU}_{d-1,1})|$ is $d$ (resp. 1) if $d > 2$ (resp. $d = 2$).

## 1.9. Preliminaries on torsors (= principal bundles)

*Definition* **1.9.1.** — Let G be an affine group scheme over a field L (resp. an L-analytic group), and let X be an L-scheme (resp. an L-analytic space). A G-*torsor* over X is a scheme (resp. an analytic space) T over X with an action $G \times T \to T$ of G on T over X such that for some surjective étale covering $X' \to X$ the fiber product $T \times_X X'$ is the trivial G-torsor over X′ (that is isomorphic to $G \times X'$).

*Remark* **1.9.2.** — Since each étale morphism of complex analytic spaces is a local isomorphism, our definition in this case coincides with the classical one.

*Lemma* **1.9.3.** — *a) If* T *is a* G-*torsor over* X, *then the map* $\varphi_T : G \times T \to T \times_X T$ $(\varphi_T(g, t) = (gt, t))$ *is an isomorphism.*

*b) Let* T *and* T′ *be two* G-*torsors over* X *and* Y *respectively. Then for each* G-*equivariant map* $f: T \to T'$ *the natural morphism* $T \to T' \times_Y X$ *is an isomorphism.*

*Proof.* — *a)* Since the problem is local for the étale topology on X (see [Mi2, Ch. I, Rem. 2.24] in the algebraic case, [Be3, Prop. 4.1.3] in the $p$-adic analytic and Remark 1.9.2 in the complex one), we may suppose that T is trivial. Then our morphism $(g, (h, x)) \mapsto ((gh, x), (h, x))$ is invertible.

*b)* For trivial torsors the statement is clear. The general case follows as in *a)*. □

*Remark* **1.9.4.** — By [Mi2, Ch. I, Rem. 2.24 and Prop. 3.26] our definition in the algebraic case is equivalent to the standard one. In particular, a G-torsor over X is affine and faithfully flat over X.

*Lemma* **1.9.5.** — *Let* X *be a separated scheme over a field* L, *let* G *and* H *be two affine group schemes over* L, *let* T *be a* G-*torsor over* X, *and let* $\pi : T \to X$ *be the natural projection.*

*a) The functor* $\mathscr{F} \mapsto \pi^* \mathscr{F}$ *defines an equivalence between the category of quasi-coherent sheaves on* X *and the category of* G-*equivariant quasi-coherent sheaves on* T, *that is, quasi-coherent sheaves on* T *with a* G-*action that lifts the action of* G *on* T.

*b) The functor* $Z \mapsto Z \times_X T$ *defines an equivalence between the following categories*:

i) *the category of vector bundles of finite rank on* X *and the category of* G-*equivariant vector bundles of finite rank on* T;

ii) *the category of* H-*torsors over* X *and the category of* G-*equivariant* H-*torsors over* T;

iii) *(if* X *is noetherian and regular) the category of* $\mathbf{P}^n$-*bundles on* X *and the category of* G-*equivariant* $\mathbf{P}^n$-*bundles on* T.

*The quasi-inverse functor is* $\tilde{Z} \mapsto G\backslash\tilde{Z}$.

*Proof.* — This is a consequence of a descent theory.

*a)* Abusing notation we will write $\mathcal{F} \times_{Y_2} Y_1$ instead of $\rho^* \mathcal{F}$ for every morphism $\rho : Y_1 \to Y_2$ and every sheaf of modules $\mathcal{F}$ on $Y_2$. Let $\tilde{\mathcal{F}}$ be a G-equivariant quasi-coherent sheaf on T. Define an isomorphism $\varphi : (\tilde{\mathcal{F}} \times_T T) \times_X T \xrightarrow{\sim} T \times_X (\tilde{\mathcal{F}} \times_T T)$ over $T \times_X T$ by the formula $\varphi(f, gt, t) = (gt, g^{-1}f, t)$ for all $g \in G$, $t \in T$ and $f \in \tilde{\mathcal{F}}_t$ (use Lemma 1.9.3). Then $\varphi$ satisfies the descent conditions of [Mi2, Prop. 2.22]. Since $T \to X$ is affine and faithfully flat, there is a unique quasi-coherent sheaf $\mathcal{F}$ on X such that $\tilde{\mathcal{F}} \cong \mathcal{F} \times_X T$. Since the construction of descent is functorial (see [Mi2, 2.19]), we obtain an equivalence of categories. Notice that $\mathcal{F} \cong G\backslash(\mathcal{F} \times_X T)$.

*b)* follows from *a)* in a standard way (use [Ha, II, Ex. 5.18, 5.17 and 7.10]). □

From now on we suppose that the reader is familiar with basic definitions of tensor categories (see [DM]).

*Notation* **1.9.6.** — For a field L, an affine group scheme (resp. an analytic group) G over L and a scheme (resp. an analytic space) X over L:

*a)* let $\mathcal{R}ep_L(G)$ be the category of finite-dimensional representations of G over L;

*b)* let $\mathcal{V}ec_X$ be the category of vector bundles of finite rank on X;

*c)* let $\mathrm{Tor}_X(G)$ be the category of G-torsors over X.

We will sometimes identify categories with the sets of their objects.

*Definition* **1.9.7.** — Let L be a field and let G be an affine group scheme over L. A G-*fibre functor with values in a separated scheme (resp. analytic space)* X over L is an exact faithful tensor functor from $\mathcal{R}ep_L(G)$ to $\mathcal{V}ec_X$.

*Remark* **1.9.8.** — If X = Spec R is affine, then $\mathcal{V}ec_X$ is equivalent to the category of finitely generated projective modules over R, hence our definition is a global version of that of [DM, 3.1].

**1.9.9.** — Let T be a G-torsor over X, then by Lemma 1.9.5, the correspondence $V \mapsto G\backslash(V \times T)$ defines a G-fibre functor with values in X. This correspondence defines a functor $\nu$ from $\mathrm{Tor}_X(G)$ to the category of G-fibre functors with values in X.

*Theorem* **1.9.10.** — *The functor* $\nu$ *determines an equivalence between* $Tor_{\mathrm{X}}(\mathrm{G})$ *and the category of* G*-fibre functors with values in* X.

*Proof.* — The local version is [DM, Thm. 2.11 and 3.2]. The gluing works because X is separated. □

**1.9.11.** Later on, we will use the following description of the quasi-inverse functor $\tau$ of $\nu$. Let $\eta$ be a G-fibre functor with values in X. For each morphism $\pi_0 : \mathrm{T}_0 \to \mathrm{X}$ we define two tensor functors $\eta_1 : \mathrm{V} \mapsto \mathrm{V} \times \mathrm{T}_0$ and $\eta_2 = \pi_0^* \circ \eta$ from $\mathcal{R}ep_{\mathrm{L}}(\mathrm{G})$ to $\mathcal{V}ec_{\mathrm{T}_0}$. Let $\mu(\mathrm{T}_0, \pi_0) := \mathrm{Isom}(\eta_2, \eta_1)$ be the set of isomorphisms of tensor functors. The action of G on the first factor of $\mathrm{V} \times \mathrm{T}_0$ defines an action of G on $\eta_1$, and a fortiori defines an action of G on $\mu(\mathrm{T}_0, \pi_0)$. Thus $\mu$ is a functor from the category of schemes over X to the category of sets with a G-action. Theorem 1.9.10 says that this functor is representable by a G-torsor $\tau(\eta)$ over X (see [DM, Thm. 2.11 and 3.2] and their proofs).

**1.9.12.** Let T be a G-torsor over X. For each $\mathrm{V} \in \mathcal{R}ep_{\mathrm{L}}(\mathrm{G})$ the identity map of T, viewed as a T-valued point of T, corresponds to a certain isomorphism $\varphi_{\mathrm{V}} : \mathrm{V} \times \mathrm{T} \xrightarrow{\sim} (\mathrm{G}\backslash(\mathrm{V} \times \mathrm{T})) \times_{\mathrm{X}} \mathrm{T}$. Then $\varphi_{\mathrm{V}}$ is the quotient of the G-equivariant isomorphism $\mathrm{Id}_{\mathrm{V}} \times \varphi_{\mathrm{T}} : \mathrm{V} \times \mathrm{G} \times \mathrm{T} \xrightarrow{\sim} \mathrm{V} \times \mathrm{T} \times_{\mathrm{X}} \mathrm{T}$ (for the diagonal action of G on the first two factors on both sides) by the action of G. Explicitely, $\varphi_{\mathrm{V}}(v, t) = ([v, t], t)$.

*Proposition* **1.9.13.** — *Let* L *be equal to* $\mathrm{K}_w$ *or to* $\mathbf{C}$ *as in* 1.3.1. *Let* X *be a projective* L*-scheme, and let* G *be a linear algebraic group over* L. *The functor* $\mathrm{T} \mapsto \mathrm{T}^{\mathrm{an}}$ *induces an equivalence between the category of* G*-torsors over* X *and the category of* $\mathrm{G}^{\mathrm{an}}$*-torsors over* $\mathrm{X}^{\mathrm{an}}$.

*Proof.* — A quasi-inverse functor can be described as follows. Let $\tilde{\pi} : \tilde{\mathrm{T}} \to \mathrm{X}^{\mathrm{an}}$ be a $\mathrm{G}^{\mathrm{an}}$-torsor. Then the map $\mathrm{V} \mapsto \mathrm{G}^{\mathrm{an}}\backslash(\mathrm{V}^{\mathrm{an}} \times \tilde{\mathrm{T}})$ defines a G-fibre functor with values in $\mathrm{X}^{\mathrm{an}}$. Since the correspondence described in Corollary 1.2.3 commutes with tensor products, the tensor categories $\mathcal{V}ec_{\mathrm{X}}$ and $\mathcal{V}ec_{\mathrm{X}^{\mathrm{an}}}$ are equivalent. Therefore Theorem 1.9.10 gives us an algebraic G-torsor $\pi : \mathrm{T} \to \mathrm{X}$.

It remains to show that there exists a canonical isomorphism $\tilde{\mathrm{T}} \xrightarrow{\sim} \mathrm{T}^{\mathrm{an}}$. By the definition of T we have for each $\mathrm{V} \in \mathcal{R}ep_{\mathrm{L}}(\mathrm{G})$ a canonical isomorphism $\psi_{\mathrm{V}} : \mathrm{G}^{\mathrm{an}}\backslash(\mathrm{V}^{\mathrm{an}} \times \tilde{\mathrm{T}}) \xrightarrow{\sim} \mathrm{G}^{\mathrm{an}}\backslash(\mathrm{V}^{\mathrm{an}} \times \mathrm{T}^{\mathrm{an}})$. We also have (as in 1.9.12) natural isomorphisms $\tilde{\mathrm{T}} \times \mathrm{V}^{\mathrm{an}} \xrightarrow{\sim} \tilde{\mathrm{T}} \times_{\mathrm{X}} (\mathrm{G}^{\mathrm{an}}\backslash(\tilde{\mathrm{T}} \times \mathrm{V}^{\mathrm{an}}))$ mapping $(t, v)$ to $(t, [t, v])$. Hence each point $t_0$ of $\tilde{\mathrm{T}}$ defines canonical isomorphisms

$$\mathrm{V}^{\mathrm{an}} \cong \{ t_0 \} \times \mathrm{V}^{\mathrm{an}} \xrightarrow{\sim} \{ t_0 \} \times_{\mathrm{X}} (\mathrm{G}^{\mathrm{an}}\backslash(\tilde{\mathrm{T}} \times \mathrm{V}^{\mathrm{an}})) : v \mapsto (t_0, [t_0, v]).$$

Since $t_0$ defines a point of $\mathrm{X}^{\mathrm{an}}$ and therefore of X, it gives us by the universal property of T (see 1.9.11) a point $\psi(t_0) \in \mathrm{T}^{\mathrm{an}}$, satisfying $\psi_{\mathrm{V}}([t_0, v]) = [\psi(t_0), v]$ for all $\mathrm{V} \in \mathcal{R}ep_{\mathrm{L}}(\mathrm{G})$.

Taking V to be a faithful representation of G, we obtain that the map (of sets) $\psi : \tilde{\mathrm{T}} \to \mathrm{T}^{\mathrm{an}}$ is $\mathrm{G}^{\mathrm{an}}$-equivariant, therefore it is one-to-one and surjective. It remains to show that the maps $\psi$ and $\psi^{-1}$ are analytic. Let us prove it, for example, for $\psi$. Let $\rho : X' \to X^{an}$ be an étale surjective covering such that $\rho^*(T^{an}) \cong G^{an} \times X'$. By [Be3, Prop. 4.1.3] in the $p$-adic case and by Remark 1.9.2 in the complex one it will suffice to show that $\rho^* \psi : \rho^*(\widetilde{T}) \to \rho^*(T^{an}) \cong G^{an} \times X'$ (or just its projection to the first factor $\pi' : \rho^*(\widetilde{T}) \to G^{an}$) is analytic. Consider the map

$$\widetilde{\psi}_V : V^{an} \times \rho^*(\widetilde{T}) \xrightarrow{\text{proj}} G^{an}\backslash[V^{an} \times \rho^*(\widetilde{T})]$$

$$\xrightarrow{\rho^*(\psi_V)} G^{an}\backslash[V^{an} \times \rho^*(T^{an})] \cong G^{an}\backslash(V^{an} \times G^{an} \times X') \cong V^{an} \times X' \xrightarrow{\text{proj}} V^{an}.$$

It is analytic, and satisfies $\widetilde{\psi}_V(v, t) = (\pi'(t))^{-1} v$. Hence $\pi'$ is analytic as well. □

*Corollary* **1.9.14.** — *Let* X *and* Y *be projective L-schemes, let* G *and* H *be algebraic groups over* L, *and let* $\psi : G \to H$ *be an algebraic group homomorphism over* L. *If* $T \in Tor_X(G)$ *and* $S \in Tor_Y(H)$, *then for any* $\psi$*-equivariant analytic map* $\widetilde{f}: T^{an} \to S^{an}$ *(that is, satisfying* $\widetilde{f}(gt) = \psi(g)\widetilde{f}(t)$ *for all* $g \in G^{an}$ *and* $t \in T^{an}$*), there is a unique algebraic morphism* $f: T \to S$ *such that* $f^{an} \cong \widetilde{f}$.

*Proof* (compare the proof of Corollary 1.2.5). — Since $\widetilde{f}$ is $\psi$-equivariant, it covers some algebraic morphism $\bar{f}: X \to Y$ (use Corollary 1.2.2). Therefore $\widetilde{f}$ factors through $S^{an} \times_{Y^{an}} X^{an} \cong (S \times_X Y)^{an}$. Hence we may suppose, replacing S by $S \times_X Y$, that $X = Y$ and that $\bar{f}$ is the identity.

Consider the H-torsor $H \times T$ over T equipped with the following G-action: $g(h, t) = (h\psi(g)^{-1}, gt)$ for all $g \in G$, $h \in H$ and $t \in T$. By Lemma 1.9.5, there exists an H-torsor $H \times_G T := G\backslash(H \times T)$ over X. Let $i$ be the composition of the embedding $t \mapsto (1, t)$ of T into $H \times T$ with the natural projection to $H \times_G T$. Then by the definition, every $\psi$-equivariant algebraic morphism $\mu : T \to S$ factors as a composition of $i$ with the unique H-equivariant map $H \times_G T \to T$ (defined by $[h, t] \mapsto h\mu(t)$). Therefore $(H \times_G T)^{an} \cong H^{an} \times_{G^{an}} T^{an}$ is an $H^{an}$-torsor over $X^{an}$ having the same functorial property.

Now we are ready to prove our corollary. From the $\psi$-equivariance of $\widetilde{f}$ we conclude that it factors uniquely as $\widetilde{f}: T^{an} \xrightarrow{i^{an}} (H \times_G T)^{an} \xrightarrow{\widetilde{f}'} S^{an}$. By the proposition, $\widetilde{f}'$ has a unique underlying algebraic morphism $f' : H \times_G T \to S$. Set $f := f' \circ i$. The uniqueness can be derived from the above considerations as in the proof of Corollary 1.2.5. □

Now we recall the notion and basic properties of connections on torsors (following [St, Ch. VI, § 1]).

*Definition* **1.9.15.** — Let X be a smooth scheme or an analytic space, and let $\pi : P \to X$ be a G-torsor. A *connection on* P is a G-equivariant vector subbundle $\mathscr{H}$ of the tangent bundle T(P) of P such that $\pi_{*|\mathscr{H}_p}$ is an isomorphism $\mathscr{H}_p \xrightarrow{\sim} T_{\pi(p)}(X)$ for each $p \in P$.

**1.9.16.** Starting from the isomorphism $\varphi_P : G \times P \xrightarrow{\sim} P \times_X P$ we obtain an isomorphism of tangent spaces $(\varphi_P)_* : T_e(G) \times T_p(P) \xrightarrow{\sim} T_p(P) \times_{T_{\pi(p)}(X)} T_p(P)$ and an identification $(X \mapsto \mathrm{proj}_1((\varphi_P)_*(X, 0))$ of $\mathcal{G} := \mathrm{Lie}(G) = T_e(G)$ with the tangent space to the fiber through $p \in P$. Therefore a connection $\mathcal{H}$ on P gives us a canonical decomposition $T_p(P) = \mathcal{G} \oplus \mathcal{H}_p$ for each $p \in P$. Now considering the projection of $T_p(P)$ onto $\mathcal{G}$ with kernel $\mathcal{H}_p$ for each $p \in P$ we get a certain $\mathcal{G}$-valued differential 1-form $\Omega = \Omega(\mathcal{H})$, called the *connection form of* $\mathcal{H}$.

*Definition* **1.9.17.** — Let $\mathcal{H}$ be a connection on a G-torsor P, whose connection form is $\Omega$. Let $h$ be the natural projection of $T_p(P)$ on $\mathcal{H}_p$ for all $p \in P$. The *curvature* of the connection $\mathcal{H}$ is the 2-form $D\Omega$ defined by $\langle X \wedge Y \mid D\Omega \rangle := \langle h(X) \wedge h(Y) \mid d\Omega \rangle$. A connection with zero curvature is called *flat.*

*Remark* **1.9.18.** — The trivial torsor $P \cong G \times X$ has a natural flat connection, consisting of vectors, tangent to X. We will call such a connection *trivial.*

*Lemma* **1.9.19.** — *Let* X *be a simply connected complex manifold, let* $\pi : P \to X$ *be a* G-*torsor, and let* $\mathcal{H}$ *be a flat connection on* P. *Then there exists a unique decomposition* $P \xrightarrow{\sim} G \times X$ *such that* $\mathcal{H}$ *corresponds to the trivial connection on* $G \times X$.

*Proof.* — By [St, Ch. VII, Thm. 1.1 and 1.2], there exists a unique G-equivariant diffeomorphism $\varphi : P \xrightarrow{\sim} G \times X$ over X which maps $\mathcal{H}$ to the trivial connection. Hence $\varphi$ induces complex isomorphism between tangent spaces $T_p(P) = \mathcal{G} \oplus \mathcal{H}_p$ and $T_{\varphi(p)}(G \times X) = \mathcal{G} \oplus T_{\pi(p)}(X)$ for each $p \in P$. In other words both $\varphi$ and $\varphi^{-1}$ are almost complex mappings between complex manifolds. [He, Ch. VIII, p. 284] then implies that $\varphi$ is biholomorphic. □

## 2. FIRST MAIN THEOREM

### 2.1. Basic examples

*Definition* **2.1.1.** — Let $K/k$ be a quadratic field extension and let D be a central simple algebra over K. We say that $\alpha : D \to D$ is an *involution of the second kind over k* if $\alpha(d_1 + d_2) = \alpha(d_1) + \alpha(d_2)$, $\alpha(d_1 d_2) = \alpha(d_2)\,\alpha(d_1)$ for all $d_1, d_2 \in D$ and the restriction of $\alpha$ to K is the conjugation over $k$.

*Notation* **2.1.2.** — For $k$, D and $\alpha$ as in Definition 2.1.1, let $G = GU(D, \alpha)$ be the algebraic group over $k$ of unitary similitudes, that is $G(R) = \{ d \in (D \otimes_k R)^\times \mid d\alpha(d) \in R^\times \}$ for each $k$-algebra R. Define the *similitudes homomorphism* $G \to \mathbf{G}_m$ by $x \mapsto x\alpha(x)$. Notice also that by the Skolem-Noether theorem the group G satisfies $PG(L) = G(L)/Z(G(L))$ for every field extension L of $k$.

**2.1.3.** *First basic example.* — Let F be a totally real field of degree $g$ over $\mathbf{Q}$, let K be a totally imaginary quadratic extension on F. Let D be a central simple algebra of dimension $d^2$ over K with an involution of the second kind $\alpha$ over F. Set $\mathrm{G} := \mathrm{GU}(\mathrm{D}, \alpha)$, and put $\mathrm{D}_u := \mathrm{D} \otimes_{\mathrm{K}} \mathrm{K}_u$ for each prime $u$ of K. Let $v$ be a (non-archimedean) prime of F that splits in K and let $w$ and $\bar{w}$ be the primes of K that lie over $v$. Then $\mathrm{D} \otimes_{\mathrm{F}} \mathrm{F}_v \cong \mathrm{D}_w \oplus \mathrm{D}_{\bar{w}}$, and the projection to the first factor together with the similitude homomorphism induce an isomorphism $\mathrm{G}(\mathrm{F}_v) \overset{\sim}{\to} \mathrm{D}_w^\times \times \mathrm{F}_v^\times$. We identify $\mathrm{G}(\mathrm{F}_v)$ with $\mathrm{D}_w^\times \times \mathrm{F}_v^\times$ by this isomorphism.

Suppose that $\mathrm{D}_w \cong \mathrm{Mat}_d(\mathrm{K}_w)$. Identifying $\mathrm{D}_w$ with $\mathrm{Mat}_d(\mathrm{K}_w)$ by some isomorphism we identify $\mathrm{G}(\mathrm{F}_v)$ with $\mathrm{GL}_d(\mathrm{K}_w) \times \mathrm{F}_v^\times$. Suppose that $\alpha$ is positive definite, that is $\mathrm{G}(\mathrm{F}_{\infty_i}) \cong \mathrm{GU}_{\mathrm{d}}(\mathbf{R})$ for all archimedean completions $\mathrm{F}_{\infty_i} \cong \mathbf{R}$ of F. Put $\mathrm{E}' := \mathrm{F}_v^\times \times \mathrm{G}(\mathbf{A}_{\mathrm{F}}^{f;v})$, then E′ is a noncompact locally profinite group. Set

$$\Gamma := \mathrm{G}(\mathrm{F}) \subset \mathrm{G}(\mathbf{A}_{\mathrm{F}}^f) = \mathrm{GL}_d(\mathrm{K}_w) \times \mathrm{E}',$$

embedded diagonally.

*Proposition* **2.1.4.** — *The subgroup* $\Gamma \subset \mathrm{G}(\mathbf{A}_{\mathrm{F}}^f) = \mathrm{GL}_d(\mathrm{K}_w) \times \mathrm{E}'$ *satisfies the assumptions of Construction* 1.5.1.

*Proof.* — *a)* is trivial.

*b)* is true, because the closure of $\mathrm{Z}(\Gamma) \cong \mathrm{K}^\times$ is cocompact in $\mathrm{Z}(\mathrm{G}(\mathbf{A}_{\mathrm{F}}^f)) \cong (\mathbf{A}_{\mathrm{K}}^f)^\times$.

*c)* Since $\mathrm{PE}' = \mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f;v})$ and $\mathrm{PG}(\mathrm{F}_v) \cong \mathrm{PGL}_d(\mathrm{K}_w)$, we have to show that $\mathrm{P}\Gamma(= \mathrm{PG}(\mathrm{F}))$ is a cocompact lattice in $\mathrm{PG}(\mathbf{A}_{\mathrm{F}}^f)$.

*Lemma* **2.1.5.** — *If* H *is an* F*-anisotropic group, then* H(F) *is a cocompact lattice in* $\mathrm{H}(\mathbf{A}_{\mathrm{F}})$.

*Proof.* — See [PR, Thm. 5.5]. □

Since PG is anisotropic over each $\mathrm{F}_{\infty_i}$, it is anisotropic over F. Hence by the lemma, PG(F) is a cocompact lattice in $\mathrm{PG}(\mathbf{A}_{\mathrm{F}})$. The compactness of the $\mathrm{PG}(\mathrm{F}_{\infty_i})$'s implies also that the projection of PG(F) to $\mathrm{PG}(\mathbf{A}_{\mathrm{F}}^f)$ is a cocompact lattice as well (see [Shi, Prop. 1.10]). Observe also that the projection $\mathrm{PG}(\mathrm{F}) \to \mathrm{PG}(\mathrm{F}_v) \cong \mathrm{PGL}_d(\mathrm{K}_w)$ is injective.

*d)* Since $\mathrm{Z}(\Gamma) \cong \mathrm{K}^\times$ and $\mathrm{Z}(\mathrm{G}(\mathbf{A}_{\mathrm{F}}^f)) \cong (\mathbf{A}_{\mathrm{K}}^f)^\times$, we have to show that the intersection of $\overline{\mathrm{K}}^\times \subset (\mathbf{A}_{\mathrm{K}}^f)^\times$ with $\mathrm{K}_w^\times \times \{1\}$ is trivial. This can be shown either by the direct computation or using the relation between global and local Artin maps (see [CF, Ch. VII, Prop. 6.2]). □

Fix a central skew field $\widetilde{\mathrm{D}}_w$ over $\mathrm{K}_w$ with invariant $1/d$. Set $\mathrm{E} =: \widetilde{\mathrm{D}}_w^\times \times \mathrm{E}'$, then Construction 1.5.1 gives us an $(\mathrm{E}, \mathrm{K}_w)$-scheme X corresponding to $\Gamma$.

**2.1.6.** *Second basic example.* — By Brauer-Hasse-Noether theorem (see [Wel, Ch. XIII, § 6]) there exists a unique central skew field $\mathrm{D}^{\mathrm{int}}$ over K which is locally isomorphic to D at all places of K except $w$ and $\bar{w}$ and has Brauer invariant $1/d$ at $w$. By Landherr theorem (see [Sc, Ch. 10, Thm. 2.4]), $\mathrm{D}^{\mathrm{int}}$ admits an involution of the second kind over F. Fix an embedding $\infty_1 : \mathrm{K} \hookrightarrow \mathbf{C}$. It induces an archimedean completion $\mathrm{F}_{\infty_1}$ of F, and we have the following

*Proposition* **2.1.7.** — *a) There exists an involution of the second kind* $\alpha^{\mathrm{int}}$ *of* $\mathrm{D}^{\mathrm{int}}$ *over* F *such that:*

i) *the pairs* $(\mathrm{D}, \alpha) \otimes_{\mathrm{F}} \mathrm{F}_u$ *and* $(\mathrm{D}^{\mathrm{int}}, \alpha^{\mathrm{int}}) \otimes_{\mathrm{F}} \mathrm{F}_u$ *are isomorphic at all places* $u$ *of* F, *except* $v$ *and* $\infty_1$*;*

ii) *the signature of* $(\mathrm{D}^{\mathrm{int}}, \alpha^{\mathrm{int}})$ *at* $\infty_1$ *is* $(d-1, 1)$.

*b) The group* $\mathrm{G}^{\mathrm{int}} := \mathrm{GU}(\mathrm{D}^{\mathrm{int}}, \alpha^{\mathrm{int}})$ *is determined uniquely (up to an isomorphism) by conditions* i), ii) *of a).*

*Proof.* — *a)* follows from [Cl, (2.2) and the discussion around it] as in [Cl, Prop. 2.3].

