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COMPLETE INTERSECTION DIMENSION

by LUCHEZAR L. AVRAMOV, VESSELIN N. GASHAROV, *and* IRENA V. PEEVA [1]

Abstract. A new homological invariant is introduced for a finite module over a commutative noetherian ring: its CI-dimension. In the local case, sharp quantitative and structural data are obtained for modules of finite CI-dimension, providing the first class of modules of (possibly) infinite projective dimension with a rich structure theory of free resolutions.

CONTENTS

INTRODUCTION

Let M be a finite module over a commutative noetherian local ring R. There always exists a minimal free resolution $\mathbf{F}$ of M over R, that is unique up to isomorphism, and defines the Betti numbers $b_n^R(M) = \operatorname{rank}_R F_n$. The structure of finite resolutions is very rigid, but little is known in the infinite case.

To some extent, this is due to intrinsic numerical difficulties: Anick [2] shows that the Betti sequence of the residue field k of R may be non-recurrent, and Avramov [6] that it has exponential growth, unless R is a *complete intersection*. In contrast, the behavior at infinity of Betti sequences over complete intersections is not so daunting: Tate [36] proves that $b_n^R(k)$ is eventually given by a polynomial, and Gulliksen [23] that each $b_n^R(M)$ is a quasi-polynomial of period 2 and degree smaller than the codimension.

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The point of view of this paper, expanding on that of Eisenbud [18] and Avramov [8], [9], is that while the beginning of a minimal free resolution is usually unstructured and (therefore) complicated, remarkable patterns emerge at infinity. Our objective is the introduction and study of a class of modules that afford a highly non-trivial, yet tractable, theory of minimal free resolutions. As necessary ingredients for such a study, we develop a conceptual framework for describing infinite resolutions, and new techniques for their analysis. They belong to *asymptotic homological algebra*, which shifts the focus from properties of an individual module to those of its entire sequence of syzygies.

In Section 1 we define a new homological invariant for a finite module M over a noetherian ring P, its *complete intersection dimension* $\mathrm{CI\text{-}dim}_P\, M$. It is finite for all modules over a complete intersection, and the example is paradigmatic, exactly as modules over a regular local ring provide the paradigm of modules of finite projective dimension. There is more to the concept: exactly as the complete intersection property of a ring is intermediate between its being Gorenstein and regular, the new dimension interpolates between the G-dimension of Auslander and Bridger [3] and the classical projective dimension:

$$\mathrm{G\text{-}dim}_P\, M \leqslant \mathrm{CI\text{-}dim}_P\, M \leqslant \mathrm{pd}_P\, M,$$

and equalities hold to the left of any finite dimension.

CI-dimension shares many basic properties with other homological dimensions. In particular, it localizes, so it is more flexible than the strictly local *virtual projective dimension* of [8], which it generalizes. Over a local ring R it satisfies an analog of the Auslander-Buchsbaum and Auslander-Bridger Equalities: if $\mathrm{CI\text{-}dim}_R\, M$ is finite, then it equals $\mathrm{depth}\, R - \mathrm{depth}_R\, M$. The finiteness of $\mathrm{CI\text{-}dim}_R\, k$ implies that the ring R is a complete intersection, giving a CI version of Serre's characterization of regularity and Auslander and Bridger's characterization of Gorensteinness.

The class of modules of finite CI-dimension contains all modules of finite projective dimension and all modules over a complete intersection. In Section 3 we show that the notion is meaningful in a much wider context, by constructing modules of finite CI-dimension and infinite projective dimension over any ring that has an embedded deformation.

An initial estimate of the size of an $\mathbf{F}$ is given by the *complexity* $\mathrm{cx}_R\, M$ of M, that is, the smallest integer d such that $\lim_{n\to\infty} b_n^R(M)/n^d = 0$, cf. [1], [8]. When $\mathrm{CI\text{-}dim}_R\, M$ is finite, so is $\mathrm{cx}_R\, M$, due essentially to Gulliksen [23]. In Section 5 we establish a uniform bound on complexities: If $\mathrm{CI\text{-}dim}_R\, M < \infty$, then

$$\mathrm{cx}_R\, M \leqslant \mathrm{edim}\, R - \mathrm{depth}\, R,$$

with strict inequality unless R is a complete intersection. Since complexity provides a polynomial scale for infinite projective dimensions, the inequality may be viewed as a version of the fact that depth R is a universal bound on the finite projective dimensions.

In sections 7 and 8 we obtain definitive results on the behavior of the Betti sequence for a module of finite CI-dimension. An important role is played by the discovery of a " marker " that signals a place after which **F** starts to exhibit asymptotically stable patterns. This *critical degree* $\operatorname{cr}\deg_R M$ is equal to the projective dimension when $\operatorname{pd}_R M < \infty$, but the finiteness of $\operatorname{cr}\deg_R M$ is a much weaker condition, and is implied by $\text{CI-dim}_R M < \infty$.

We prove that the Betti sequence of a module of finite CI-dimension is either constant or strictly increasing after $\operatorname{cr}\deg_R M$ steps, that asymptotically there are equalities

$$b_n^R(M) = o(n^{d-1}) \quad \text{and} \quad b_n^R(M) - b_{n-1}^R(M) = O(n^{d-2}) \quad \text{with } d = \operatorname{cx}_R M,$$

and that the order cannot be improved in the second one. These facts are obtained as consequences of a very precise statement on the form of the Poincaré series of M. They are new even over complete intersections, and provide the last step in a proof that over local rings of small codimension, or linked in few steps to complete intersections, all Betti series are eventually non-decreasing.

When $R = Q/(\mathbf{x})$ for some Q-regular sequence $\mathbf{x}$, Shamash [33] and Eisenbud [18] produce a not necessarily minimal *standard resolution* of M over R, by using higher order structures on a minimal resolution of M over Q. In Section 6 we investigate how tightly a standard resolution approximates the minimal one. Contrary to many expectations, we prove that this approximation is almost always very weak.

Previous work on modules over a ring R with a deformation Q has extensively used Gulliksen's finiteness theorem [23]: if $\operatorname{Ext}_Q^*(M, N)$ is finite over R for some R-module N, then $\operatorname{Ext}_R^*(M, N)$ is finite over a ring $R[\chi]$ of cohomology operators. To study these operators we develop in Section 4 a novel type of change-of-rings spectral sequence, and obtain a perfect converse to Gulliksen's theorem. We also use it in Section 9 to construct obstructions to the validity of a conjecture [18] on the existence of graded $R[\chi]$-module structure on minimal free resolutions, and produce modules for which they do not vanish.

This paper brings new tools to the study of resolutions over commutative rings.

Much of our analysis is based on the *Yoneda product* structure of $\operatorname{Ext}_R^*(M, k)$ as a graded left $\operatorname{Ext}^*(k, k)$- and right $\operatorname{Ext}_R^*(M, M)$-*bimodule*. In general these algebras are neither finitely generated nor commutative. We expand on a result of Mehta [30] to show that if $\text{CI-dim}_R M < \infty$, then $\operatorname{Ext}_R^*(M, k)$ is finite over the subalgebra of $\operatorname{Ext}_R^*(M, M)$ generated by central elements of degree two. Furthermore, by [12] it is also finite over the subalgebra of $\operatorname{Ext}^*(k, k)$ generated by the degree 2 part of the center of the *homotopy Lie algebra* $\pi^*(R)$: this is crucial in obtaining the universal bound on $\operatorname{cx}_R M$, as deep results of Félix *et al.* [20] on the radical of $\pi^*(R)$ can be applied.

Non-commutative ring theory is used to produce modules of finite CI-dimension. In Section 2 we develop the new concept of a *quantum regular sequence* of endomorphisms of a module, that generalizes the classical notion of regular sequence. To study such sequences we introduce constructions over quantum symmetric algebras, in particular an extension of Manin's [28] quantum Koszul complex. The application of quantum techniques in the commutative context is so efficient as to suggest possibilities of further interaction.

1. Homological dimensions

In this section, R is a commutative noetherian ring and M is a finite R-module.

A primary motive for the introduction of a new homological dimension is to describe a class of modules over a local ring with tractable minimal free resolutions. There are two basic ways of changing a local ring R to a local ring R′ by a local homomorphism (that is, one that maps the maximal ideal of R into that of R′) without introducing complications into the homological structure of M. If $R \to R'$ is a *local flat extension*, then the structure of a minimal free resolution of M over R is essentially the same as that of $M' = M \otimes_R R'$ over R′. If $R' \to R$ is a *(codimension c) deformation*, by which we mean a surjective local homomorphism with kernel generated by a (length c) regular sequence, then the resolution of M over R′ can only be simpler than over R.

We describe a notion that incorporates these two classes of maps.

(1.1) A *(codimension c) quasi-deformation* of R is a diagram of local homomorphisms $R \to R' \leftarrow Q$, with $R \to R'$ a flat extension and $R' \leftarrow Q$ a (codimension c) deformation. When M is an R-module and $R \to R' \leftarrow Q$ is a quasi-deformation, we set $M' = M \otimes_R R'$.

The next definition describes the modules that have finite projective dimension " up to quasi-deformation ". As we are targeting properties of minimal resolutions, the new homological dimension is introduced locally. We write $(R, \mathfrak{m}, k)$ or $(R, \mathfrak{m})$ to denote a local ring R with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

(1.2) *Complete intersection dimension.* For a module $M \neq 0$ over a local ring R, set

$$\text{CI-dim}_R M = \inf\{ \operatorname{pd}_Q M' - \operatorname{pd}_Q R' \mid R \to R' \leftarrow Q \text{ is a quasi-deformation} \},$$

and complement this by $\text{CI-dim}_R 0 = 0$. The *CI-dimension* (a shorthand for *complete intersection dimension*) of a module M over a noetherian ring R is defined to be the number

$$\text{CI-dim}_R M = \sup\{ \text{CI-dim}_{R_\mathfrak{m}} M_\mathfrak{m} \mid \mathfrak{m} \in \operatorname{Max} R \}.$$

The choice of terminology is motivated by the next two theorems. Recall that R is *a complete intersection* if the defining ideal of some Cohen presentation of the $\mathfrak{m}$-adic completion $\hat{R}$ as a quotient of a regular ring can be generated by a regular sequence; when this is the case, any Cohen presentation has the corresponding property.

(1.3) *Theorem. — Let* $(R, \mathfrak{m}, k)$ *be a local ring.*

If R *is a complete intersection, then each* R*-module* M *has finite* CI*-dimension.*

If $\text{CI-dim}_R k < \infty$, *then* R *is a complete intersection.*

This result should be viewed in the context of homological characterizations of other local properties. By the Auslander-Buchsbaum-Serre theorem if R is regular, then $\mathrm{pd}_R\, M < \infty$ for each R-module M, and $\mathrm{pd}_R\, k < \infty$ implies R is regular. A similar description of Gorenstein rings is due to Auslander and Bridger [5, (4.20)]. It uses their generalization of the classical projective dimension $\mathrm{pd}_R\, M$, based on properties of the functor $M^* = \mathrm{Hom}_R(M, R)$ and of the canonical biduality map $\beta_M : M \to M^{**}$.

More precisely, $\mathrm{G\text{-}dim}_R\, M = 0$ means that M is reflexive (that is, β_M is bijective), and $\mathrm{Ext}^i(M, R) = 0 = \mathrm{Ext}^i(M^*, R)$ for $i \neq 0$. In general, the *Gorenstein dimension* (or: G-*dimension*) of M is the infimum $\mathrm{G\text{-}dim}_R\, M$ of those n for which there exists an exact sequence $0 \to P_n \to \ldots \to P_1 \to P_0 \to M \to 0$ with $\mathrm{G\text{-}dim}_R\, P_i = 0$ for $i \in \mathbf{Z}$.

The place of CI-dimension in the hierarchy of homological dimensions is determined by

(1.4) *Theorem. — For each finite module* M *over a noetherian ring* R *there are inequalities*

$$\mathrm{G\text{-}dim}_R\, M \leqslant \mathrm{CI\text{-}dim}_R\, M \leqslant \mathrm{pd}_R\, M.$$

If some of these dimensions is finite, then it is equal those to its left.
If R *is local and* $\mathrm{CI\text{-}dim}_R\, M < \infty$, *then* $\mathrm{CI\text{-}dim}_R\, M = \mathrm{depth}\, R - \mathrm{depth}_R\, M$.

The rank of the n-th free module in the (unique up to isomorphism) minimal free resolution of a finite module M over a local ring R is known as the *n-th Betti number of* M over R. Betti numbers can be computed from the equality $b_n^R(M) = \mathrm{rank}_k\, \mathrm{Ext}_R^n(M, k)$, and are often most conveniently studied through the generating function $P_M^R(t) = \sum_{n=0}^{\infty} b_n^R(M)\, t^n$, known as the *Poincaré series* of M over R. We include a variation on known results.

(1.5) *Lemma. — For a codimension c quasi-deformation* $R \to R' \leftarrow Q$, *an* R-*module* M, *and* $M' = M \otimes_R R'$, *there are coefficientwise (in)equalities of formal power series*

$$P_M^R(t) = P_{M'}^{R'}(t);$$
$$P_{M'}^{R'}(t) \leqslant P_{M'}^{Q}(t)\,(1 - t^2)^{-c};$$
$$P_{M'}^{Q}(t) \leqslant P_{M'}^{R'}(t)\,(1 + t)^c.$$

Proof. — For the equality observe that if $\mathbf{F}$ is a minimal free resolution of M over R, then $\mathbf{F} \otimes_R R'$ is such a resolution of M′ over R′. For the inequalities it suffices to treat the case $c = 1$. The standard change-of-rings spectral sequence

$${}_2E^{pq} = \mathrm{Ext}_{R'}^p(M', \mathrm{Ext}_Q^q(R', \ell)) \Rightarrow \mathrm{Ext}_Q^{p+q}(M', \ell),$$

with ℓ denoting the residue field of R′, degenerates to a long exact sequence

$$\ldots \to \mathrm{Ext}_{R'}^{n-2}(M', \ell) \to \mathrm{Ext}_{R'}^{n}(M', \ell) \to \mathrm{Ext}_Q^n(M', \ell) \to \mathrm{Ext}_{R'}^{n-1}(M', \ell) \to \ldots$$

For each $n \geqslant 0$ we get $b_n^{R'}(M') \leqslant \sum_{i \geqslant 0} b_{n-2i}^{Q}(M')$ and $b_n^Q(M') \leqslant b_n^{R'}(M') + b_{n-1}^{R'}(M')$: these are transcripts of the desired inequalities. □

Proof of Theorem (1.3). — In view of the Auslander-Buchsbaum-Serre characterization of regular local rings, the first assertion follows immediately from the definition in (1.2).