*b)* follows immediately from [Sc, Ch. 10, Thm. 6.1]. □

Let $\mathrm{G}^{\mathrm{int}}$ be as in the proposition. Then embedding $\infty_1$ defines an isomorphism $\mathrm{D}^{\mathrm{int}} \otimes_{\mathrm{K}} \mathrm{K}_{\infty_1} \xrightarrow{\sim} \mathrm{Mat}_d(\mathbf{C})$, and we identify $\mathrm{PG}^{\mathrm{int}}(\mathrm{F}_{\infty_1})$ with $\mathrm{PGU}_{d-1,1}(\mathbf{R})$ by the induced isomorphism. Set $\mathrm{G}^{\mathrm{int}}(\mathrm{F})_+ := \mathrm{G}^{\mathrm{int}}(\mathrm{F}) \cap \mathrm{G}^{\mathrm{int}}(\mathrm{F}_{\infty_1})^0$. Then $\mathrm{G}^{\mathrm{int}}(\mathrm{F})_+ = \mathrm{G}^{\mathrm{int}}(\mathrm{F})$ if $d > 2$, and $[\mathrm{G}^{\mathrm{int}}(\mathrm{F}) : \mathrm{G}^{\mathrm{int}}(\mathrm{F})_+] = 2$ if $d = 2$. Set $\mathrm{E}^{\mathrm{int}} := \mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^f)$ and let $\mathrm{E}_0^{\mathrm{int}} \subset \mathrm{E}^{\mathrm{int}}$ be the closure of $\mathrm{Z}(\mathrm{G}^{\mathrm{int}}(\mathrm{F})) \subset \mathrm{E}^{\mathrm{int}}$. Embed diagonally $\mathrm{G}^{\mathrm{int}}(\mathrm{F})$ into $\mathrm{G}^{\mathrm{int}}(\mathrm{F}_{\infty_1}) \times \mathrm{E}^{\mathrm{int}}$ and define $\Gamma^{\mathrm{int}}$ to be the image of $\mathrm{G}^{\mathrm{int}}(\mathrm{F})_+$ under the natural projection to

$$\mathrm{PG}^{\mathrm{int}}(\mathrm{F}_{\infty_1}) \times (\mathrm{E}^{\mathrm{int}}/\mathrm{E}_0^{\mathrm{int}}) = \mathrm{PGU}_{d-1,1}(\mathbf{R}) \times (\mathrm{E}^{\mathrm{int}}/\mathrm{E}_0^{\mathrm{int}}).$$

*Proposition* **2.1.8.** — *The subgroup* $\Gamma^{\mathrm{int}}$ *is a cocompact lattice in*

$$\mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times (\mathrm{E}^{\mathrm{int}}/\mathrm{E}_0^{\mathrm{int}})$$

*and it has an injective projection to the first factor.*

*Proof.* — Notice that the natural projection $\mathrm{E}^{\mathrm{int}}/\mathrm{E}_0^{\mathrm{int}} \to \mathrm{E}^{\mathrm{int}}/\mathrm{Z}(\mathrm{E}^{\mathrm{int}}) = \mathrm{PE}^{\mathrm{int}}$ induces an isomorphism $\Gamma^{\mathrm{int}} \xrightarrow{\sim} \mathrm{PG}^{\mathrm{int}}(\mathrm{F})_+ \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{PE}^{\mathrm{int}}$ and that the group $\mathrm{Z}(\mathrm{E}^{\mathrm{int}})/\mathrm{E}_0^{\mathrm{int}} \cong (\mathbf{A}_{\mathrm{K}}^f)^\times/\overline{\mathrm{K}^\times}$ is compact. Therefore it will suffice to prove that $\mathrm{PG}^{\mathrm{int}}(\mathrm{F})$ is a cocompact lattice with an injective projection to the first factor of $\mathrm{PG}^{\mathrm{int}}(\mathrm{F}_{\infty_1}) \times \mathrm{PE}^{\mathrm{int}}$. This can be proved by exactly the same considerations as in the proof of Proposition 2.1.4, *c).* □

By the proposition, $\Gamma^{\mathrm{int}}$ satisfies the assumptions of Construction 1.3.6, so it determines an $(\mathrm{E}^{\mathrm{int}}/\mathrm{E}_0^{\mathrm{int}}, \mathbf{C})$-scheme $\widetilde{\mathrm{X}}^{\mathrm{int}}$, which can be regarded as an $(\mathrm{E}^{\mathrm{int}}, \mathbf{C})$-scheme with a trivial action of $\mathrm{E}_0^{\mathrm{int}}$.

*Remark* **2.1.9.** — For each $\mathrm{S} \in \mathscr{F}(\mathrm{E}^{\mathrm{int}})$ we have the following isomorphisms

$$\begin{aligned}(\widetilde{\mathrm{X}}_{\mathrm{S}}^{\mathrm{int}})^{\mathrm{an}} &\cong \mathrm{S}\backslash[\mathrm{B}^{d-1} \times (\mathrm{E}^{\mathrm{int}}/\mathrm{E}_0^{\mathrm{int}})]/\Gamma^{\mathrm{int}} \\ &\cong (\mathrm{S}\cdot\overline{\mathrm{Z}(\mathrm{G}^{\mathrm{int}}(\mathrm{F}))}\backslash[\mathrm{B}^{d-1} \times \mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^f)]/\mathrm{G}^{\mathrm{int}}(\mathrm{F})_+ \\ &\cong (\mathrm{S}\cdot\mathrm{Z}(\mathrm{G}^{\mathrm{int}}(\mathrm{F}))\backslash[\mathrm{B}^{d-1} \times \mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^f)]/\mathrm{G}^{\mathrm{int}}(\mathrm{F})_+ \\ &\cong \mathrm{S}\backslash[\mathrm{B}^{d-1} \times \mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^f)]/\mathrm{G}^{\mathrm{int}}(\mathrm{F})_+.\end{aligned}$$

## 2.2. First Main Theorem

*Definition* **2.2.1.** — An isomorphism $\Phi : \mathrm{E} \xrightarrow{\sim} \mathrm{E}^{\mathrm{int}}$ is called *admissible* if it is a product of $\mathrm{G}(\mathbf{A}_{\mathrm{F}}^{f;v}) \xrightarrow{\sim} \mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f;v})$, induced by some $\mathbf{A}_{\mathrm{F}}^{f;v}$-linear algebra isomorphism $\mathrm{D} \otimes_{\mathrm{F}} \mathbf{A}_{\mathrm{F}}^{f;v} \xrightarrow{\sim} \mathrm{D}^{\mathrm{int}} \otimes_{\mathrm{F}} \mathbf{A}_{\mathrm{F}}^{f;v}$ (compare Proposition 2.1.7), and the composition map $\widetilde{\mathrm{D}}_w^{\times} \times \mathrm{F}_v^{\times} \xrightarrow{\sim} (\mathrm{D}_w^{\mathrm{int}})^{\times} \times \mathrm{F}_v^{\times} \xrightarrow{\sim} \mathrm{G}^{\mathrm{int}}(\mathrm{F}_v)$, constructed from some algebra isomorphism $\widetilde{\mathrm{D}}_w \xrightarrow{\sim} \mathrm{D}^{\mathrm{int}} \otimes_{\mathrm{K}} \mathrm{K}_w$ as in 2.1.3.

**2.2.2.** Fix a field isomorphism $\mathbf{C} \xrightarrow{\sim} \mathbf{C}_p$, whose composition with embedding $\infty_1 : \mathrm{K} \hookrightarrow \mathbf{C}$ (chosen in 2.1.6) is the natural embedding $\mathrm{K} \hookrightarrow \mathrm{K}_w \hookrightarrow \mathbf{C}_p$. Identifying $\mathbf{C}$ with $\mathbf{C}_p$ by means of this isomorphism we can view, in particular, $\mathrm{K}_w$ as a subfield of $\mathbf{C}$.

*First Main Theorem* **2.2.3.** — *For some admissible isomorphism* $\Phi : \mathrm{E} \xrightarrow{\sim} \mathrm{E}^{\mathrm{int}}$ *there exists a* $\Phi$*-equivariant isomorphism* $f_\Phi$ *from the* $(\mathrm{E}, \mathbf{C})$*-scheme* $\mathrm{X}_{\mathbf{C}}$ *to the* $(\mathrm{E}^{\mathrm{int}}, \mathbf{C})$*-scheme* $\widetilde{\mathrm{X}}^{\mathrm{int}}$.

**2.2.4.** Let $\mathrm{E}_0$ be the kernel of the action of E on X, and put $\widetilde{\mathrm{E}} := \mathrm{E}/\mathrm{E}_0$. By Corollary 1.6.4 there exists a subgroup $\Delta \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times \widetilde{\mathrm{E}}$ such that the $(\widetilde{\mathrm{E}}, \mathbf{C})$-scheme $\mathrm{X}_{\mathbf{C}}$ corresponds to $\Delta$ by the real case of Construction 1.3.6. By Proposition 1.5.3, each admissible isomorphism $\Phi : \mathrm{E} \xrightarrow{\sim} \mathrm{E}^{\mathrm{int}}$ satisfies $\Phi(\mathrm{E}_0) = \mathrm{E}_0^{\mathrm{int}}$. Hence $\Phi$ induces an isomorphism $\overline{\Phi} : \widetilde{\mathrm{E}} \xrightarrow{\sim} \mathrm{E}^{\mathrm{int}}/\mathrm{E}_0^{\mathrm{int}}$.

*Theorem* **2.2.5.** — *There exists an admissible isomorphism* $\Phi : \mathrm{E} \xrightarrow{\sim} \mathrm{E}^{\mathrm{int}}$ *and an inner automorphism* $\varphi$ *of* $\mathrm{PGU}_{d-1,1}$ *such that* $(\varphi \times \overline{\Phi})(\Delta) = \Gamma^{\mathrm{int}}$.

*Lemma* **2.2.6.** — *Theorem* 2.2.5 *implies the First Main Theorem.*

*Proof.* — Theorem 2.2.5 implies that there exists a $\Phi$-equivariant analytic isomorphism $\widetilde{f}_\Phi : (\mathrm{X}_{\mathbf{C}})^{\mathrm{an}} \xrightarrow{\sim} (\widetilde{\mathrm{X}}^{\mathrm{int}})^{\mathrm{an}}$. From the $\Phi$-equivariance we obtain analytic isomorphism $\widetilde{f}_{\Phi,\mathrm{S}} : (\mathrm{X}_{\mathrm{S},\mathbf{C}})^{\mathrm{an}} \xrightarrow{\sim} (\widetilde{\mathrm{X}}^{\mathrm{int}}_{\Phi(\mathrm{S})})^{\mathrm{an}}$ for each $\mathrm{S} \in \mathscr{F}(\mathrm{E})$. Corollary 1.2.2 provides us with an algebraic isomorphism $f_{\Phi,\mathrm{S}} : \mathrm{X}_{\mathrm{S},\mathbf{C}} \xrightarrow{\sim} \widetilde{\mathrm{X}}^{\mathrm{int}}_{\Phi(\mathrm{S})}$ satisfying $(f_{\Phi,\mathrm{S}})^{\mathrm{an}} \cong \widetilde{f}_{\Phi,\mathrm{S}}$. Taking their inverse limit we obtain a $\Phi$-equivariant isomorphism $f_\Phi := \varprojlim_{\mathrm{S}} f_{\Phi,\mathrm{S}} : \mathrm{X}_{\mathbf{C}} \xrightarrow{\sim} \widetilde{\mathrm{X}}^{\mathrm{int}}$. □

Thus we have reduced our First Main Theorem to a purely group-theoretic statement. For proving it we need to know more information about $\Delta$. First we introduce some auxiliary notation.

**2.2.7.** Let $\Delta' \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{PE}$ and $\Delta'' \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{PE}'$ be the images of $\Delta$ under the natural projections. Since the groups $\mathrm{E}_0 \backslash \mathrm{Z}(\mathrm{E})$ and $\mathrm{E}_0 \backslash \widetilde{\mathrm{D}}_w^{\times} \cdot \mathrm{Z}(\mathrm{E})$ are compact, Lemma 1.3.11 shows that subgroups $\Delta'$ and $\Delta''$ correspond by the real case of Construction 1.3.6 to the $(\mathrm{PE}, \mathbf{C})$-scheme $\mathrm{X}'_{\mathbf{C}} := \mathrm{Z}(\mathrm{E}) \backslash \mathrm{X}_{\mathbf{C}}$ and to the $(\mathrm{PE}', \mathbf{C})$-scheme $\mathrm{X}''_{\mathbf{C}} := (\widetilde{\mathrm{D}}_w^{\times} \times \mathrm{Z}(\mathrm{E})) \backslash \mathrm{X}_{\mathbf{C}}$ respectively. The same lemma implies also that the natural projections $\Delta \to \Delta'$ and $\Delta \to \Delta''$ are isomorphisms.

Let $\mathrm{E}'_0$ be the image of $\mathrm{E}_0$ under the canonical projection to $\mathrm{E}'$. Let $\Gamma'$ be the image of $\Gamma$ under the projection $\mathrm{GL}_d(\mathrm{K}_w) \times \mathrm{E}' \to \mathrm{GL}_d(\mathrm{K}_w) \times (\mathrm{E}'_0\backslash\mathrm{E}')$. Then, by Proposition 1.5.3 *c)*, the group $\Gamma'$ corresponds by the $p$-adic case of Construction 1.3.6 to the $(\mathrm{E}'_0\backslash\mathrm{E}', \mathrm{K}_w)$-scheme $\mathrm{X}''' := \widetilde{\mathrm{D}}_w^\times\backslash\mathrm{X}$. Recall also that, by Proposition 1.5.3 *d)*, the $(\mathrm{PE}', \mathrm{K}_w)$-scheme $\mathrm{X}'' = (\widetilde{\mathrm{D}}_w^\times \times \mathrm{Z}(\mathrm{E}))\backslash\mathrm{X}$ is obtained from the subgroup $\mathrm{P}\Gamma \subset \mathrm{PGL}_d(\mathrm{K}_w) \times \mathrm{PE}'$ by the $p$-adic case of Construction 1.3.6.

For each subset $\Theta$ of $\Delta$, $\Delta'$ or $\Delta''$ (resp. of $\Gamma$, $\Gamma'$ or $\mathrm{P}\Gamma$) we denote by $\Theta_\infty$ (resp. $\Theta_\mathrm{G}$) and $\Theta_\mathrm{E}$ its projections to the first and to the second factors respectively (compare 1.1.8).

Our next task is to establish the connection between $\Delta$ and $\Gamma$. The next key proposition is the modifications of [Ch2, Prop. 2.6]. In it we apply Ihara's technique of elliptic elements to relate elements in $\Delta$ and in $\Gamma$.

*Proposition* **2.2.8.** — *For each* $\delta \in \Delta$ *with elliptic projection* $\delta_\infty \in \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$, *there exist* $\gamma \in \Gamma$ *and* $\gamma_\mathrm{D} \in \widetilde{\mathrm{D}}_w^\times$ *with* $(\gamma_\mathrm{G}, \gamma_\mathrm{D}) \in \mathrm{GL}_d(\mathrm{K}_w) \times \widetilde{\mathrm{D}}_w^\times$ *elliptic (with respect to its action on* $\Sigma^d_{\mathrm{K}_w}$*) and a representative* $\widetilde{\delta} = (\widetilde{\delta}_\infty, \widetilde{\delta}_\mathrm{E}) \in \mathrm{GU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{E}$ *of* $\delta$ *satisfying the following conditions*:

*a) the elements* $(\gamma_\mathrm{D}, \gamma_\mathrm{E})$ *and* $\widetilde{\delta}_\mathrm{E}$ *are conjugate in* E;
*b) the characteristic polynomials of* $\widetilde{\delta}_\infty$ *and* $\gamma_\mathrm{G}$ *are equal.*

*Conversely, for each* $\gamma \in \Gamma$ *and* $\gamma_\mathrm{D} \in \widetilde{\mathrm{D}}_w^\times$ *with* $(\gamma_\mathrm{G}, \gamma_\mathrm{D}) \in \mathrm{GL}_d(\mathrm{K}_w) \times \widetilde{\mathrm{D}}_w^\times$ *elliptic, there exist* $\delta \in \Delta$ *with elliptic projection* $\delta_\infty \in \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ *and a representative* $\widetilde{\delta} \in \mathrm{GU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{E}$ *of* $\delta$ *satisfying conditions a) and b).*

*Proof.* — If an element $\delta_\infty \in \Delta_\infty$ is elliptic, then $\delta_\infty$ has a fixed elliptic point P on $\mathrm{B}^{d-1}$. The action of $\delta_\infty$ on $\mathrm{B}^{d-1}$ coincides with the action of $\delta_\mathrm{E}$ on $\mathrm{B}^{d-1} \cong \mathrm{B}^{d-1} \times \{1\} \subset (\mathrm{B}^{d-1} \times \widetilde{\mathrm{E}})/\Delta \cong (\mathrm{X}_\mathbf{C})^{\mathrm{an}}$, therefore P, viewed as a point of $(\mathrm{X}_\mathbf{C})^{\mathrm{an}}$ (or of $\mathrm{X}(\mathbf{C})$), is an elliptic point of $\delta_\mathrm{E}$. Using the isomorphism $\mathbf{C} \xrightarrow{\sim} \mathbf{C}_p$, chosen above, P can be considered as a point of $\mathrm{X}(\mathbf{C}_p)$, hence as a point of the $p$-adic pro-analytic space $\mathrm{X}^{\mathrm{an}}$. There exists an element $g \in \mathrm{E}$ such that the point $\mathrm{P}' := g(\mathrm{P})$ lies in $\mathscr{Y} := \rho_1(\Sigma^d_{\mathrm{K}_w})$ in the notation of Corollary 1.5.4.

Let $\pi$ be the natural projection $\mathrm{X} \to \mathrm{X}'''$. Choose a representative $\widetilde{g} \in \mathrm{E}$ of $g\,\delta_\mathrm{E}\,g^{-1} \in \widetilde{\mathrm{E}}$. Since $\widetilde{g}$ fixes P′, it fixes the projection $\mathrm{P}'' := \pi(\mathrm{P}') \in (\mathrm{X}'''_{\mathbf{C}_p})^{\mathrm{an}}$. Hence $\widetilde{g}$ stabilizes the connected component $\Omega^d_{\mathrm{K}_w} \widehat{\otimes}_{\mathrm{K}_w} \mathbf{C}_p \times \{1\} \subset (\mathrm{X}'''_{\mathbf{C}_p})^{\mathrm{an}}$ containing P″. By Proposition 1.5.3 *c)*, the image of $\widetilde{g}$ under the canonical projection $\mathrm{E} \to \mathrm{E}'_0\backslash\mathrm{E}'$ belongs to the projection of $\Gamma'$ to $\mathrm{E}'_0\backslash\mathrm{E}'$. We can therefore choose $\gamma \in \Gamma$ whose projection to $\mathrm{E}'_0\backslash\mathrm{E}'$ coincides with that of $\widetilde{g} \in \mathrm{E}$. Therefore $\widetilde{g}\,\gamma_\mathrm{E}^{-1}$ belongs to $\widetilde{\mathrm{D}}_w^\times \times \mathrm{E}'_0 = \widetilde{\mathrm{D}}_w^\times\,\mathrm{E}_0$. Hence there exists a $\gamma_\mathrm{D} \in \widetilde{\mathrm{D}}_w^\times$ such that $\widetilde{g}(\gamma_\mathrm{D}^{-1}, \gamma_\mathrm{E}^{-1}) \in \mathrm{E}_0$. It follows that $(\gamma_\mathrm{D}, \gamma_\mathrm{E}) \in \mathrm{E}$ is also a representative of $g\,\delta_\mathrm{E}\,g^{-1}$.

The action of $(\gamma_\mathrm{D}, \gamma_\mathrm{E})$ on the tangent space of $\mathrm{P}' \in \mathscr{Y}$ is conjugate to the action of $\delta_\mathrm{E}$ on the tangent space of P, therefore P′ is an elliptic point of $(\gamma_\mathrm{D}, \gamma_\mathrm{E})$. Since $\rho_1$ is étale, one-to-one (use Corollary 1.5.4) and $\widetilde{\mathrm{D}}_w \times \Gamma$-equivariant, the action of $(\gamma_\mathrm{D}, \gamma_\mathrm{E})$ on the tangent space of $\mathrm{P}' \in \mathscr{Y}$ coincides with the action of $(\gamma_\mathrm{G}, \gamma_\mathrm{D})$ on the tangent space of $\rho_1^{-1}(\mathrm{P}') \in \Sigma^d_{\mathrm{K}_w}$. Therefore $\rho_1^{-1}(\mathrm{P}')$ is an elliptic point of $(\gamma_{\mathrm{G}}, \gamma_{\mathrm{D}})$. It follows that the action of $(\gamma_{\mathrm{G}}, \gamma_{\mathrm{D}})$ on the tangent space of $\rho^{-1}(\mathrm{P}') \in \Sigma^d_{\mathrm{K}_w}$ is conjugate to the action of $\delta_\infty$ on the tangent space of $\mathrm{P} \in \mathrm{B}^{d-1}$. Using the étalness of the projection $\Sigma^d_{\mathrm{K}_w} \to \Omega^d_{\mathrm{K}_w}$ we conclude from Lemma 1.7.2 that there exists a representative $\delta'_\infty \in \mathrm{GU}_{d-1,1}(\mathbf{R})^0$ of $\delta_\infty$ such that the characteristic polynomials of $\delta'_\infty$ and $\gamma_{\mathrm{G}}$ are equal. Hence $\widetilde{\delta} := (\delta'_\infty, g^{-1}(\gamma_{\mathrm{D}}, \gamma_{\mathrm{E}})\, g)$ is the required representative of $\delta$.

The proof of the opposite direction is very similar, but much easier technically. If an element $(\gamma_{\mathrm{G}}, \gamma_{\mathrm{D}}) \in \Gamma_{\mathrm{G}} \times \widetilde{\mathrm{D}}^\times_w$ is elliptic, then it has an elliptic point $\mathrm{Q} \in \Sigma^d_{\mathrm{K}_w}$. Hence $\mathrm{Q}' := \rho_1(\mathrm{Q}) \in \mathrm{X}^{\mathrm{an}}$ is an elliptic point of $(\gamma_{\mathrm{D}}, \gamma_{\mathrm{E}}) \in \mathrm{E}$. Hence $\mathrm{Q}'$ can be considered as a point of the complex analytic space $(\mathrm{X}_{\mathbf{C}})^{\mathrm{an}} \cong (\mathrm{B}^{d-1} \times \widetilde{\mathrm{E}})/\Delta$. Choose a representative $(x, g) \in \mathrm{B}^{d-1} \times \mathrm{E}$ of $\mathrm{Q}'$. Then the element $g(\gamma_{\mathrm{D}}, \gamma_{\mathrm{E}})\, g^{-1} \in \mathrm{E}$ fixes $\mathrm{Q}'' := g(\mathrm{Q}') \in \mathrm{B}^{d-1} \times \{1\}$, hence it stabilizes the connected component

$$\mathrm{B}^{d-1} \times \{1\} \subset (\mathrm{X}_{\mathbf{C}})^{\mathrm{an}}.$$

It follows that the image of $g(\gamma_{\mathrm{D}}, \gamma_{\mathrm{E}})\, g^{-1}$ under the projection of E to $\widetilde{\mathrm{E}}$ belongs to $\Delta_{\mathrm{E}}$. The rest of the proof is exactly the same as in the other direction. □

*Corollary* **2.2.9.** — *For each* $\delta \in \Delta$ *with elliptic projection* $\delta_\infty \in \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$, *there exists a representative*

$$\widetilde{\delta} = (\widetilde{\delta}_\infty, \widetilde{\delta}_v, \widetilde{f}_v, \widetilde{\delta}^{f;v}) \in \mathrm{GU}_{d-1,1}(\mathbf{R})^0 \times \widetilde{\mathrm{D}}^\times_w \times \mathrm{F}^\times_v \times \mathrm{G}(\mathbf{A}^{f;v}_{\mathrm{F}})$$

*such that*

*a) if we view* K *as a subset of* $\mathbf{C}$, *of* $\mathrm{K}_w$ *and of* $\mathrm{K} \otimes_{\mathrm{F}} \mathbf{A}^{f;v}_{\mathrm{F}}$ *respectively, then the characteristic polynomials of* $\widetilde{\delta}_\infty$, $\widetilde{\delta}_v$ *and* $\widetilde{\delta}^{f;v}$ *have their coefficients in* K *and coincide;*

*b)* $\widetilde{f}_v$ *and the similitude factor of* $\widetilde{\delta}^{f;v}$ *belong to* F, *viewed as a subset of* $\mathrm{F}^\times_v$ *and of* $(\mathbf{A}^{f;v}_{\mathrm{F}})^\times$ *respectively, and coincide.*

*Proof.* — Take $\gamma$ and $\widetilde{\delta}$ as in the proposition. Then the statement follows from Proposition 1.7.5. □

*Proposition* **2.2.10.** — *We have the inclusion* $\overline{\Delta'_{\mathrm{E}}} \supset (\mathrm{S}\widetilde{\mathrm{D}}^\times_w \cap \mathrm{T}_1) \times \mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathbf{A}^{f;v}_{\mathrm{F}}))$.

*Proof.* — Let $\mathrm{X}'_0$ be the connected component of $\mathrm{X}'_{\mathbf{C}}$ such that $(\mathrm{X}'_0)^{\mathrm{an}} \supset \mathrm{B}^{d-1} \times \{1\}$. Then by Proposition 1.3.8 *c)*, $\overline{\Delta'_{\mathrm{E}}} = \mathrm{Stab}_{\mathrm{PE}'}(\mathrm{X}'_0)$. Proposition 1.4.6 implies that the group $\mathrm{S}\widetilde{\mathrm{D}}^\times_w \cap \mathrm{T}_1$ acts trivially on the set of connected components of $\mathrm{X}'_{\mathbf{C}}$, therefore it remains to show only that $\overline{\Delta''_{\mathrm{E}}} \supset \mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathbf{A}^{f;v}_{\mathrm{F}}))$. To prove it we first observe that by the strong approximation theorem (see, for example, [Ma, Ch. II, Thm. 6.8]), the closure $\overline{\mathrm{PG}(\mathrm{F})}$ of $\mathrm{PG}(\mathrm{F})$ in $\mathrm{PE}' = \mathrm{PG}(\mathbf{A}^{f;v}_{\mathrm{F}})$ contains $\mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathbf{A}^{f;v}_{\mathrm{F}}))$. So the proposition follows from

*Lemma* **2.2.11.** — *We have* $\overline{\Delta''_{\mathrm{E}}} = \overline{\mathrm{PG}(\mathrm{F})}$.

*Proof.* — Proposition 1.3.8 *c)* we see that $\overline{\Delta''_{\mathrm{E}}}$ is the stabilizer of the connected component $\mathrm{Y}_\infty$ of $\mathrm{X}''_{\mathbf{C}}$ such that $(\mathrm{Y}_\infty)^{\mathrm{an}} \supset \mathrm{B}^{d-1} \times \{1\}$ and $\overline{\mathrm{PG(F)}}$ is the stabilizer of the connected component $\mathrm{Y}_p$ of $\mathrm{X}''_{\mathbf{C}_p}$ such that $(\mathrm{Y}_p)^{\mathrm{an}} \supset \Omega^d_{\mathrm{K}_w} \times \{1\}$. Since the group PE′ acts transitively on the set of geometrically connected components of X″, the subgroups $\overline{\Delta''_{\mathrm{E}}}$ and $\overline{\mathrm{PG(F)}}$ are conjugate in PE′. Since $\overline{\Delta''_{\mathrm{E}}}$ contains $\mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathbf{A}_{\mathrm{F}}^{f;v}))$, it is normal. So we are done. □

## 2.3. Computation of $\mathbf{Q}$(Tr Ad)

In the next subsection a field $\mathbf{Q}$(Tr Ad) (generated by the traces of the adjoint representation) will be a field of definition of a certain algebraic group.

*Remark* **2.3.1.** — If $g \in \mathrm{GL}_d$, then by direct computation we obtain that $\mathrm{Tr}\,\mathrm{Ad}\,g = \mathrm{Tr}\,g \cdot \mathrm{Tr}(g^{-1})$. Hence for $g \in \mathrm{PGL}_d$ we have $\mathrm{Tr}\,\mathrm{Ad}\,g = \mathrm{Tr}\,\tilde{g} \cdot \mathrm{Tr}(\tilde{g}^{-1}) - 1$ for each representative $\tilde{g} \in \mathrm{GL}_d$ of $g$.

*Proposition* **2.3.2.** — *We have* $\mathbf{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Delta_\infty) = \mathrm{F} \overset{\infty_1}{\hookrightarrow} \mathbf{R}$.

*Proof.* — It follows from Proposition 2.2.8, Proposition 1.7.5 and Remark 2.3.1 that $\mathbf{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\delta_\infty \mid \delta_\infty \in \Delta_\infty \text{ is elliptic}) = \mathbf{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\gamma_{\mathrm{G}} \mid \gamma_{\mathrm{G}} \in \mathrm{P}\Gamma_{\mathrm{G}} \subset \mathrm{PGL}_d(\mathrm{K}_w) \text{ is elliptic})$. Let F′ be the last-named field. Then $\mathrm{F}' \subset \mathrm{F}$, since $\mathrm{P}\Gamma = \mathrm{PG(F)}$ and since PG is an algebraic group defined over F. It follows from the weak approximation theorem that for each non-archimedean prime $u \neq v$ of F, the closure of the projection to $\mathrm{PG}(\mathrm{F}_u)$ of the set $\{\gamma \in \mathrm{P}\Gamma \mid \gamma_{\mathrm{G}} \text{ is elliptic}\}$ contains an open non-empty subset of $\mathrm{PG}(\mathrm{F}_u)$. (Recall that the closure of $\mathrm{P}\Gamma_{\mathrm{G}}$ in $\mathrm{PGL}_d(\mathrm{K}_w)$ contains $\mathrm{PSL}_d(\mathrm{K}_w)$ by Proposition 1.1.10, and that the set of elliptic elements of $\mathrm{PSL}_d(\mathrm{K}_w)$ is open and non-empty by Proposition 1.7.3.) Therefore F′ is dense in each non-archimedean completion $\mathrm{F}_u$ of F for $u \neq v$. Thus F′ splits completely in F at almost all places. Hence $\mathrm{F}' = \mathrm{F}$ (see [La, Ch. VII, § 4, Thm. 9]). This part of the proof is completely identical with Cherednik's proof of [Ch2, Prop. 2.7].

Now we want to prove that $\mathbf{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Delta_\infty) = \mathbf{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\delta_\infty \mid \delta_\infty \in \Delta_\infty \text{ is elliptic})$. Since the group $\mathrm{PGU}_{d-1,1}$ is absolutely simple, the representation

$$\mathrm{Ad} : \mathrm{PGU}_{d-1,1}(\mathbf{R}) \to \mathrm{GL}(\mathrm{Lie}(\mathrm{PGU}_{d-1,1}(\mathbf{R}))) \cong \mathrm{GL}_{d^2-1}(\mathbf{R})$$

is absolutely irreducible. Therefore our statement is a consequence of the following general

*Lemma* **2.3.3.** — *Let* $\rho$ *be an absolutely irreducible algebraic representation of* $\mathrm{PGU}_{d-1,1}$ *and let* $\tilde{\Delta}$ *be a dense subgroup of* $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$. *Then*

$$\mathbf{Q}(\mathrm{Tr}(\rho(\tilde{\Delta}))) = \mathbf{Q}(\mathrm{Tr}\,\rho(\delta) \mid \delta \in \tilde{\Delta} \textit{ is elliptic}).$$

*Proof.* — Let L be the last-named field. If $g \in \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ is elliptic and $g^r$ is not elliptic for some $r \in \mathbf{Z} - \{0\}$, then by Lemma 1.7.2, $g$ belongs to some Zariski closed proper subset of $\mathrm{PGU}_{d-1,1}$. Therefore for each $\mathrm{N} \in \mathbf{N}$, there exists an open subset $\mathrm{W} \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ such that for $g \in \mathrm{W}$ and $r \in \mathbf{Z}$ satisfying $1 \leqslant |r| \leqslant \mathrm{N}$, the element $g^r$ is elliptic. Choose $g \in \mathrm{W}$. By the continuity of multiplication, there exists an open neighbourhood $\mathrm{U} \subset \mathrm{W}$ of $g$ such that for $g_1, \ldots, g_k \in \mathrm{U}$, and $n_1, \ldots, n_k \in \mathbf{Z}$, satisfying $n_1 + \ldots + n_k \neq 0$, $|n_1| + \ldots + |n_k| \leqslant \mathrm{N}$, the element $g_1^{n_1} \cdot \ldots \cdot g_k^{n_k}$ is elliptic. Take $\mathrm{N} = 6m^2$, where $m$ is the dimension of $p$.

Since $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ is a connected real Lie group, it is generated by U. The subgroup $\widetilde{\Delta}$ is dense in $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ by Proposition 1.1.10, therefore $\widetilde{\Delta} \cap \mathrm{U}$ generates the group $\widetilde{\Delta}$ (see [Ma, Ch. IX, Lem. 3.3]). Since the restriction of $\rho$ to the Zariski dense subgroup $\widetilde{\Delta}$ is absolutely irreducible, Burnside's theorem (see [Wa, vol. II, Ch. XVII, 130]) implies that $\mathscr{D} := \dim_{\mathbf{R}}(\mathrm{Span}_{\mathbf{R}}(\rho(\widetilde{\Delta}))) = m^2$.

Set $\widetilde{\Delta}^0 := \{1\} \subset \widetilde{\Delta}$, and for each positive integer $n$ set

$$\widetilde{\Delta}^n := \{ g_1^{n_1} \cdot \ldots \cdot g_k^{n_k} \mid g_i \in \Delta_\infty \cap \mathrm{U}, |n_1| + \ldots + |n_k| \leqslant n \} \subset \widetilde{\Delta}.$$

Denote $\dim_{\mathbf{R}}(\mathrm{Span}_{\mathbf{R}}(\rho(\widetilde{\Delta}^n)))$ by $\mathscr{D}_n$. Since $\widetilde{\Delta} = \bigcup_n \widetilde{\Delta}^n$, we have

$$1 = \mathscr{D}_0 \leqslant \mathscr{D}_1 \leqslant \ldots \leqslant \mathscr{D}_n \leqslant \ldots \leqslant \mathscr{D} = \sup_n \mathscr{D}_n.$$

Moreover, if $\mathscr{D}_n = \mathscr{D}_{n+1}$ for some $n$, then $\mathscr{D}_n = \mathscr{D}_{n+1} = \ldots = \mathscr{D}$. Therefore $\mathscr{D}_{m^2-1} = m^2$. Hence there exist elements $\bar{\delta}_i \in \widetilde{\Delta}^{m^2-1}$, $i = 1, \ldots, m^2$ such that $\{ \rho(\bar{\delta}_i) \}_i$ constitute a basis for $\mathrm{Mat}_{m^2}(\mathbf{R})$. Choose any $g \in \widetilde{\Delta} \cap \mathrm{U}$ and take $\delta_i := g^{m^2+1} \bar{\delta}_i$. Then $\{ \rho(\delta_i) \}^i$ still constitutes a basis for $\mathrm{Mat}_{m^2}(\mathbf{R})$. Each $\delta_i$ is of the form $g_1^{n_1} \cdot \ldots \cdot g_k^{n_k}$, where the $g_i$'s belong to $\widetilde{\Delta} \cap \mathrm{U}$ and the $n_i$'s satisfy $n_1 + \ldots + n_k \geqslant 2$ and $|n_1| + |n_2| + \ldots + |n_k| \leqslant 2m^2$. In particular, each $\delta_i$ is elliptic, therefore $\mathrm{Tr}\, \rho(\delta_i) \in \mathrm{L}$.

*Lemma* **2.3.4.** — *If for some $\delta \in \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ the elements $\delta\delta_i$ are elliptic for all $i = 1, \ldots, m^2$, then $\rho(\delta)$ can be written as a linear combination of the $\rho(\delta_i)$'s with coefficients in* L.

*Proof.* — Let $e_1, \ldots, e_{m^2}$ be the dual basis of $\{ \rho(\delta_i) \}_i$ relative to the bilinear form $(x, y) \mapsto \mathrm{Tr}(xy)$. If $\delta$ is as in the lemma, then $\mathrm{Tr}\, \rho(\delta\delta_i) = \mathrm{Tr}(\rho(\delta)\, \rho(\delta_i)) \in \mathrm{L}$ for all $i = 1, \ldots, m^2$. Hence $\rho(\delta)$ can be written as a linear combination of the $e_i$'s with coefficients in L. Therefore it is enough to prove that each $e_i$ can be written as a linear combination of the $\rho(\delta_i)$'s with coefficients in L. The last condition is equivalent to the condition that each $\rho(\delta_i)$ can be written as a linear combination of the $e_i$'s with coefficients in L. Thus, as we mentioned above, to complete the proof it is enough to show that each $\delta_i$ satisfies the conditions of the lemma. This follows directly from the definition of the $\delta_i$'s and of U. □

The choice of the $\delta_i$'s assures that for every $\delta \in \widetilde{\Delta} \cap (\mathrm{U} \cup \mathrm{U}^{-1})$ the elements $\delta\delta_i$

are elliptic for all $i = 1, \dots, m^2$. Therefore the above lemma implies that $\rho(\delta)$ can be written as a linear combination of the $\rho(\delta_i)$'s with coefficients in L. The set $\mathrm{U} \cap \widetilde{\Delta}$ generates the group $\widetilde{\Delta}$, hence for every $\delta \in \widetilde{\Delta}$, the linear transformation $\rho(\delta)$ can be written as a polynomial in the $\rho(\delta_i)$'s with coefficients in L. For any $i, j, k \in \{1, \dots, m^2\}$ the elements $\delta_i\, \delta_j\, \delta_k$ are elliptic, therefore by the lemma each $\rho(\delta_i\, \delta_j) = \rho(\delta_i)\, \rho(\delta_j)$ can be written as a linear combination of the $\rho(\delta_k)$'s with coefficients in L. Hence every polynomial in the $\rho(\delta_i)$'s with coefficients in L, can be written as a linear combination of the $\rho(\delta_i)$'s with coefficients in L. In particular, this is true for each $\rho(\delta)$ with $\delta \in \widetilde{\Delta}$. Hence $\mathbf{Q}(\mathrm{Tr}\, \rho(\widetilde{\Delta})) \subset \mathrm{L}$. □

*Corollary* **2.3.5.** — *Suppose that a subgroup* $\widetilde{\Delta} \subset \Delta_\infty$ *is Zariski dense in* $\mathrm{PGU}_{d-1,1}$ *and that* $\Delta_\infty \subset \mathrm{Comm}_{\mathrm{PGU}_{d-1,1}(\mathbf{R})}(\widetilde{\Delta})$. *Then* $\mathbf{Q}(\mathrm{Tr\, Ad}\, \widetilde{\Delta}) = \mathbf{Q}(\mathrm{Tr\, Ad}\, \Delta_\infty)$ $(= \mathrm{F})$.

*Proof.* — Set $\mathrm{L} := \mathbf{Q}(\mathrm{Tr\, Ad}\, \widetilde{\Delta})$, then there exists an L-form V of $\mathrm{Lie}(\mathrm{PGU}_{d-1,1}(\mathbf{R}))$ preserved by $\mathrm{Ad}\, \widetilde{\Delta}$ (see [Ma, Ch. VIII, Prop. 3.22]). Take any $\delta \in \Delta_\infty$. Then some subgroup of finite index $\widetilde{\Delta}'$ of $\widetilde{\Delta}$ satisfies $\delta \widetilde{\Delta}'\, \delta^{-1} \subset \widetilde{\Delta}$, hence $(\mathrm{Ad}\, \delta)\, (\mathrm{Ad}\, \widetilde{\Delta}')\, (\mathrm{Ad}\, \delta)^{-1}(\mathrm{V}) = \mathrm{V}$.

Since the subgroup $\widetilde{\Delta}'$ is also Zariski dense in $\mathrm{PGU}_{d-1,1}$, Burnside's theorem implies that $\mathrm{Ad}\, \widetilde{\Delta}'$ generates End V as an L-vector space. Therefore

$$(\mathrm{Ad}\, \delta)\, (\mathrm{End\, V})\, (\mathrm{Ad}\, \delta)^{-1} \subset \mathrm{End\, V}.$$

In other words, $\mathrm{Ad}(\mathrm{Ad}\, \delta)\, (\mathrm{End\, V}) = \mathrm{End\, V}$. Let H be the Zariski closure of $\mathrm{Ad}\, \widetilde{\Delta} \subset \mathrm{GL}(\mathrm{V})$. Then H is an L-form of $\mathrm{Ad\, PGU}_{d-1,1}$, hence $\mathrm{Lie\, H} \subset \mathrm{End\, V}$ is an L-form of $\mathrm{Lie}(\mathrm{Ad\, PGU}_{d-1,1})$. In particular, $\mathrm{Lie\, H} = \mathrm{End\, V} \cap \mathrm{Lie}(\mathrm{Ad\, PGU}_{d-1,1})$, therefore $\mathrm{Ad}(\mathrm{Ad}\, \Delta_\infty)\, (\mathrm{Lie\, H}) = \mathrm{Lie\, H}$. Since $\mathrm{PGU}_{d-1,1}$ is adjoint, the homomorphism $\mathrm{ad} := \mathrm{Ad}_* : \mathrm{Lie\, PGU}_{d-1,1} \to \mathrm{Lie}(\mathrm{Ad\, PGU}_{d-1,1})$ is an isomorphism. Therefore $\widetilde{\mathrm{V}} := \mathrm{ad}^{-1}(\mathrm{Lie\, H})$ is an L-form of $\mathrm{Lie}(\mathrm{PGU}_{d-1,1})$ and $\mathrm{Ad}\, \Delta_\infty \subset \mathrm{GL}(\widetilde{\mathrm{V}})$. It follows that $\mathbf{Q}(\mathrm{Tr\, Ad}\, \Delta_\infty) \subset \mathrm{L}$. □

## 2.4. Proof of arithmeticity

**2.4.1.** Consider the subgroup $\Delta' \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{PE} \subset \mathrm{PGU}_{d-1,1}(\mathbf{R}) \times \mathrm{PE}$, defined in 2.2.7. For a finite place $u$ of F let $\mathrm{G}_u$ be $\mathrm{PG}_{\mathrm{F}_u}$ for $u \neq v$ and $\mathrm{PGL}_1(\widetilde{\mathrm{D}}_w)$, viewed as an algebraic group over $\mathrm{F}_v \cong \mathrm{K}_w$, for $u = v$. In what follows it will be also convenient to introduce a formal symbol $\infty$ and to write $\mathrm{F}_\infty$ instead of $\mathbf{R}$ and $\mathrm{G}_\infty$ instead of $\mathrm{PGU}_{d-1,1}$ (the algebraic group over $\mathrm{F}_\infty \cong \mathbf{R}$).

Let M be a finite set of non-archimedean primes of F, containing $v$ for simplicity of notation. Set $\overline{\mathrm{M}} := \mathrm{M} \cup \infty$ and choose $\mathrm{S} \in \mathscr{F}(\mathrm{PG}(\mathbf{A}_\mathrm{F}^{f;\mathrm{M}}))$. For each subset M′ of $\overline{\mathrm{M}}$, denote $\prod_{u \in \mathrm{M}'} \mathrm{G}_u(\mathrm{F}_u)$ by $\mathrm{G}_{\mathrm{M}'}$. Denote also the projection of $\Delta' \cap (\mathrm{G}_{\overline{\mathrm{M}}} \times \mathrm{S})$ to $\mathrm{G}_{\overline{\mathrm{M}}}$ by $\Delta^\mathrm{S}$. Let $\Delta^\mathrm{S}_\infty$ (resp. $\Delta^\mathrm{S}_\mathrm{E}$) be the projection of $\Delta^\mathrm{S}$ to $\mathrm{G}_\infty(\mathrm{F}_\infty)$ (resp. to $\mathrm{G}_\mathrm{M}$). For $u \in \overline{\mathrm{M}}$ and $\delta \in \Delta^\mathrm{S}$ denote the projection of $\delta$ to $\mathrm{G}_u(\mathrm{F}_u)$ by $\delta_u$.

*Definition* **2.4.2.** — A lattice $\Gamma \subset G_{\overline{M}}$ is called *irreducible* if for every proper non-empty subset $M' \subset \overline{M}$ the subgroup $(\Gamma \cap G_{M'})\,(\Gamma \cap G_{\overline{M}-M'})$ is of infinite index in $\Gamma$ (compare [Ma, p. 133]).

*Definition* **2.4.3.** — We say that a lattice $\Gamma$ of $G_{\overline{M}}$ has *property* (QD′) if the closure of $\Gamma G_\infty(F_\infty)$ in $G_{\overline{M}}$ has finite index.

*Remark* **2.4.4.** — Since the group $PGU_{d-1,1}$ is isotropic over $\mathbf{R}$, it follows from [Ma, p. 290, Rem. (v)] that if $\Gamma$ has property (QD′), then it has property (QD) in the sense of Margulis (see [Ma, p. 289]).

*Proposition* **2.4.5.** — *The subgroup* $\Delta^S \subset G_{\overline{M}}$ *is a finitely generated cocompact irreducible lattice, which is of infinite index in* $\mathrm{Comm}_{G_{\overline{M}}}(\Delta^S)$ *and has property* (QD′).

*Proof.* — Observe that $PGU_{d-1,1}(\mathbf{R}) \times PE = G_{\overline{M}} \times PG(\mathbf{A}_F^{f;M})$ and that $\Delta'$ is a cocompact lattice in $G_{\overline{M}} \times PG(\mathbf{A}_F^{f;M})$ having injective projection to $PGU_{d-1,1}(\mathbf{R})$, hence to $G_{\overline{M}}$. It follows from Lemma 1.1.9 that $\Delta^S \subset G_{\overline{M}}$ is a cocompact lattice, which is of infinite index in $\mathrm{Comm}_{G_{\overline{M}}}(\Delta^S)$. By Proposition 2.2.10 the closure of $G_\infty(F_\infty)\Delta'$ in $G_{\overline{M}} \times PG(\mathbf{A}_F^{f;M})$ contains $G_\infty(F_\infty) \times (S\widetilde{D}_w^\times \cap T_1) \times P(G^{der}(\mathbf{A}_F^{f;v}))$. Hence the closure of $G_\infty(F_\infty)\Delta'$ in $G_{\overline{M}}$ contains $G_\infty(F_\infty) \times (S\widetilde{D}_w^\times \cap T_1) \times \prod_{u \in M-\{v\}} P(G^{der}(F_u))$.

In particular, $\Delta^S$ has property (QD′). Let $M'$ be a non-empty subset of M. Then $\Delta^S \cap G_{M'} = \{1\}$, because the projection of $\Delta'$ to $PGU_{d-1,1}(\mathbf{R})$ is injective. Suppose that $\Delta^S$ is not irreducible, then $[\Delta^S : (\Delta^S \cap G_{\overline{M}-M'})] < \infty$. Hence

$$\overline{[G_\infty(F_\infty)\,\Delta^S} : \overline{(G_\infty(F_\infty)\,\Delta^S) \cap G_{\overline{M}-M'}}] < \infty.$$

Since $\overline{G_\infty(F_\infty)\Delta^S} \supset G_\infty(F_\infty) \times (S\widetilde{D}_w^\times \cap T_1) \times \prod_{u \in M-\{v\}} P(G^{der}(F_u))$ and

$$\overline{(G_\infty(F_\infty)\Delta^S) \cap G_{\overline{M}-M'}} \subset G_{\overline{M}-M'},$$

we get a contradiction. Since $\Delta^S$ is a cocompact lattice in $G_{\overline{M}}$, it is finitely generated (see [Ma, Ch. IX, 3.1 (v)]). □

**2.4.6.** Now we are going to use the results of Margulis (see [Ma]). By [Ma, Ch. VIII, Prop. 3.22], there exists a basis in $\mathrm{Lie}(PGU_{d-1,1}(\mathbf{R}))$ such that all transformations in $\mathrm{Ad}\,\Delta_\infty$ are written in this basis as matrices with entries in $\mathbf{Q}(\mathrm{Tr}\,\mathrm{Ad}\,\Delta_\infty) = F \subset F_{\infty_1} \cong \mathbf{R}$. Define a homomorphism $\varphi : G_\infty \to GL_{d^2-1}$ rational over $\mathbf{R}$ by assigning to $g \in G_\infty$ the matrix of $\mathrm{Ad}\,g$ in the above basis. It follows that $\varphi(\Delta_\infty) \subset GL_{d^2-1}(F)$. Let H be the Zariski closure of $\varphi(\Delta_\infty)$; then H is an algebraic group, defined over F and $\varphi(\Delta_\infty) \subset H(F)$. Since $\Delta_\infty$ is Zariski dense in $G_\infty$ and since the group $G_\infty = PGU_{d-1,1}$ is adjoint, $\varphi$ induces an isomorphism $PGU_{d-1,1} \xrightarrow{\sim} H_{F_{\infty_1}}$. In particular, H is an F-form of $PGU_{d-1,1}$.

By Proposition 2.4.5, $\Delta^S$ satisfies the conditions of Theorem (B) of [Ma, p. 298], therefore it is arithmetic in the sense of [Ma, p. 292]. The group $\Delta^S_\infty$ is Zariski dense in $G_\infty$ (see [Ma, Ch. IX, Lem. 2.1]) and $\Delta_\infty \subset \mathrm{Comm}_{G_\infty(F_\infty)}(\Delta^S_\infty)$. Hence $\varphi(\Delta^S_\infty)$ is Zariski dense in H and $\mathbf{Q}(\mathrm{Tr\,Ad}\,\Delta^S_\infty) = F$ (by Corollary 2.3.5).

It follows (see Margulis' proof [Ma, p. 307-311]) that the following conditions are satisfied:

*a)* The group $H(F_{\infty_i})$ is compact for each $i = 2, \dots, g$.

*b)* There exists a unique bijection I from M to a (finite) set of non-archimedean primes of F satisfying the following property: for each $u \in M$ there exists a continuous isomorphism $\omega_u : F_u \xrightarrow{\sim} F_{I(u)}$ and an $\omega_u$-algebraic isomorphism $\tau_u : G_u \xrightarrow{\sim} H$ (that is $\tau_u$ becomes an isomorphism of algebraic groups over $F_u$ after the identification of $F_u$ with $F_{I(u)}$ by means of $\omega_u$) such that $\tau_u(\delta_u) = \varphi(\delta_\infty) \in H(F) \subset H(F_{I(u)})$ for all $\delta \in \Delta^S$. Since the subgroup $\Delta^S_u$ is Zariski dense in $G_u$ (see [Ma, Ch. IX, Lem. 2.1]), $\tau_u$ is unique.

*c)* Let $\tau_M : \prod_{u \in M} G_u(F_u) \xrightarrow{\sim} \prod_{u \in M} H(F_{I(u)})$ be the product of the $\tau_u$'s. Put $\mathcal{O}_{F, I(M)} := \{f \in F \mid f \in \mathcal{O}_{F_u}$ for each finite prime $u \notin I(M)$ of $F\}$. Then the subgroup $\tau(\Delta^S_E) \subset H(F)$ is commensurable with $H(\mathcal{O}_{F, I(M)})$.

Taking M larger and larger we conclude from *b)* that there exists a unique one-to-one surjective map I of the set of all non-archimedean primes of F into itself such that for each prime $u$ of F there exists a continuous isomorphism $\omega_u : F_u \xrightarrow{\sim} F_{I(u)}$ and a unique $\omega_u$-algebraic isomorphism $\tau_u : G_u \xrightarrow{\sim} H$ such that $\tau_u(\delta_u) = \varphi(\delta_\infty) \in H(F) \subset H(F_{I(u)})$ for all $\delta \in \Delta'$. The maps $\tau_u$ combined together for all non-archimedean primes $u$ of F give us a continuous isomorphism $\tau : \prod_u G_u(F_u) \xrightarrow{\sim} \prod_u H(F_u)$ such that

$$\tau(\Delta'_E) \subset H(F) \subset H(\mathbf{A}^f_F) \subset \prod_u H(F_u).$$

By *c)*, the subgroup $\tau(\Delta'_E \cap S)$ is commensurable with $H(\mathcal{O}_F)$ for each $S \in \mathcal{F}(PE)$.

## 2.5. Determination of H

**2.5.1.** Recall that H is an F-form of $PGU_{d-1,1}$. In particular, it is a form of $PGL_d$. By the classification of simple algebraic groups (see [Ti]), there exists a quadratic extension $F'$ of F and a central simple algebra $D'$ over $F'$ of dimension $d^2$ (defined up to a replacement $D' \mapsto (D')^{\mathrm{opp}}$) with an involution of the second kind $\alpha'$ over F such that $H \cong PGU(D', \alpha')$. Moreover, $F'$ is uniquely determined if $d > 2$ and can be chosen arbitrary if $d = 2$. We denote the group $GU(D', \alpha')$ by $G'$ and will not distinguish between H and $PG'$.

*Claim* **2.5.2.** — *For each non-archimedean prime $u$ of* F, *we have* $I(u) = u$ *and* $\omega_u$ *is the identity.*

*Proof.* — Since the map $\tau_u : G_u \xrightarrow{\sim} H$ is $\omega_u$-algebraic, we have

$$\mathrm{Tr\,Ad}(\tau_u(g)) = \omega_u(\mathrm{Tr\,Ad}(g))$$

for each $g \in G_u(F_u)$. Hence for each $\delta \in \Delta'$ we have

$$\mathrm{Tr\,Ad}(\varphi \times \tau)(\delta) = (\mathrm{Tr\,Ad}(\delta_\infty); \dots, \omega_u(\mathrm{Tr\,Ad}(\delta_u)), \dots) \\ \in (F_{\infty_1}; \dots, F_{I(u)}, \dots).$$

Recall that $(\varphi \times \tau)(\Delta') \subset H(F)$, hence $\mathrm{Tr\,Ad}((\varphi \times \tau)\,\Delta') \subset F$. On the other hand, Corollary 2.2.9 implies that $\mathrm{Tr\,Ad}(\delta) \in F \subset F_{\infty_1} \times \mathbf{A}_F^f$ for each $\delta \in \Delta'$ with elliptic $\delta_\infty$. In particular, for such $\delta$'s we have $\mathrm{Tr\,Ad}(\delta_\infty) = \mathrm{Tr\,Ad}(\delta_u) \in F$ for each $u$. Since we showed in the proof of Proposition 2.3.2 that $\mathbf{Q}(\mathrm{Tr\,Ad}(\delta_\infty) \mid \delta_\infty \text{ is elliptic}) = F$, we conclude from the above that the restriction of each $\omega_u : F_u \xrightarrow{\sim} F_{I(u)}$ to F is the identity. Since each $\omega_u$ is continuous, the claim follows. □

**2.5.3.** Next we will show that in the case $d > 2$ we have $F' = K$. Indeed, if a prime $u$ of F splits in K, then $PG'(F_u) \cong G_u(F_u) \cong P\bar{D}_u^\times$ for some central simple algebra $\bar{D}_u$ over $F_u$. It follows that $u$ splits in F′. By [La, Ch. VII, § 4, Thm. 9], $F' = K$. As we mentioned before, we may take $F' = K$ also in the case $d = 2$.

*Proposition* **2.5.4.** — *The map* $\tau$ *induces a continuous isomorphism* $PE \xrightarrow{\sim} H(\mathbf{A}_F^f)$.

*Proof.* — Since $PE \cong P\tilde{D}_w^\times \times PE'$ and $H(\mathbf{A}_F^f) \cong H(F_v) \times H(\mathbf{A}_F^{f;v})$, we need only to show that $\tau^v : \prod_{u \neq v} G_u(F_u) \xrightarrow{\sim} \prod_{u \neq v} H(F_u)$ induces a continuous isomorphism $PE' \xrightarrow{\sim} H(\mathbf{A}_F^{f;v})$.