Conversely, let $R \to R' \leftarrow Q$ be a quasi-deformation such that the Q-module $R'/\mathfrak{m}R' = (R/\mathfrak{m}) \otimes_R R'$ has finite projective dimension. By (1.5) the sequence $b_n^R(R/\mathfrak{m})$ is bounded by a polynomial in n, hence [24; (2.3)] shows that R is a complete intersection. □

Recall from [8] that a module over a local ring R has finite *virtual projective dimension* if its completion has finite projective dimension over some deformation of $\hat{R}$. It is not known whether localizations of a module of finite virtual projective dimension also have this property. However, such a module has finite CI-dimension, and we show next that so do its localizations. Finiteness of CI-dimension is thus not only a more general property, but also a more natural one.

(1.6) *Proposition. — For any multiplicatively closed subset* U *of* R *there is an inequality* $\text{CI-dim}_{U^{-1}R}(U^{-1} M) \leqslant \text{CI-dim}_R M$, *and furthermore*

$$\text{CI-dim}_R M = \sup\{ \text{CI-dim}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R M \}.$$

Proof. — It suffices to establish the last formula, so we need to show that $\text{CI-dim}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \text{CI-dim}_R M$ for R local and $\mathfrak{p} \in \operatorname{Supp}_R M$. Let $R \to R' \leftarrow Q$ be a quasi-deformation. By faithful flatness, pick in R' a prime $\mathfrak{p}'$ lying over $\mathfrak{p}$, let $\mathfrak{q}$ be its inverse image in Q, and note that the induced diagram $R_\mathfrak{p} \to R'_{\mathfrak{p}'} \leftarrow Q_\mathfrak{q}$ is a quasi-deformation. The (in)equalities

$$\operatorname{pd}_{Q_\mathfrak{q}} M'_\mathfrak{q} \leqslant \operatorname{pd}_Q M' \quad \text{and} \quad \operatorname{pd}_{Q_\mathfrak{q}} R'_\mathfrak{q} = \operatorname{pd}_Q R'$$

finish the proof, since $M'_\mathfrak{q} \cong M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R'_{\mathfrak{p}'}$ and $R'_\mathfrak{q} \cong R'_{\mathfrak{p}'}$ as $Q_\mathfrak{q}$-modules. □

(1.7) *Lemma. — If* R *is a local ring,* $R \to R' \leftarrow Q$ *is a quasi-deformation, and* M *is a non-zero* R*-module, then*

$$\operatorname{pd}_Q M' - \operatorname{pd}_Q R' = \begin{cases} \operatorname{depth} R - \operatorname{depth}_R M & \textit{if } \operatorname{pd}_Q M' < \infty; \\ \infty & \textit{if } \operatorname{pd}_Q M' = \infty. \end{cases}$$

Proof. — Note that

$$\operatorname{depth}_{R'} M' - \operatorname{depth}_R M = \operatorname{depth} R' - \operatorname{depth} R = \operatorname{depth}(R'/\mathfrak{m}R'),$$

cf. e.g. [29; (23.3)]. Thus, when $\operatorname{pd}_Q M'$ is finite the Auslander-Buchsbaum Equality yields

$$\begin{aligned} \operatorname{pd}_Q M' - \operatorname{pd}_Q R' &= (\operatorname{depth} Q - \operatorname{depth}_Q M') - (\operatorname{depth} Q - \operatorname{depth}_Q R') \\ &= \operatorname{depth}_Q R' - \operatorname{depth}_Q M' = \operatorname{depth} R' - \operatorname{depth}_{R'} M' \\ &= \operatorname{depth} R - \operatorname{depth}_R M. \quad \square \end{aligned}$$

(1.8) *Syzygies.* — The n-th syzygy of a finite module M over a local ring R is defined uniquely up to isomorphism by $\mathrm{Syz}_n^{\mathrm{R}}(\mathrm{M}) = \mathrm{Coker}\, \partial_{n+1}$, where $(\mathbf{F}, \partial)$ is a minimal free resolution of M. The exact sequence of R-modules with $\partial_n(\mathrm{F}_n) \subseteq \mathfrak{m}\mathrm{F}_{n-1}$ for $n \geqslant 1$

$$(1.8.1) \qquad 0 \longrightarrow \mathrm{Syz}_n^{\mathrm{R}}(\mathrm{M}) \longrightarrow \mathrm{F}_{n-1} \xrightarrow{\partial_{n-1}} \mathrm{F}_{n-2} \longrightarrow \ldots \longrightarrow \mathrm{F}_1 \xrightarrow{\partial_1} \mathrm{F}_0 \longrightarrow \mathrm{M} \longrightarrow 0$$

shows that when $\mathrm{R} \to \mathrm{R}'$ is a local flat extension, then

$$(1.8.2) \qquad \mathrm{Syz}_n^{\mathrm{R}'}(\mathrm{M}') \cong \mathrm{Syz}_n^{\mathrm{R}}(\mathrm{M}) \otimes_{\mathrm{R}} \mathrm{R}'.$$

(1.9) *Lemma.* — *If* $\mathrm{M} \neq 0$ *is a finite module over a local ring* R, *then*

$$(1.9.1) \qquad \text{CI-dim}_{\mathrm{R}}\, \mathrm{Syz}_n^{\mathrm{R}}(\mathrm{M}) = \max\{\, \text{CI-dim}_{\mathrm{R}}\, \mathrm{M} - n, 0 \,\} \quad \text{for } n \geqslant 0.$$

If furthermore $\text{CI-dim}_{\mathrm{R}}\, \mathrm{M}$ *is finite, then the following also hold*:

$$(1.9.2) \qquad \operatorname{depth}_{\mathrm{R}} \mathrm{Syz}_n^{\mathrm{R}}(\mathrm{M}) = \min\{\, \operatorname{depth}_{\mathrm{R}} \mathrm{M} + n, \operatorname{depth} \mathrm{R} \,\} \quad \text{for } 0 \leqslant n < \mathrm{pd}_{\mathrm{R}}\, \mathrm{M}.$$

Proof. — If $\mathrm{R} \to \mathrm{R}' \leftarrow \mathrm{Q}$ is a quasi-deformation, then it is easily seen from (1.8) that $\text{CI-dim}_{\mathrm{R}}\, \mathrm{Syz}_n^{\mathrm{R}}(\mathrm{M})$ is infinite if and only if $\text{CI-dim}_{\mathrm{R}}\, \mathrm{M}$ is. For the rest of the proof we assume that $\text{CI-dim}_{\mathrm{R}}\, \mathrm{M}$ is finite. In view of (1.7), then

$$\text{CI-dim}_{\mathrm{R}}\, \mathrm{M} = \operatorname{depth} \mathrm{R} - \operatorname{depth}_{\mathrm{R}} \mathrm{M},$$

hence it suffices to prove (1.9.2). Changing notation, we may also assume $\mathrm{pd}_{\mathrm{Q}}\, \mathrm{M}' < \infty$. If $\mathbf{x} = x_1, \ldots, x_c$ is a regular sequence generating $\mathrm{Ker}(\mathrm{Q} \to \mathrm{R}')$, then $(\mathbf{x})\, \mathrm{M}' = 0$ implies

$$(1.9.3) \qquad \mathrm{pd}_{\mathrm{Q}}\, \mathrm{M}' \geqslant \operatorname{grade}_{\mathrm{Q}} \mathrm{M}' \geqslant c = \mathrm{pd}_{\mathrm{Q}}\, \mathrm{R}'.$$

Now (1.7) yields $\operatorname{depth}_{\mathrm{R}} \mathrm{M} \leqslant \operatorname{depth} \mathrm{R}$, so we conclude by counting depths in (1.8.1). □

Proof of Theorem (1.4). — Both the projective dimension and the Gorenstein dimensions can be computed locally, in the sense that there are equalities

$$\mathrm{pd}_{\mathrm{R}}\, \mathrm{M} = \sup\{\, \mathrm{pd}_{\mathrm{R}_{\mathfrak{p}}}\, \mathrm{M}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}_{\mathrm{R}}\, \mathrm{M} \,\};$$

$$\text{G-dim}_{\mathrm{R}}\, \mathrm{M} = \sup\{\, \text{G-dim}_{\mathrm{R}_{\mathfrak{p}}}\, \mathrm{M}_{\mathfrak{p}} \mid \mathfrak{p} \in \mathrm{Supp}_{\mathrm{R}}\, \mathrm{M} \,\}:$$

this is classical for the first dimension and proved in [5; (4.15)] for the second one. As the CI-dimension has the corresponding property by (1.6), it suffices to treat the local case.

If the projective dimension of M over R is finite, then so is its complete intersection dimension, because the constant quasi-deformation $\mathrm{R} \xrightarrow{=} \mathrm{R} \xleftarrow{=} \mathrm{R}$ appears in the determination of $\text{CI-dim}_{\mathrm{R}}\, \mathrm{M}$. By expressing $\mathrm{pd}_{\mathrm{R}}\, \mathrm{M}$ from the Auslander-Buchsbaum Equality and $\text{CI-dim}_{\mathrm{R}}\, \mathrm{M}$ from (1.7), we see that they are actually equal.

Suppose next that $n = \text{CI-dim}_R M$ is finite. By (1.9) the CI-dimension of $\text{Syz}^R_n(M)$ is zero and its depth is equal to that of R. Thus, after changing notation, we may assume that $\text{CI-dim}_R M = 0$ and $\text{depth}_R M = \text{depth } R$, and we want to show that $\text{G-dim}_R M = 0$.

Choose a codimension c quasi-deformation $R \to R' \leftarrow Q$ with $\text{pd}_Q M' < \infty$. By the Auslander-Buchsbaum Equality we then have $\text{pd}_Q M' = \text{pd}_Q R' = c$, and the inequalities (1.9.3) imply that the Q-module M′ is perfect of projective dimension c.

For any R′-module L there are well-known change of ring isomorphisms due to Rees:

(1.9.4) $$\text{Ext}^n_{R'}(L, R') \cong \text{Ext}^{n+c}_Q(L, Q) \quad \text{for } n \in \mathbf{Z}.$$

With $L = M'$ they show that $\text{Ext}^n_{R'}(M', R') = 0$ for $n > 0$, and that the Q-module $M'^* = \text{Hom}_{R'}(M', R')$ is perfect of projective dimension c and naturally isomorphic to $\text{Ext}^c_Q(M', Q)$. With $L = M'^*$ the Rees formulas yield $\text{Ext}^n_{R'}(M'^*, R') = 0$ for $n > 0$, and $M'^{**} \cong \text{Ext}^c_Q(\text{Ext}^c_Q(M', Q), Q)$. As M′ is perfect, the last module is isomorphic to M′, hence $\text{G-dim}_{R'} M' = 0$. In view of the faithful flatness of R′ over R, it follows easily from the definition that $\text{G-dim}_{R'} M' = 0$ implies $\text{G-dim}_R M = 0$, as desired.

We have now shown that $\text{G-dim}_R M \leqslant \text{CI-dim}_R M$. Thus, if the CI-dimension of M is finite, then so is its G-dimension. By [5; (4.13.b)] we then have $\text{G-dim}_R M = \text{depth } R - \text{depth}_R M$, and (1.7) shows that this is precisely $\text{CI-dim}_R M$. □

We establish change of module and ring formulas for CI-dimension under certain types of ring homomorphisms. The arguments are based on the following construction.

(1.10) *Compositions.* — Let $R \xrightarrow{\rho'} R' \xleftarrow{\rho} Q$ and $Q \xrightarrow{\kappa'} Q' \xleftarrow{\kappa} P$ be quasi-deformations. The lower row of the diagram

$$\begin{tikzcd} & & P \arrow[d, "\kappa"] \\ & Q \arrow[r, "\kappa'"] \arrow[d, "\rho"'] & Q' \arrow[d, "\rho' \otimes_Q Q'"] \\ R \arrow[r, "\rho'"'] & R' \arrow[r, "R' \otimes_Q \kappa'"'] & R' \otimes_Q Q' \end{tikzcd}$$

is then a composition of local flat extensions, and its right hand column is a composition of deformations. Thus, the resulting diagram $R \to R' \otimes_Q Q' \leftarrow P$ is a quasi-deformation, that we call the *composition* of the initial two. This operation is associative and yields a category whose objects are local rings and whose morphisms are quasi-deformations up to isomorphism, with the identity morphism of R given by the *constant quasi-deformation* $R \xrightarrow{=} R \xleftarrow{=} R$.

(1.11) *Lemma.* — *If* $R \to R' \leftarrow Q$ *is a codimension* c *quasi-deformation and* M *is a finite* R*-module, then*

$$\text{CI-dim}_R M \leqslant \text{CI-dim}_{R'} M' \leqslant \text{CI-dim}_Q M' - c.$$

When one of the quantities is finite it is equal to those to its left.

Proof. — The first inequality is immediate from the observation that $R \to R' \xleftarrow{=} R'$ is a quasi-deformation, and that quasi-deformations form a category. When $\text{CI-dim}_{R'} M'$ is finite, by using (1.7) we get $\text{CI-dim}_{R'} M' = \text{CI-dim}_R M$.

If $Q \to Q' \leftarrow P$ is a quasi-deformation, and $R \to R'' = R' \otimes_Q Q' \leftarrow P$ is its composition with $R \to R' \leftarrow Q$, then in view of the isomorphism

$$M \otimes_R R'' \cong (M \otimes_R R') \otimes_{R'} R''$$

we get an inequality $\text{CI-dim}_{R'} M' \leqslant \text{CI-dim}_Q M'$. When the right hand side is finite, (1.7) shows it is equal to

$$\operatorname{depth} Q - \operatorname{depth}_Q M' = \operatorname{depth} R + c - \operatorname{depth}_R M' = \text{CI-dim}_{R'} M' + c. \quad \square$$

(1.12) *Proposition.* — *Given a sequence* $\mathbf{y} = y_1, \ldots, y_g$ *of elements of a noetherian ring* R, *set* $\overline{R} = R/(\mathbf{y})R$ *and* $\overline{M} = M/(\mathbf{y})M$. *If* $\mathbf{y}$ *is* M*-regular, then*

(1.12.1) $$\text{CI-dim}_R \overline{M} \leqslant \text{CI-dim}_R M + g$$

with equality when $\mathbf{y}$ *is in the Jacobson radical of* R. *If* $\mathbf{y}$ *is* R*-regular, then*

(1.12.2) $$\text{CI-dim}_{\overline{R}} \overline{M} \leqslant \text{CI-dim}_R M \quad \textit{when } \mathbf{y} \textit{ is } M\textit{-regular};$$

(1.12.3) $$\text{CI-dim}_{\overline{R}} M \leqslant \text{CI-dim}_R M - g \quad \textit{when } (\mathbf{y})M = 0,$$

with equalities when $\mathbf{y}$ *is in the Jacobson radical of* R *and* $\text{CI-dim}_R M$ *is finite.*

Proof. — In view of (1.6) and an obvious induction, it suffices to prove the claims when $(R, \mathfrak{m})$ is local, $g = 1$, and $y_1 = y$ is a non-zero-divisor in $\mathfrak{m}$.

Consider a quasi-deformation $R \to R' \leftarrow Q$. If y is M-regular, then the exact sequence $0 \to M \xrightarrow{y} M \to N \to 0$ induces an exact sequence of R'-modules

$$0 \to M' \xrightarrow{y} M' \to N' \to 0.$$

Thus, $N' \cong M'/yM'$ and $\operatorname{pd}_Q N' = \operatorname{pd}_Q M' + 1$, so (1.12.1) follows.

Assume next that y is R-regular. If $(\mathbf{y})M = 0$, then $\overline{M} = M$, and (1.12.3) is obtained by applying (1.11) to the quasi-deformation $\overline{R} \xrightarrow{=} \overline{R} \leftarrow R$. If y is M-regular and $\text{CI-dim}_R M$ is finite, then the equality in (1.12 2) is a consequence of the other two formulas. $\square$

(1.13) *Proposition.* — *Let* M *be a module over a noetherian ring* R.

(1) *If* $\varphi : \mathrm{R} \to \mathrm{S}$ *is a faithfully flat homomorphism of rings, then*

$$\text{CI-dim}_{\mathrm{R}}\, \mathrm{M} \leqslant \text{CI-dim}_{\mathrm{S}}(\mathrm{M} \otimes_{\mathrm{R}} \mathrm{S})$$

with equality when $\text{CI-dim}_{\mathrm{S}}(\mathrm{M} \otimes_{\mathrm{R}} \mathrm{S})$ *is finite.*

(2) *If* $\mathfrak{a} \subsetneq \mathrm{R}$ *is an ideal,* R^* *is the* $\mathfrak{a}$*-adic completion of* R *and* M^* *is that of* M, *then*

$$\text{CI-dim}_{\mathrm{R}^*}\, \mathrm{M}^* \leqslant \text{CI-dim}_{\mathrm{R}}\, \mathrm{M}$$

with equality when $\mathfrak{a}$ *is contained in the Jacobson radical* $\mathfrak{j}(\mathrm{R})$ *of* R.

Proof. — (1) Pick $\mathfrak{q} \in \operatorname{Spec} \mathrm{S}$ and set $\mathfrak{p} = \mathfrak{q} \cap \mathrm{R}$. The induced local homomorphism $\varphi_{\mathfrak{q}} : \mathrm{R}_{\mathfrak{p}} \to \mathrm{S}_{\mathfrak{q}}$ is flat, the $\mathrm{S}_{\mathfrak{q}}$-modules $\mathrm{M}_{\mathfrak{p}} \otimes_{\mathrm{R}_{\mathfrak{p}}} \mathrm{S}_{\mathfrak{q}}$ and $(\mathrm{M} \otimes_{\mathrm{R}} \mathrm{S})_{\mathfrak{q}}$ are canonically isomorphic, hence (1.11) yields $\text{CI-dim}_{\mathrm{R}_{\mathfrak{p}}} \mathrm{M}_{\mathfrak{p}} \leqslant \text{CI-dim}_{\mathrm{S}_{\mathfrak{q}}}(\mathrm{M} \otimes_{\mathrm{R}} \mathrm{S})_{\mathfrak{q}}$, with equality when the right hand side is finite. As each prime of R is the contraction of one of S, the desired (in)equality follows.

(2) Any maximal ideal of R^* is of the form $\mathfrak{m}^* = \mathfrak{m}\mathrm{R}^*$ for some maximal ideal $\mathfrak{m} \subseteq \mathrm{R}$ with $\mathfrak{m} \supseteq \mathfrak{a}$. Let $\mathrm{R}_{\mathfrak{m}} \to \mathrm{R}' \leftarrow \mathrm{Q}$ be a quasi-deformation and let $\mathfrak{b}$ be the inverse image in Q of $\mathfrak{a}\mathrm{R}'$. Ideal-adic completions yield a quasi-deformation $\mathrm{R}^*_{\mathfrak{m}^*} \to \mathrm{R}'^* \leftarrow \mathrm{Q}^*$. As $\operatorname{pd}_{\mathrm{Q}^*}(\mathrm{M}^*_{\mathfrak{m}^*} \otimes_{\mathrm{R}^*_{\mathfrak{m}^*}} \mathrm{R}'^*) = \operatorname{pd}_{\mathrm{Q}}(\mathrm{M}_{\mathfrak{m}} \otimes_{\mathrm{R}_{\mathfrak{m}}} \mathrm{R}')$ and $\operatorname{pd}_{\mathrm{Q}^*} \mathrm{R}'^* = \operatorname{pd}_{\mathrm{Q}} \mathrm{R}'$, we get the inequality. If $\mathfrak{a} \subseteq \mathfrak{j}(\mathrm{R})$, then R^* is faithfully flat over R, so (1) gives a converse inequality. □

(1.14) *Residue field extensions.* — Let $\mathrm{R} \to \mathrm{R}' \leftarrow \mathrm{Q}$ be a quasi-deformation, let k' be the residue field of R', and let $k' \subseteq \ell$ be a field extension. By [15; Appendice] there is a local flat extension $\mathrm{Q} \to \mathrm{Q}'$, such that $\mathrm{Q}' \otimes_{\mathrm{Q}} k' = \ell$. Composition of the original quasi-deformation with $\mathrm{Q} \to \mathrm{Q}' \xleftarrow{=} \mathrm{Q}'$ yields a quasi-deformation $\mathrm{R} \to \mathrm{R}'' \leftarrow \mathrm{Q}'$, such that ℓ is the residue field of R'', and $\operatorname{pd}_{\mathrm{Q}}(\mathrm{M} \otimes_{\mathrm{R}} \mathrm{R}') = \operatorname{pd}_{\mathrm{Q}'}(\mathrm{M} \otimes_{\mathrm{R}} \mathrm{R}'')$ for each R-module M.

It follows that $\text{CI-dim}_{\mathrm{R}}\, \mathrm{M}$ can be computed by using quasi-deformations in which the local rings Q have infinite, or even algebraically closed, residue fields. In view of (1.13.2), one can further restrict to those quasi-deformations for which the ring Q is also complete.

2. Quantum regular sequences

In this section E is a module over a commutative ring Q. We define a notion of E-regular sequence of endomorphisms of E, that extends the classical notion of E-regular sequence of elements of Q, and reduces to it when the endomorphisms are homotheties.

We say a square matrix $\boldsymbol{q} = (q_{ij})$ of elements of Q is a *commutation matrix* if $q_{ij}\, q_{ji} = 1$ and $q_{jj} = 1$ for $1 \leqslant i, j \leqslant c$. A family $\{ \varphi_1, \ldots, \varphi_c \}$ of endomorphisms of a Q-module E is said to be *quantum commuting* if there is a $c \times c$ commutation matrix $\boldsymbol{q}$

such that $\varphi_i \varphi_j = q_{ij} \varphi_j \varphi_i$ for all i, j. Powers of elements of a quantum commuting family also form such a family, with a new commutation matrix.

We denote $(\varphi_{j_1}, \ldots, \varphi_{j_s}) \mathrm{E}$ the Q-submodule $\mathrm{Im}(\varphi_{j_1}) + \ldots + \mathrm{Im}(\varphi_{j_s})$ of E, and set $(\boldsymbol{\varphi}) \mathrm{E} = (\varphi_1, \ldots, \varphi_c) \mathrm{E}$. If the family is quantum commuting, then for arbitrary integers $j, j_1, \ldots, j_s$ between 1 and c, the endomorphism φ_j maps the $(\varphi_{j_1}, \ldots, \varphi_{j_s}) \mathrm{E}$ into itself. Thus, each φ_j induces on any of the modules $\mathrm{E}/(\varphi_{j_1}, \ldots, \varphi_{j_s}) \mathrm{E}$ an endomorphism, also denoted φ_j. A *sequence* $\boldsymbol{\varphi} = \varphi_1, \ldots, \varphi_c$ of quantum commuting endomorphisms is said to be E-*regular*, if $\mathrm{E} \neq (\boldsymbol{\varphi}) \mathrm{E}$ and φ_j is an injection on $\mathrm{E}/(\varphi_1, \ldots, \varphi_{j-1}) \mathrm{E}$ for $j = 1, \ldots, c$. Note that $x_1, \ldots, x_c \in \mathrm{Q}$ form a (classical) E-regular sequence precisely when the commuting sequence of endomorphisms $x_1 \,\mathrm{id}_\mathrm{E}, \ldots, x_c \,\mathrm{id}_\mathrm{E}$ is E-regular.

(2.1) *Proposition. — Let $n_1, \ldots, n_c$ be positive integers. A sequence $\varphi_1, \ldots, \varphi_c$ of quantum commuting endomorphisms is* E*-regular if and only if so is the sequence $\varphi_1^{n_1}, \ldots, \varphi_c^{n_c}$.*

Proof. — When the endomorphisms commute, [29; (16.1)] establishes the "only if" part, and the argument carries over. As a first step, it shows that if $u_1, \ldots, u_c \in \mathrm{E}$ satisfy $\varphi_1(u_1) + \ldots + \varphi_c(u_c) = 0$, then $u_j \in (\boldsymbol{\varphi}) \mathrm{E}$ for $j = 1, \ldots, c$. Suppose the sequence $\varphi_1^{n_1}, \ldots, \varphi_c^{n_c}$ is E-regular and note that $(\varphi_1, \ldots, \varphi_c)^{n_1 + \cdots + n_c} \mathrm{E} \subseteq (\varphi_1^{n_1}, \ldots, \varphi_c^{n_c}) \mathrm{E}$ implies $(\boldsymbol{\varphi}) \mathrm{E} \neq \mathrm{E}$. To establish the regularity of $\varphi_1, \ldots, \varphi_c$ it suffices to show that $\varphi_1, \varphi_2^{n_2}, \ldots, \varphi_c^{n_c}$ is E-regular. We assume $n_1 \geqslant 2$ and argue by induction on c.

When $c = 1$ all we have to show is that φ_1 is injective, and this follows from the injectivity of $\varphi_1^{n_1}$. Let $c \geqslant 2$ and suppose that

$$\varphi_c^{n_c}(u) = \varphi_1(u_1) + \varphi_2^{n_2}(u_2) + \ldots + \varphi_{c-1}^{n_{c-1}}(u_{c-1})$$

for appropriate $u, u_1, u_2, \ldots, u_{c-1}$ in E. Applying $\varphi_1^{n_1 - 1}$ to this expression we get

$$\varphi_c^{n_c}(q_c\, \varphi_1^{n_1-1}(u)) = \varphi_1^{n_1}(u_1) + \varphi_2^{n_2}(q_2\, \varphi_1^{n_1-1}(u_2)) + \ldots$$
$$+ \varphi_{c-1}^{n_{c-1}}(q_{c-1}\, \varphi_1^{n_1-1}(u_{c-1})),$$

where $q_j = (q_{1j})^{(n_1-1) n_j}$ is a unit in Q for $j = 2, \ldots, c$. Since $\varphi_1^{n_1}, \ldots, \varphi_c^{n_c}$ is E-regular, there are $v_1, \ldots, v_{c-1} \in \mathrm{E}$ for which $\varphi_1^{n_1-1}(u) = \varphi_1^{n_1}(v_1) + \ldots + \varphi_{c-1}^{n_{c-1}}(v_{c-1})$, that is:

$$\varphi_1^{n_1-1}(u - \varphi_1(v_1)) = \varphi_2^{n_2}(v_1) + \ldots + \varphi_{c-1}^{n_{c-1}}(v_{c-1}).$$

Since $\varphi_1, \varphi_2^{n_2}, \ldots, \varphi_{c-1}^{n_{c-1}}$ is E-regular by the induction hypothesis, the "only if" part shows that $\varphi_1^{n_1-1}, \varphi_2^{n_2}, \ldots, \varphi_{c-1}^{n_{c-1}}$ has the same property. Thus, there exist $w_1, \ldots, w_{c-1} \in \mathrm{E}$ such that $u - \varphi_1(v_1) = \varphi_1^{n_1-1}(w_1) + \ldots + \varphi_{c-1}^{n_{c-1}}(w_{c-1})$. Rewriting the last equality as

$$u = \varphi_1(v_1 + \varphi_1^{n_1-2}(w_1)) + \varphi_2^{n_2}(w_2) + \ldots + \varphi_{c-1}^{n_{c-1}}(w_{c-1}) \in (\boldsymbol{\varphi}) \mathrm{E},$$

we see that the sequence $\varphi_1, \varphi_2^{n_2}, \ldots, \varphi_c^{n_c}$ is E-regular, as desired. $\square$

(2.2) *Quantum symmetric algebras.* — Let $\boldsymbol{q} = (q_{ij})$ be a $c \times c$ commutation matrix with entries in Q. Following Manin [28], we call the Q-algebra $Q_{\boldsymbol{q}}[X]$ generated by a set $X = \{X_1, \ldots, X_c\}$ whose elements are subject only to the relations $X_i X_j = q_{ij} X_j X_i$ for $1 \leqslant i, j \leqslant c$, a *quantum symmetric* Q*-algebra* (with commutation matrix $\boldsymbol{q}$). For a c-tuple $J = (j_1, \ldots, j_c) \in \mathbf{N}^c$, set $|J| = j_1 + \ldots + j_c$. Clearly, the algebra $Q_{\boldsymbol{q}}[X]$ is $\mathbf{N}^c$-graded, with component of multidegree J the free Q-module with basis element $X^J = \prod_{i=1}^c X_i^{j_i}$.

We use without further comment the coincidence of the *left*, *right*, and *two-sided* ideals in $Q_{\boldsymbol{q}}[X]$ generated by some set of monomials in the indeterminates. Any such ideal is called a *monomial ideal.* These are the only ideals homogeneous with respect to the multigrading of $Q_{\boldsymbol{q}}[X]$. Such an ideal $\mathfrak{A}$ is cofinite—in the sense that the Q-module $Q_{\boldsymbol{q}}[X]/\mathfrak{A}$ is finite—precisely when it contains positive powers of all the variables.

The following properties are easily established by tracking multi-degrees.

(2.2.1) The monomial ideals of $Q_{\boldsymbol{q}}[X]$ are generated by finite sets of monomials. As in [17], where monomials in commuting indeterminates are treated, we consider the lattice with respect to sums and intersections of the (cofinite) monomial ideals in $Q_{\boldsymbol{q}}[X]$, and note that it is distributive. It follows that any (cofinite) monomial ideal is the intersection of a finite family of ideals generated by positive powers of (all) the indeterminates.