First we claim that $\tau^v$ induces a continuous map from $\overline{\Delta'} \subset PE'$ to $H(\mathbf{A}_F^{f;v})$. In fact, let a sequence $\{\delta_n\}_n \subset \Delta'_E$ converge to $g \in PE'$. Then the sequence $\{\delta_n \delta_{n+1}^{-1}\}_n$ converges to 1. Therefore for each $S \in \mathscr{F}(PE')$ there exists $N_S \in \mathbf{N}$ such that $\delta_n \delta_{n+1}^{-1} \in \Delta''_E \cap S$ (hence $\tau^v(\delta_n \delta_{n+1}^{-1}) \in \tau^v(\Delta''_E \cap S)$) for all $n \geqslant N_S$. Since $\tau^v(\Delta''_E \cap S)$ is commensurable with $H(\mathcal{O}_F)$, it is contained in a compact subset of $H(\mathbf{A}_F^{f;v})$. Therefore the sequence $\{\tau^v(\delta_n \delta_{n+1}^{-1})\}_n \subset H(\mathbf{A}_F^{f;v})$ has a limit point. Let $h$ be some limit point of $\{\tau^v(\delta_n \delta_{n+1}^{-1})\}_n$, and let $\{\tau^v(\delta_{n_i} \delta_{n_i+1}^{-1})\}_i$ be a subsequence, converging to $h$. Then for each prime $u \neq v$ of F we have

$$h_u = \lim_{i \to \infty} \tau_u((\delta_{n_i} \delta_{n_i+1}^{-1})_u) = \tau_u(\lim_{i \to \infty} (\delta_{n_i} \delta_{n_i+1}^{-1})_u) = 1,$$

because $\tau_u$ is continuous. It follows that 1 is the only limit point of $\{\tau^v(\delta_n \delta_{n+1}^{-1})\}_n$, therefore the sequence $\{\tau^v(\delta_n)\,\tau^v(\delta_{n+1}^{-1})\}_n$ converges to 1. Now by similar arguments we see that the sequence $\{\tau^v(\delta_n)\}_n$ converges to $\tau^v(g) \in H(\mathbf{A}_F^{f;v})$.

Moreover, the same arguments also imply that if we show that $\tau^v(PE') = H(\mathbf{A}_F^{f;v})$, then the continuity of $\tau^v$ and of $(\tau^v)^{-1}$ will follow automatically.

Observe that for each non-archimedean place $u$ we have $G(F_u)^{\mathrm{der}} = G^{\mathrm{der}}(F_u)$ (resp. $G'(F_u)^{\mathrm{der}} = (G')^{\mathrm{der}}(F_u)$) (see [PR, 1.3.4 and Thm. 6.5] in the anisotropic and [PR, Thm. 7.1 and 7.5] in the isotropic cases respectively). Therefore $\tau^v$ induces an isomorphism of derived groups $\prod_{u \neq v} P(G^{\mathrm{der}}(F_u)) \xrightarrow{\sim} \prod_{u \neq v} P((G')^{\mathrm{der}}(F_u))$.

By Proposition 2.2.10, $\overline{\Delta''_{\mathrm{E}}} \supset \mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathbf{A}_{\mathrm{F}}^{f;v})) = \mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f;v}) \cap \prod_{u \neq v} \mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathrm{F}_u))$. Hence by the facts shown above,

$$\tau^v(\mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathbf{A}_{\mathrm{F}}^{f;v}))) \subset \mathrm{PG}'(\mathbf{A}_{\mathrm{F}}^{f;v}) \cap \prod_{u \neq v} \mathrm{P}((\mathrm{G}')^{\mathrm{der}}(\mathrm{F}_u)) = \mathrm{P}((\mathrm{G}')^{\mathrm{der}}(\mathbf{A}_{\mathrm{F}}^{f;v})).$$

In particular, $\tau^v(\prod_{u \neq v} \mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathcal{O}_{\mathrm{F}_u}))) = \prod_{u \neq v} \tau_u(\mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathcal{O}_{\mathrm{F}_u}))) \subset \mathrm{P}((\mathrm{G}')^{\mathrm{der}}(\mathbf{A}_{\mathrm{F}}^{f;v}))$. It follows that $\tau_u(\mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathcal{O}_{\mathrm{F}_u})) \subset \mathrm{P}((\mathrm{G}')^{\mathrm{der}}(\mathcal{O}_{\mathrm{F}_u}))$ for almost all $u \neq v$. Since each $\tau_u$ is algebraic, the subgroups $\tau_u(\mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathcal{O}_{\mathrm{F}_u}))$ and $\mathrm{P}((\mathrm{G}')^{\mathrm{der}}(\mathcal{O}_{\mathrm{F}_u}))$ are conjugate (hence equal) for almost all $u \neq v$. It follows that $\tau^v(\mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathbf{A}_{\mathrm{F}}^{f;v}))) = \mathrm{P}((\mathrm{G}')^{\mathrm{der}}(\mathbf{A}_{\mathrm{F}}^{f;v}))$.

Therefore to complete the proof it will suffice to show that $\mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f;v})$ (resp. $\mathrm{PG}'(\mathbf{A}_{\mathrm{F}}^{f;v})$) is the normalizer of $\mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathbf{A}_{\mathrm{F}}^{f;v}))$ in the product $\prod_{u \neq v} \mathrm{PG}(\mathrm{F}_u)$, and similarly for $\mathrm{PG}'$. Since $\mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f;v})$ is the restricted topological product of the $\mathrm{PG}(\mathrm{F}_u)$'s with respect to the $\mathrm{PG}(\mathcal{O}_{\mathrm{F}_u})$'s, it remains to show that the normalizer of $\mathrm{P}(\mathrm{G}^{\mathrm{der}}(\mathcal{O}_{\mathrm{F}_u}))$ in $\mathrm{PG}(\mathrm{F}_u)$ is $\mathrm{PG}(\mathcal{O}_{\mathrm{F}_u})$ for almost all $u$. This can be done by direct calculation. □

We will use the same letter $\tau$ to denote the isomorphism between PE and $\mathrm{PG}'(\mathbf{A}_{\mathrm{F}}^{f;v})$.

**2.5.5.** Notice that a regular function $t := \mathrm{Tr}^d/\det$ on $\mathrm{GL}_d$ defines a function on $\mathrm{PGL}_d$. Moreover, an algebraic automorphism $\psi$ of $\mathrm{PGL}_d$ is inner if and only if it satisfies $t \circ \psi = t$. Therefore there is a unique choice of an algebra D′ defining G′ (see 2.5.1) such that the function $t' := \mathrm{Tr}^d/\det$ on PG′, defined by the natural embedding $\mathrm{G}'(\mathrm{F}) \hookrightarrow \mathrm{D}'$, satisfies $t' \circ \varphi = t$.

*Proposition* **2.5.6.** — *We have* $\mathrm{D}' \cong \mathrm{D}^{\mathrm{int}}$, $\mathrm{G}' \cong \mathrm{G}^{\mathrm{int}}$ *and* $\tau$ *is induced by some admissible isomorphism.*

*Proof.* — By Corollary 2.2.9, for each $\delta \in \Delta'$ with elliptic $\delta_\infty$ we have $t(\delta_\infty) = t(\delta_v) = t(\delta^{f;v}) \in \mathrm{K}$. Since $(\varphi \times \tau)(\delta) \in \mathrm{PG}'(\mathrm{F}) \subset \mathrm{PG}'(\mathrm{F}_{\infty_1} \times \mathbf{A}_{\mathrm{F}}^{f})$, we have $t((\varphi \times \tau)\,\delta) \in \mathrm{K}$. By our assumption, $t(\varphi(\delta_\infty)) = t(\delta_\infty)$ for all $\delta \in \Delta'$. Hence for each $\delta \in \Delta'$ with elliptic $\delta_\infty$ we have $t(\tau_u(\delta_u)) = t(\delta_u)$ for each non-archimedean prime $u$ of F.

Recall that the algebraic isomorphism $\tau_u : \mathrm{PG}(\mathrm{F}_u) \to \mathrm{PG}'(\mathrm{F}_u)$ for $u \neq v$ is induced either by an $\mathrm{F}_u$-linear isomorphism $\mathrm{D} \otimes_{\mathrm{F}} \mathrm{F}_u \xrightarrow{\sim} \mathrm{D}' \otimes_{\mathrm{F}} \mathrm{F}_u$ or by an $\mathrm{F}_u$-linear isomorphism $\mathrm{D} \otimes_{\mathrm{F}} \mathrm{F}_u \xrightarrow{\sim} (\mathrm{D}')^{\mathrm{opp}} \otimes_{\mathrm{F}} \mathrm{F}_u$, composed with an inverse map $(g \mapsto g^{-1})$. In the first case we have $t(\tau_u(g_u)) = t(g_u)$ for all $g_u \in \mathrm{G}_u(\mathrm{F}_u)$, and in the second one $t(\tau_u(g_u)) = t(g_u^{-1})$ for all $g_u \in \mathrm{G}_u(\mathrm{F}_u)$.

To exclude the second possibility we need to show the existence of a $\delta \in \Delta'$ with elliptic $\delta_\infty$ such that $t(\delta_\infty) \neq t(\delta_\infty^{-1})$. Since the condition $t(g) = t(g^{-1})$ is Zariski closed and non-trivial and since the closure of all elliptic elements of $\Delta_\infty \in \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ contains an open non-empty set, we are done.

It follows that D′ is locally isomorphic to $\mathrm{D}^{\mathrm{int}}$ at every non-archimedean place of K, except possibly at $w$ and $\overline{w}$, and that the map $\tau^v : \mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f;v}) \xrightarrow{\sim} \mathrm{PG}'(\mathbf{A}_{\mathrm{F}}^{f;v})$ is induced by some admissible isomorphism. To prove the statement for the $v$-component we copy the above proof replacing $\mathrm{PG}(\mathrm{F}_u)$ by $\mathrm{PGL}_1(\widetilde{\mathrm{D}}_w)$ and $\mathrm{D} \otimes_{\mathrm{F}} \mathrm{F}_u$ by $\widetilde{\mathrm{D}}_w \oplus \widetilde{\mathrm{D}}_w^{\mathrm{opp}}$.

Since $\mathrm{D}'$ and $\mathrm{D}^{\mathrm{int}}$ are locally isomorphic at all places, they are isomorphic. We showed before that $\mathrm{PG}'(\mathrm{F}_{\infty_1}) \cong \mathrm{PGU}_{d-1,1}(\mathbf{R})$ and that for each $i = 2, \ldots, g$ the group $\mathrm{PG}'(\mathrm{F}_{\infty_i})$ is compact and, therefore, is isomorphic to $\mathrm{PGU}_d(\mathbf{R})$. Proposition 2.1.7 *b)* then implies that $\mathrm{G}' \cong \mathrm{G}^{\mathrm{int}}$. □

From now on we identify $\mathrm{G}'$ with $\mathrm{G}^{\mathrm{int}}$.

## 2.6. Completion of the proof

Our next task is to prove the following

*Proposition* **2.6.1.** — *We have* $(\varphi \times \tau)\,(\Delta') = \mathrm{PG}^{\mathrm{int}}(\mathrm{F})_+$.

*Proof.* — First observe that

$$(\varphi \times \tau)\,(\Delta')_\infty = \varphi(\Delta_\infty) \subset \varphi(\mathrm{PGU}_{d-1,1}(\mathbf{R})^0) = \mathrm{PG}^{\mathrm{int}}(\mathrm{F}_{\infty_1})^0,$$

therefore $(\varphi \times \tau)\,(\Delta') \subset \mathrm{PG}^{\mathrm{int}}(\mathrm{F})_+$ and $(\varphi \times \tau^v)\,(\Delta'') \subset \mathrm{PG}^{\mathrm{int}}(\mathrm{F})_+$. Since the projection of $\mathrm{PG}^{\mathrm{int}}(\mathrm{F})$ to $\mathrm{PG}^{\mathrm{int}}(\mathrm{F}_{\infty_1}) \times \mathrm{PG}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f;v})$ is injective, it remains to show that

$$\text{(2.1)} \qquad [\mathrm{PG}^{\mathrm{int}}(\mathrm{F}) : \mathrm{PG}^{\mathrm{int}}(\mathrm{F})_+] = [\mathrm{PG}^{\mathrm{int}}(\mathrm{F}) : (\varphi \times \tau^v)\,(\Delta'')].$$

We are going to use of Kottwitz' results described in 1.8. Recall that $\mathrm{PG}^{\mathrm{int}}$ is an inner form of PG. Let $\omega_{\mathrm{PG}}$ and $\omega_{\mathrm{PG}^{\mathrm{int}}}$ be non-zero invariant differential forms of top degree on PG and $\mathrm{PG}^{\mathrm{int}}$ respectively, connected with one another by some inner twist as in 1.8.2. They define invariant measures $|\omega_{\mathrm{PG}}|$ and $|\omega_{\mathrm{PG}^{\mathrm{int}}}|$ on $\mathrm{PG}(\mathrm{F}_u)$ and $\mathrm{PG}^{\mathrm{int}}(\mathrm{F}_u)$ for every completion $\mathrm{F}_u$ of F and product measures on $\mathrm{PG}(\mathbf{A}_{\mathrm{F}})$ and $\mathrm{PG}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}})$ respectively (see [We2, Ch. 2]). It follows from Weil's conjecture on Tamagawa numbers and from Ono's result (see Ono's appendix to [We2]) that

$$\text{(2.2)} \qquad |\omega_{\mathrm{PG}^{\mathrm{int}}}|\,(\mathrm{PG}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}})/\mathrm{PG}^{\mathrm{int}}(\mathrm{F})) = |\omega_{\mathrm{PG}}|\,(\mathrm{PG}(\mathbf{A}_{\mathrm{F}})/\mathrm{PG}(\mathrm{F})).$$

*Lemma* **2.6.2.** — *Let* A *and* B *be locally compact groups, let* S *be a compact and open subgroup of* A *and let* Γ *be a lattice in* A × B *with injective projection to* B. *Then for every right invariant measures* $\mu_{\mathrm{A}}$ *on* A *and* $\mu_{\mathrm{B}}$ *on* B *we have*

$$(\mu_{\mathrm{A}} \times \mu_{\mathrm{B}})\,([\mathrm{A} \times \mathrm{B}]/\Gamma) = \mu_{\mathrm{A}}(\mathrm{S}) \cdot \mu_{\mathrm{B}}([(\mathrm{S}\backslash\mathrm{A}) \times \mathrm{B}]/\Gamma).$$

*Proof.* — Let $\Gamma_{\mathrm{A}}$ be the projection of Γ to A. Choose representatives $\{a_i\}_{i \in \mathrm{I}}$ of the double classes $\mathrm{S}\backslash\mathrm{A}/\Gamma_{\mathrm{A}}$. For each $i \in \mathrm{I}$ let $\Gamma_i$ be the projection of the subgroup $(a_i^{-1}\mathrm{S}a_i \times \mathrm{B}) \cap \Gamma$ to B. Then $\Gamma_i$ is a lattice in B, therefore there exists a measurable subset $\mathrm{U}_i$ of B such that B is the disjoint union $\coprod_{\gamma \in \Gamma_i} \mathrm{U}_i\gamma$. Since Γ has an injective projection to B, we have $\mathrm{A} \times \mathrm{B} = \coprod_{\gamma \in \Gamma} \coprod_{i \in \mathrm{I}} (\mathrm{S}a \times \mathrm{U}_i)\,\gamma$. Then

$$\begin{aligned}(\mu_{\mathrm{A}} \times \mu_{\mathrm{B}})\,([\mathrm{A} \times \mathrm{B}]/\Gamma) &= \sum_i \mu_{\mathrm{A}}(\mathrm{S}a_i) \cdot \mu_{\mathrm{B}}(\mathrm{U}_i) = \mu_{\mathrm{A}}(\mathrm{S}) \cdot \sum_i \mu_{\mathrm{B}}(\mathrm{U}_i) \\ &= \mu_{\mathrm{A}}(\mathrm{S}) \cdot \sum_i \mu_{\mathrm{B}}(a_i \times \mathrm{U}_i) = \mu_{\mathrm{A}}(\mathrm{S}) \cdot \mu_{\mathrm{B}}([(\mathrm{S}\backslash\mathrm{A}) \times \mathrm{B}]/\Gamma). \quad \square\end{aligned}$$

By the lemma, for each $\mathrm{S} \in \mathscr{F}(\mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f;v}))$ the left hand side of (2.2) is equal to

$$\text{(2.3)} \qquad \prod_{i=2}^{g} |\,\omega_{\mathrm{PG}^{\mathrm{int}}}\,|\,(\mathrm{PG}^{\mathrm{int}}(\mathrm{F}_{\infty_i}))\cdot|\,\omega_{\mathrm{PG}^{\mathrm{int}}}\,|\,(\mathrm{PG}^{\mathrm{int}}(\mathrm{F}_v))\cdot|\,\omega_{\mathrm{PG}^{\mathrm{int}}}\,|\,(\tau^v(\mathrm{S})) \\ \cdot|\,\omega_{\mathrm{PG}^{\mathrm{int}}}\,|\,(\tau^v(\mathrm{S})\backslash[\mathrm{PG}^{\mathrm{int}}(\mathrm{F}_{\infty_1}) \times \mathrm{PG}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f;v})]/\mathrm{PG}^{\mathrm{int}}(\mathrm{F}))$$

and the right hand side of (2.2) is equal to

$$\text{(2.4)} \qquad \prod_{i=1}^{g} |\,\omega_{\mathrm{PG}}\,|\,(\mathrm{PG}(\mathrm{F}_{\infty_i}))\cdot|\,\omega_{\mathrm{PG}}\,|(\mathrm{S})\cdot|\,\omega_{\mathrm{PG}}\,|\,(\mathrm{S}\backslash\mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f})/\mathrm{PG}(\mathrm{F})).$$

By definition, $|\,\omega_{\mathrm{PG}^{\mathrm{int}}}\,|\,(\mathrm{PG}^{\mathrm{int}}(\mathrm{F}_{\infty_i})) = |\,\omega_{\mathrm{PG}}\,|\,(\mathrm{PG}(\mathrm{F}_{\infty_i}))$ for each $i = 2, \ldots, d$ and $|\,\omega_{\mathrm{PG}^{\mathrm{int}}}\,|\,(\tau^v(\mathrm{S})) = |\,\omega_{\mathrm{PG}}\,|\,(\mathrm{S})$ for each $\mathrm{S} \in \mathscr{F}(\mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f;v}))$.

Since the expressions of (2.3) and (2.4) are equal, Proposition 1.8.5 and Remark 1.8.6 imply that

$$\text{(2.5)} \qquad \mu_{\mathrm{PG}^{\mathrm{int}}_{\mathrm{F}_{\infty_1}}}(\tau^v(\mathrm{S})\backslash[\mathrm{PG}^{\mathrm{int}}(\mathrm{F}_{\infty_1}) \times \mathrm{PG}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f;v})]/\mathrm{PG}^{\mathrm{int}}(\mathrm{F})_+) \\ = d\cdot\mu_{\mathrm{PG}_{\mathrm{F}_v}}(\mathrm{S}\backslash\mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f})/\mathrm{PG}(\mathrm{F}))$$

(" + " was added to multiply the left hand side by 2 when $d = 2$).

If S is sufficiently small, then for each $a \in \mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f;v})$ the group $a^{-1}\,\mathrm{S}a \cap \mathrm{PG}(\mathrm{F})$ is torsion-free by Proposition 1.1.10. Let $\mathrm{Y}_{a^{-1}\mathrm{S}a}$ be the projective variety over $\mathrm{K}_w$ such that $\mathrm{Y}^{\mathrm{an}}_{a^{-1}\mathrm{S}a} \cong (a^{-1}\,\mathrm{S}a \cap \mathrm{PG}(\mathrm{F}))_v\backslash\Omega^d_{\mathrm{K}_w}$. By Kurihara's result (see [Ku, Thm. 2.2.8]) $c_{d-1}(\mathrm{T}_{\mathrm{Y}_{a^{-1}\mathrm{S}a}}) = \chi_{\mathrm{E}}(a^{-1}\,\mathrm{S}a \cap \mathrm{PG}(\mathrm{F}))\cdot c_{d-1}(\mathrm{T}_{\mathbf{P}^{d-1}})$, where $c_{d-1}(\mathrm{T}_{\mathrm{Y}_{a^{-1}\mathrm{S}a}})$ (resp. $c_{d-1}(\mathrm{T}_{\mathbf{P}^{d-1}})$) is the $(d-1)$-st Chern class of the tangent bundle of $\mathrm{Y}_{a^{-1}\mathrm{S}a}$ (resp. $\mathbf{P}^{d-1}$). Notice that $c_{d-1}(\mathrm{T}_{\mathbf{P}^{d-1}}) = d$, hence $c_{d-1}(\mathrm{T}_{\mathrm{Y}_{a^{-1}\mathrm{S}a}}) = d\cdot\mu_{\mathrm{PG}_{\mathrm{F}_v}}((a^{-1}\,\mathrm{S}a \cap \mathrm{PG}(\mathrm{F}))_v\backslash\mathrm{PG}(\mathrm{F}_v))$.

Since $(\mathrm{Y}_{a^{-1}\mathrm{S}a,\mathbf{C}})^{\mathrm{an}} \cong \Delta''_{a^{-1}\mathrm{S}a}\backslash\mathrm{B}^{d-1}$, we have

$$c_{d-1}(\mathrm{T}_{\mathrm{Y}_{a^{-1}\mathrm{S}a}}) = c_{d-1}(\mathrm{T}_{(\mathrm{Y}_{a^{-1}\mathrm{S}a})\mathbf{C}}) = \chi_{\mathrm{E}}(\Delta''_{a^{-1}\mathrm{S}a}\backslash\mathrm{B}^{d-1})$$

(see for example [BT, Prop. 11.24 and (20.10.6)]). The last expression is equal to $\chi_{\mathrm{E}}(\Delta''_{a^{-1}\mathrm{S}a}) = \mu_{\mathrm{PGU}_{d-1,1}}(\Delta''_{a^{-1}\mathrm{S}a}\backslash\mathrm{PGU}_{d-1,1}(\mathbf{R}))$. This shows that for each $a \in \mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f;v})$ we have

$$d\cdot\mu_{\mathrm{PG}_{\mathrm{F}_v}}((a^{-1}\,\mathrm{S}a \cap \mathrm{PG}(\mathrm{F}))_v\backslash\mathrm{PG}(\mathrm{F}_v)) = \mu_{\mathrm{PGU}_{d-1,1}}(\Delta''_{a^{-1}\mathrm{S}a}\backslash\mathrm{PGU}_{d-1,1}(\mathbf{R})).$$

Summing this equality for $a$ running over a set of representatives of double classes in $\mathrm{S}\backslash\mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f;v})/\mathrm{PG}(\mathrm{F})$, we obtain that

$$d\cdot\mu_{\mathrm{PG}_{\mathrm{F}_v}}(\mathrm{S}\backslash\mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f})/\mathrm{PG}(\mathrm{F})) \\ = \mu_{\mathrm{PGU}_{d-1,1}}(\mathrm{S}\backslash[\mathrm{PGU}_{d-1,1}(\mathbf{R}) \times \mathrm{PG}(\mathbf{A}_{\mathrm{F}}^{f;v})]/\Delta'').$$

Since the right hand side of the last expression is equal to

$$\mu_{\mathrm{PG}^{\mathrm{int}}_{\mathrm{F}_{\infty_1}}}(\tau^v(\mathrm{S})\backslash[\mathrm{PG}^{\mathrm{int}}(\mathrm{F}_{\infty_1}) \times \mathrm{PG}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f;v})]/(\varphi \times \tau^v)\,(\Delta'')),$$

we conclude (2.1) from (2.5). □

**2.6.3.** By Proposition 2.5.6 there exists an admissible isomorphism $\Phi : \mathrm{E} \xrightarrow{\sim} \mathrm{E}^{\mathrm{int}}$, inducing the isomorphism $\tau : \mathrm{PE} \xrightarrow{\sim} \mathrm{PE}^{\mathrm{int}}$. Choose $\delta \in \Delta$ with elliptic $\delta_\infty \in \Delta_\infty$ and $\mathrm{Tr}\,\mathrm{Ad}(\delta_\infty) \neq -1$. Choose its representative $\widetilde{\delta} \in \mathrm{GU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{E}$ as in Corollary 2.2.9. Then $(\mathrm{Tr}\,\widetilde{\delta})\,(\mathrm{Tr}\,\widetilde{\delta}^{-1}) \in \mathrm{K}^\times$. Let $\delta'$ be the projection of $\widetilde{\delta}$ to $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{E}$. Set $\widetilde{\gamma} := (\varphi \times \Phi)\,(\delta')$, and let $\widetilde{\gamma}_{\mathrm{E}}$ be its projection to E.

By the definition of admissible maps, $\mathrm{Tr}(\widetilde{\gamma}_{\mathrm{E}}) \in \mathrm{K}^\times$. Let $\bar{\delta}$ be the image of $\delta$ in $\Delta'$, then $\bar{\gamma} := (\tau \times \varphi)\,(\bar{\delta})$ belongs to $\mathrm{PG}^{\mathrm{int}}(\mathrm{F})_+$. Let $\gamma' \in \mathrm{G}^{\mathrm{int}}(\mathrm{F})_+$ be some representative of $\bar{\gamma}$, then $\widetilde{\gamma}_{\mathrm{E}}^{-1}\,\gamma'_{\mathrm{E}} \in \mathrm{Z}(\mathrm{E}^{\mathrm{int}})$. Therefore $\widetilde{\gamma}_{\mathrm{E}}^{-1}\,\gamma'_{\mathrm{E}} = (\mathrm{Tr}\,\widetilde{\gamma}_{\mathrm{E}})^{-1}(\mathrm{Tr}\,\gamma'_{\mathrm{E}}) \in \mathrm{K}^\times = \mathrm{Z}(\mathrm{G}^{\mathrm{int}}(\mathrm{F}))$. Thus $\widetilde{\gamma}_{\mathrm{E}}$ and $\gamma'_{\mathrm{E}}$ have equal projections to $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times (\mathrm{E}^{\mathrm{int}}/\mathrm{E}^{\mathrm{int}})$, hence $(\varphi \times \bar{\Phi})\,(\delta) \in \Gamma^{\mathrm{int}}$.

The condition $\{\, \delta_\infty$ is elliptic and $\mathrm{Tr}\,\mathrm{Ad}(\delta_\infty) \neq -1 \,\}$ is open and non-empty, therefore the above $\delta$'s generate the whole group $\Delta \cong \Delta_\infty$ (see [Ma, Ch. IX, Lem. 3.3]). It follows that $(\varphi \times \bar{\Phi})\,(\Delta) \subset \Gamma^{\mathrm{int}}$. Since the projection $\pi : \Gamma^{\mathrm{int}} \to \mathrm{PG}^{\mathrm{int}}(\mathrm{F})_+$ is an isomorphism, Proposition 2.6.1 implies that $(\varphi \times \bar{\Phi})\,(\Delta) = \Gamma^{\mathrm{int}}$. This completes the proof of Theorem 2.2.5 and of the First Main Theorem.

## 3. THE THEOREM ON THE $p$-ADIC UNIFORMIZATION

The First Main Theorem implies that for some admissible isomorphism $\Phi : \mathrm{E} \xrightarrow{\sim} \mathrm{E}^{\mathrm{int}}$ there exists a $\Phi$-equivariant $\mathbf{C}$-rational isomorphism $f_\Phi : \mathrm{X}_{\mathbf{C}} \xrightarrow{\sim} \widetilde{\mathrm{X}}^{\mathrm{int}}$. Therefore for some $\mathbf{C}/\mathrm{K}_w$-descent $\mathrm{X}^{\mathrm{int}}$ of the $(\mathrm{E}^{\mathrm{int}}, \mathbf{C})$-scheme $\widetilde{\mathrm{X}}^{\mathrm{int}}$, $f_\Phi$ induces a $\mathrm{K}_w$-rational isomorphism $\mathrm{X} \xrightarrow{\sim} \mathrm{X}^{\mathrm{int}}$. To describe $\mathrm{X}^{\mathrm{int}}$ we need some preparations, following [Del] (see also [Mil]).