(2.2.2) The lattice of (cofinite) monomial ideals is closed under colons $\mathfrak{A} : \mathfrak{B} = \{ u \in Q_{\boldsymbol{q}}[X] \mid \mathfrak{B} u \subseteq \mathfrak{A} \}$. In particular, setting $X_i^{n_i - s_i} = 1$ if $n_i \leqslant s_i$, one has equalities

$$(X_1^{n_1}, \ldots, X_c^{n_c}) : (\prod_{i=1}^{c} X_i^{s_i}) = (X_1^{n_1 - s_1}, \ldots, X_c^{n_c - s_c});$$

$$(X_1^{n_1}, \ldots, X_c^{n_c}) : (X_1^{s_1}, \ldots, X_c^{s_c}) = (\prod_{i=1}^{c} X_i^{n_i - s_i}) + (X_1^{n_1}, \ldots, X_c^{n_c}).$$

(2.3) *Quantum Koszul complexes.* — Let $\boldsymbol{\varphi} = \{ \varphi_1, \ldots, \varphi_c \}$ be a quantum commuting family of endomorphisms of the Q-module E, with commutation matrix $\boldsymbol{q}$. Consider a graded free Q-module $\mathbf{Y}$ with Y_n free on a basis $\{ Y_H \mid H \subseteq \{1, \ldots, c\}, \operatorname{card}(H) = n \}$. Let $h_1, \ldots, h_n$ be the elements of H listed in increasing order, and define a degree -1 endomorphism ∂ of the graded Q-module $\mathbf{Y} \otimes_Q E$ by the formula:

$$\partial(Y_H \otimes v) = \sum_{i=1}^{n} (-1)^{i-1} (\prod_{j=1}^{i-1} q_{h_i h_j}) \, Y_{H \setminus h_i} \otimes \varphi_{h_i}(v). \tag{2.3.1}$$

A direct computation that uses the quantum commutativity of the sequence $\varphi_1, \ldots, \varphi_c$ shows that $\partial^2 = 0$. We denote $\mathbf{K}(\boldsymbol{\varphi}; E)$ the complex $(\mathbf{Y} \otimes_Q E, \partial)$, call it the *quantum Koszul complex* of the family $\{ \varphi_1, \ldots, \varphi_c \}$, and set $H_*(\boldsymbol{\varphi}; E) = H_* \mathbf{K}(\boldsymbol{\varphi}; E)$.

(2.3.2) *Example.* — *When* $\mathbf{x} = \{x_1, \ldots, x_c\}$ *is a set of elements of* Q, *the quantum Koszul complex* $\mathbf{K}(\boldsymbol{\varphi}; \mathrm{E})$ *on the family* $\boldsymbol{\varphi} = \{x_1 \,\mathrm{id}_{\mathrm{E}}, \ldots, x_c \,\mathrm{id}_{\mathrm{E}}\}$ *of commuting endomorphisms of* E *coincides with the classical Koszul complex* $\mathbf{K}(\mathbf{x}; \mathrm{E}) = \mathbf{K}(\mathbf{x}; \mathrm{Q}) \otimes_{\mathrm{Q}} \mathrm{E}$.

As in the classical case, vanishing of Koszul homology is related to regularity.

(2.3.3) *Proposition.* — *For a quantum commuting family* $\boldsymbol{\varphi} = \{\varphi_1, \ldots, \varphi_c\}$ *of endomorphisms of a* Q-*module* E, *there is an isomorphism* $\mathrm{H}_0(\boldsymbol{\varphi}; \mathrm{E}) \cong \mathrm{E}/(\boldsymbol{\varphi})\,\mathrm{E}$.

If the sequence $\boldsymbol{\varphi}$ *is* E-*regular, then* $\mathrm{H}_n(\boldsymbol{\varphi}; \mathrm{E}) = 0$ *for* $n \neq 0$.

Proof. — The expression for the zeroth homology is clear from the definitions.

If $\boldsymbol{\varphi}$ is regular, then $\boldsymbol{\varphi}' = \varphi_1, \ldots, \varphi_{c-1}$ is a quantum commuting E-regular sequence. By induction, we may assume $\mathrm{H}_n(\boldsymbol{\varphi}'; \mathrm{E}) = 0$ when $n \neq 0$. For each $\mathrm{H} \subseteq \{1, \ldots, c\}$ set

$$\alpha(\mathrm{Y}_{\mathrm{H}} \otimes v) = \mathrm{Y}_{\mathrm{H}} \otimes v; \quad \beta(\mathrm{Y}_{\mathrm{H}} \otimes v) = \begin{cases} (-1)^{n-1}\,\mathrm{Y}_{\mathrm{H}\setminus c} \otimes v & \text{if } c \in \mathrm{H}; \\ 0 & \text{if } c \notin \mathrm{H}. \end{cases}$$

It is easy to see that $0 \to \mathbf{K}(\boldsymbol{\varphi}'; \mathrm{E}) \xrightarrow{\alpha} \mathbf{K}(\boldsymbol{\varphi}; \mathrm{E}) \xrightarrow{\beta} \Sigma\mathbf{K}(\boldsymbol{\varphi}'; \mathrm{E}) \to 0$ is an exact sequence of complexes of Q-modules. The associated homology exact sequence yields equalities $\mathrm{H}_n(\boldsymbol{\varphi}; \mathrm{E}) = 0$ for $n \neq 0, 1$ and an exact sequence:

$$0 \to \mathrm{H}_1(\boldsymbol{\varphi}; \mathrm{E}) \to \mathrm{E}/(\boldsymbol{\varphi}')\,\mathrm{E} \xrightarrow{\varphi_c} \mathrm{E}/(\boldsymbol{\varphi}')\,\mathrm{E} \to \mathrm{E}/(\boldsymbol{\varphi})\,\mathrm{E} \to 0.$$

As $\boldsymbol{\varphi}$ is regular, the homomorphism φ_c is injective, hence $\mathrm{H}_1\,\mathbf{K}(\boldsymbol{\varphi}; \mathrm{E}) = 0$. □

(2.3.4) *Example.* — *The sequence* X *of endomorphisms of* $\mathrm{Q}_q[\mathrm{X}]$, *induced by* left *multiplication with* $\mathrm{X}_1, \ldots, \mathrm{X}_c$, *is quantum commuting and* $\mathrm{Q}_q[\mathrm{X}]$-*regular, hence* $\mathrm{H}_n(\mathrm{X}; \mathrm{Q}_q[\mathrm{X}])$ *vanishes in degrees* $n \neq 0$, *and equals* Q *in degree* 0 *by* (2.3.3). *Observe that in this case our quantum Koszul complex coincides with that of Manin* [28].

Given a quantum commuting family $\{\varphi_1, \ldots, \varphi_c\} \subset \mathrm{End}_{\mathrm{Q}}(\mathrm{E})$ with commutation matrix $\boldsymbol{q}$, there is a unique homomorphism of Q-algebras $\mathrm{Q}_q[\mathrm{X}] \to \mathrm{End}_{\mathrm{Q}}(\mathrm{E})$ that maps X_j to φ_j for $1 \leqslant j \leqslant c$, and so E becomes a left $\mathrm{Q}_q[\mathrm{X}]$-module. A submodule of E of the form $\mathfrak{A}\mathrm{E}$ for some monomial ideal $\mathfrak{A}$ in $\mathrm{Q}_q[\mathrm{X}]$ is said to be *monomial*. It is clear that sums of monomial submodules are monomial. Under additional conditions, intersections and colon submodules $(\mathrm{P} : \mathfrak{B}) = \{v \in \mathrm{E} \mid \mathfrak{B}v \subset \mathrm{P}\}$ also display a similar stability.

(2.4) *Theorem.* — *Let* $\boldsymbol{\varphi} = \varphi_1, \ldots, \varphi_c$ *be a quantum commuting sequence of endomorphisms of* E *with commutation matrix* $\boldsymbol{q}$, *and let* $\mathfrak{A}$ *be a cofinite monomial ideal in* $\mathrm{Q}_q[\mathrm{X}]$.

If the sequence $\boldsymbol{\varphi}$ *is* E-*regular, then* $\mathrm{Tor}_n^{\mathrm{Q}_q[\mathrm{X}]}(\mathrm{Q}_q[\mathrm{X}]/\mathfrak{A}, \mathrm{E}) = 0$ *for* $n > 0$, *the natural map* $\mathfrak{A} \otimes_{\mathrm{Q}_q[\mathrm{X}]} \mathrm{E} \to \mathfrak{A}\mathrm{E}$ *is bijective, and for each cofinite monomial ideal* $\mathfrak{B}$ *there are equalities* $(\mathfrak{A} \cap \mathfrak{B})\,\mathrm{E} = (\mathfrak{A}\mathrm{E}) \cap (\mathfrak{B}\mathrm{E})$ *and* $(\mathfrak{A}\mathrm{E} : \mathfrak{B}) = (\mathfrak{A} : \mathfrak{B})\,\mathrm{E}$.

Proof. — The first assertion is proved by induction on the rank r of the free Q-module $Q_q[X]/\mathfrak{A}$. If $r = 1$ then $\mathfrak{A}$ is the ideal $(X_1, \ldots, X_c)$. As $\mathbf{K}(X; Q_q[X])$ is a complex of free *right* $Q_q[X]$-modules, by (2.3.4) it provides a free resolution of $Q = Q_q[X]/(X)$, hence

$$\mathrm{Tor}_n^{Q_q[X]}(Q, E) = H_n(\mathbf{K}(X; Q_q[X]) \otimes_{Q_q[X]} E) = H_n(\boldsymbol{\varphi}; E),$$

and by (2.3.3) the last group vanishes for $n > 0$.

When $r > 1$ there is a sequence $Q_q[X] = \mathfrak{A}_0 \supset \ldots \supset \mathfrak{A}_r = \mathfrak{A}$ of monomial ideals such that $\mathfrak{A}_i/\mathfrak{A}_{i+1} \cong Q$ for $0 \leqslant i < r$. For $n > 0$ the modules at both ends of the exact sequence

$$\mathrm{Tor}_n^{Q_q[X]}(Q, E) \to \mathrm{Tor}_n^{Q_q[X]}(Q_q[X]/\mathfrak{A}, E) \to \mathrm{Tor}_n^{Q_q[X]}(Q_q[X]/\mathfrak{A}_{r-1}, E)$$

are trivial by induction, hence so is the middle one. Using this fact for $n = 1$ we see that the short exact sequence $0 \to \mathfrak{A} \to Q_q[X] \to Q_q[X]/\mathfrak{A} \to 0$ induces a canonical isomorphism $\mathfrak{A} \otimes_{Q_q[X]} E \to \mathfrak{A}E$, as desired.

To compute the intersection of monomial submodules, consider the short exact sequence of right $Q_q[X]$-modules $0 \to \mathfrak{A} \cap \mathfrak{B} \to \mathfrak{A} \oplus \mathfrak{B} \to \mathfrak{A} + \mathfrak{B} \to 0$. It induces a commutative diagram with exact rows and bijective (by the first part of the theorem) vertical maps:

$$\begin{array}{ccccccc}
(\mathfrak{A} \cap \mathfrak{B}) \otimes_{Q_q[X]} E & \longrightarrow & (\mathfrak{A} \otimes_{Q_q[X]} E) \oplus (\mathfrak{B} \otimes_{Q_q[X]} E) & \longrightarrow & (\mathfrak{A} + \mathfrak{B}) \otimes_{Q_q[X]} E & \longrightarrow & 0 \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
(\mathfrak{A} \cap \mathfrak{B})\, E & \xrightarrow{\ \gamma\ } & \mathfrak{A}E \oplus \mathfrak{B}E & \xrightarrow{\ \delta\ } & (\mathfrak{A} + \mathfrak{B})\, E & \longrightarrow & 0
\end{array}$$

As $(\mathfrak{A} + \mathfrak{B})\, E = \mathfrak{A}E + \mathfrak{B}E$, we see that $\mathrm{Ker}\,\delta = (\mathfrak{A}E) \cap (\mathfrak{B}E)$. On the other hand, as $\mathrm{Tor}_1^{Q_q[X]}(\mathfrak{A} + \mathfrak{B}, E) \cong \mathrm{Tor}_2^{Q_q[X]}(Q_q[X]/(\mathfrak{A} + \mathfrak{B}), E)$ and the last module vanishes by the first part of the theorem, we see that γ is injective. The exactness of the bottom row now shows $\mathrm{Im}\,\gamma = \mathrm{Ker}\,\delta$, that is, $(\mathfrak{A} \cap \mathfrak{B})\, E = \mathrm{Ker}\,\gamma = (\mathfrak{A}E) \cap (\mathfrak{B}E)$, as desired.

Finally, we compute the colon submodule of a monomial submodule and a monomial ideal, and start with the special case when $\mathfrak{A} = (X_1^{n_1}, \ldots, X_c^{n_c})$ and $\mathfrak{B} = (\prod_{i=1}^c X_i^{s_i})$. In view of (2.2.2) and the obvious inclusion $(\mathfrak{A} : \mathfrak{B})\, E \subseteq (\mathfrak{A}E : \mathfrak{B})$, we have to show that if $(\prod_{i=1}^c \varphi_i^{s_i})(v) \in (\varphi_1^{n_1}, \ldots, \varphi_c^{n_c})\, E$ then v is in $(\varphi_1^{n_1 - s_1}, \ldots, \varphi_c^{n_c - s_c})\, E$, where $\varphi_i^{n_i - s_i} = \mathrm{id}_E$ if $n_i \leqslant s_i$. The assertion is trivial when $c = 0$, so we assume by induction that it has been established for less than c quantum commuting variables; also, since there is nothing to prove otherwise, we assume that $n_i > s_i$ for $1 \leqslant i \leqslant c$. The sequence $\varphi_2^{n_2}, \ldots, \varphi_c^{n_c}$ being regular on $E/\varphi_1^{n_1}(E)$ by (2.1), we conclude by the induction hypothesis that $\varphi_1^{s_1}(v) = \varphi_1^{n_1}(w_1) + \sum_{i=2}^c \varphi_i^{n_i - s_i}(w_i)$ for appropriate $w_i \in E$. The regularity of $\varphi_1^{s_1}, \varphi_2^{n_2 - s_2}, \ldots, \varphi_c^{n_c - s_c}$ implies $v - \varphi_1^{n_1 - s_1}(w_1) \in (\varphi_2^{n_2 - s_2}, \ldots, \varphi_c^{n_c - s_c})\, E$, as required.

In general, we know from (2.2) that $\mathfrak{A} = \bigcap_{a=1}^s \mathfrak{A}_a$ with each $\mathfrak{A}_a$ generated by positive powers of the variables X_i, and that $\mathfrak{B} = (B_1, \ldots, B_t)$ for appropriate monomials B_b. By the special case of the formula for colons and the formula for intersections, we get

$$(\mathfrak{A}E : \mathfrak{B}) = \bigcap_{a,b} (\mathfrak{A}_a E : B_b) = \bigcap_{a,b} ((\mathfrak{A}_a : B_b) E)$$
$$= (\bigcap_{a,b} (\mathfrak{A}_a : B_b)) E = (\mathfrak{A} : \mathfrak{B}) E.$$

The proof of the theorem is now complete. □

3. Constructions of modules of finite CI-dimension

The simplest type of modules of finite CI-dimension are those with finite projective dimension. It is easily seen that there are rings, over which these are the only ones. For instance, if R is Cohen-Macaulay and its multiplicity $e(R)$ has the minimal possible value, equal to $\operatorname{edim} R - \dim R + 1$, then each R-module M has

$$b_n^R(M) = b_m^R(M)\,(\operatorname{edim} R)^{n-m} \quad \text{for } n \geqslant m = \operatorname{depth} R - \operatorname{depth}_R M + 1\,;$$

it follows from (5.3) that $\text{CI-dim}_R\, M = \infty$ whenever $e(R) \geqslant 3$ and $\operatorname{pd}_R M = \infty$. This raises the problem to determine which local rings admit modules with finite CI-dimension and infinite projective dimension.