### 3.1. Technical preliminaries

In this subsection we recall basic notions related to Shimura varieties and give their explicit description in the cases we are interested in.

**3.1.1.** First we realize $\widetilde{\mathrm{X}}^{\mathrm{int}}$ as a Shimura variety. Set $\mathrm{H}^{\mathrm{int}} := \mathrm{R}_{\mathrm{F}/\mathbf{Q}}\,\mathrm{G}^{\mathrm{int}}$. Then $\mathrm{H}^{\mathrm{int}}$ is a reductive group over $\mathbf{Q}$ such that $\mathrm{H}^{\mathrm{int}}(\mathbf{A}^f) = \mathrm{G}^{\mathrm{int}}(\mathbf{A}^f_{\mathrm{F}})$ and $\mathrm{H}^{\mathrm{int}}(\mathbf{R}) = \prod_{i=1}^{r} \mathrm{G}^{\mathrm{int}}(\mathrm{F}_{\infty_i})$. Put $\mathrm{S} := \mathrm{R}_{\mathbf{C}/\mathbf{R}}\,\mathbf{G}_m$ and let $h$ be a homomorphism $\mathrm{S} \to \mathrm{H}^{\mathrm{int}}_{\mathbf{R}}$ such that for each $z \in \mathbf{C}^\times \cong \mathrm{S}(\mathbf{R})$ we have

$$h(z) = (\mathrm{diag}(1, \ldots, 1, z/\bar{z})^{-1}; \mathrm{I}_d; \ldots; \mathrm{I}_d) \in \prod_{i=1}^{r} \mathrm{G}^{\mathrm{int}}(\mathrm{F}_{\infty_i}),$$

using the identification of $\mathrm{G}^{\mathrm{int}}(\mathrm{F}_{\infty_i})$ with $\mathrm{GU}_{d-1,1}(\mathbf{R})$ chosen in 2.1.6. Then the conjugacy class $\mathrm{M}^{\mathrm{int}}$ of $h$ in $\mathrm{H}^{\mathrm{int}}(\mathbf{R})$ is isomorphic to $\mathrm{B}^{d-1}$ if $d > 2$ and to $\mathbf{C} - \mathbf{R}$ if $d = 2$. Then the pair $(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$ satisfies Deligne's axioms (see [Del, 1.5 and 2.1] or [Mil, II, 2.1]), and the Shimura variety $\mathrm{M}_{\mathbf{C}}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$, corresponding to $(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$, is isomorphic to $\widetilde{\mathrm{X}}^{\mathrm{int}}$.

**3.1.2.** For each pair $(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$ as above there is a number field $\mathrm{E}(\mathrm{H}^{\mathrm{int}}, \mathrm{M})^{\mathrm{int}} \subset \mathbf{C}$, called the reflex field of $(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$, which is defined as follows (compare [Del, 1.2, 1.3 and 3.7]). The group $\mathrm{Hom}_{\mathbf{C}}(\mathrm{S}_{\mathbf{C}}, (\mathbf{G}_m)_{\mathbf{C}})$ is a free abelian group of rank 2 with generators $z$ and $\bar{z}$ such that if $i : \mathrm{S}(\mathbf{R}) \hookrightarrow \mathrm{S}(\mathbf{C})$ is the natural inclusion, then for each $w \in \mathbf{C}^{\times} \cong \mathrm{S}(\mathbf{C})$ we have $z \circ i(w) = z$ and $\bar{z} \circ i(w) = \overline{w}$. Let $r : (\mathbf{G}_m)_{\mathbf{C}} \to \mathrm{S}_{\mathbf{C}}$ be the algebraic homomorphism such that $(z^p \bar{z}^q) \circ r(x) = x^p$. Then $\mathrm{E}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$ is the field of definition of the conjugacy class of the composition map $r'' : (\mathbf{G}_m)_{\mathbf{C}} \xrightarrow{r} \mathrm{S}_{\mathbf{C}} \xrightarrow{h} \mathrm{H}^{\mathrm{int}}_{\mathbf{C}}$.

*Proposition* **3.1.3.** — *We have* $\mathrm{E}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}}) = \mathrm{K}$, *the latter being viewed as a subfield of* $\mathbf{C}$ *through the embedding* $\infty_1$ *chosen in* 2.1.6.

*Proof.* — Note that $\mathrm{H}^{\mathrm{int}}(\mathbf{C})$ is naturally embedded into $\mathrm{GL}_d(\mathbf{C})^{2a}$ so that each factor corresponds to an embedding of K into $\mathbf{C}$. Supposing that the first and the second factors corresponds to our fixed embedding and to its complex conjugate respectively we have

$$r''(z) = (\mathrm{diag}(1, \ldots, 1, z^{-1}); \mathrm{diag}(1, \ldots, 1, z); \mathrm{I}_d; \ldots; \mathrm{I}_d)$$

for each $z \in \mathbf{C}^{\times}$. Therefore the reflex field $\mathrm{E}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$ contains $\mathrm{K} \subset \mathbf{C}$. On the other hand, the Skolem-Noether theorem implies that for each $\sigma \in \mathrm{Aut}_{\mathrm{K}}(\mathbf{C})$ the homomorphism $\sigma(r'')$ is conjugate to $r''$. This implies the assertion. □

**3.1.4.** Let $\mathrm{T} \subset \mathrm{H}^{\mathrm{int}}$ be a maximal torus of $\mathrm{H}^{\mathrm{int}}$, defined over $\mathbf{Q}$, such that some conjugate $h' \in \mathrm{M}^{\mathrm{int}}$ of $h$ in $\mathrm{H}^{\mathrm{int}}(\mathbf{R})$ factors through $\mathrm{T}_{\mathbf{R}}$. Then we have a natural embedding $i_{\mathrm{T}} : \mathrm{M}_{\mathbf{C}}(\mathrm{T}, h') \hookrightarrow \mathrm{M}_{\mathbf{C}}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$, where $\mathrm{M}_{\mathbf{C}}(\mathrm{T}, h')$ is the Shimura variety corresponding to $(\mathrm{T}, h')$. Since T is commutative, the reflex field $\mathrm{E}_{\mathrm{T}} := \mathrm{E}(\mathrm{T}, h')$ of $(\mathrm{T}, h')$ is the field of definition of the morphism $r'' : (\mathbf{G}_m)_{\mathbf{C}} \xrightarrow{r} \mathrm{S}_{\mathbf{C}} \xrightarrow{h'} \mathrm{T}_{\mathbf{C}}$. Hence $r''$ defines a morphism of algebraic groups over $\mathbf{Q}$

$$r' : \mathrm{E}^*_{\mathrm{T}} := \mathrm{R}_{\mathrm{E}_{\mathrm{T}}/\mathbf{Q}}(\mathbf{G}_m) \xrightarrow{\mathrm{R}_{\mathrm{E}_{\mathrm{T}}/\mathbf{Q}}(r'')} \mathrm{R}_{\mathrm{E}_{\mathrm{T}}/\mathbf{Q}}(\mathrm{T}) \xrightarrow{\mathrm{N}_{\mathrm{E}_{\mathrm{T}}/\mathbf{Q}}} \mathrm{T}.$$

Notice that $\mathrm{E}_{\mathrm{T}} \supset \mathrm{E}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$. Let $\theta_{\mathrm{E}_{\mathrm{T}}}$ be the Artin isomorphism of global class field theory sending the uniformizer to the arithmetic Frobenius automorphism. Let $\lambda_{\mathrm{T}} : \mathrm{Gal}(\mathrm{E}^{\mathrm{ab}}_{\mathrm{T}}/\mathrm{E}_{\mathrm{T}}) \to \mathrm{T}(\mathbf{A}^f)/\overline{\mathrm{T}(\mathbf{Q})}$ be the composition map

$$\begin{aligned} \mathrm{Gal}(\mathrm{E}^{\mathrm{ab}}_{\mathrm{T}}/\mathrm{E}_{\mathrm{T}}) &\overset{\theta^{-1}_{\mathrm{E}_{\mathrm{T}}}}{\xrightarrow{\sim}} \mathrm{E}^*_{\mathrm{T}}(\mathbf{R})^0 \backslash \mathrm{E}^*_{\mathrm{T}}(\mathbf{A})/\overline{\mathrm{E}^*_{\mathrm{T}}(\mathbf{Q})} \\ &\xrightarrow{r'} \mathrm{T}(\mathbf{R})^0 \backslash \mathrm{T}(\mathbf{A})/\overline{\mathrm{T}(\mathbf{Q})} \xrightarrow{\mathrm{proj}} \mathrm{T}(\mathbf{A}^f)/\overline{\mathrm{T}(\mathbf{Q})}. \end{aligned}$$

For each $\mathrm{E}' \supset \mathrm{E}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$ we denote the composition map

$$\mathrm{Gal}(\mathrm{E}^{\mathrm{ab}}_{\mathrm{T}} \cdot \mathrm{E}'/\mathrm{E}') \xrightarrow{\mathrm{res}} \mathrm{Gal}(\mathrm{E}^{\mathrm{ab}}_{\mathrm{T}}/\mathrm{E}_{\mathrm{T}}) \xrightarrow{\lambda_{\mathrm{T}}} \mathrm{T}(\mathbf{A}^f)/\overline{\mathrm{T}(\mathbf{Q})}$$

by $\lambda_{\mathrm{T}, \mathrm{E}'}$.

*Lemma* **3.1.5.** — *Each maximal torus* $\mathrm{T}$ *of* $\mathrm{H}^{\mathrm{int}}$, *defined over* $\mathbf{Q}$, *is equal to the intersection of* $\mathrm{H}^{\mathrm{int}}$ *with* $\mathrm{R}_{\mathrm{L}/\mathbf{Q}}\,\mathbf{G}_m$ *for a unique maximal commutative subfield* $\mathrm{L}$ *of* $\mathrm{D}^{\mathrm{int}}$. *(In such a situation we will call* $\mathrm{T}$ *an* L-*torus.) In this case,* $\alpha^{\mathrm{int}}$ *induces a nontrivial automorphism of* $\mathrm{L}$, *and the subgroup* $\mathrm{T}(\mathbf{Q}) \subset \mathrm{L}^\times \cong \mathrm{R}_{\mathrm{L}/\mathrm{K}}\,\mathbf{G}_m(\mathrm{K})$ *is Zariski dense in* $\mathrm{R}_{\mathrm{L}/\mathrm{K}}\,\mathbf{G}_m$.

*Proof.* — Let $\mathrm{L}$ be the subalgebra of $\mathrm{D}^{\mathrm{int}}$ spanned over $\mathrm{K}$ by $\mathrm{T}(\mathbf{Q}) \subset \mathrm{H}^{\mathrm{int}}(\mathbf{Q}) \subset \mathrm{D}^{\mathrm{int}}$, then $\mathrm{L}$ is a commutative subfield and $\mathrm{T}(\mathbf{Q}) \subset \mathrm{H}^{\mathrm{int}}(\mathbf{Q}) \cap \mathrm{R}_{\mathrm{L}/\mathbf{Q}}\,\mathbf{G}_m(\mathbf{Q})$. Since $\mathrm{T}$ is connected and $\mathbf{Q}$ is infinite and perfect, the subgroup $\mathrm{T}(\mathbf{Q})$ is Zariski dense in $\mathrm{T}$ (see [Bo, Ch. V, Cor. 13.3]). It follows that $\mathrm{T} \subset \mathrm{H}^{\mathrm{int}} \cap \mathrm{R}_{\mathrm{L}/\mathbf{Q}}\,\mathbf{G}_m$. Since $\mathrm{T}$ is maximal, $\mathrm{L}$ have to be maximal and $\mathrm{T} = \mathrm{H}^{\mathrm{int}} \cap \mathrm{R}_{\mathrm{L}/\mathbf{Q}}\,\mathbf{G}_m$.

For each $g \in \mathrm{T}(\mathbf{Q})$ we have $\alpha^{\mathrm{int}}(g) \in g^{-1}\,\mathrm{F}^\times \subset \mathrm{T}(\mathbf{Q})$, so that $\alpha^{\mathrm{int}}(\mathrm{L}) = \mathrm{L}$. To prove the last assertion we observe that there exists a maximal F-rational subtorus $\mathrm{T}'$ of $\mathrm{G}^{\mathrm{int}}$ such that $\mathrm{T} = \mathrm{R}_{\mathrm{F}/\mathbf{Q}}(\mathrm{T}')$. Then the subgroup $\mathrm{T}(\mathbf{Q}) = \mathrm{T}'(\mathrm{F})$ is Zariski dense in $\mathrm{T}'_{\mathrm{K}} \cong \mathrm{R}_{\mathrm{L}/\mathrm{K}}\,\mathbf{G}_m \times (\mathbf{G}_m)_{\mathrm{K}}$. Hence its projection to $\mathrm{R}_{\mathrm{L}/\mathrm{K}}\,\mathbf{G}_m$ is also Zariski dense. □

**3.1.6.** Now we want to calculate the reflex field $\mathrm{E}_{\mathrm{T}}$. Observe that

$$\mathrm{L} \otimes_{\mathbf{Q}} \mathbf{C} \subset \mathrm{D}^{\mathrm{int}} \otimes_{\mathbf{Q}} \mathbf{C} \cong \mathrm{Mat}_d(\mathbf{C})^{2g}.$$

Possibly after a conjugation we may assume that $\mathrm{L} \otimes_{\mathbf{Q}} \mathbf{C}$ is the subalgebra of diagonal matrices of $\mathrm{Mat}_d(\mathbf{C})^{2g}$. Then each diagonal entry of each of the $2g$ copies of $\mathrm{Mat}_d(\mathbf{C})$ corresponds to an embedding of $\mathrm{L}$ into $\mathbf{C}$, and the map $r'' : (\mathbf{G}_m)_{\mathbf{C}} \to \mathrm{T}_{\mathbf{C}}$ is as follows:

$$r''(z) = (\mathrm{diag}(1, \ldots, 1, z^{-1}); \mathrm{diag}(1, \ldots, 1, z); \mathrm{I}_d; \ldots; \mathrm{I}_d).$$

Let $\iota_1$ be the embedding $\mathrm{L} \hookrightarrow \mathbf{C}$, corresponding to the right low entry of the first matrix, then the right low entry of the second matrix corresponds to the embedding $\bar{\iota}_1 := \iota_1 \circ \alpha^{\mathrm{int}}$. Now we embed $\mathrm{L}$ into $\mathbf{C}$ via $\iota_1$.

*Proposition* **3.1.7.** — *We have* $\mathrm{E}_{\mathrm{T}} = \mathrm{L} \subset \mathbf{C}$, *and* $r' : \mathrm{E}^*_{\mathrm{T}} \to \mathrm{T}$ *is characterized by* $r'(l) = l^{-1}\cdot\alpha^{\mathrm{int}}(l)$ *for each* $l \in \mathrm{E}^*_{\mathrm{T}}(\mathbf{Q}) \subset \mathrm{L}^\times$.

*Proof.* — As was noted before, $\mathrm{E}_{\mathrm{T}} \supset \mathrm{E}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$. Hence by Proposition 3.1.3, $\mathrm{E}_{\mathrm{T}} \supset \mathrm{K}$. By the definition, $\sigma(r''(z)) = r''(\sigma(z))$ for each $\sigma \in \mathrm{Aut}_{\mathrm{E}_{\mathrm{T}}}(\mathbf{C})$, hence the group $\mathrm{Aut}_{\mathrm{E}_{\mathrm{T}}}(\mathbf{C})$ must stabilize $\iota_1$, so that $\mathrm{E}_{\mathrm{T}} \supset \mathrm{L}$. Finally, it is clear that $r''$ is defined over $\mathrm{L} \subset \mathbf{C}$. For each $l \in \mathrm{E}_{\mathrm{T}} = \mathrm{L}$ we have

$$\begin{aligned} r'(l) &= \mathrm{N}_{\mathrm{L}/\mathbf{Q}}(\mathrm{diag}(1, \ldots, 1, l^{-1}); \mathrm{diag}(1, \ldots, 1, l); \mathrm{I}_d; \ldots; \mathrm{I}_d) \\ &= l^{-1}\cdot\alpha^{\mathrm{int}}(l). \quad \square \end{aligned}$$

Set $\mathrm{L}_w := \mathrm{L} \otimes_{\mathrm{K}} \mathrm{K}_w \subset \mathrm{D}^{\mathrm{int}}_w := \mathrm{D}^{\mathrm{int}} \otimes_{\mathrm{K}} \mathrm{K}_w$. Since $\mathrm{D}^{\mathrm{int}}_w$ is a division algebra, $\mathrm{L}_w$ is a field extension of $\mathrm{K}_w$ of degree $d$, and $\mathrm{L}_w = \mathrm{L}\cdot\mathrm{K}_w$.

*Lemma* **3.1.8.** — *The following relations hold:*

*a)* $\mathrm{E}_{\mathrm{T}}\cdot\mathrm{K}_w = \mathrm{L}_w$;
*b)* $\mathrm{E}^{\mathrm{ab}}_{\mathrm{T}}\cdot\mathrm{K}_w = (\mathrm{L}_w)^{\mathrm{ab}}$.

*Proof.* — *a)* was proved above.

*b)* The group $\mathrm{Gal}(\mathrm{E}_{\mathrm{T}}^{\mathrm{ab}}\cdot\mathrm{K}_w/\mathrm{E}_{\mathrm{T}}\cdot\mathrm{K}_w)$ is abelian, hence $\mathrm{L}_w \subset \mathrm{E}_{\mathrm{T}}^{\mathrm{ab}}\cdot\mathrm{K}_w \subset \mathrm{L}_w^{\mathrm{ab}}$. By the class field theory, the composition of the canonical projections

$$\mathrm{Gal}((\mathrm{L}_w)^{\mathrm{ab}}/\mathrm{L}_w) \to \mathrm{Gal}(\mathrm{E}_{\mathrm{T}}^{\mathrm{ab}}\cdot\mathrm{K}_w/\mathrm{E}_{\mathrm{T}}\cdot\mathrm{K}_w) \to \mathrm{Gal}(\mathrm{E}_{\mathrm{T}}^{\mathrm{ab}}/\mathrm{E}_{\mathrm{T}}) \cong \mathrm{Gal}(\mathrm{L}^{\mathrm{ab}}/\mathrm{L})$$

is injective (use, for example, [CF, Ch. VII, Prop. 6.2]). Therefore we have the required equality. □

*Proposition* **3.1.9.** — *For each* $l \in \mathrm{L}_w^\times \subset (\mathrm{D}_w^{\mathrm{int}})^\times$ *the element*

$$(l^{-1}, 1, 1) \in (\mathrm{D}_w^{\mathrm{int}})^\times \times \mathrm{F}_v^\times \times \mathrm{G}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f;v}) \cong \mathrm{H}^{\mathrm{int}}(\mathbf{A}^f)$$

*belongs to* $\mathrm{T}(\mathbf{A}^f)$, *and its equivalence class in* $\mathrm{T}(\mathbf{A}^f)/\overline{\mathrm{T}(\mathbf{Q})}$ *is* $\lambda_{\mathrm{T},\mathrm{K}_w}(\theta_{\mathrm{L}_w}(l))$.

*Proof.* — The statement follows immediately from the explicit formulas of Proposition 3.1.7 using the connection between local and global Artin maps. □

*Definition* **3.1.10.** — A point $x \in \mathrm{M}_{\mathbf{C}}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})(\mathbf{C})$ is called (T-)*special* if $x \in i_{\mathrm{T}}(\mathrm{M}_{\mathbf{C}}(\mathrm{T}, h)(\mathbf{C}))$.

*Remark* **3.1.11.** — The group $\mathrm{T}(\mathbf{A}^f)$ acts naturally on the set of T-special points and the group $\mathrm{T}(\mathbf{Q})$ acts on it trivially. Hence by continuity the closure $\overline{\mathrm{T}(\mathbf{Q})} \subset \mathrm{T}(\mathbf{A}^f)$ acts trivially on the set of T-special points, therefore the action of $\mathrm{T}(\mathbf{A}^f)/\overline{\mathrm{T}(\mathbf{Q})}$ on it is well-defined.

*Definition* **3.1.12.** — Let $\mathrm{K}' \supset \mathrm{E}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$ be a subfield of $\mathbf{C}$. A $\mathbf{C}/\mathrm{K}'$-descent of the $(\mathrm{H}^{\mathrm{int}}(\mathbf{A}^f), \mathbf{C})$-scheme $\mathrm{M}_{\mathbf{C}}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$ is called *weakly-canonical* if for each maximal torus $\mathrm{T} \hookrightarrow \mathrm{H}^{\mathrm{int}}$ as above, each T-special point $x$ is defined over $\mathrm{E}_{\mathrm{T}}^{\mathrm{ab}}\cdot\mathrm{K}'$, and for each $\sigma \in \mathrm{Gal}(\mathrm{E}_{\mathrm{T}}^{\mathrm{ab}}\cdot\mathrm{K}'/\mathrm{E}_{\mathrm{T}}\cdot\mathrm{K}')$ we have $\sigma(x) = \lambda_{\mathrm{T},\mathrm{K}'}(\sigma)(x)$.

*Remark* **3.1.13.** — Our definition of the canonical model coincides with that of [Mi3], which differs from those of [De2] and [Mi1] (see the discussion in [Mi3, 1.10]). The seeming difference (by sign) between our reciprocity map and that of [Mi3] is due to the fact that we consider left action of the adelic group whereas Milne considers right action.

*Proposition* **3.1.14.** — *For each field* $\mathrm{K}'$ *satisfying* $\mathrm{E}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}}) \subset \mathrm{K}' \subset \mathbf{C}$ *there exists a unique (up to an isomorphism) weakly-canonical* $\mathbf{C}/\mathrm{K}'$*-descent of the* $(\mathrm{H}^{\mathrm{int}}(\mathbf{A}^f), \mathbf{C})$*-scheme* $\mathrm{M}_{\mathbf{C}}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$.

*Proof.* — Uniqueness is proved in [De1, 5.4], for the existence see [De1, 6.4] or [Mi1, II, Thm. 5.5]. □

By Proposition 3.1.3, we have $\mathrm{E}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}}) = \mathrm{K} \subset \mathrm{K}_w \subset \mathbf{C}$ (in our convention 2.2.2). Hence by Proposition 3.1.14, the $(\mathrm{E}^{\mathrm{int}}, \mathbf{C})$-scheme $\widetilde{\mathrm{X}}^{\mathrm{int}}$ has a unique weakly-canonical $\mathbf{C}/\mathrm{K}_w$-descent $\mathrm{X}^{\mathrm{int}}$.

## 3.2. Theorem on the $p$-adic uniformization

Now we are ready to formulate our

*Second Main Theorem* **3.2.1.** — *For each admissible isomorphism* $\Phi : \mathrm{E} \xrightarrow{\sim} \mathrm{E}^{\mathrm{int}}$ *there exists a* $\Phi$*-equivariant isomorphism* $f_\Phi$ *from the* $(\mathrm{E}, \mathrm{K}_w)$*-scheme* $\mathrm{X}$ *to the* $(\mathrm{E}^{\mathrm{int}}, \mathrm{K}_w)$*-scheme* $\mathrm{X}^{\mathrm{int}}$.

*Corollary* **3.2.2.** — *After the identification of* $\mathrm{E}$ *with* $\mathrm{E}^{\mathrm{int}}$ *by means of* $\Phi$ *we have for each* $\mathrm{S} \in \mathscr{F}(\mathrm{E})$ *of the form* $\mathrm{T}_n \times \mathrm{S}'$, *where* $\mathrm{S}' \in \mathscr{F}(\mathrm{E}')$, *an isomorphism of* $\mathrm{K}_w$*-analytic spaces* $\varphi_{\mathrm{S}} : (\mathrm{X}^{\mathrm{int}}_{\mathrm{S}})^{\mathrm{an}} \xrightarrow{\sim} \mathrm{GL}_d(\mathrm{K}_w)\backslash(\Sigma^{d,n}_{\mathrm{K}_w} \times (\mathrm{S}'\backslash \mathrm{G}(\mathbf{A}^f_{\mathrm{F}})/\mathrm{G}(\mathrm{F})))$. *These isomorphisms commute with the natural projections for* $\mathrm{T} \supset \mathrm{S}$ *and with the action of* $\mathrm{E} \overset{\Phi}{\xrightarrow{\sim}} \mathrm{E}^{\mathrm{int}}$.

*Proof* (of the Second Main Theorem) :

*Step 1.* — We want to prove that for $\Phi$ and $f_\Phi$ as in the First Main Theorem, the $\mathbf{C}/\mathrm{K}_w$-descent of $\widetilde{\mathrm{X}}^{\mathrm{int}}$ corresponding to $\mathrm{X}$ is weakly-canonical. For this we have to show that for each maximal torus $\mathrm{T} \hookrightarrow \mathrm{H}^{\mathrm{int}}$ as in 3.1.4 and each T-special point $x = f_\Phi(y) \in \mathrm{M}_{\mathbf{C}}(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})(\mathbf{C}) = \widetilde{\mathrm{X}}^{\mathrm{int}}$ we have:

*a)* $y \in \mathrm{X}(\mathbf{C}_p)$ is defined over $\mathrm{E}^{\mathrm{ab}}_{\mathrm{T}} \cdot \mathrm{K}_w$;
*b)* $\sigma(y) = \Phi^{-1}(\lambda_{\mathrm{T},\mathrm{M}}(\sigma))(y)$ for each $\sigma \in \mathrm{Gal}(\mathrm{E}^{\mathrm{ab}}_{\mathrm{T}} \cdot \mathrm{K}_w/\mathrm{E}_{\mathrm{T}} \cdot \mathrm{K}_w)$.

By Proposition 3.1.9, Lemma 3.1.8 and the definition of admissible map, it will suffice to show that when $\mathrm{L}_w$ is embedded into $\widetilde{\mathrm{D}}_w$ by means of the isomorphism $\mathrm{D}^{\mathrm{int}}_w \xrightarrow{\sim} \widetilde{\mathrm{D}}_w$ from Definition 2.2.1 we have

**(3.1)** i) every point $y \in \mathrm{X}(\mathbf{C}_p)$, fixed by $\Phi^{-1}(\mathrm{T}(\mathbf{Q}))$, is rational over $(\mathrm{L}_w)^{\mathrm{ab}}$;
ii) $\theta_{\mathrm{L}_w}(l)(y) = l^{-1}(y)$ for each $l \in \mathrm{L}^\times_w \subset \widetilde{\mathrm{D}}^\times_w \cong \widetilde{\mathrm{D}}^\times_w \times \{1\} \subset \widetilde{\mathrm{D}}^\times_w \times \mathrm{E}'$.

Let $(x, a) \in \Sigma^d_{\mathrm{K}_w} \times \mathrm{E}'$ be a representative of $y \in \mathrm{X}(\mathbf{C}_p)$. Then $(\sigma(x), a)$ is a representative of $\sigma(y)$ for each (not necessarily continuous) $\sigma \in \mathrm{Aut}_{\mathrm{K}_w}(\mathbf{C}_p)$. Recall that for each embedding $\mathrm{L}_w \hookrightarrow \mathrm{Mat}_d(\mathrm{K}_w)$ there exists an $(\mathrm{L}^\times_w \times \mathrm{L}^\times_w)$-equivariant $\mathrm{L}_w$-rational embedding $\tilde{\iota} : \Sigma^1_{\mathrm{L}_w} \hookrightarrow \Sigma^d_{\mathrm{K}_w}$.

*Proposition* **3.2.3.** — *There exists an embedding* $\mathrm{L}_w \hookrightarrow \mathrm{Mat}_d(\mathrm{K}_w)$ *such that the image of the corresponding* $\tilde{\iota} : \Sigma^1_{\mathrm{L}_w} \hookrightarrow \Sigma^d_{\mathrm{K}_w}$ *contains* $x$.