In [16] Buchweitz *et al.* have produced an interesting class of maximal Cohen-Macaulay modules over a complete hypersurface ring. Their method easily extends to provide modules with periodic of period 2 free resolutions over any quotient of a local ring $(R, \mathfrak{m})$ by a non-zero-divisor in $\mathfrak{m}^2$, a fact noted independently by Herzog *et al.* [25].

In this section we produce a plentiful supply of modules of finite CI-dimension and infinite projective dimension over any quotient of a noetherian ring by a decomposable, in a very weak sense, regular sequence. The input comes from different sources: the technique of quantum regular sequences developed in Section 2; a procedure of Shamash [33] and Eisenbud [18] that yields an R-free resolution starting from a Q-free one; and, to a much larger extent than [16], the rich internal structure of exterior algebras.

(3.1) *Theorem.* — *Let Q be a noetherian ring, let $\mathfrak{n} = (a_1, \dots, a_m)$ be a proper ideal in Q, and let $\mathbf{x} = x_1, \dots, x_c$ be a Q-regular sequence contained in $\mathfrak{n}^2$.*

Over $R = Q/(\mathbf{x})$ there exists a sequence of modules $\{M_r^{[c]}\}_{r \in \mathbf{Z}}$ such that whenever $s > r$ the module $M_s^{[c]}$ is an $(s-r)$-th syzygy of $M_r^{[c]}$, and for each $r \in \mathbf{Z}$ there are equalities $\operatorname{pd}_Q(M_r^{[c]}) = c$ and $\text{CI-dim}_R(M_r^{[c]}) = 0$. If Q is local, then also

$$b_n^Q(M_0^{[c]}) = 2^{cm}\binom{c}{n} \quad \textit{for } n \in \mathbf{Z},$$

$$\textit{and} \quad b_n^R(M_r^{[c]}) = \begin{cases} 2^{cm}\dbinom{-n-r+c-2}{c-1} & \textit{for } 0 \leqslant n < -r; \\[2ex] 2^{cm}\dbinom{n+r+c-1}{c-1} & \textit{for } n \geqslant \max\{0, -r\}. \end{cases}$$

The modules above are obtained from an unbounded complex of free R-modules $\mathbf{T}$. We denote by $\Sigma\mathbf{T}$ the *shifted complex*: $(\Sigma\mathbf{T})_n = \mathrm{T}_{n-1}$, and $\partial_n^{\Sigma\mathbf{T}} = -\partial_{n-1}^{\mathbf{T}}$.

(3.2) *Theorem.* — *Let* $\mathfrak{a} = (a_1, \ldots, a_m)$ *and* $\mathfrak{b}$ *be ideals in a commutative ring* Q, *and let* $\mathbf{x} = x_1, \ldots, x_c$ *be a Q-regular sequence contained in* $\mathfrak{ab}$. *Over the ring* $\mathrm{R} = \mathrm{Q}/(\mathbf{x})$ *there exists an exact complex of free modules* $(\mathbf{T}, \partial)$ *such that* $\partial(\mathbf{T}) \subseteq (\mathfrak{a} + \mathfrak{b})\,\mathbf{T}$ (*and hence* $\mathbf{T}$ *is not split if* $\mathfrak{a} + \mathfrak{b} \neq \mathrm{Q}$), *there is a chain isomorphism* $\mathrm{Hom}_{\mathrm{R}}(\mathbf{T}, \mathrm{R}) \cong \Sigma\mathbf{T}$, *and*

$$\mathrm{rank}_{\mathrm{R}}\,\mathrm{T}_n = \begin{cases} 2^{cm}\dbinom{n+c-1}{c-1} & \textit{for } n \geqslant 0; \\[2ex] 2^{cm}\dbinom{-n+c-2}{c-1} & \textit{for } n < 0. \end{cases}$$

Furthermore, $\mathrm{M}(\mathbf{a}, \mathbf{x}) = \mathrm{Im}\,\partial_0$ *has a Q-free resolution* $(\mathbf{K}, \partial)$ *with* $\partial(\mathbf{K}) \subseteq (\mathfrak{a} + \mathfrak{b})\,\mathbf{K}$ *and*

$$\mathrm{rank}_{\mathrm{Q}}\,\mathrm{K}_n = 2^{cm}\binom{c}{n} \quad \textit{for } n \in \mathbf{Z}.$$

We present various stages of the construction of the complex $\mathbf{T}$ in a series of steps.

(3.3) *The module* $\mathrm{M}(\mathbf{a}, \mathbf{x})$. — Write $x_j = \sum_{r=1}^m b_{jr}\,a_r$ with $b_{jr} \in \mathfrak{b}$ for $1 \leqslant j \leqslant c$. Let E be the free module of rank 2^{cm} underlying the (non-graded) exterior algebra $\bigoplus_{h=0}^{cm} \wedge^h \mathrm{L}$ on a Q-module L with basis $e_1, \ldots, e_{cm}$. Let ε_h be left wedge multiplication by e_h on E and let θ_h be the Q-linear skew derivation of E with $\theta_h(e_h) = 1$ and $\theta_h(e_i) = 0$ for $i \neq h$. Set

$$\partial_j = \sum_{r=1}^{m} a_r\,\theta_{(j-1)m+r}, \qquad \delta_j = \sum_{r=1}^{m} b_{jr}\,\varepsilon_{(j-1)m+r},$$

and consider the sequence $\boldsymbol{\varphi} = \varphi_1, \ldots, \varphi_c$ of endomorphisms of E given by $\varphi_j = \partial_j + \delta_j$.

Set $\mathrm{M}(\mathbf{a}, \mathbf{x}) = \mathrm{E}/(\boldsymbol{\varphi})\,\mathrm{E}$. As a module over Q, it has a free presentation

$$\mathrm{E}^c \xrightarrow{(\varphi_1, \ldots, \varphi_c)} \mathrm{E} \xrightarrow{\pi} \mathrm{M}(\mathbf{a}, \mathbf{x}) \longrightarrow 0,$$

where $\psi = (\varphi_1, \ldots, \varphi_c)$ and π is the canonical projection. The inclusions

$$(\mathbf{x})\,\mathrm{E} = (x_1, \ldots, x_c)\,\mathrm{E} = (\varphi_1^2, \ldots, \varphi_c^2)\,\mathrm{E} \subseteq (\varphi_1, \ldots, \varphi_c)\,\mathrm{E} = (\boldsymbol{\varphi})\,\mathrm{E}$$

show that $\mathrm{M}(\mathbf{a}, \mathbf{x}) = \mathrm{E}/(\boldsymbol{\varphi})\,\mathrm{E}$ has a natural structure of R-module.

When $c = 1$ we recover the construction of [16; (2.3)].

(3.4) *Lemma.* — *The sequence of endomorphisms* $\boldsymbol{\varphi}$ *is quantum commuting, with*

$$\varphi_i\,\varphi_j = -\varphi_j\,\varphi_i \qquad \varphi_j^2 = x_j\,\mathrm{id}_{\mathrm{E}} \qquad \varphi_j(\mathrm{E}) \subseteq (\mathfrak{a} + \mathfrak{b})\,\mathrm{E}$$

for $1 \leqslant i, j \leqslant c$ *with* $i \neq j$. *If the sequence* $\mathbf{x}$ *is Q-regular, then* $\boldsymbol{\varphi}$ *is* E-*regular.*

Proof. — The inclusions $\varphi_j(\mathrm{E}) \subseteq (\mathfrak{a} + \mathfrak{b})\,\mathrm{E}$ are clear from the construction. The multiplication table for the φ_j's easily follows from the relations

$$\partial_j \partial_j = 0 \qquad \delta_j \delta_j = 0 \qquad \partial_j \delta_j + \delta_j \partial_j = x_j \,\mathrm{id}_{\mathrm{L}}$$
$$\partial_i \partial_j = -\partial_j \partial_i \qquad \delta_i \delta_j = -\delta_j \delta_i \qquad \partial_j \delta_i = -\delta_i \partial_j$$

that hold for $1 \leqslant i, j \leqslant c$ with $i \neq j$. Indeed, the first column results from the fact that both $\partial_i \partial_j + \partial_j \partial_i$ and $\partial_j \partial_j$ are degree -2 derivations of E which vanish on the algebra generators, hence are identically zero; the second one reflects the strict skew-commutativity of the exterior algebra; the third one follows from the next computation, in which $f_j = \sum_{r=1}^{m} b_{jr} \varepsilon_{(j-1)m+r}$ and e stands for an arbitrary element of E:

$$\partial_j \delta_i(e) + \delta_i \partial_j(e) = \partial_j(f_i \wedge e) + f_i \wedge \partial_j(e) = \partial_j(f_i)\, e = \begin{cases} x_j e & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

If $\mathbf{x}$ is Q-regular, then the E-regularity of $\boldsymbol{\varphi}$ follows from (2.1) and (2.3.2). □

(3.5) *Grassmann duality.* — An excellent reference for this is [14; § 11], from where we borrow a sign convention: for subsets J and K of $\mathrm{I} = \{1, \ldots, cm\}$, set $\rho_{\mathrm{JK}} = (-1)^p$, where p equals the number of pairs $(j, k) \in \mathrm{J} \times \mathrm{K}$ with $j > k$. When $\mathrm{J} = \{j_1, \ldots, j_s\}$, with elements listed in increasing order, set $e_{\mathrm{J}} = e_{j_1} \wedge \ldots \wedge e_{j_s} \in \bigwedge \mathrm{L}$. Note that $\{e_{\mathrm{J}}\}_{\mathrm{J} \subseteq \mathrm{I}}$ is a Q-basis of E, and let $\{e^*_{\mathrm{K}}\}_{\mathrm{K} \subseteq \mathrm{I}}$ be the dual basis of $\mathrm{E}^* = \mathrm{Hom}_{\mathrm{Q}}(\mathrm{E}, \mathrm{Q})$.

The Q-linear homomorphism $\omega : \mathrm{E} \to \mathrm{E}^*$ defined in these bases by

$$\omega(e_{\mathrm{J}}) = (-1)^{\binom{s}{2}} \rho_{\mathrm{JK}}\, e^*_{\mathrm{K}}, \qquad \text{where } \mathrm{K} = \mathrm{I} \setminus \mathrm{J},$$

is clearly bijective. Furthermore, it is symmetric or antisymmetric, in the sense that

$$\textbf{(3.5.1)} \qquad \omega = (-1)^{\binom{cm}{2}} \omega^* \beta_{\mathrm{E}},$$

where $\omega^* = \mathrm{Hom}_{\mathrm{Q}}(\omega, \mathrm{Q})$ and $\beta_{\mathrm{E}} : \mathrm{E} \to \mathrm{E}^{**}$ is the canonical biduality map. To see it, use

$$\omega(e_{\mathrm{J}}) = (-1)^{\binom{s}{2}} \rho_{\mathrm{JK}}\, e^*_{\mathrm{K}}, \qquad \omega^* \beta_{\mathrm{E}}(e_{\mathrm{J}}) = (-1)^{\binom{cm-s}{2}} \rho_{\mathrm{KJ}}\, e^*_{\mathrm{K}},$$

in conjunction with the relation $\rho_{\mathrm{JK}} \rho_{\mathrm{KJ}} = (-1)^{s(cm-s)}$ and the congruence

$$\binom{s}{2} + \binom{cm}{2} + \binom{cm-s}{2} + s(cm-s) \equiv 0 \bmod (2).$$

Next we exhibit a beautiful reflexivity of $\mathrm{M}(\mathbf{a}, \mathbf{x})$. With bars denoting reduction modulo $(\mathbf{x})$ set $\mathrm{F} = \overline{\mathrm{E}} = \mathrm{E}/(\mathbf{x})\,\mathrm{E}$ and $\overline{\mathrm{E}^*} = \mathrm{F}^* = \mathrm{Hom}_{\mathrm{R}}(\mathrm{F}, \mathrm{R})$ via the canonical isomorphism.

(3.6) *Lemma.* — *The homomorphism* $\tau = \omega \prod_{j=1}^{c} \varphi_j : \mathrm{E} \to \mathrm{E}^*$ *induces a homomorphism*

$$\alpha : \mathrm{M}(\mathbf{a}, \mathbf{x}) \to \mathrm{M}(\mathbf{a}, \mathbf{x})^*$$

of R*-modules, that is bijective if the sequence* $\mathbf{x}$ *is* Q*-regular.*

Proof. — For a start, note that for $1 \leqslant i, j \leqslant c$ and $\varphi_j^* = \mathrm{Hom}_Q(\varphi_j, Q)$ there are equalities

$$(\bar{\varphi}_j)^2 = 0, \qquad \bar{\varphi}_i \, \bar{\varphi}_j = - \bar{\varphi}_j \, \bar{\varphi}_i, \qquad \bar{\omega}\bar{\varphi}_j = \bar{\varphi}_j^* \, \bar{\omega}. \tag{3.6.1}$$

The first two are clear from (3.4). For the third, consider θ_h from (3.3) and set $\theta_j^* = \mathrm{Hom}_Q(\theta_j, Q)$. If $k_1, \ldots, k_t$ are the elements of K listed in increasing order, then $\omega\theta_h(e_J) = 0 = \theta_h^* \, \omega(e_J)$ for $h \notin J$, while for $h = j_{u+1} \in J$ with $k_v < h < k_{v+1}$ we get

$$\begin{aligned} \omega\theta_h(e_J) &= (-1)^{u + \binom{s-1}{2}} \rho_{(J \setminus h)(K \cup h)} \, e^*_{K \cup h} \\ &= (-1)^{u + \binom{s}{2} - (s-1)} (-1)^{(s-u-1)+v} \rho_{JK} \, e^*_{K \cup h} \\ &= (-1)^{v + \binom{s}{2}} \rho_{JK} \, e^*_{K \cup h} = \theta_h^* \, \omega(e_J). \end{aligned}$$

Thus, $\omega\theta_j = \theta_j^* \, \omega$. By a similar computation, $\omega\varepsilon_h(e_J) = \varepsilon_h^* \, \omega(e_J)$ for $\varepsilon_h^* = \mathrm{Hom}_Q(\varepsilon_h, Q)$.