*Proof.* — Let $x' \in \Omega^d_{\mathrm{K}_w}$ be the projection of $x$. Then

$$y' := [(x', a)] \in (\widetilde{\mathrm{D}}^\times_w\backslash \mathrm{X})^{\mathrm{an}} \cong (\Omega^d_{\mathrm{K}_w} \times (\mathrm{E}')^{\mathrm{disc}})/\mathrm{G}(\mathrm{F})\ \overline{\mathrm{Z}(\mathrm{G}(\mathrm{F}))}$$

(use Proposition 1.5.3) is the projection of $y$. Since $\Phi^{-1}(\mathrm{T}(\mathbf{Q}))$ stabilizes $y$, it also stabilizes $y'$, therefore the projection of $a^{-1}\,\Phi^{-1}(\mathrm{T}(\mathbf{Q}))\,a$ to $\mathrm{E}'$ is contained in $\mathrm{G}(\mathrm{F})\ \overline{\mathrm{Z}(\mathrm{G}(\mathrm{F}))} \subset \mathrm{E}'$. In other words, for each $t \in \mathrm{T}(\mathbf{Q}) \subset \mathrm{D}^{\mathrm{int}}$ we have

$$\mathrm{pr}_{\mathrm{E}'}(a^{-1}\,\Phi^{-1}(t)\,a) = g \cdot z$$

for some $g \in \mathrm{G}(\mathrm{F})$ and some $z \in \overline{\mathrm{Z}(\mathrm{G}(\mathrm{F}))}$.

Since $\Phi$ is induced by some algebra isomorphism $\mathrm{D}(\mathbf{A}_{\mathrm{F}}^{f;v}) \xrightarrow{\sim} \mathrm{D}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f;v})$, we have $\operatorname{Tr} t = \operatorname{Tr}(a^{-1}\Phi^{-1}(t)\,a) = (\operatorname{Tr} g)\,z$. Therefore for $t$'s with non-zero trace we get $z = (\operatorname{Tr} t)(\operatorname{Tr} g)^{-1} \in \mathrm{K}^{\times} \subset (\mathbf{A}_{\mathrm{K}}^{f;w,\bar{w}})^{\times}$. This means that $\mathrm{pr}_{\mathrm{E}'}(a^{-1}\Phi^{-1}(t)\,a) \in \mathrm{G}(\mathrm{F}) \subset \mathrm{E}'$. As the set of all $t$'s in $\mathrm{T}(\mathbf{Q}) \subset \mathrm{D}^{\mathrm{int}}$ with $\operatorname{Tr} t \neq 0$ generates $\mathrm{L} \subset \mathrm{D}^{\mathrm{int}}$ as an algebra, the map $l \mapsto \mathrm{pr}_{\mathrm{E}'}(a^{-1}\Phi^{-1}(l)\,a)$ defines embeddings $\mathrm{L} \hookrightarrow \mathrm{D}$ and $\mathrm{L}_w \hookrightarrow \mathrm{D} \otimes_{\mathrm{K}} \mathrm{K}_w \cong \mathrm{Mat}_d(\mathrm{K}_w)$.

This shows that $\mathrm{pr}_{\mathrm{E}'}(a^{-1}\Phi^{-1}(\mathrm{T}(\mathbf{Q}))\,a) \subset \mathrm{G}(\mathrm{F}) \subset \mathrm{E}'$, so that

$$a^{-1}\Phi^{-1}(\mathrm{T}(\mathbf{Q}))\,a \subset \widetilde{\mathrm{D}}_w^{\times} \times \Gamma_{\mathrm{E}} \subset \widetilde{\mathrm{D}}_w^{\times} \times \mathrm{E}' = \mathrm{E}.$$

Hence $\Phi^{-1}(\mathrm{T}(\mathbf{Q}))$ preserves $\rho_a(\Sigma_{\mathrm{K}_w}^d) \subset (\mathrm{X}_{\mathbf{C}_p})^{\mathrm{an}}$ (in the notation of Corollary 1.5.4). Moreover, it follows from the definition of the embeddings $\mathrm{L}_w \hookrightarrow \widetilde{\mathrm{D}}_w$ and $\mathrm{L}_w \hookrightarrow \mathrm{Mat}_d(\mathrm{K}_w)$ that for each $t \in \mathrm{T}(\mathbf{Q}) \subset \mathrm{L}^{\times} \subset \mathrm{L}_w^{\times}$ the image of $a^{-1}\Phi^{-1}(t)\,a \subset \widetilde{\mathrm{D}}_w^{\times} \times \Gamma_{\mathrm{E}}$ under the canonical map $\widetilde{\mathrm{D}}_w^{\times} \times \Gamma_{\mathrm{E}} \xrightarrow{\sim} \widetilde{\mathrm{D}}_w^{\times} \times \Gamma_{\mathrm{G}} \subset \widetilde{\mathrm{D}}_w^{\times} \times \mathrm{GL}_d(\mathrm{K}_w)$ is equal to $(t, t)$. Since $y$ is a fixed point $\Phi^{-1}(\mathrm{T}(\mathbf{Q}))$, we conclude from the above that $(t, t)\,(x) = x$ for every $t \in \mathrm{T}(\mathbf{Q})$. Noticing that $\mathrm{T}(\mathbf{Q})$ is Zariski dense in $\mathrm{R}_{\mathrm{L}/\mathrm{K}}\,\mathbf{G}_m$ by Lemma 3.1.5 and that $\mathrm{R}_{\mathrm{L}/\mathrm{K}}\,\mathbf{G}_m \otimes_{\mathrm{K}} \mathrm{K}_w \cong \mathrm{R}_{\mathrm{L}_w/\mathrm{K}_w}\,\mathbf{G}_m$, Lemma 1.4.5 completes the proof. □

Since $\tilde{\iota}$ is $(\mathrm{L}_w^{\times} \times \mathrm{L}_w^{\times})$-equivariant and $\mathrm{L}_w$-rational, the proposition together with Lemma 1.4.3 imply (3.1). In other words, we have proved that for some admissible isomorphism $\Phi : \mathrm{E} \xrightarrow{\sim} \mathrm{E}^{\mathrm{int}}$ there exists a $\Phi$-equivariant $\mathrm{K}_w$-linear isomorphism $f_{\Phi} : \mathrm{X} \xrightarrow{\sim} \mathrm{X}^{\mathrm{int}}$.

*Step 2.* — Let $\Psi$ be another admissible isomorphism $\mathrm{E} \xrightarrow{\sim} \mathrm{E}^{\mathrm{int}}$. The definition of admissibility together with the theorem of Skolem-Noether imply that $\Psi \circ \Phi^{-1} : \mathrm{E}^{\mathrm{int}} \xrightarrow{\sim} \mathrm{E}^{\mathrm{int}}$ is an inner automorphism, so that there exists $g_{\Psi} \in \mathrm{E}^{\mathrm{int}}$ such that $\Psi \circ \Phi^{-1}(g) = g_{\Psi}\,g g_{\Psi}^{-1}$ for all $g \in \mathrm{E}^{\mathrm{int}}$. Take $f_{\Psi} : \mathrm{X} \xrightarrow{f_{\Phi}} \mathrm{X}^{\mathrm{int}} \xrightarrow{g_{\Psi}} \mathrm{X}^{\mathrm{int}}$. Then for each $g \in \mathrm{E}$ we have

$$\begin{aligned} f_{\Psi} \circ g = g_{\Psi} \circ f_{\Phi} \circ g = g_{\Psi} \circ \Phi(g) \circ f_{\Phi} &= (g_{\Psi} \circ \Phi(g) \circ g_{\Psi}^{-1}) \circ (g_{\Psi} \circ f_{\Phi}) \\ &= (\Psi \circ \Phi^{-1})\,(\Phi(g)) \circ f_{\Psi} = \Psi(g) \circ f_{\Psi}, \end{aligned}$$

that is $f_{\Psi}$ is a $\Psi$-equivariant isomorphism. This completes the proof of the Second Main Theorem. □

## 4. *p*-ADIC UNIFORMIZATION OF AUTOMORPHIC VECTOR BUNDLES

In the previous section we proved that the Shimura varieties corresponding to the pairs $(\mathrm{H}^{\mathrm{int}}, \mathrm{M}^{\mathrm{int}})$ have $p$-adic uniformization. Our next task is to show the analogous result for automorphic vector bundles.

### 4.1. Equivariant vector bundles

**4.1.1.** Set $\mathrm{H} := \mathrm{R}_{\mathrm{F}/\mathbf{Q}}\,\mathrm{G}$. Then for some algebraic group $\widetilde{\mathrm{H}}$ over $\mathrm{K}_w$ we have natural isomorphisms $\mathrm{H}_{\mathrm{K}_w} \cong \mathrm{GL}_d \times \widetilde{\mathrm{H}}$, $\mathrm{PH}_{\mathrm{K}_w} \cong \mathrm{PGL}_d \times \mathrm{P}\widetilde{\mathrm{H}}$ and $\mathrm{PH}_{\mathrm{K}_w}^{\mathrm{int}} \cong \mathrm{PGL}_1(\mathrm{D}_w^{\mathrm{int}}) \times \mathrm{P}\widetilde{\mathrm{H}}$, where the first factors correspond to the natural embedding $F \hookrightarrow K_w$. Using these decompositions let $PH_{K_w}$ acts on $\mathbf{P}^{d-1}_{K_w}$ through the natural action of the first factor and the trivial action of the second one, and let $H(K_w)$, $PH(K_w)$ and $PH^{int}(\mathbf{R})^0 \cong PGU_{d-1,1}(\mathbf{R})^0 \times PGU_d(\mathbf{R})^{g-1}$ act similarly on $\Sigma^d_{K_w}$, on $\Omega^d_{K_w}$ and on $B^{d-1}$ respectively. Let $\beta_{\mathbf{R}}$ be the natural embedding $B^{d-1} \hookrightarrow (\mathbf{P}^{d-1}_{\mathbf{C}})^{an}$, and let $\beta_w$ (resp. $\beta_{w,n}$) be the composition of the natural projection $\Sigma^d_{K_w} \to \Omega^d_{K_w}$ (resp. $\Sigma^{d,n}_{K_w} \to \Omega^d_{K_w}$) and the natural embedding $\Omega^d_{K_w} \hookrightarrow (\mathbf{P}^{d-1}_{K_w})^{an}$.

Let $\pi \in K_w$ be a uniformizer, let $\widetilde{\Pi}$ be an element of $GL_d(K_w)$ satisfying $\widetilde{\Pi}^d = \pi$, and let $\Pi' \in PGL_d(K_w)$ be the projection of $\widetilde{\Pi}$. Set

$$\Pi := (\Pi', 1) \in PGL_d(K_w) \times \widetilde{PH}(K_w) \cong PH(K_w).$$

Let $K^{(d)}_w$ be the unramified field extension of $K_w$ of degree $d$. Since the Brauer invariant of $D^{int}_w$ is $1/d$, the group $PH^{int}_{K_w}$ is isomorphic to the quotient of $PH_{K_w} \otimes_{K_w} K^{(d)}_w$ by the equivalence relation $Fr(x) \sim \Pi^{-1} x \Pi$, where $Fr \in Gal(K^{(d)}_w/K_w)$ is the Frobenius automorphism. For each scheme Y over $K_w$ on which $PH_{K_w}$ acts $K_w$-rationally define a twist $Y^{tw} := (Fr(x) \sim \Pi^{-1} x) \backslash Y \otimes_{K_w} K^{(d)}_w$. Then $Y \otimes_{K_w} K^{(d)}_w \cong Y^{tw} \otimes_{K_w} K^{(d)}_w$ and the natural action of $PH^{int}_{K_w}$ on it is $K_w$-rational.

Let W be a $PH_{K_w}$-equivariant vector bundle on $\mathbf{P}^{d-1}_{K_w}$, that is a vector bundle on $\mathbf{P}^{d-1}_{K_w}$ equipped with an action of the group $PH_{K_w}$, lifting its action on $\mathbf{P}^{d-1}_{K_w}$. Then $(W^{tw}, p^{tw})$ is a $PH^{int}_{K_w}$-equivariant vector bundle on $(\mathbf{P}^{d-1}_{K_w})^{tw}$, and $\beta^*_{\mathbf{R}}((W^{tw}_{\mathbf{C}})^{an})$ (resp. $\beta^*_w(W^{an})$, $\beta^*_{w,n}(W^{an})$) is a $PH^{int}(\mathbf{R})^0$- (resp. $H(K_w)$-)equivariant analytic vector bundle on $B^{d-1}$ (resp. $\Sigma^d_{K_w}$, $\Sigma^{d,n}_{K_w}$).

For each $S \in \mathscr{F}(E)$ (resp. $S \in \mathscr{F}(E^{int})$) consider a double quotient

$$\widetilde{V}_S := S \backslash [\beta^*_w(W^{an}) \times E']/\Gamma \text{ (resp. } \widetilde{\widetilde{V}}{}^{int}_S := S \backslash [\beta^*_{\mathbf{R}}(W^{tw}_{\mathbf{C}})^{an} \times E^{int}]/\Gamma^{int}).$$

*Proposition* **4.1.2.** — *For each* $S \in \mathscr{F}(E)$ (*resp.* $S \in \mathscr{F}(E^{int})$) $\widetilde{V}_S$ (*resp.* $\widetilde{\widetilde{V}}{}^{int}_S$) *has a natural structure of an affine scheme* $V_S$ *over* $X_S$ (*resp.* $\widetilde{V}^{int}_S$ *over* $\widetilde{X}^{int}_S$). *Moreover,* $V_S$ (*resp.* $\widetilde{V}^{int}_S$) *is a vector bundle on* $X_S$ (*resp.* $\widetilde{X}^{int}_S$) *if* S *is sufficiently small.*

*Proof.* — We give the proof in the $p$-adic case. The complex case is similar, but easier.

I) First we take S of the form $T_n \times S'$ with sufficiently small $S' \in \mathscr{F}(E')$. Then $\widetilde{V}_S$ is a finite disjoint union of quotients of the form $\Gamma_{aS'a^{-1}} \backslash \beta^*_{w,n}(W^{an})$ with some $a \in E'$. Since the projection $\Sigma^{d,n}_{K_w} \to \Omega^d_{K_w}$ factors through each $\Gamma_{aS'a^{-1},0} \backslash \Sigma^{d,n}_{K_w}$ (in the notation of the proof of Proposition 1.5.2), the quotient $\Gamma_{aS'a^{-1},0} \backslash \beta^*_{w,n}(W^{an})$ is naturally an analytic vector bundle on $\Gamma_{aS'a^{-1},0} \backslash \Sigma^{d,n}_{K_w}$. Now (as in the proof of Proposition 1.5.2) the quotient vector bundle $P\Gamma_{a\bar{S}a^{-1}} \backslash (\Gamma_{aS'a^{-1},0} \backslash \beta^*_{w,n}(W^{an})) \cong \Gamma_{aS'a^{-1}} \backslash \beta^*_{w,n}(W^{an})$ on $\Gamma_{aS'a^{-1}} \backslash \Sigma^{d,n}_{K_w}$ is obtained by gluing. For the algebraization we use Corollary 1.2.3 *a)*.

II) For each $T \in \mathscr{F}(E)$ there exists a normal subgroup of the form $S = T_n \times S'$, where $S' \in \mathscr{F}(E')$ is sufficiently small. Then by the same considerations as in Proposition 1.3.7, $V_T$ can be defined as $(T/S) \backslash V_S$ (using Corollary 1.2.3 *a)*).

III) Suppose that $V_{S_1}$ and $V_{S_2}$, constructed in I) and II), are vector bundles on $X_{S_1}$ and $X_{S_2}$ respectively for $S_1 \subset S_2$ in $\mathscr{F}(E)$. Then the natural morphism $f : V_{S_1} \to V_{S_2} \times_{X_{S_2}} X_{S_1}$ of vector bundles on $X_{S_1}$ induces an isomorphism on each fiber. Hence it is an isomorphism.

IV) Suppose that $T \subset S$ in $\mathscr{F}(E)$ and that $V_S$ is a vector bundle on $X_S$. Choose a normal subgroup $S_0 \in \mathscr{F}(E)$ of $T$ such that $V_{S_0}$ is a vector bundle on $X_{S_0}$. Then $V_T = (T/S_0)\backslash V_{S_0} \cong (T/S_0)\backslash V_S \times_{X_S} X_{S_0} \cong V_S \times_{X_S} ((T/S_0)\backslash X_{S_0}) \cong V_S \times_{X_S} X_T$, so $V_T$ is a vector bundle on $X_T$. □

**4.1.3.** Choose $S \in \mathscr{F}(E)$ (resp. $S \in \mathscr{F}(E^{\mathrm{int}})$) sufficiently small. Then $V_S$ (resp. $\widetilde{V}_S^{\mathrm{int}}$) is a vector bundle on $X_S$ (resp. $\widetilde{X}_S^{\mathrm{int}}$). Thus $V := V_S \times_{X_S} X$ (resp. $\widetilde{V}^{\mathrm{int}} := \widetilde{V}_S^{\mathrm{int}} \times_{\widetilde{X}_S^{\mathrm{int}}} \widetilde{X}^{\mathrm{int}}$) is a vector bundle on $X$ (resp. $\widetilde{X}^{\mathrm{int}}$). By Step III) of the proof, $V$ (resp. $\widetilde{V}^{\mathrm{int}}$) does not depend on $S$. Each $g \in E$ (resp. $g \in E^{\mathrm{int}}$) defines an isomorphism $V_S^{\mathrm{an}} \xrightarrow{\sim} V_{gSg^{-1}}^{\mathrm{an}}$ (resp. $(\widetilde{V}_S^{\mathrm{int}})^{\mathrm{an}} \xrightarrow{\sim} (\widetilde{V}_{gSg^{-1}}^{\mathrm{int}})^{\mathrm{an}}$). Therefore by Corollary 1.2.3 *a)*, $g$ defines an isomorphism $V_S \xrightarrow{\sim} V_{gSg^{-1}}$ (resp. $\widetilde{V}_S^{\mathrm{int}} \xrightarrow{\sim} \widetilde{V}_{gSg^{-1}}^{\mathrm{int}}$). The product of this isomorphism and the action of $g$ on $X$ (resp. $\widetilde{X}^{\mathrm{int}}$) gives us an isomorphism $g : V = V_S \times_{X_S} X \xrightarrow{\sim} V_{gSg^{-1}} \times_{X_{gsg^{-1}}} X = V$ (resp. $g : \widetilde{V}^{\mathrm{int}} \xrightarrow{\sim} \widetilde{V}^{\mathrm{int}}$). Thus we have constructed an algebraic action of $E$ (resp. of $E^{\mathrm{int}}$) on $V$ (resp. $\widetilde{V}^{\mathrm{int}}$), satisfying $S\backslash V \cong V_S$ for all $S \in \mathscr{F}(E)$ (resp. $S\backslash\widetilde{V}^{\mathrm{int}} \cong \widetilde{V}_S^{\mathrm{int}}$ for all $S \in \mathscr{F}(E^{\mathrm{int}})$). Moreover, $V = \varprojlim_S V_S$ and $\widetilde{V}^{\mathrm{int}} = \varprojlim_S \widetilde{V}_S^{\mathrm{int}}$.

By [Mil], there exists a unique canonical model $V^{\mathrm{int}}$ of $\widetilde{V}^{\mathrm{int}}$ over $K_w$ (the definition of the canonical model will be explained in the last paragraph of the proof of Proposition 4.3.1) such that $V^{\mathrm{int}}$ is an $E^{\mathrm{int}}$-equivariant vector bundle on $X^{\mathrm{int}}$.

Our main task is to prove the following

*Third Main Theorem* **4.1.4.** — *For any admissible isomorphism* $\Phi : E \xrightarrow{\sim} E^{\mathrm{int}}$, *each isomorphism* $f_\Phi$ *from the First or the Second Main Theorem can be lifted to a* $\Phi$*-equivariant isomorphism* $f_{\Phi, V} : V \xrightarrow{\sim} V^{\mathrm{int}}$.

We will prove this theorem, using standard principal bundles (= torsors) (see [Mil, Ch. III, § 3]).

## 4.2. Equivariant torsors

**4.2.1.** For each $S \in \mathscr{F}(E)$ (resp. $S \in \mathscr{F}(E^{\mathrm{int}})$) consider the double quotient $\widetilde{P}_S := S\backslash[\Sigma_{K_w}^d \times (PH_{K_w})^{\mathrm{an}} \times E']/\Gamma$ (resp. $\widetilde{P}_S^{\mathrm{int}} := S\backslash[B^{d-1} \times (PH_{\mathbf{C}}^{\mathrm{int}})^{\mathrm{an}} \times E^{\mathrm{int}}]/\Gamma^{\mathrm{int}}$).

*Proposition* **4.2.2.** — *For each* $S \in \mathscr{F}(E)$ (*resp.* $S \in \mathscr{F}(E^{\mathrm{int}})$) $\widetilde{P}_S$ (*resp.* $\widetilde{\widetilde{P}}_S^{\mathrm{int}}$) *has a natural structure of an affine scheme* $P_S$ *over* $X_S$ (*resp.* $\widetilde{P}_S^{\mathrm{int}}$ *over* $\widetilde{X}_S^{\mathrm{int}}$). *Moreover,* $P_S$ *is a* $PH_{K_w}$*-torsor over* $X_S$ (*resp.* $\widetilde{P}_S^{\mathrm{int}}$ *is a* $PH_{\mathbf{C}}^{\mathrm{int}}$*-torsor over* $\widetilde{X}_S^{\mathrm{int}}$ *if* $S$ *is sufficiently small*).

The proof is almost identical to that of Proposition 4.1.2 (using Proposition 1.9.13 and Lemma 1.9.3 instead of Corollary 1.2.3 *a)* and arguments of step III) respectively). □

**4.2.3.** Arguing as in 4.1.3 and using Corollary 1.9.14 we obtain an E-equivariant $\mathrm{PH}_{\mathrm{K}_w}$-torsor $\mathrm{P} = \varprojlim_{\mathrm{S}} \mathrm{P}_{\mathrm{S}}$ over X (resp. an $\mathrm{E}^{\mathrm{int}}$-equivariant $\mathrm{PH}_{\mathbf{C}}^{\mathrm{int}}$-torsor $\widetilde{\mathrm{P}}^{\mathrm{int}} = \varprojlim_{\mathrm{S}} \widetilde{\mathrm{P}}_{\mathrm{S}}^{\mathrm{int}}$ over $\widetilde{\mathrm{X}}^{\mathrm{int}}$). By [Mil, III, Thm. 4.3], there exists a unique canonical model $\mathrm{P}^{\mathrm{int}}$ of $\widetilde{\mathrm{P}}^{\mathrm{int}}$ over $\mathrm{K}_w$ (the definition will be explained in Corollary 4.7.2) such that $\mathrm{P}^{\mathrm{int}}$ is an $\mathrm{E}^{\mathrm{int}}$-equivariant $\mathrm{PH}_{\mathrm{K}_w}^{\mathrm{int}}$-torsor over $\mathrm{X}^{\mathrm{int}}$. Let $\pi : \mathrm{P} \to \mathrm{X}$ and $\pi^{\mathrm{int}} : \mathrm{P}^{\mathrm{int}} \to \mathrm{X}^{\mathrm{int}}$ be the natural projections. Denote also the natural projection from the $\mathrm{PH}_{\mathrm{K}_w}^{\mathrm{int}}$-torsor $\mathrm{P}^{\mathrm{tw}}$ to X by $\pi^{\mathrm{tw}}$.

*Fourth Main Theorem* **4.2.4.** — *For any admissible isomorphism* $\Phi : \mathrm{E} \xrightarrow{\sim} \mathrm{E}^{\mathrm{int}}$, *each isomorphism* $f_\Phi$ *from the First or the Second Main Theorems can be lifted to a* $\Phi$*-equivariant isomorphism* $f_{\Phi,\mathrm{P}} : \mathrm{P}^{\mathrm{tw}} \xrightarrow{\sim} \mathrm{P}^{\mathrm{int}}$ *of* $\mathrm{PH}_{\mathrm{K}_w}^{\mathrm{int}}$*-torsors.*

## 4.3. Connection between the Main Theorems

*Proposition* **4.3.1.** — *The Fourth Main Theorem implies the third one.*

*Proof.* — Consider the pro-analytic maps

$$\widetilde{\rho}' : [\Sigma_{\mathrm{K}_w}^{d} \times (\mathrm{PH}_{\mathrm{K}_w})^{\mathrm{an}} \times (\mathrm{E}')^{\mathrm{disc}}]/\Gamma \to (\mathbf{P}_{\mathrm{K}_w}^{d-1})^{\mathrm{an}}$$

and

$$(\tilde{\tilde{\rho}}')^{\mathrm{int}} : [\mathrm{B}^{d-1} \times (\mathrm{PH}_{\mathbf{C}}^{\mathrm{int}})^{\mathrm{an}} \times (\mathrm{E}^{\mathrm{int}})^{\mathrm{disc}}]/\Gamma^{\mathrm{int}} \to (\mathbf{P}_{\mathbf{C}}^{d-1})^{\mathrm{an}}$$

given by $\widetilde{\rho}'(x, g, e) = g\beta_w(x)$ and $(\tilde{\tilde{\rho}}')^{\mathrm{int}}(x, g, e) = g\beta_{\mathbf{C}}(x)$. Then $\widetilde{\rho}'$ (resp. $(\tilde{\tilde{\rho}}')^{\mathrm{int}}$) is $(\mathrm{PH}_{\mathrm{K}_w})^{\mathrm{an}}$- (resp. $(\mathrm{PH}_{\mathbf{C}}^{\mathrm{int}})^{\mathrm{an}}$-) equivariant and commutes with the action of E (resp. $\mathrm{E}^{\mathrm{int}}$). Hence it defines an equivariant analytic map $\widetilde{\rho} : \mathrm{P}^{\mathrm{an}} \to (\mathbf{P}_{\mathrm{K}_w}^{d-1})^{\mathrm{an}}$ (resp. $\tilde{\tilde{\rho}}^{\mathrm{int}} : (\widetilde{\mathrm{P}}^{\mathrm{int}})^{\mathrm{an}} \to (\mathbf{P}_{\mathbf{C}}^{d-1})^{\mathrm{an}}$).

*Proposition* **4.3.2.** — *There exists a unique algebraic morphism* $\rho : \mathrm{P} \to \mathbf{P}_{\mathrm{K}_w}^{d-1}$ *(resp.* $\widetilde{\rho}^{\mathrm{int}} : \widetilde{\mathrm{P}}^{\mathrm{int}} \to \mathbf{P}_{\mathbf{C}}^{d-1}$*) such that* $\rho^{\mathrm{an}} \cong \widetilde{\rho}$ *(resp.* $(\widetilde{\rho}^{\mathrm{int}})^{\mathrm{an}} \cong \tilde{\tilde{\rho}}^{\mathrm{int}}$*).*

*Proof.* — We prove the statement for $\rho$ (in the second case the proof is exactly the same). We have to show that the graph $\mathrm{Gr}(\widetilde{\rho}) \subset \mathrm{P}^{\mathrm{an}} \times (\mathbf{P}_{\mathrm{K}_w}^{d-1})^{\mathrm{an}}$ corresponds to an algebraic subscheme. For each $\mathrm{S} \in \mathscr{F}(\mathrm{E})$ let $\widetilde{\rho}_{\mathrm{S}} : \mathrm{P}_{\mathrm{S}}^{\mathrm{an}} \to (\mathbf{P}_{\mathrm{K}_w}^{d-1})^{\mathrm{an}}$ be the morphism induced by $\widetilde{\rho}$. Since $\mathrm{Gr}(\widetilde{\rho}) = \varprojlim_{\mathrm{S}} \mathrm{Gr}(\widetilde{\rho}_{\mathrm{S}}) \subset (\varprojlim_{\mathrm{S}} \mathrm{P}_{\mathrm{S}}^{\mathrm{an}}) \times (\mathbf{P}_{\mathrm{K}_w}^{d-1})^{\mathrm{an}}$, it remains to show that the graph $\mathrm{Gr}(\widetilde{\rho}_{\mathrm{S}}) \subset \mathrm{P}_{\mathrm{S}}^{\mathrm{an}} \times (\mathbf{P}_{\mathrm{K}_w}^{d-1})^{\mathrm{an}}$ corresponds to a unique algebraic subvariety for each S sufficiently small.

Take S so small that $\mathrm{X}_{\mathrm{S}}$ is smooth, then by Lemma 1.9.5 *b)* there exists a quotient $\mathrm{Q}_{\mathrm{S}} := \mathrm{PH}_{\mathrm{K}_w} \backslash (\mathrm{P}_{\mathrm{S}} \times \mathbf{P}_{\mathrm{K}_w}^{d-1})$ by the diagonal action of $\mathrm{PH}_{\mathrm{K}_w}$. Moreover, $\mathrm{Q}_{\mathrm{S}}$ is a $\mathbf{P}^{d-1}$-bundle on $\mathrm{X}_{\mathrm{S}}$, hence it is projective over $\mathrm{K}_w$. Let $\alpha : \mathrm{P}_{\mathrm{S}} \times \mathbf{P}_{\mathrm{K}_w}^{d-1} \to \mathrm{Q}_{\mathrm{S}}$ be the natural projection. Since $\widetilde{\rho}_{\mathrm{S}}$ is $(\mathrm{PH}_{\mathrm{K}_w})^{\mathrm{an}}$-equivariant, $\mathrm{Gr}(\widetilde{\rho}_{\mathrm{S}})$ is invariant under the diagonal action of $(\mathrm{PH}_{\mathrm{K}_w})^{\mathrm{an}}$. Therefore the quotient $\widetilde{\mathrm{Q}} := (\mathrm{PH}_{\mathrm{K}_w})^{\mathrm{an}}\backslash\mathrm{Gr}(\widetilde{\rho}_{\mathrm{S}})$ is a closed analytic subspace of $\mathrm{Q}_{\mathrm{S}}^{\mathrm{an}}$, so that it is algebraic (see Corollary 1.2.2). It follows that its inverse image $\alpha^{-1}(\widetilde{\mathrm{Q}}) = \mathrm{Gr}(\widetilde{\rho}_{\mathrm{S}})$ is also algebraic. The uniqueness is clear. □

*Claim* **4.3.3.** — *The map* $\overset{\approx}{\rho}{}^{\mathrm{int}}$ *is the only* $(\mathrm{PH}_{\mathbf{C}}^{\mathrm{int}})^{\mathrm{an}} \times \mathrm{E}^{\mathrm{int}}$*-equivariant analytic map from* $(\widetilde{\mathrm{P}}^{\mathrm{int}})^{\mathrm{an}}$ *to* $(\mathbf{P}_{\mathbf{C}}^{d-1})^{\mathrm{an}}$.

*Proof.* — Let $\rho' : (\widetilde{\mathrm{P}}^{\mathrm{int}})^{\mathrm{an}} \to (\mathbf{P}_{\mathbf{C}}^{d-1})^{\mathrm{an}}$ be any such map. Composing it with the natural $(\mathrm{PH}_{\mathbf{C}}^{\mathrm{int}})^{\mathrm{an}} \times \mathrm{E}^{\mathrm{int}}$-equivariant projection

$$[\mathrm{B}^{d-1} \times (\mathrm{PH}_{\mathbf{C}}^{\mathrm{int}})^{\mathrm{an}} \times (\mathrm{E}^{\mathrm{int}})^{\mathrm{disc}}]/\Gamma^{\mathrm{int}} \to (\mathrm{P}^{\mathrm{int}})^{\mathrm{an}}$$

and, identifying a complex analytic space with the set of its $\mathbf{C}$-rational points, we obtain a $\mathrm{PH}^{\mathrm{int}}(\mathbf{C}) \times \mathrm{E}^{\mathrm{int}}$-equivariant analytic map

$$\rho'' : [\mathrm{B}^{d-1} \times \mathrm{PH}^{\mathrm{int}}(\mathbf{C}) \times (\mathrm{E}^{\mathrm{int}})^{\mathrm{disc}}]/\Gamma^{\mathrm{int}} \to (\mathbf{P}_{\mathbf{C}}^{d-1})^{\mathrm{an}}.$$

Let $\rho_0$ be the restriction of $\rho$ to $\mathrm{B}^{d-1} \xrightarrow{\sim} \mathrm{B}^{d-1} \times \{1\} \times \{1\}$. Then $\rho''(x, g, e) = g\rho_0(x)$ for all $x \in \mathrm{B}^{d-1}$, $g \in (\mathrm{PH}_{\mathbf{C}}^{\mathrm{int}})^{\mathrm{an}}$ and $e \in \mathrm{E}^{\mathrm{int}}$. Therefore $\gamma\rho_0(x) = \rho_0(\gamma x)$ for all $\gamma \in \Gamma^{\mathrm{int}}$ and $x \in \mathrm{B}^{d-1}$. Since the subgroup $\Gamma^{\mathrm{int}}$ is dense in $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$, we obtain by continuity that $\gamma\rho_0(x) = \rho_0(\gamma x)$ for all $\gamma \in \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ and $x \in \mathrm{B}^{d-1}$. In particular, for the origin $0 \in \mathrm{B}^{d-1}$ we get $\mathrm{Stab}_{\mathrm{PGU}_{d-1,1}(\mathbf{R})^0}(0) \subset \mathrm{Stab}_{\mathrm{PGU}_{d-1,1}(\mathbf{R})^0}(\rho_0(0))$. The subgroup $\mathrm{Stab}_{\mathrm{PGU}_{d-1,1}(\mathbf{R})^0}(0)$ stabilizes precisely one point $(0 : \ldots : 0 : 1) \in \mathbf{P}^{d-1}(\mathbf{C})$ if $d > 2$ and two points $(0 : 1)$ and $(1 : 0)$ in $\mathbf{P}^1(\mathbf{C})$ if $d = 2$. The case $\rho_0(0) = (1 : 0)$ is impossible, because identifying $\mathbf{P}^1(\mathbf{C})$ with $\overline{\mathbf{C}} = \mathbf{C} \cup \infty$ by $(x : y) \mapsto x/y$ we would get in this case $\rho_0(z) = 1/\bar{z}$ for all $z \in \mathrm{B}^1$, contradicting the analyticity of $\rho_0$. We conclude that $\rho_0(0) = (0 : \ldots : 0 : 1)$. Hence $\rho_0 = \beta_{\mathbf{R}}$ and $\rho' = \overset{\approx}{\rho}{}^{\mathrm{int}}$. □

**4.3.4.** Next we show that the map $\widetilde{\rho}^{\mathrm{int}} : \mathrm{P}_{\mathbf{C}}^{\mathrm{int}} \to (\mathbf{P}_{\mathrm{K}_w}^{d-1})_{\mathbf{C}}^{\mathrm{tw}}$ is $\mathrm{K}_w$-rational. Recall that the map $\widetilde{\rho}^{\mathrm{int}}$ is $\mathrm{PH}_{\mathbf{C}}^{\mathrm{int}}$-equivariant and that the actions of the group $\mathrm{PH}_{\mathrm{K}_w}^{\mathrm{int}}$ on both $\mathrm{P}^{\mathrm{int}}$ and $(\mathbf{P}_{\mathrm{K}_w}^{d-1})^{\mathrm{tw}}$ are $\mathrm{K}_w$-rational. Therefore for each $\sigma \in \mathrm{Aut}(\mathbf{C}/\mathrm{K}_w)$ the analytic map $\sigma(\widetilde{\rho}^{\mathrm{int}})^{\mathrm{an}}$ is $(\mathrm{PH}_{\mathbf{C}}^{\mathrm{int}})^{\mathrm{an}} \times \mathrm{E}^{\mathrm{int}}$-equivariant, hence it coincides with $(\widetilde{\rho}^{\mathrm{int}})^{\mathrm{an}} = \overset{\approx}{\rho}{}^{\mathrm{int}}$. By the uniqueness of the algebraic structure, $\sigma(\widetilde{\rho}^{\mathrm{int}}) = \widetilde{\rho}^{\mathrm{int}}$.

It follows that $\widetilde{\rho}^{\mathrm{int}}$ defines a $\mathrm{PH}_{\mathrm{K}_w}^{\mathrm{int}} \times \mathrm{E}^{\mathrm{int}}$-equivariant map $\rho^{\mathrm{int}} : \mathrm{P}^{\mathrm{int}} \to (\mathbf{P}_{\mathrm{K}_w}^{d-1})^{\mathrm{tw}}$. Notice also that $\rho$ defines a $\mathrm{PH}_{\mathrm{K}_w}^{\mathrm{int}} \times \mathrm{E}$-equivariant map $\rho^{\mathrm{tw}} : \mathrm{P}^{\mathrm{tw}} \to (\mathbf{P}_{\mathrm{K}_w}^{d-1})^{\mathrm{tw}}$.

Suppose that the Fourth Main Theorem holds, then

*Lemma* **4.3.5.** — *We have* $\rho^{\mathrm{int}} \circ f_{\Phi,\mathrm{P}} = \rho^{\mathrm{tw}}$.

*Proof.* — By the claim, $\overset{\approx}{\rho}{}^{\mathrm{int}} \cong (\rho_{\mathbf{C}}^{\mathrm{int}})^{\mathrm{an}}$ is equal to $(\rho_{\mathbf{C}}^{\mathrm{tw}} \circ (f_{\Phi,\mathrm{P}})_{\mathbf{C}}^{-1})^{\mathrm{an}}$. From the uniqueness of algebraic structures we conclude that $\rho_{\mathbf{C}}^{\mathrm{int}} = \rho_{\mathbf{C}}^{\mathrm{tw}} \circ (f_{\Phi,\mathrm{P}})_{\mathbf{C}}^{-1}$. Now we descent to $\mathrm{K}_w$ as in 4.3.4. □

It follows from the definitions that $\rho^*(\mathrm{W}) \cong \pi^*(\mathrm{V})$ (hence $\rho^*(\mathrm{W})^{\mathrm{tw}} \cong (\pi^{\mathrm{tw}})^*(\mathrm{V})$) and $(\widetilde{\rho}^{\mathrm{int}})^*(\mathrm{W}^{\mathrm{tw}}_{\mathbf{C}}) \cong (\pi^{\mathrm{int}}_{\mathbf{C}})^*(\widetilde{\mathrm{V}}^{\mathrm{int}})$. Lemma 1.9.5 allows us to define $\mathrm{V}^{\mathrm{int}}$ by the requirement that $(\pi^{\mathrm{int}})^*(\mathrm{V}^{\mathrm{int}}) \cong (\rho^{\mathrm{int}})^*(\mathrm{W}^{\mathrm{tw}})$. (By the definition, this is the canonical model of $\widetilde{\mathrm{V}}^{\mathrm{int}}$ on $\mathrm{X}^{\mathrm{int}}$.) Lemma 4.3.5 implies that $f_{\Phi, \mathrm{P}}$ can be lifted to the $\Phi$-equivariant isomorphism $\rho^*(\mathrm{W})^{\mathrm{tw}} \cong (\rho^{\mathrm{tw}})^*(\mathrm{W}^{\mathrm{tw}}) \xrightarrow{\sim} (\rho^{\mathrm{int}})^*(\mathrm{W}^{\mathrm{tw}})$, commuting with the $\mathrm{PH}^{\mathrm{int}}_{\mathrm{K}_w}$-action. This gives us the $\mathrm{PH}^{\mathrm{int}}_{\mathrm{K}_w}$-equivariant isomorphism $(\pi^{\mathrm{tw}})^*(\mathrm{V}) \xrightarrow{\sim} (\pi^{\mathrm{int}})^*(\mathrm{V}^{\mathrm{int}})$. Hence the Third Main Theorem follows from Lemma 1.9.5. □

*Remark* **4.3.6.** — Tannakian arguments can be used to show (see Theorem 1.9.10 and the discussion around it) that the Third Main Theorem implies the Fourth one. We will not use this implication.

### 4.4. Reduction of the problem

**4.4.1.** Now we start the proof of the Fourth Main Theorem. For simplicity of notation we identify E with $\mathrm{E}^{\mathrm{int}}$ by means of $\Phi$ and X with $\mathrm{X}^{\mathrm{int}}$ by means of $f_{\Phi}$. Recall that $\mathrm{P}_{\mathrm{S},\mathbf{C}}$ is a $\mathrm{PH}_{\mathbf{C}}$-torsor over $\mathrm{X}_{\mathrm{S},\mathbf{C}}$ for all sufficiently small $\mathrm{S} \in \mathscr{F}(\mathrm{E})$, hence $(\mathrm{P}_{\mathrm{S},\mathbf{C}})^{\mathrm{an}}$ is a $(\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$-torsor over $(\mathrm{X}_{\mathrm{S},\mathbf{C}})^{\mathrm{an}}$ and $(\mathrm{P}_{\mathbf{C}})^{\mathrm{an}} = (\mathrm{P}_{\mathrm{S},\mathbf{C}})^{\mathrm{an}} \times_{(\mathrm{X}_{\mathrm{S},\mathbf{C}})^{\mathrm{an}}} (\mathrm{X}_{\mathbf{C}})^{\mathrm{an}}$ is a $(\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$-torsor over $(\mathrm{X}_{\mathbf{C}})^{\mathrm{an}} \cong [\mathrm{B}^{d-1} \times (\mathrm{E}^{\mathrm{int}}/\mathrm{E}^{\mathrm{int}}_0)^{\mathrm{disc}}]/\mathrm{P}\Gamma^{\mathrm{int}}$. Set $\mathrm{Y} := (\pi^{\mathrm{an}})^{-1}(\mathrm{B}^{d-1} \times \{1\}) \subset (\mathrm{P}_{\mathbf{C}})^{\mathrm{an}}$. Then Y is a $(\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$-torsor over $\mathrm{B}^{d-1}$. Recall that $\mathrm{E}_0 = \mathrm{E}^{\mathrm{int}}_0$ acts trivially on P, hence $(\mathrm{P}_{\mathbf{C}})^{\mathrm{an}} \cong (\mathrm{Y} \times (\mathrm{E}^{\mathrm{int}}/\mathrm{E}^{\mathrm{int}}_0)^{\mathrm{disc}})/\mathrm{P}\Gamma^{\mathrm{int}}$.

*Proposition* **4.4.2.** — *There exists a homomorphism* $j : \mathrm{P}\Gamma^{\mathrm{int}} \to \mathrm{PH}(\mathbf{C})$ *and an isomorphism* $(\mathrm{P}_{\mathbf{C}})^{\mathrm{an}} \xrightarrow{\sim} (\mathrm{B}^{d-1} \times (\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}} \times (\mathrm{E}^{\mathrm{int}}/\mathrm{E}^{\mathrm{int}}_0)^{\mathrm{disc}})/\mathrm{P}\Gamma^{\mathrm{int}}$ *such that* $(x, h, g)\,\gamma = (\gamma_\infty^{-1} x, hj(\gamma), g\gamma_{\mathrm{E}})$ *for all* $x \in \mathrm{B}^{d-1}$, $h \in (\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$, $g \in \mathrm{E}^{\mathrm{int}}/\mathrm{E}^{\mathrm{int}}_0$ *and* $\gamma \in \mathrm{P}\Gamma^{\mathrm{int}}$.

*Proof.* — The proposition asserts that there exists a decomposition

$$\mathrm{Y} \cong \mathrm{B}^{d-1} \times (\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$$

such that the group $\mathrm{P}\Gamma^{\mathrm{int}}$ acts on $\mathrm{B}^{d-1} \times (\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$ by the product of actions on factors.

The trivial connection on $\Sigma^d_{\mathrm{K}_w} \times (\mathrm{PH}_{\mathrm{K}_w})^{\mathrm{an}} \to \Sigma^d_{\mathrm{K}_w}$ is $\Gamma \times \widetilde{\mathrm{D}}^{\times}_w$-invariant, therefore it defines a natural E-invariant flat connection $\widetilde{\mathscr{H}}$ on the $(\mathrm{PH}_{\mathrm{K}_w})^{\mathrm{an}}$-torsor $[\Sigma^d_{\mathrm{K}_w} \times (\mathrm{PH}_{\mathrm{K}_w})^{\mathrm{an}} \times \mathrm{E}^{\mathrm{disc}}]/\Gamma$ over $[\Sigma^d_{\mathrm{K}_w} \times \mathrm{E}^{\mathrm{disc}}]/\Gamma$. Since for all sufficiently small $\mathrm{S} \in \mathscr{F}(\mathrm{E})$ the projection $(\Sigma^d_{\mathrm{K}_w} \times (\mathrm{PH}_{\mathrm{K}_w})^{\mathrm{an}} \times \mathrm{E}^{\mathrm{disc}})/\Gamma \to \mathrm{P}^{\mathrm{an}}_{\mathrm{S}}$ is étale, it induces an isomorphism of tangent spaces up to an extension of scalars. Hence $\widetilde{\mathscr{H}}$ induces a flat connection $\widetilde{\mathscr{H}}_{\mathrm{S}}$ on $\mathrm{P}^{\mathrm{an}}_{\mathrm{S}}$. By the definition, $\widetilde{\mathscr{H}}_{\mathrm{S}}$ is a $(\mathrm{PH}_{\mathrm{K}_w})^{\mathrm{an}}$-invariant analytic vector subbundle of $(\mathrm{T}_{\mathrm{P}_{\mathrm{S}}})^{\mathrm{an}}$, therefore Lemma 1.9.5 and Corollary 1.2.3 imply the existence of a unique flat connection $\mathscr{H}_{\mathrm{S}}$ on $\mathrm{P}_{\mathrm{S}}$ such that $\widetilde{\mathscr{H}}_{\mathrm{S}} \cong \mathscr{H}^{\mathrm{an}}_{\mathrm{S}}$. Since the projection $\pi_{\mathrm{S}} : \mathrm{P} \to \mathrm{P}_{\mathrm{S}}$ is étale, $\mathscr{H}_{\mathrm{S}}$ defines a unique flat connection $\mathscr{H}$ on P satisfying $(\pi_{\mathrm{S}})_*(\mathscr{H}) = \mathscr{H}_{\mathrm{S}}$. Moreover, $\mathscr{H}$ is E-equivariant and does not depend on S.

The connection $\mathscr{H}$ determines flat connections $\mathscr{H}_{\mathbf{C}}$ on $\mathrm{P}_{\mathbf{C}}$ and $(\mathscr{H}_{\mathbf{C}})^{\mathrm{an}}$ on $(\mathrm{P}_{\mathbf{C}})^{\mathrm{an}}$.

Let $\mathcal{H}'$ be the restriction of $(\mathcal{H}_{\mathbf{C}})^{\mathrm{an}}$ to Y. Then $\mathcal{H}'$ is a $\mathrm{P}\Gamma^{\mathrm{int}}$-invariant flat connection on the $(\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$-principal bundle Y over the simply connected complex manifold $\mathrm{B}^{d-1}$. By Lemma 1.9.19, there exists a decomposition $\mathrm{Y} \cong \mathrm{B}^{d-1} \times (\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$ such that the corresponding action of $\mathrm{P}\Gamma^{\mathrm{int}}$ on $\mathrm{B}^{d-1} \times (\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$ preserves the trivial connection.

For each $\gamma \in \mathrm{P}\Gamma^{\mathrm{int}}$ let $\widetilde{\gamma} : \mathrm{B}^{d-1} \times (\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}} \to (\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$ be the analytic map such that $\gamma(x, h) = (\gamma(x), \widetilde{\gamma}(x, h))$ for all $x \in \mathrm{B}^{d-1}$ and $h \in (\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$. Since the action of $\mathrm{P}\Gamma^{\mathrm{int}}$ preserves the trivial connection, we have $\partial\widetilde{\gamma}/\partial x \equiv 0$ for each $\gamma \in \mathrm{P}\Gamma^{\mathrm{int}}$. Hence analytic $\widetilde{\gamma}$'s depend only on $h$. Since the action of $\mathrm{P}\Gamma^{\mathrm{int}}$ commutes with the action of $(\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$, we have $\widetilde{\gamma}(h) = h\widetilde{\gamma}(1)$ for all $h \in (\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$ and $\gamma \in \mathrm{P}\Gamma^{\mathrm{int}}$. Therefore the map $\gamma \mapsto \widetilde{\gamma}(1)^{-1}$ is the required homomorphism. □

*Theorem* **4.4.3.** — *There exists an inner isomorphism (= inner twisting)*

$$\Phi_{\mathbf{C}} : \mathrm{PH}_{\mathbf{C}} \xrightarrow{\sim} \mathrm{PH}^{\mathrm{int}}_{\mathbf{C}}$$

*such that* $j \circ \Phi_{\mathbf{C}} : \mathrm{P}\Gamma^{\mathrm{int}} = \mathrm{PG}^{\mathrm{int}}(\mathrm{F})_{+} \to \mathrm{PH}^{\mathrm{int}}(\mathbf{C}) \cong \mathrm{PG}^{\mathrm{int}}(\mathbf{C} \otimes_{\mathbf{Q}} \mathrm{F})$ *is induced by the natural (diagonal) embedding* $\mathrm{F} \hookrightarrow \mathbf{C} \otimes_{\mathbf{Q}} \mathrm{F} \cong \mathbf{C}^{g}$.

*Remark* **4.4.4.** — Algebraization considerations as in Lemma 2.2.6 (using Proposition 1.9.13 instead of Corollary 1.2.2) show that Theorem 4.4.3 implies the existence of a $\Phi$-equivariant isomorphism $\mathrm{P}^{\mathrm{tw}}_{\mathbf{C}} \xrightarrow{\sim} \widetilde{\mathrm{P}}^{\mathrm{int}}$, lifting $\widetilde{f_{\Phi}}$.

## 4.5. Proof of density

To prove Theorem 4.4.3 we will use Margulis' results. For this we first show that the subgroup $j(\mathrm{P}\Gamma^{\mathrm{int}})$ is sufficiently large. We start with the following technical

*Lemma* **4.5.1.** — *Let $n$ and $d$ be positive integers. For each $i = 1, \dots, n$ we denote by* $\mathrm{pr}_i$ *the projection to the i-th factor.*

*a) Let $\mathcal{G}_1, \dots, \mathcal{G}_n$ be Lie algebras, and let $\mathcal{H}$ be an ideal in the Lie algebra $\mathcal{G} = \prod_{i=1}^{n} \mathcal{G}_i$. Then $\mathcal{H} \supset \prod_{i=1}^{n}[\mathrm{pr}_i \mathcal{H}, \mathcal{G}_i]$.*

*b) Let $\Delta$ be a subgroup of* $\mathrm{PGL}_d(\mathbf{C})^n$. *Suppose that* $\mathrm{pr}_i(\Delta)$ *is infinite for every $i = 1, \dots, n$. If* $\widetilde{\Delta} := \mathrm{Comm}_{\mathrm{PGL}_d(\mathbf{C})^n}(\Delta)$ *is Zariski dense in* $(\mathrm{PGL}_d)^n$, *then the same is true for* $\Delta$.

*c) If a subgroup* $\Delta \subset \mathrm{PGU}_d(\mathbf{R})^n$ *is Zariski dense (in* $(\mathrm{PGU}_d)^n$*), then it is dense.*

*Proof.* — *a)* If $x = (x_1, \dots, x_n) \in \prod_{i=1}^{n} \mathcal{G}_i$ belongs to $\mathcal{H}$, then

$$[x, y_i] = (0, \dots, [x_i, y_i], \dots, 0) = [\mathrm{pr}_i\, x, y_i] \in \mathcal{H} \quad \text{for all } y_i \in \mathcal{G}_i.$$

*b)* Let J be the Zariski closure of $\Delta$ in $(\mathrm{PGL}_d)^n$, then $\delta \mathrm{J}\, \delta^{-1} \cap \mathrm{J}$ is an algebraic subgroup of finite index in J for each $\delta \in \widetilde{\Delta}$. Hence $\delta \mathrm{J}^0 \delta^{-1} = \mathrm{J}^0$. In particular, the subgroup $\mathrm{Ad}\, \widetilde{\Delta}$ stabilizes $\mathrm{Lie}\, \mathrm{J}^0 \subset \mathrm{Lie}(\mathrm{PGL}_a)^n$. Since $\widetilde{\Delta}$ is Zariski dense in $(\mathrm{PGL}_d)^n$, the Lie algebra $\mathrm{Lie}\, \mathrm{J}^0 = \mathrm{Lie}\, \mathrm{J}$ is an ideal in $\mathrm{Lie}(\mathrm{PGL}_d)^n$. By our assumption, $\mathrm{pr}_i(\mathrm{J})$ is an infinite algebraic group for each $i = 1, \dots, n$, therefore $\mathrm{pr}_i(\mathrm{Lie}\, \mathrm{J}) \neq 0$ is an ideal in a simple Lie algebra $\mathrm{Lie}(\mathrm{PGL}_d)$. Therefore *a)* implies that $\mathrm{Lie}\,\mathrm{J} = \mathrm{Lie}(\mathrm{PGL}_d)^n$. Since the group $(\mathrm{PGL}_d)^n$ is connected, $\mathrm{J} = (\mathrm{PGL}_d)^n$.

*c)* Let M be the closure of $\Delta$ in $\mathrm{PGU}_d(\mathbf{R})^n$. Then M is a Lie subgroup of the Lie group $\mathrm{PGU}_d(\mathbf{R})^n$. Hence Lie M is an Ad M-invariant subspace of $\mathrm{Lie}(\mathrm{PGU}_d(\mathbf{R}))^n$. Since the adjoint representation is algebraic, Lie M is an ideal in $\mathrm{Lie}(\mathrm{PGU}_d(\mathbf{R}))^n$. Since M is compact, it has a finite number of connected components. Hence $\mathrm{M}^0$ is also Zariski dense, therefore it is not contained in $\mathrm{PGU}_d(\mathbf{R})^{i-1} \times \{1\} \times \mathrm{PGU}_d(\mathbf{R})^{n-i}$ for any $i = 1, \dots, n$. It follows that $\mathrm{Lie}\,\mathrm{M} = \mathrm{Lie}\,\mathrm{M}^0$ is not contained in

$$\mathrm{Lie}(\mathrm{PGU}_d(\mathbf{R}))^{i-1} \times \{0\} \times \mathrm{Lie}(\mathrm{PGU}_d(\mathbf{R}))^{n-i},$$

so that $\mathrm{pr}_i(\mathrm{Lie}\,\mathrm{M}) \neq 0$. Now the assertion follows exactly in the same way as in *b)*. □

*Proposition* **4.5.2.** — *The subgroup* $j(\mathrm{P}\Gamma^{\mathrm{int}})$ *is Zariski dense in* $\mathrm{PH}_{\mathbf{C}}$.

*Proof.* — Let $\mathrm{G}' \subset \mathrm{PH}_{\mathbf{C}}$ be the Zariski closure of $j(\mathrm{P}\Gamma^{\mathrm{int}})$. Then

$$\widetilde{\mathrm{R}} := (\mathrm{B}^{d-1} \times (\mathrm{G}')^{\mathrm{an}} \times (\mathrm{PG}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f;v}))^{\mathrm{disc}})/\mathrm{P}\Gamma^{\mathrm{int}}$$

is a $\mathrm{PG}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f;v})$-invariant $(\mathrm{G}')^{\mathrm{an}}$-subtorsor of the $(\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}}$-torsor

$$\begin{aligned}(\mathrm{P}''_{\mathbf{C}})^{\mathrm{an}} &= ((\mathrm{G}^{\mathrm{int}}(\mathrm{F}_v) \times \mathrm{Z}(\mathrm{E}^{\mathrm{int}}))\backslash \mathrm{P}_{\mathbf{C}})^{\mathrm{an}} \\ &\cong [\mathrm{B}^{d-1} \times (\mathrm{PH}_{\mathbf{C}})^{\mathrm{an}} \times (\mathrm{PG}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f;v}))^{\mathrm{disc}}]/\mathrm{P}\Gamma^{\mathrm{int}}\end{aligned}$$

over $(\mathrm{X}''_{\mathbf{C}})^{\mathrm{an}} \cong [\mathrm{B}^{d-1} \times (\mathrm{PG}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f;v}))^{\mathrm{disc}}]/\mathrm{P}\Gamma^{\mathrm{int}}$. Hence by Proposition 1.9.13 there exists an algebraic $\mathrm{G}'$-subtorsor R of $\mathrm{P}''_{\mathbf{C}}$ such that $\mathrm{R}^{\mathrm{an}} \cong \widetilde{\mathrm{R}}$. Using our identification of $\mathbf{C}_p$ with $\mathbf{C}$, we obtain a closed analytic subspace

$$(\mathrm{R}_{\mathbf{C}_p})^{\mathrm{an}} \subset (\mathrm{P}''_{\mathbf{C}_p})^{\mathrm{an}} \cong (\Omega^d_{\mathrm{K}_w} \widehat{\otimes}_{\mathrm{K}_w} \mathbf{C}_p \times (\mathrm{PH}_{\mathbf{C}_p})^{\mathrm{an}} \times (\mathrm{PE}')^{\mathrm{disc}})/\mathrm{P}\Gamma.$$

Recall that $\mathrm{P}\Gamma = \mathrm{PH}(\mathbf{Q})$ is naturally embedded into $\mathrm{PH}(\mathbf{C}_p)$.