With homomorphisms $\psi : E^c \to E$ and $\xi : E \to E^c$ defined by

$$\psi(e_1, \ldots, e_c) = \varphi_1(e_1) + \ldots + \varphi_c(e_c) \quad \text{and} \quad \xi(e) = (\varphi_1(e), \ldots, \varphi_c(e)),$$

we now have a diagram of R-modules

$$\begin{tikzcd} F^c \arrow[r, "\bar{\psi}"] \arrow[d, "\bar{\omega}^c"', "\cong"] & F \arrow[r, "\prod_{j=1}^c \bar{\varphi}_j"] \arrow[d, "\bar{\omega}", "\cong"'] \arrow[dr, "\bar{\tau}"] & F \arrow[r, "\bar{\xi}"] \arrow[d, "\bar{\omega}"', "\cong"] & F^c \arrow[d, "\bar{\omega}^c", "\cong"'] \\ (F^*)^c \arrow[r, "\bar{\xi}^*"'] & F^* \arrow[r, "\prod_{j=1}^c \bar{\varphi}_j^*"'] & F^* \arrow[r, "\bar{\psi}^*"'] & (F^*)^c \end{tikzcd} \tag{3.6.2}$$

From (3.6.1) we see that the diagram is commutative and that its rows are complexes of R-modules. It follows that $\bar{\tau}$ induces the desired homomorphism

$$\alpha : M(\mathbf{a}, \mathbf{x}) = \mathrm{Coker}\, \psi = \mathrm{Coker}\, \bar{\psi} \to \mathrm{Ker}\, \bar{\psi}^* = M(\mathbf{a}, \mathbf{x})^*.$$

The sequences $\boldsymbol{\varphi} \subset \mathrm{End}_R(E)$ and $\boldsymbol{\varphi}^* = \varphi_1^*, \ldots, \varphi_c^* \subset \mathrm{End}_R(E^*)$ quantum commute and the squares of their elements are homotheties by $x_1, \ldots, x_c$ by (3.4). Thus, $\boldsymbol{\varphi}$ and $\boldsymbol{\varphi}^*$ are regular by (2.1), so (2.6.2) has exact rows by (2.4) and (2.2.2), hence α is bijective. □

We study the homological algebra of $M(\mathbf{a}, \mathbf{x})$, first over the ring Q.

(3.7) *Lemma.* — *If the sequence* $\mathbf{x}$ *is Q-regular, then the quantum Koszul complex* $\mathbf{K}(\boldsymbol{\varphi}; E)$ *is a complex of free Q-modules which satisfies*

$$H_0(\boldsymbol{\varphi}; E) = M(\mathbf{a}, \mathbf{x}) \quad \textit{and} \quad H_n(\boldsymbol{\varphi}; E) = 0 \;\; \textit{for } n > 0$$

$$\partial(\mathbf{K}(\boldsymbol{\varphi}; E)) \subseteq (\mathfrak{a} + \mathfrak{b})\, \mathbf{K}(\boldsymbol{\varphi}; E);$$

$$\mathrm{rank}_Q\, K_n = 2^{cm} \binom{c}{n} \;\; \textit{for } n \in \mathbf{Z}.$$

In particular, $\mathrm{pd}_Q\, M(\mathbf{a}, \mathbf{x}) \leqslant c$, *and equality holds when* $\mathfrak{a} + \mathfrak{b} \neq Q$.

Proof. — By (2.3), we have $\mathbf{K}(\boldsymbol{\varphi};\mathrm{E})^{\natural} = \mathbf{Y}\otimes_{\mathrm{Q}}\mathrm{E}$ where $\mathbf{Y}$ is a graded Q-module whose n-th homogeneous component has a basis $\{y_{\mathrm{H}}\}$ indexed by the n-element subsets H of $\{1, \dots, cm\}$. Thus, $\mathbf{K}(\boldsymbol{\varphi};\mathrm{E})$ is a complex of free modules with $\operatorname{rank}_{\mathrm{Q}}\mathrm{K}_n = 2^{cm}\binom{c}{n} = q_n$. As $\boldsymbol{\varphi}$ is E-regular by (3.4), $\mathbf{K}(\boldsymbol{\varphi};\mathrm{E})$ is a Q-free resolution of $\mathrm{M}(\mathbf{a},\mathbf{x})$ by (2.3.3), so $\operatorname{pd}_{\mathrm{Q}}\mathrm{M}(\mathbf{a},\mathbf{x})\leqslant c$. Now (2.3.1) and (3.3) yield

$$\partial(\mathbf{K}(\boldsymbol{\varphi};\mathrm{E}))\subseteq(\mathfrak{a}+\mathfrak{b})\,\mathbf{K}(\boldsymbol{\varphi};\mathrm{E}),$$

hence

$$\operatorname{Ext}^n_{\mathrm{Q}}(\mathrm{M}(\mathbf{a},\mathbf{x}),\mathrm{Q}/(\mathfrak{a}+\mathfrak{b}))\cong(\mathrm{Q}/(\mathfrak{a}+\mathfrak{b}))^{q_n},$$

and the last module is different from zero when $\mathfrak{a}+\mathfrak{b}\neq\mathrm{Q}$ and $0\leqslant n\leqslant c$. □

In order to produce an R-free resolution of $\mathrm{M}(\mathbf{a},\mathbf{x})$ starting from its Q-free resolution $\mathbf{K}(\boldsymbol{\varphi};\mathrm{E})$, we use a construction of Shamash and Eisenbud, that we recall below.

(3.8) *Higher homotopies.* — We use the multi-grading convention from (2.2) and abuse notation by writing (0) for $(0,\dots,0)\in\mathbf{N}^c$, and (j) for the c-tuple $(0,\dots,0,1,0,\dots,0)$ in which 1 appears in the j-th place. Let $(\mathbf{E},\partial)$ be a complex of Q-modules. A family $\boldsymbol{\sigma}=\{\sigma^{\mathrm{J}}\in\operatorname{Hom}_{\mathrm{Q}}(\mathbf{E},\mathbf{E})_{2|\mathrm{J}|-1}\}_{\mathrm{J}\in\mathbf{N}^c}$ is called a *system of higher homotopies* on $\mathbf{E}$ if

$$\sigma^{(0)}=\partial;$$
$$\sigma^{(0)}\sigma^{(j)}+\sigma^{(j)}\sigma^{(0)}=x_j\,\mathrm{id}_{\mathbf{E}}\quad\text{for } 1\leqslant j\leqslant c;$$
$$\sum_{\mathrm{J}'+\mathrm{J}''=\mathrm{J}}\sigma^{\mathrm{J}'}\sigma^{\mathrm{J}''}=0\quad\text{for } \mathrm{J}\in\mathbf{N}^c\ \text{ with } |\mathrm{J}|\geqslant 2.$$

Partly responsible for the name is the following observation: the first two conditions above imply that for $j=1,\dots,c$ the endomorphism $\sigma^{(j)}$ of $\mathbf{E}$ is a homotopy between multiplication by x_j and the zero map. In particular, $(\mathbf{x})$ annihilates $\mathrm{H}_*\mathbf{E}$. Conversely, if M is a module over $\mathrm{R}=\mathrm{Q}/(\mathbf{x})$, and $(\mathbf{E},\partial)$ is a free resolution of M as a Q-module, then basic homological algebra exhibits a family of homotopies $\{\sigma^{(j)}: x_j\,\mathrm{id}_{\mathbf{E}}\sim 0_{\mathbf{E}}\}_{1\leqslant j\leqslant c}$. If $(\mathbf{x})$ is a proper ideal which contains a non-zero-divisor, then by [18; (8.1)] this family can be extended to a system $\boldsymbol{\sigma}=\{\sigma^{\mathrm{J}}\}_{\mathrm{J}\in\mathbf{N}^c}$ of higher homotopies for $\mathbf{x}$.

(3.9) *Lemma.* — *The complex* $\mathbf{K}(\boldsymbol{\varphi};\mathrm{E})$ *admits a system* $\boldsymbol{\sigma}$ *of higher homotopies such that*

$$\sigma^{(j)}\sigma^{(j)}=0\quad\textit{and}\quad\sigma^{(i)}\sigma^{(j)}=-\sigma^{(j)}\sigma^{(i)}\quad\textit{for } 1\leqslant i\leqslant j\leqslant c;$$
$$\sigma^{\mathrm{J}}(\mathbf{K}(\boldsymbol{\varphi};\mathrm{E}))\subseteq(\mathfrak{a}+\mathfrak{b})\,\mathbf{K}(\boldsymbol{\varphi};\mathrm{E})\quad\textit{when } |\mathrm{J}|\leqslant 1,$$

and

$$\sigma^{\mathrm{J}}=0\quad\textit{when } |\mathrm{J}|\geqslant 2.$$

Proof. — With the notation of the proof of (3.7), for $j=1,\dots,c$ set

$$\sigma^{(j)}(y_{\mathrm{H}}\otimes u)=\begin{cases} y_{\mathrm{H}\cup j}\otimes\varphi_j(u) & \text{if } j\notin\mathrm{H};\\ 0 & \text{if } j\in\mathrm{H}.\end{cases}$$

A direct computation, that uses (2.3.1) and the anticommutativity (3 4) of the sequence $\boldsymbol{\varphi}$, shows that the degree 1 endomorphism $\sigma^{(j)}$ of $\mathbf{K}(\boldsymbol{\varphi};\mathrm{E})=\mathbf{Y}\otimes_{\mathrm{Q}}\mathrm{E}$ is a square zero homotopy from x_j id to the zero map, and that it anticommutes with $\sigma^{(i)}$. It is clear that such a set of homotopies can be extended to a system $\boldsymbol{\sigma}$ of higher homotopies on $\mathbf{K}(\boldsymbol{\varphi}; \mathrm{E})$ by setting $\sigma^{\mathrm{J}} = 0$ when $|\mathrm{J}| \geqslant 2$. For $|\mathrm{J}| = 1$ we have $\sigma^{\mathrm{J}}(\mathbf{K}(\boldsymbol{\varphi}; \mathrm{E})) \subseteq (\mathfrak{a} + \mathfrak{b})\, \mathbf{K}(\boldsymbol{\varphi}; \mathrm{E})$ by (3.3) and the definition above. The desired inclusion for the $\sigma^{(0)}$ comes from (3.7). □

(3.10) *Standard resolutions.* — Let $\mathrm{Q}[\chi] = \mathrm{Q}[\chi_1, \ldots, \chi_c]$ be a polynomial ring graded by assigning to the indeterminates $\chi_1, \ldots, \chi_c$ *lower* degree -2, let $\boldsymbol{\sigma}$ be a system of higher homotopies on a bounded below complex of Q-modules, $\mathbf{E}$, and set $\widetilde{\mathbf{E}}\{\boldsymbol{\sigma}\} = \mathrm{Hom}_{\mathrm{Q}}(\mathrm{Q}[\chi], \mathbf{E})$.

For $\gamma \in \mathrm{Hom}_{\mathrm{Q}}(\mathrm{Q}[\chi], \mathbf{E})$, $y \in \mathrm{Q}[\chi]$, and $\chi^{\mathrm{J}} = \chi_1^{j_1} \ldots \chi_c^{j_c}$, all but a finite number of summands in the expression $\sum_{\mathrm{J} \in \mathbf{N}^c} \sigma^{\mathrm{J}}\, \gamma(\chi^{\mathrm{J}} y)$ vanish for degree reasons. Thus,

$$\textbf{(3.10.1)} \qquad \widetilde{\partial} = \sum_{\mathrm{J} \in \mathbf{N}^c} \sigma^{\mathrm{J}} \circ \mathrm{Hom}_{\mathrm{Q}}(\chi^{\mathrm{J}}, \mathbf{E}) : \widetilde{\mathbf{E}}\{\boldsymbol{\sigma}\} \to \widetilde{\mathbf{E}}\{\boldsymbol{\sigma}\}$$

is a degree -1 homomorphism of graded Q-modules. A direct computation shows that

$$\textbf{(3.10.2)} \qquad (\mathbf{E}\{\boldsymbol{\sigma}\}, \partial) = (\widetilde{\mathbf{E}}\{\boldsymbol{\sigma}\} \otimes_{\mathrm{Q}} \mathrm{R}, \widetilde{\partial} \otimes_{\mathrm{Q}} \mathrm{R})$$

is a DG module over the graded ring $\mathrm{R}[\chi] = \mathrm{Q}[\chi] \otimes_{\mathrm{Q}} \mathrm{R}$.

If $\mathbf{E}$ is a complex of free Q-modules, then $\mathbf{E}\{\boldsymbol{\sigma}\}$ is one of free R-modules, and

$$\textbf{(3.10.3)} \qquad \mathrm{Hilb}^{\mathrm{R}}_{\mathbf{E}\{\boldsymbol{\sigma}\}}(t) = \mathrm{Hilb}^{\mathrm{Q}}_{\mathbf{E}}(t)\, \mathrm{Hilb}^{\mathrm{Q}}_{\mathrm{Q}[\chi]}(t) = \mathrm{Hilb}^{\mathrm{Q}}_{\mathbf{E}}(t)\, (1 - t^2)^{-c},$$

where $\mathrm{Hilb}^{\mathrm{Q}}_{\mathbf{E}}(t)$ denotes the generating function $\sum_{n \in \mathbf{Z}} (\mathrm{rank}_{\mathrm{Q}}\, \mathrm{E}_n)\, t^n$.

The basic property of this construction is given by [18; (7.2)], that extends [33; § 3]:

(3.10.4) *There is a natural isomorphism* $\mathrm{H}_0\, \mathbf{E}\{\boldsymbol{\sigma}\} \cong \mathrm{M}$. *If* $\mathbf{E}$ *is a* Q*-free resolution of* M, *and the sequence* $\mathbf{x}$ *is* Q*-regular, then* $\mathbf{E}\{\boldsymbol{\sigma}\}$ *is an* R*-free resolution of* M. □

A resolution $\mathbf{E}\{\boldsymbol{\sigma}\}$ as above is called a *standard* R*-free resolution* of M constructed from a system of higher homotopies $\boldsymbol{\sigma}$ on a Q-free resolution $\mathbf{E}$. Now we describe the homological algebra of $\mathrm{M}(\mathbf{a}, \mathbf{x})$ as an R-module.

(3.11) *Lemma.* — *If the sequence* $\mathbf{x}$ *is* Q*-regular,* $\mathbf{E} = \mathbf{K}(\boldsymbol{\varphi}; \mathrm{E})$ *is the quantum Koszul complex, and* $\{\boldsymbol{\sigma}\}$ *is its system of higher homotopies* (3.9), *then the complexes of free* R*-modules* $\mathbf{F} = \mathbf{E}\{\boldsymbol{\sigma}\}$ *and* $\mathbf{F}^* = \mathrm{Hom}_{\mathrm{R}}(\mathbf{F}, \mathrm{R})$ *satisfy*

$$\mathrm{H}_0\, \mathbf{F} = \mathrm{M}(\mathbf{a}, \mathbf{x}); \qquad \mathrm{H}_0(\mathbf{F}^*) = \mathrm{M}(\mathbf{a}, \mathbf{x})^*;$$
$$\mathrm{H}_n\, \mathbf{F} = 0 \ \ \textit{for}\ n > 0; \qquad \mathrm{H}_n(\mathbf{F}^*) = 0 \ \ \textit{for}\ n < 0;$$
$$\partial(\mathbf{F}) \subseteq (\mathfrak{a} + \mathfrak{b})\, \mathbf{F}; \qquad \partial(\mathbf{F}^*) \subseteq (\mathfrak{a} + \mathfrak{b})\, \mathbf{F}^*;$$
$$\mathrm{rank}_{\mathrm{R}}\, \mathrm{F}_n = 2^{cm} \binom{n + c - 1}{c - 1} \ \ \textit{for}\ n \geqslant 0;$$
$$\mathrm{rank}_{\mathrm{R}}\, \mathrm{F}^*_n = 2^{cm} \binom{-n + c - 1}{c - 1} \ \ \textit{for}\ n \leqslant 0.$$

In particular, $\mathrm{pd}_{\mathrm{R}}\, \mathrm{M}(\mathbf{a}, \mathbf{x}) = \infty$ *when* $\mathfrak{a} + \mathfrak{b} \neq \mathrm{Q}$.

Proof. — The preceding proposition computes the homology of $\mathbf{F}$.

It follows that $\mathrm{H}_0(\mathbf{F}^*) = \mathrm{M}(\mathbf{a}, \mathbf{x})^*$, and that $\mathrm{H}_{-n}(\mathbf{F}^*) \cong \mathrm{Ext}^n_{\mathrm{R}}(\mathrm{M}(\mathbf{a}, \mathbf{x}), \mathrm{R})$ for $n > 0$. As the Q-regular sequence $\mathbf{x}$ of length c annihilates $\mathrm{M}(\mathbf{a}, \mathbf{x})$, (1.9.4) yields

$$\mathrm{Ext}^n_{\mathrm{R}}(\mathrm{M}(\mathbf{a}, \mathbf{x}), \mathrm{R}) \cong \mathrm{Ext}^{n+c}_{\mathrm{Q}}(\mathrm{M}(\mathbf{a}, \mathbf{x}), \mathrm{Q}) \quad \text{for } n \in \mathbf{Z}.$$

As $\mathrm{pd}_{\mathrm{Q}}\, \mathrm{M}(\mathbf{a}, \mathbf{x}) \leqslant c$ by (3.7), these modules vanish when $n > 0$.

In view of (3.7), the equality of power series (3.10.3) becomes

$$\mathrm{Hilb}^{\mathrm{R}}_{\mathbf{F}}(t) = 2^{cm}(1+t)^c(1-t^2)^{-c} = 2^{cm}(1-t)^{-c},$$

which is another way to say that $r_n = \mathrm{rank}_{\mathrm{R}}\, \mathrm{F}_n$ is equal to $2^{cm}\binom{n+c-1}{c-1}$ for $n \in \mathbf{Z}$.

Now (3.10.1), (3.10.2), (3.9) imply $\partial(\mathbf{F}) \subseteq (\mathfrak{a} + \mathfrak{b})\,\mathbf{F}$, hence $\partial(\mathbf{F}^*) \subseteq (\mathfrak{a} + \mathfrak{b})\,\mathbf{F}^*$. Thus, $\mathrm{Ext}^n_{\mathrm{R}}(\mathrm{M}(\mathbf{a}, \mathbf{x}), \mathrm{R}/(\mathfrak{a} + \mathfrak{b})) \cong (\mathrm{R}/(\mathfrak{a} + \mathfrak{b}))^{r_n}$, so $\mathrm{pd}_{\mathrm{R}}\, \mathrm{M}(\mathbf{a}, \mathbf{x}) = \infty$ when $\mathfrak{a} + \mathfrak{b} \neq \mathrm{Q}$. $\square$

We are finally ready to assemble the information obtained so far for the

Proof of Theorem (3.2). — Identify the differential ∂_1 of the complex $(\mathbf{F}, \partial)$ with the homomorphism $\bar{\psi} : \mathrm{F}^c \to \mathrm{F}$ of (3.3). Accordingly, identify the differential ∂''_0 of $(\mathbf{F}, \partial)^*$ with $-\bar{\psi}^* : \mathrm{F}^* \to (\mathrm{F}^c)^*$. As $\bar{\tau}\,\partial_1 = 0 = \partial''_0\,\bar{\tau}$ due to the commutativity of (3.6.2), setting

$$(\mathbf{T}, \partial)_{\geqslant 0} = (\mathbf{F}, \partial) \qquad \partial_0 = \bar{\tau} \qquad (\mathbf{T}, \partial)_{\leqslant -1} = \Sigma^{-1}((\mathbf{F}, \partial)^*)$$

we obtain a complex of free R-modules $(\mathbf{T}, \partial)$.

By inception, it comes along with an exact sequence of complexes

$$0 \to \mathbf{F} \to \mathbf{T} \to \Sigma^{-1}(\mathbf{F}^*) \to 0.$$

The associated long homology exact sequence degenerates to isomorphisms

$$\mathrm{H}_n\,\mathbf{T} \cong \begin{cases} \mathrm{H}_n\,\mathbf{F} & \text{for } n \geqslant 1; \\ \mathrm{H}_{n+1}\,\mathbf{F}^* & \text{for } n \leqslant -2, \end{cases}$$

and an exact sequence $0 \to \mathrm{H}_0\,\mathbf{T} \to \mathrm{M}(\mathbf{a}, \mathbf{x}) \xrightarrow{\alpha} \mathrm{M}(\mathbf{a}, \mathbf{x})^* \to \mathrm{H}_{-1}\,\mathbf{T} \to 0$. The exactness of $\mathbf{T}$ now follows from (3.6) and (3.11). The latter statement also yields the desired expression for $t_n = \mathrm{rank}_{\mathrm{R}}\,\mathrm{T}_n$, and an inclusion $\partial(\mathbf{T}) \subseteq (\mathfrak{a} + \mathfrak{b})\,\mathbf{T}$. Thus, with $\mathrm{N} = \mathrm{Q}/(\mathfrak{a} + \mathfrak{b})$ we get $\mathrm{H}_n(\mathbf{T} \otimes_{\mathrm{Q}} \mathrm{N}) \cong \mathrm{N}^{t_n}$ for and $n \in \mathbf{Z}$, so $\mathbf{T}$ is not split if $\mathfrak{a} + \mathfrak{b} \neq \mathrm{Q}$.

It remains to see that $\mathbf{T}$ is self-dual up to shift. We have expressions

$$\partial'_n = \begin{cases} -\partial_{n-1} & \text{if } n > 1; \\ -\bar{\tau} & \text{if } n = 1; \\ (-1)^{1-n}\,\partial^*_{1-n} & \text{if } n < 1; \end{cases} \qquad \partial''_n = \begin{cases} -\partial^{**}_{n-1} & \text{if } n > 1; \\ \bar{\tau}^* & \text{if } n = 1; \\ (-1)^{1-n}\,\partial^*_{1-n} & \text{if } n < 1, \end{cases}$$

for the differentials ∂' of $\Sigma\mathbf{T}$ and ∂'' of $\mathrm{Hom}_{\mathrm{R}}(\mathbf{T}, \mathrm{R})$. Using (3.5.1) and (3.6.1) we get

$$\bar{\tau}^* \beta_{\mathrm{F}} = \prod_{j=0}^{c-1} \bar{\varphi}^*_{c-j}\, \bar{\omega}^* \beta_{\mathrm{F}} = (-1)^{\binom{cm}{2}} \prod_{j=0}^{c-1} \bar{\varphi}^*_{c-j}\, \bar{\omega}$$

$$= (-1)^{\binom{cm}{2}}\, \bar{\omega} \prod_{j=0}^{c-1} \bar{\varphi}_{c-j} = (-1)^{\binom{cm}{2}+\binom{c}{2}}\, \bar{\tau}.$$

It follows that the maps $\zeta_{n+1} : \mathrm{T}_n \to \mathrm{Hom}_{\mathrm{R}}(\mathrm{T}_{-n}, \mathrm{R})$, defined by

$$\zeta_{n+1} = \begin{cases} (-1)^{1+\binom{c}{2}+\binom{cm}{2}}\, \beta_{\mathrm{T}_n} & \text{when } n > 0; \\ \mathrm{id}_{\mathrm{T}_n} & \text{when } n \leqslant 0, \end{cases}$$

provide a chain isomorphism of complexes $\zeta : \Sigma\mathbf{T} \to \mathrm{Hom}_{\mathrm{R}}(\mathbf{T}, \mathrm{R})$.

The proof of Theorem (3.2) is now complete. □

Proof of Theorem (3.1). — By (1.6) we may assume R is local. The formulas for the Betti numbers of $\mathrm{M}_r = \mathrm{M}_r^{[c]}$ follow from (3.2) applied with $\mathfrak{a} = \mathfrak{b} = \mathfrak{n}$. In particular, the Q-module M_0 has finite projective dimension. This conclusion propagates to all M_r, as M_0 is a syzygy of M_r when $r < 0$, and has M_r as a syzygy when $r > 0$. Since each M_r is an infinite syzygy over R, we have $\mathrm{depth}_{\mathrm{R}}\, \mathrm{M}_r = \mathrm{depth}\, \mathrm{R}$, and consequently $\mathrm{pd}_{\mathrm{Q}}\, \mathrm{M}_r = \mathrm{pd}_{\mathrm{Q}}\, \mathrm{R} = c$. It follows that $\text{CI-dim}_{\mathrm{R}}\, \mathrm{M}_r$ is finite for all r, and hence zero by (1.9.1). □

4. Cohomology operators

In this section Q denotes a commutative ring, $\mathbf{x} = x_1, \ldots, x_c$ is a Q-regular sequence, $\rho : \mathrm{Q} \to \mathrm{Q}/(\mathbf{x}) = \mathrm{R}$ is the canonical projection, and M, N are R-modules.

(4.1) *Eisenbud operators.* — If $(\mathbf{F}, \partial)$ is a free resolution of a module M over the ring $\mathrm{R} = \mathrm{Q}/(\mathbf{x})$, then a pair $(\widetilde{\mathbf{F}}, \widetilde{\partial})$ consisting of a graded free Q-module $\widetilde{\mathbf{F}}$ and its degree -1 endomorphism $\widetilde{\partial}$, such that $(\mathbf{F}, \partial) = (\widetilde{\mathbf{F}} \otimes_{\mathrm{Q}} \mathrm{R}, \widetilde{\partial} \otimes_{\mathrm{Q}} \mathrm{R})$, is called a *lifting* of the complex $(\mathbf{F}, \partial)$. The relation $\partial^2 = 0$ yields an inclusion $\widetilde{\partial}^2(\widetilde{\mathbf{F}}) \subseteq (\mathbf{x})\, \widetilde{\mathbf{F}}$, hence for $1 \leqslant j \leqslant c$ there are endomorphisms $\widetilde{t}_j(\mathrm{Q}, \mathbf{x}, \widetilde{\mathbf{F}}, \widetilde{\partial}) \in \mathrm{Hom}^2_{\mathrm{Q}}(\widetilde{\mathbf{F}}, \widetilde{\mathbf{F}})$, such that $\widetilde{\partial}^2 = \Sigma_{j=1}^{c}\, x_j\, \widetilde{t}_j(\mathrm{Q}, \mathbf{x}, \widetilde{\mathbf{F}}, \widetilde{\partial})$.

Any lifting as above defines a *family of Eisenbud operators*

$$\{\, t_j = t_j(\mathrm{Q}, \mathbf{x}, \mathbf{F}) = \widetilde{t}_j(\mathrm{Q}, \widetilde{\mathbf{x}}, \mathbf{F}, \widetilde{\partial}) \otimes_{\mathrm{Q}} \mathrm{R} \in \mathrm{Hom}^2_{\mathrm{R}}(\mathbf{F}, \mathbf{F})\, \}_{1 \leqslant j \leqslant c}.$$

The t_j's are chain maps that are defined uniquely up to homotopy, and they commute up to homotopy, cf. [18; § 1]. Thus, for each R-module N, they define a family $\chi = \chi_1, \ldots, \chi_c$ of commuting *cohomology operators* $\chi_j = \mathrm{H}^*(\mathrm{Hom}_{\mathrm{R}}(t_j, \mathrm{N}))$. In this way, $\mathrm{H}^*\, \mathrm{Hom}_{\mathrm{R}}(\mathbf{F}, \mathrm{N}) = \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N})$ becomes a graded module over a polynomial algebra $\mathcal{S}^* = \mathrm{R}[\chi]$. This structure may be computed from any system of Eisenbud operators on any R-free resolution.

In particular, if $\boldsymbol{\sigma}$ is a system of higher homotopies (3.8) on a Q-free resolution $\mathbf{E}$ of M, then $\widetilde{\mathbf{E}}\{\boldsymbol{\sigma}\} = \mathrm{Hom}_Q(Q[\chi], \mathbf{E})$ with the endomorphism $\widetilde{\partial}$ from (3.10.2) is a lifting of the standard resolution $(\mathbf{E}\{\boldsymbol{\sigma}\}, \partial)$ of M over R. Computing with it, one easily sees that the map $\widetilde{t}_j(Q, \mathbf{x}, \widetilde{\mathbf{E}}\{\boldsymbol{\sigma}\}, \widetilde{\partial})$ may be chosen to be multiplication with χ_j, cf. [18; (7.2)].

By [12] the action of the χ_j coincides with that of the operators studied by Gulliksen, who shows [23; (2.3)] that when the R-module $\mathrm{Ext}_Q^*(M, N)$ is noetherian, so is the $R[\chi]$-module $\mathrm{Ext}_R^*(M, N)$. A direct proof for the Eisenbud operators is given in [8; (2.1)], and a partial converse in [8; (3.10)]. Here is the full converse.

(4 2) *Theorem. — If* $\mathrm{Ext}_R^*(M, N)$ *is a noetherian graded* $R[\chi]$*-module, then* $\mathrm{Ext}_Q^*(M, N)$ *is a noetherian graded* R*-module.*

We also characterize the regularity of a family of Eisenbud operators.

(4.3) *Theorem. — The kernel of the canonical change of rings homomorphism*

(4.3.1) $$\rho_{MN}^* = \mathrm{Ext}_\rho^*(M, N) : \mathrm{Ext}_R^*(M, N) \to \mathrm{Ext}_Q^*(M, N)$$

contains $(\boldsymbol{\chi})\,\mathrm{Ext}_R^*(M, N)$, *hence it factors through a homomorphism*

(4.3.2) $$\kappa_{MN}^* : \frac{\mathrm{Ext}_R^*(M, N)}{(\boldsymbol{\chi})\,\mathrm{Ext}_R^*(M, N)} \to \mathrm{Ext}_Q^*(M, N).$$

Furthermore, the following conditions are equivalent:

(i) *The sequence* $\boldsymbol{\chi} = \chi_1, \ldots, \chi_c$ *is* $\mathrm{Ext}_R^*(M, N)$*-regular.*
(ii) *The change of rings homomorphism* ρ_{MN}^* *is surjective.*
(iii) *The reduced change of rings homomorphism* κ_{MN}^* *is bijective.*

The proofs depend on a very useful change of rings spectral sequence. For group cohomology, a sequence with similar flavor is obtained by Benson and Carlson [13; § 4].