*Lemma* **4.5.3.** — *The subgroup generated by the elements of* $\mathrm{P}\Gamma$ *with elliptic projections to* $\mathrm{PGL}_d(\mathrm{K}_w)$ *is Zariski dense in* $\mathrm{PH}_{\mathbf{C}_p}$.

*Proof.* — The subgroup of $\mathrm{PGL}_d(\mathrm{K}_w)$ generated by the set of all elliptic elements is open and normal, because a conjugate of an elliptic element is elliptic. Hence it contains $\mathrm{PSL}_d(\mathrm{K}_w)$. The subgroup $\mathrm{P}\Gamma_{\mathrm{G}} \cap \mathrm{PSL}_d(\mathrm{K}_w)$ is dense in $\mathrm{PSL}_d(\mathrm{K}_w)$. Therefore, by [Ma, Ch. IX, Lem. 3.3], the subgroup of $\mathrm{P}\Gamma_{\mathrm{G}}$ generated by all elliptic elements of $\mathrm{P}\Gamma_{\mathrm{G}}$ contains $\mathrm{P}\Gamma_{\mathrm{G}} \cap \mathrm{PSL}_d(\mathrm{K}_w)$. In particular, it has finite index in $\mathrm{P}\Gamma_{\mathrm{G}} = \mathrm{PH}(\mathbf{Q})$. Since PH is connected, the statement follows from [Bo, Ch. V, Cor. 18.3]. □

If $\mathrm{G}' \neq \mathrm{PH}_{\mathbf{C}}$, then by the lemma there exists $\gamma \in \mathrm{P}\Gamma$ with elliptic projection to $\mathrm{PGU}_{d-1,1}(\mathbf{R})$ whose image $\gamma_p \in \mathrm{PH}(\mathbf{C}_p)$ does not belong to $\mathrm{G}'(\mathbf{C}_p)$. Let $x \in \Omega^d_{\mathrm{K}_w} \widehat{\otimes}_{\mathrm{K}_w} \mathbf{C}_p \times \{1\} \subset (\mathrm{X}''_{\mathbf{C}_p})^{\mathrm{an}}$ be an elliptic point of $\gamma_{\mathrm{E}} \in \mathrm{PE}'$, and let $\widetilde{x}$ be an arbitrary point of $(\mathrm{R}_{\mathbf{C}_p})^{\mathrm{an}}$, lying over $x$. Then $\gamma_{\mathrm{E}}(\widetilde{x}) = \gamma_p(\widetilde{x})$ is another point of $(\mathrm{R}_{\mathbf{C}_p})^{\mathrm{an}}$, lying over $x$. Hence $\gamma_p$ must belong to $\mathrm{G}'(\mathbf{C}_p)$, contradicting to our choice of $\gamma$. □

Recall that we defined in Proposition 4.3.2 the algebraic $\mathrm{PH}_{\mathrm{K}_w} \times \mathrm{E}$-equivariant map $\rho : \mathrm{P} \to \mathbf{P}^{d-1}_{\mathrm{K}_w}$. Identify $\mathrm{PH}_{\mathbf{C}}$ with $(\mathrm{PGL}_d)^g$ in such a way that the first factor corresponds to the embedding $\infty_1 : \mathrm{K} \hookrightarrow \mathbf{C}$. Denote by $j_k : \mathrm{P}\Gamma^{\mathrm{int}} \to \mathrm{PGL}_d(\mathbf{C})$ the composition of $j$ with the projection $\mathrm{pr}_k$ of $\mathrm{PGL}_d(\mathbf{C})^g$ to its $k$-th factor. Denote also by $\mathrm{pr}^i$ the projection of $\mathrm{PGL}_d(\mathbf{C})^g$ to the product of all factors except the $i$-th. We will sometimes identify $\mathrm{P}\Gamma^{\mathrm{int}}$ with its projection $\mathrm{P}\Gamma^{\mathrm{int}}_\infty \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$.

*Proposition* **4.5.4.** — *The subgroup* $j_1(\mathrm{P}\Gamma^{\mathrm{int}})$ *is not relatively compact in* $\mathrm{PGL}_d(\mathbf{C})$.

*Proof.* — If not, then $j_1(\mathrm{P}\Gamma^{\mathrm{int}})$ is contained in some maximal compact subgroup of $\mathrm{PGL}_d(\mathbf{C})$ (see for example [PR, Prop. 3.11]). After a suitable conjugation we may assume that $j_1(\mathrm{P}\Gamma^{\mathrm{int}}) \subset \mathrm{PGU}_d(\mathbf{R})$. By Proposition 4.5.2, $j_1(\mathrm{P}\Gamma^{\mathrm{int}})$ is Zariski dense in $\mathrm{PGL}_d$, hence it is infinite. Therefore, by Lemma 4.5.1, $j_1(\mathrm{P}\Gamma^{\mathrm{int}})$ is dense in $\mathrm{PGU}_d(\mathbf{R})$.

Consider the map $\rho : \mathrm{P}(\mathbf{C}) \to \mathbf{P}^{d-1}(\mathbf{C})$ and its restriction $\rho_0$ to $\mathrm{B}^{d-1} \times \{ 1 \} \cong \mathrm{B}^{d-1}$. Then, as in the proof of Claim 4.3.3, $\rho_0 : \mathrm{B}^{d-1} \to \mathbf{P}^{d-1}(\mathbf{C})$ satisfies $\rho_0(\gamma x) = j_1(\gamma)\, \rho_0(x)$ for all $x \in \mathrm{B}^{d-1}$ and $\gamma \in \mathrm{P}\Gamma^{\mathrm{int}}$. The group $\mathrm{PGU}_d(\mathbf{R})$ acts transitively on $\mathbf{P}^{d-1}(\mathbf{C})$, hence $\rho_0(\mathrm{B}^{d-1})$ is dense in $\mathbf{P}^{d-1}(\mathbf{C})$.

Now we want to prove that $j_1 : \mathrm{P}\Gamma^{\mathrm{int}} \to \mathrm{PGU}_d(\mathbf{R})$ can be extended to a continuous homomorphism $\widetilde{j}_1 : \mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \to \mathrm{PGU}_d(\mathbf{R})$. For each $g \in \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ choose a sequence $\{ \gamma_n \}_n \subset \mathrm{P}\Gamma^{\mathrm{int}} \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ converging to $g$. Since $\mathrm{PGU}_d(\mathbf{R})$ is compact, there exists a subsequence $\{ \gamma_{n_k} \}_k \subset \{ \gamma_n \}_n$ such that $\{ j_1(\gamma_{n_k}) \}_k$ converges to some $a \in \mathrm{PGU}_d(\mathbf{R})$. Then $\rho_0(gx) = \lim_k \rho_0(\gamma_{n_k}(x)) = (\lim_k j_1(\gamma_{n_k}))\, \rho_0(x) = a\rho_0(x)$ for all $x \in \mathrm{B}^{d-1}$. It follows that $a = a(g)$ depends only on $g$, since $\rho_0(\mathrm{B}^{d-1})$ is dense in $\mathbf{P}^{d-1}(\mathbf{C})$ and since the group $\mathrm{PGL}_d(\mathbf{C})$ acts faithfully on $\mathbf{P}^{d-1}(\mathbf{C})$. In particular, $a(g)$ does not depend on the choice of $\{ \gamma_n \}_n$ and $a(g) = \lim_n j_1(\gamma_n)$. It follows that $\widetilde{j}_1 := a$ is the required extension.

Since $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$ is simple and $j_1(\mathrm{P}\Gamma^{\mathrm{int}})$ is dense, $\widetilde{j}_1$ must be injective and surjective. Hence is it an isomorphism, a contradiction. □

*Proposition* **4.5.5.** — *For each* $i = 1, \ldots, g$ *the homomorphism* $j_i : \mathrm{P}\Gamma^{\mathrm{int}} \to \mathrm{PGL}_d(\mathbf{C})$ *is injective.*

*Proof.* — Suppose that for some $i$ the subgroup $\Delta_i := \mathrm{Ker}(j_i)$ is non-trivial. Then $\Delta_i$ is a normal subgroup of $\mathrm{P}\Gamma^{\mathrm{int}} \cong \mathrm{P}\Gamma^{\mathrm{int}}_\infty \subset \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$. Hence the closure of $(\Delta_i)_\infty$ is a non-trivial normal subgroup of a simple group $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$. Therefore the projection $(\Delta_i)_\infty$ is dense in $\mathrm{PGU}_{d-1,1}(\mathbf{R})^0$. Hence there exists an element $\delta \in \Delta_i$ with elliptic projection $\delta_\infty \in \mathrm{PGU}_{d-1,1}(\mathbf{R})^0$. Therefore the element $(j(\delta), \delta_{\mathrm{E}}) \in \mathrm{PGL}_d(\mathbf{C})^g \times \mathrm{PE}'$ has a fixed point $[y, g, e] \in \mathrm{P}''(\mathbf{C}_p) = (\Omega^d_{\mathrm{K}_w}(\mathbf{C}_p) \times \mathrm{PGL}_d(\mathbf{C}_p)^g \times \mathrm{PE}')/\mathrm{P}\Gamma$. Hence $(g^{-1} j(\delta)\, g, e^{-1}\, \delta_{\mathrm{E}}\, e)$ stabilizes $[y, 1, 1] \in \mathrm{P}''(\mathbf{C}_p)$. It follows that $e^{-1}\, \delta_{\mathrm{E}}\, e = \gamma_{\mathrm{E}}$ for some $\gamma \in \mathrm{P}\Gamma = \mathrm{PH}(\mathbf{Q})$ and $g^{-1} j(\delta)\, g \in \mathrm{PH}(\mathbf{C}_p)$ is the image of $\gamma$. Hence $j_k(\delta) \neq 1$ for all $k$, contradicting to our assumption. □

## 4.6. Use of rigidity

Now we are going to use the following theorem of Margulis [Ma, Ch. VII, Thm. 5.6].

*Theorem* **4.6.1.** — *Let* $L$ *be a local field, let* $J$ *be a connected absolutely simple adjoint* $L$*-group, and let* $A$ *be a finite set. For each* $\alpha \in A$ *let* $k_\alpha$ *be a local field and let* $G_\alpha$ *be an adjoint absolutely simple* $k_\alpha$*-isotropic group. Set* $G := \prod_{\alpha \in A} G_\alpha(k_\alpha)$. *Let* $\Gamma$ *be an irreducible lattice in* $G$ *and let* $\Lambda$ *be a subgroup of* $\mathrm{Comm}_G(\Gamma)$. *Suppose that* $\mathrm{rank}\, G := \sum_{\alpha \in A} \mathrm{rank}_{k_\alpha} G_\alpha \geqslant 2$.

*If the image of a homomorphism* $\tau : \Lambda \to J(L)$ *is Zariski dense in* $J$ *and not relatively compact in* $J(L)$, *then there exists a unique* $\alpha \in A$, *a continuous homomorphism* $\theta : k_\alpha \to L$ *and a unique* $\theta$*-algebraic isomorphism* $\eta : G_\alpha \xrightarrow{\sim} J$ *such that* $\tau(\lambda) = \eta(\theta(\mathrm{pr}_\alpha(\lambda)))$ *for all* $\lambda \in \Lambda$.

**4.6.2.** We use the notation of 2.4.1 with $\Delta' = P\Gamma^{\mathrm{int}}$. Take any $M$ and $S$ such that $\mathrm{rank}\, G_{\overline{M}} \geqslant 2$. Then by Proposition 2.4.5, $\Gamma := \Delta^S$ is an irreducible lattice in $G_{\overline{M}}$. We will try to apply Theorem 4.6.1 in the following situation. Take $G = G_{\overline{M}}$, $\Lambda$ be the projection of $\Delta'$ to $G_{\overline{M}}$, $L = \mathbf{C}$, $J = (\mathrm{PGL}_d)_{\mathbf{C}}$ and $\tau$ be the homomorphism $j_i : P\Gamma^{\mathrm{int}} \to \mathrm{PGL}_d(\mathbf{C})$ for some $i \in \{1, \dots, g\}$. Consider first $i = 1$. By Proposition 4.5.2 and Proposition 4.5.4, $\tau = j_1$ satisfies the conditions of Theorem 4.6.1, hence there exists an algebraic isomorphism $\eta_1 : (\mathrm{PGU}_{d-1,1})_{\mathbf{C}} \xrightarrow{\sim} (\mathrm{PGL}_d)_{\mathbf{C}}$ such that $j_1(\gamma) = \eta_1(\gamma_\infty)$ for all $\gamma \in P\Gamma^{\mathrm{int}}$.

Now take $i \geqslant 2$. Suppose that $j_i(P\Gamma^{\mathrm{int}})$ is not relatively compact. Then using again Proposition 4.5.2 we conclude from Theorem 4.6.1 that there exists an algebraic isomorphism $\eta_i : (\mathrm{PGU}_{d-1,1})_{\mathbf{C}} \xrightarrow{\sim} (\mathrm{PGL}_d)_{\mathbf{C}}$ such that $j_i(\gamma) = \eta_i(\gamma_\infty)$ for all $\gamma \in P\Gamma^{\mathrm{int}}$. In particular, $j(P\Gamma^{\mathrm{int}})$ is not Zariski dense in $(\mathrm{PGL}_d)^g$. This contradicts to Proposition 4.5.2. Therefore after a suitable conjugation we may assume that $j_i(P\Gamma^{\mathrm{int}}) \subset \mathrm{PGU}_d(\mathbf{R})$ for all $i = 2, \dots, g$.

It follows that up to an algebraic automorphism of $(\mathrm{PGL}_d)^g$,

$$j(P\Gamma^{\mathrm{int}}) \subset \mathrm{PGU}_{d-1,1}(\mathbf{R}) \times \mathrm{PGU}_d(\mathbf{R})^{g-1} \cong \mathrm{PH}^{\mathrm{int}}(\mathbf{R})$$

and that $j_1$ is the natural embedding $\mathrm{PG}^{\mathrm{int}}(F)_+ \hookrightarrow \mathrm{PG}^{\mathrm{int}}(F_{\infty_1})$. Therefore $j$ together with the natural embedding $\mathrm{PG}^{\mathrm{int}}(F)_+ \hookrightarrow \mathrm{PG}^{\mathrm{int}}(\mathbf{A}_F^{f;v})$ embed $P\Gamma^{\mathrm{int}}$ into

$$\mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{PGU}_d(\mathbf{R})^{g-1} \times \mathrm{PG}^{\mathrm{int}}(\mathbf{A}_F^{f;v}).$$

*Lemma* **4.6.3.** — *The closure of the projection of* $P\Gamma^{\mathrm{int}}$ *to* $\mathrm{PGU}_d(\mathbf{R})^{g-1} \times \mathrm{PG}^{\mathrm{int}}(\mathbf{A}_F^{f;v})$ *contains* $\mathrm{PGU}_d(\mathbf{R})^{g-1} \times \mathrm{P}((\mathrm{G}^{\mathrm{int}})^{\mathrm{der}}(\mathbf{A}_F^{f;v}))$.

*Proof.* — Let $(g_\infty, g_f)$ be an element of $\mathrm{PGU}_d(\mathbf{R})^{g-1} \times \mathrm{P}((\mathrm{G}^{\mathrm{int}})^{\mathrm{der}}(\mathbf{A}_F^{f;v}))$, let $U \subset \mathrm{PGU}_d(\mathbf{R})^{g-1}$ be an open neighbourhood of $g_\infty$, and let $S \in \mathcal{F}(\mathrm{PG}^{\mathrm{int}}(\mathbf{A}_F^{f;v}))$. We have to show that $P\Gamma^{\mathrm{int}} \cap (\mathrm{PGU}_{d-1,1}(\mathbf{R}) \times U \times g_f S) \neq \emptyset$. By the strong approximation theorem there exists a $\gamma \in P\Gamma^{\mathrm{int}}$ whose projection to $\mathrm{PG}^{\mathrm{int}}(\mathbf{A}_F^{f;v})$ belongs to $g_f S$. Let $\gamma'$ be the projection of $\gamma^{-1}$ to $\mathrm{PGU}_d(\mathbf{R})^{g-1}$.

Since $j(\mathrm{P}\Gamma^{\mathrm{int}})$ belongs to the commensurator of $\Delta_{\mathrm{S}} := j(\mathrm{P}\Gamma^{\mathrm{int}}_{\mathrm{S}})$ in $\mathrm{PGL}_d(\mathbf{C})^g$, Proposition 4.5.2, Proposition 4.5.5 and Lemma 4.5.1, *b), c)* imply that $\mathrm{pr}^1(\Delta_{\mathrm{S}})$ is dense in $\mathrm{PGU}_d(\mathbf{R})^{g-1}$. It follows that there exists $\delta \in \mathrm{P}\Gamma^{\mathrm{int}}$ whose projection to $\mathrm{PGU}_d(\mathbf{R})^{g-1} \times \mathrm{PG}^{\mathrm{int}}(\mathbf{A}_{\mathrm{F}}^{f;v})$ belongs to $\gamma' \mathrm{U} \times \mathrm{S}$. Then $\gamma\delta$ belongs to

$$\mathrm{P}\Gamma^{\mathrm{int}} \cap (\mathrm{PGU}_{d-1,1}(\mathbf{R})^0 \times \mathrm{U} \times g_f \mathrm{S}). \quad \square$$

**4.6.4.** Now we proceed as in the proof of Theorem 2.2.5. Let M and S be as in 2.4.1, and let $\mathrm{P}\Gamma^{\mathrm{int}}_{\mathrm{S}}$ be the projection of $\mathrm{P}\Gamma^{\mathrm{int}}_{(\mathrm{S})} := \mathrm{P}\Gamma^{\mathrm{int}} \cap \mathrm{PGU}_d(\mathbf{R})^{g-1} \times \mathrm{G}_{\overline{\mathrm{M}}} \times \mathrm{S}$ to $\mathrm{PGU}_d(\mathbf{R})^{g-1} \times \mathrm{G}_{\overline{\mathrm{M}}}$. The proof of Proposition 2.4.5 holds in our case, hence $\mathrm{P}\Gamma^{\mathrm{int}}_{\mathrm{S}}$ is arithmetic. It follows that there exists a permutation $\sigma$ of the set $\{2, \ldots, g\}$ such that for every $i = 2, \ldots, g$ there exists a unique algebraic isomorphism $r_i : \mathrm{PG}_{\mathrm{F}_{\infty_i}} \xrightarrow{\sim} \mathrm{PGU}_d$ satisfying $r_i(\gamma) = j_{\sigma(i)}(\gamma)$ for each $\gamma \in \mathrm{P}\Gamma^{\mathrm{int}}_{(\mathrm{S})}$. In particular, $\sigma$ and the $r_i$'s do not depend on M and S. Since $\mathrm{P}\Gamma^{\mathrm{int}} = \bigcup_{\mathrm{M},\mathrm{S}} \mathrm{P}\Gamma^{\mathrm{int}}_{(\mathrm{S})}$, we then have $r_i(\gamma) = j_{\sigma(i)}(\gamma)$ for all $i \in \{2, \ldots, g\}$ and $\gamma \in \mathrm{P}\Gamma^{\mathrm{int}}$. This shows the existence of an algebraic isomorphism $\Phi_{\mathrm{C}}$ which will satisfy Theorem 4.4.3 if we show that it is inner. But this can be immediately shown by the standard argument using elliptic elements and function $t$ defined in 2.5.5 (compare for example the proofs of Proposition 2.5.6 and Proposition 4.5.5).

## 4.7. Rationality question

Consider the $(\mathrm{PH}^{\mathrm{int}}_{\mathrm{C}})^{\mathrm{an}}$-torsor $(\widetilde{\mathrm{P}}^{\mathrm{int}})^{\mathrm{an}} \cong [\mathrm{B}^{d-1} \times (\mathrm{PH}^{\mathrm{int}}_{\mathrm{C}})^{\mathrm{an}} \times (\mathrm{E}^{\mathrm{int}}/\mathrm{E}^{\mathrm{int}}_0)^{\mathrm{disc}}]/\mathrm{P}\Gamma^{\mathrm{int}}$ over $(\widetilde{\mathrm{X}}^{\mathrm{int}})^{\mathrm{an}}$. As in the $p$-adic case, it has a canonical flat connection $\widetilde{\widetilde{\mathscr{H}}}^{\mathrm{int}}$. The same considerations as in the $p$-adic case (see the proof of Proposition 4.4.2) show that there exists a unique connection $\widetilde{\mathscr{H}}^{\mathrm{int}}$ on $\widetilde{\mathrm{P}}^{\mathrm{int}}$ such that $(\widetilde{\mathscr{H}}^{\mathrm{int}})^{\mathrm{an}} \cong \widetilde{\widetilde{\mathscr{H}}}^{\mathrm{int}}$. It follows from the proofs of Proposition 4.4.2 and Theorem 4.4.3 that $(\widetilde{f}_{\Phi,\mathrm{P}})_*(\mathscr{H}_{\mathrm{C}}) = \widetilde{\mathscr{H}}^{\mathrm{int}}$.

*Lemma* **4.7.1.** — *If an analytic automorphism* $\varphi : (\widetilde{\mathrm{P}}^{\mathrm{int}})^{\mathrm{an}} \xrightarrow{\sim} (\widetilde{\mathrm{P}}^{\mathrm{int}})^{\mathrm{an}}$ *commutes with the action of* $(\mathrm{PH}^{\mathrm{int}}_{\mathrm{C}})^{\mathrm{an}} \times \mathrm{E}^{\mathrm{int}}$, *preserves* $\widetilde{\mathscr{H}}^{\mathrm{int}}$ *and induces the identity map on* $(\widetilde{\mathrm{X}}^{\mathrm{int}})^{\mathrm{an}} = (\mathrm{PH}_{\mathrm{C}})^{\mathrm{an}}\backslash(\widetilde{\mathrm{P}}^{\mathrm{int}})^{\mathrm{an}}$, *then* $\varphi$ *is the identity.*

*Proof.* — Recall that $(\widetilde{\mathrm{P}}^{\mathrm{int}})^{\mathrm{an}} \cong [\mathrm{B}^{d-1} \times (\mathrm{PH}^{\mathrm{int}}_{\mathrm{C}})^{\mathrm{an}} \times (\mathrm{E}^{\mathrm{int}}/\mathrm{E}^{\mathrm{int}}_0)^{\mathrm{disc}}]/\mathrm{P}\Gamma^{\mathrm{int}}$. Since $\varphi$ induces the identity map on $(\widetilde{\mathrm{X}}^{\mathrm{int}})^{\mathrm{an}}$, there exists a holomorphic map $\psi : \mathrm{B}^{d-1} \to (\mathrm{PH}^{\mathrm{int}}_{\mathrm{C}})^{\mathrm{an}}$ such that $\varphi[x, 1, 1] = [x, \psi(x), 1]$ for all $x \in \mathrm{B}^{d-1}$. Since $\varphi$ preserves $\widetilde{\mathscr{H}}^{\mathrm{int}}$, we have $\partial\psi/\partial x \equiv 0$. Hence $\psi$ is a constant, say $a$. Then $\varphi[x, h, e] = [x, ha, e]$ for all $x \in \mathrm{B}^{d-1}$, $h \in (\mathrm{PH}^{\mathrm{int}}_{\mathrm{C}})^{\mathrm{an}}$ and $e \in \mathrm{E}^{\mathrm{int}}$. In particular,

$$\varphi[x, 1, 1] = \varphi[\gamma_\infty^{-1} x, j(\gamma), \gamma_{\mathrm{E}}] = [\gamma_\infty^{-1} x, j(\gamma)\, a, \gamma_{\mathrm{E}}] = [x, j(\gamma)\, aj(\gamma)^{-1}, 1]$$

for all $\gamma \in \mathrm{P}\Gamma^{\mathrm{int}}$. Therefore $j(\gamma)\, aj(\gamma)^{-1} = a$ for all $\gamma \in \Gamma^{\mathrm{int}}$. Since $\mathrm{P}\Gamma^{\mathrm{int}}$ is Zariski dense in $\mathrm{PH}^{\mathrm{int}}_{\mathrm{C}}$, $a = 1$. $\square$

*Corollary* **4.7.2.** — *The torsor* $\widetilde{\mathrm{P}}^{\mathrm{int}}$ *has a unique* $\mathrm{E}^{\mathrm{int}}$*-equivariant structure* $\mathrm{P}^{\mathrm{int}}$ *of a* $\mathrm{PH}^{\mathrm{int}}_{\mathrm{K}_w}$*-torsor over* $\mathrm{X}^{\mathrm{int}}$ *such that there exists a connection* $\mathscr{H}^{\mathrm{int}}$ *on* $\mathrm{P}^{\mathrm{int}}$ *satisfying* $\mathscr{H}^{\mathrm{int}}_{\mathbf{C}} \cong \widetilde{\mathscr{H}}^{\mathrm{int}}$. ($\mathrm{P}^{\mathrm{int}}$ *is called the canonical model of* $\widetilde{\mathrm{P}}^{\mathrm{int}}$ *over* $\mathrm{X}^{\mathrm{int}}$.)

*Proof.* — The existence is proved in [Mil, III, § 3]. Suppose that P′ and P″ are two structures satisfying the above conditions. Let $f: \mathrm{P}'_{\mathbf{C}} \xrightarrow{\sim} \widetilde{\mathrm{P}}^{\mathrm{int}} \xrightarrow{\sim} \mathrm{P}''_{\mathbf{C}}$ be the natural isomorphism. For each $\sigma \in \mathrm{Aut}_{\mathrm{K}_w}(\mathbf{C})$ set $\varphi_\sigma := \sigma(f)^{-1} \circ f$. Then the automorphism $(\varphi_\sigma)^{\mathrm{an}}$ of $(\mathrm{P}'_{\mathbf{C}})^{\mathrm{an}} \cong (\widetilde{\mathrm{P}}^{\mathrm{int}})^{\mathrm{an}}$ satisfies the assumptions of the lemma. Hence $(\varphi_\sigma)^{\mathrm{an}}$ is the identity, so that $\sigma(f) = f$ for all $\sigma \in \mathrm{Aut}_{\mathrm{K}_w}(\mathbf{C})$. It follows that P′ = P″. □

To finish the proof of the Fourth (and the Third) Main Theorem it remains to show that the homomorphism $\widetilde{f}_{\Phi,\mathrm{P}} : \mathrm{P}^{\mathrm{tw}}_{\mathbf{C}} \xrightarrow{\sim} \mathrm{P}^{\mathrm{int}}_{\mathbf{C}}$ is $\mathrm{K}_w$-linear. Since

$$(\widetilde{f}_{\Phi,\mathrm{P}})_*(\mathscr{H}^{\mathrm{tw}}_{\mathbf{C}}) = (\widetilde{f}_{\Phi,\mathrm{P}})_*(\mathscr{H}_{\mathbf{C}}) = \mathscr{H}^{\mathrm{int}}_{\mathbf{C}},$$

this follows from the lemma by the same considerations as the corollary.

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Y. V.,
Department of Mathematics,
University of Toronto,
Toronto, Ontario M5S 3G3,
Canada.

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