(4.4) *Theorem. — There exists a fourth quadrant spectral sequence*

(4.4.1) $$\{{}^r d_p^q : {}^r\mathrm{E}_p^q \to {}^r\mathrm{E}_{p-r}^{q-r+1}\}^{r\geqslant 1} \Rightarrow \mathrm{E}^{q-p} = \mathrm{Ext}_Q^{q-p}(M, N)$$

whose initial terms are expressed in terms of the classical Koszul complex $\mathbf{K}(\boldsymbol{\chi}; -)$ *on the family* $\boldsymbol{\chi} = \chi_1, \ldots, \chi_c$ *and its homology* $\mathrm{H}_*(\boldsymbol{\chi}; -)$ *as follows:*

$$^1\mathrm{E}_p^q = \mathrm{K}_p(\boldsymbol{\chi}; \mathrm{Ext}_R^*(M, N))^q, \quad ^2\mathrm{E}_p^q = \mathrm{H}_p(\boldsymbol{\chi}; \mathrm{Ext}_R^*(M, N))^q.$$

The spectral sequence has the following vanishing lines and convergence properties:

(4.4.2) ${}^r\mathrm{E}_p^q = 0$ *when* $p < 0$ *or* $p > c$, *hence* ${}^{c+1}\mathrm{E}_p^q = {}^{c+2}\mathrm{E}_p^q = \ldots = {}^\infty\mathrm{E}_p^q$;

(4.4.3) ${}^r\mathrm{E}_p^q = 0$ *when* $q < 0$, *hence there exist vertical edge homomorphisms*

$${}^r v_{MN}^q : {}^r\mathrm{E}_0^q \twoheadrightarrow {}^\infty\mathrm{E}_0^q \hookrightarrow \mathrm{E}^q = \mathrm{Ext}_Q^q(M, N),$$

and ${}^1 v_{MN}^q = \rho_{MN}^q$ *and* ${}^2 v_{MN}^q = \kappa_{MN}^q$, *the maps from* (4.3.1) *and* (4.3.2);

(4.4.4) ${}^{r}\mathrm{E}_p^q = 0$ *when* $2p > q$, *hence there exist diagonal edge homomorphisms*

$$\delta^p_{\mathrm{MN}} : \mathrm{Ext}^p_{\mathrm{Q}}(\mathrm{M}, \mathrm{N}) = \mathrm{E}^p \twoheadrightarrow {}^{\infty}\mathrm{E}_p^{2p} \hookrightarrow {}^{2}\mathrm{E}_p^{2p} = \mathrm{H}_p(\boldsymbol{\chi}; \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N}))^{2p}.$$

We isolate parts of the argument for (4.4) in the form of a lemma and a remark.

(4.5) *Lemma.* — *For each* R*-module* N *there is a natural isomorphism of* DG $\mathrm{R}[\chi]$*-modules*:

$$(\mathrm{Hom}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, \mathrm{N})^{\#}, \partial)$$
$$\cong (\mathrm{R}[\chi] \otimes_{\mathrm{R}} \mathrm{Hom}_{\mathrm{Q}}(\mathbf{E}, \mathrm{N})^{\#}, \sum_{\mathrm{J} \in \mathbf{N}^c} \chi^{\mathrm{J}} \otimes_{\mathrm{R}} \mathrm{Hom}_{\mathrm{Q}}(\sigma^{\mathrm{J}}, \mathrm{N})).$$

Proof. — The natural isomorphisms of graded modules

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, \mathrm{N})^{\#} &\cong \mathrm{Hom}_{\mathrm{Q}}(\mathrm{Hom}_{\mathrm{Q}}(\mathrm{Q}[\chi], \mathbf{E}^{\#}), \mathrm{N})\\
&\cong \mathrm{Hom}_{\mathrm{Q}}(\mathrm{Hom}_{\mathrm{Q}}(\mathrm{Q}[\chi], \mathrm{Q}) \otimes_{\mathrm{Q}} \mathbf{E}^{\#}, \mathrm{N})\\
&\cong \mathrm{Hom}_{\mathrm{Q}}(\mathrm{Hom}_{\mathrm{Q}}(\mathrm{Q}[\chi], \mathrm{Q}), \mathrm{Hom}_{\mathrm{Q}}(\mathbf{E}^{\#}, \mathrm{N}))\\
&\cong \mathrm{Hom}_{\mathrm{R}}(\mathrm{Hom}_{\mathrm{R}}(\mathrm{R}[\chi], \mathrm{R}), \mathrm{Hom}_{\mathrm{Q}}(\mathbf{E}^{\#}, \mathrm{N}))\\
&\cong \mathrm{Hom}_{\mathrm{R}}(\mathrm{Hom}_{\mathrm{R}}(\mathrm{R}[\chi], \mathrm{R}), \mathrm{R}) \otimes_{\mathrm{R}} \mathrm{Hom}_{\mathrm{Q}}(\mathbf{E}^{\#}, \mathrm{N})\\
&\cong \mathrm{R}[\chi] \otimes_{\mathrm{R}} \mathrm{Hom}_{\mathrm{Q}}(\mathbf{E}, \mathrm{N})^{\#}
\end{aligned}$$

are compatible with the action of $\mathrm{R}[\chi]$. The formula for the differential on the right hand side is obtained by transporting that on $\mathrm{Hom}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, \mathrm{N})$ via these isomorphisms. □

(4.6) *The Koszul* DG *module.* — The Koszul complex $\mathbf{K}(\boldsymbol{\chi}; \mathrm{R}[\chi])$ is a complex of graded $\mathrm{R}[\chi]$-modules. By means of the standard process of condensation, cf. [27; § X.9], it gives rise to a DG $\mathrm{R}[\chi]$-module $\mathbf{K} = \bigoplus_{n \geqslant 0} \Sigma^{-2n} \mathrm{K}_n(\boldsymbol{\chi}; \mathrm{R}[\chi])$, that has a natural filtration by the DG submodules $\mathbf{K}_{(p)} = \bigoplus_{n=0}^{p} \Sigma^{-2n} \mathrm{K}_n(\boldsymbol{\chi}; \mathrm{R}[\chi])$.

Proof of Theorem (4.4). — The spectral sequence (4.4.1) is obtained from the filtration of the DG $\mathrm{R}[\chi]$-module $\mathbf{D} = \mathrm{Hom}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, \mathrm{N}) \otimes_{\mathrm{R}[\chi]} \mathbf{K}$ induced by $\{\mathbf{K}_{(p)}\}_{p \geqslant 0}$. The lemma shows that its first page is the Koszul complex on $\chi_1, \ldots, \chi_c$, with the bigrading

$${}^{1}\mathrm{E}_p^q = \mathrm{K}_p(\boldsymbol{\chi}; \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N}))^q.$$

To get the vanishing lines announced in (4.4.2), (4.4.3), and (4.4.4) simply note that the variables χ_i have degree -2 and filtration level 1, hence in the bigraded R-module ${}^{1}\mathrm{E}^{\#} = \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N}) \otimes_{\mathrm{R}} \bigwedge^*(\mathrm{R}^c)$, the generators of the exterior algebra lie in ${}^{1}\mathrm{E}_1^{-2}$.

To compute the limit of (4.4.1) consider the *filtration bête* of $\mathrm{Hom}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, \mathrm{N})$: its p-th level consist of the homomorphisms of degree $\leqslant p$. The 0-th page of the spectral sequence $\{{}^r\bar{\mathrm{E}}\}^{r\geqslant 0}$ it defines on $\mathbf{D}$ is the Koszul complex $\mathbf{K}(\boldsymbol{\chi}; \mathrm{Hom}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, \mathrm{N})^{\#})$ bigraded by

$$^0\bar{\mathrm{E}}^p_q = \mathrm{K}_q(\boldsymbol{\chi}; \mathrm{Hom}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, \mathrm{N})^{\#})^p.$$

It follows from the lemma that $\chi_1, \ldots, \chi_c$ is a regular sequence on $\mathrm{Hom}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, \mathrm{N})^{\#}$, hence the Koszul homology vanishes except in wedge degree zero, where we have

$$({}^1\bar{\mathrm{E}}^p_0)^{\#} = \frac{\mathrm{Hom}^p_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, \mathrm{N})^{\natural}}{(\boldsymbol{\chi})\,\mathrm{Hom}^{p-2}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, \mathrm{N})^{\#}} \cong \mathrm{Hom}^p_{\mathrm{Q}}(\mathbf{E}, \mathrm{N})^{\#}.$$

Thus, ${}^1\bar{\mathrm{E}}$ is the complex $\mathrm{Hom}_{\mathrm{Q}}(\mathbf{E}, \mathrm{N})$ concentrated in the line $q = 0$, hence

$$\mathrm{H}^*\,\mathbf{D} = {}^{\infty}\bar{\mathrm{E}}^*_0 = {}^2\bar{\mathrm{E}}^*_0 = \mathrm{H}^*\,\mathrm{Hom}_{\mathrm{Q}}(\mathbf{E}, \mathrm{N}) \cong \mathrm{Ext}^*_{\mathrm{Q}}(\mathrm{M}, \mathrm{N}).$$

It remains to identify the vertical edge homomorphisms in (4.4.3). Consider the inclusion

$$\mathbf{C} = (\boldsymbol{\chi})\,\mathrm{Hom}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, \mathrm{N}) \oplus \bigoplus_{n\geqslant 1} (\mathrm{Hom}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, \mathrm{N}) \otimes_{\mathrm{R}[\chi]} \Sigma^{-2n}\,\mathrm{K}_n(\boldsymbol{\chi}; \mathrm{R}[\chi])) \subset \mathbf{D}$$

of DG $\mathrm{R}[\chi]$-modules. The preceding argument shows that $\mathrm{H}_*\,\mathbf{C} = 0$, hence the projection $\mathbf{D} \to \mathrm{Hom}_{\mathrm{Q}}(\mathbf{E}, \mathrm{N})$ identifies $\mathrm{H}^*\,\mathbf{D}$ with $\mathrm{Ext}^*_{\mathrm{Q}}(\mathrm{M}, k)$. Thus, the vertical edge homomorphism ${}^r v^q_{\mathrm{MN}}$ is the map induced in homology by the inclusion

$$\bigoplus\nolimits_{p\leqslant r} \Sigma^{-2p}\,\mathrm{K}_p(\boldsymbol{\chi}; \mathrm{R}[\chi]) \subseteq \mathbf{D}.$$

From the lemma, we get ${}^1v^*_{\mathrm{MN}} = \rho^*_{\mathrm{MN}}$. Furthermore, as ${}^1v^*_{\mathrm{MN}}$ is the composition of ${}^1\mathrm{E}^*_0 \twoheadrightarrow$ with ${}^2v^*_{\mathrm{MN}} : {}^2\mathrm{E}^*_0 \to \mathrm{E}^*$, and ${}^2\mathrm{E}^*_0 = \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, k)/(\boldsymbol{\chi})\,\mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, k)$, we see that ρ^*_{MN} annihilates $(\boldsymbol{\chi})\,\mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, k)$, and that ${}^2v^*_{\mathrm{MN}}$ is the induced map κ^*_{MN}. □

Proof of Theorem (4.2). — For each p the graded $\mathrm{R}[\chi]$-module $\mathrm{H}_p(\boldsymbol{\chi}; \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N}))$ is noetherian and annihilated by $(\boldsymbol{\chi})$, hence is noetherian over $\mathrm{R}[\chi]/(\boldsymbol{\chi}) = \mathrm{R}$. This property is inherited by the columns on each page of the finitely convergent spectral sequence (4.4). Its stable page is the graded object associated to a finite filtration of $\mathrm{Ext}^*_{\mathrm{Q}}(\mathrm{M}, \mathrm{N})$, hence the latter R-module is noetherian. □

Proof of Theorem (4.3). — The map κ^*_{MN} has been constructed in the proof of (4.4).

(i) ⇒ (iii). When the sequence $\chi_1, \ldots, \chi_c$ is $\mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N})$-regular ${}^2\mathrm{E}^*_p = 0$ for $p \neq 0$ in the spectral sequence (4.4), so (4.4.3) shows that κ^*_{MN} is bijective.

(iii) ⇒ (ii) is obvious.

(ii) ⇒ (i). When $c = 1$ the page ${}^2\mathrm{E}$ of the spectral sequence (4.4.1) lives in just two columns, hence it reduces to an exact triangle of graded modules:

$$\begin{tikzcd} \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N}) \arrow[rr, "\chi"] & & \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N}) \arrow[dl, "\rho^*_{\mathrm{MN}}"] \\ & \mathrm{Ext}^*_{\mathrm{Q}}(\mathrm{M}, \mathrm{N}) \arrow[ul, "\Delta"] & \end{tikzcd}$$

in which Δ is a homomorphism of upper degree -1. The desired implication follows.

For the rest of the argument we assume that c is at least 2, and that the implication holds for regular sequences of length less than c.

Set $Q' = Q/(x_1, \ldots, x_{c-1})$, and let $\rho' : Q \to Q'$ and $\rho'' : Q' \to R$ be the canonical projections. Note that if $\boldsymbol{\sigma}$ is a system of higher homotopies for $\mathbf{x}$ on a Q-free resolution $\mathbf{E}$ of M then $\boldsymbol{\sigma}' = \{ \sigma^J \in \boldsymbol{\sigma} \mid J = (j_1, \ldots, j_{c-1}, 0) \in \mathbf{N}^c \}$ is a system of higher homotopies for $\mathbf{x}' = x_1, \ldots, x_{c-1}$ on $\mathbf{E}$. Let $\boldsymbol{\chi}' = \chi_1', \ldots, \chi_{c-1}'$ be the family of operators on $\mathrm{Ext}^*_{Q'}(M, N)$ defined by the standard resolution $\mathbf{E}\{\boldsymbol{\sigma}'\}$. The morphisms of complexes of R-modules

$$\begin{aligned}\mathrm{Hom}_R(\mathbf{E}\{\boldsymbol{\sigma}\}, N) &\to \mathrm{Hom}_R(\mathbf{E}\{\boldsymbol{\sigma}'\} \otimes_{Q'} R, N)\\ &= \mathrm{Hom}_{Q'}(\mathbf{E}\{\boldsymbol{\sigma}'\}, N) \to \mathrm{Hom}_Q(\mathbf{E}, N)\end{aligned}$$

show that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Ext}^*_R(M, N) \arrow[r] \arrow[d, "\rho''^*_{MN}"'] & \dfrac{\mathrm{Ext}^*_R(M, N)}{(\chi_1, \ldots, \chi_{c-1})\, \mathrm{Ext}^*_R(M, N)} \arrow[r] \arrow[d] & \dfrac{\mathrm{Ext}^*_R(M, N)}{(\boldsymbol{\chi})\, \mathrm{Ext}^*_R(M, N)} \arrow[d, "\kappa^*_{MN}"] \\
\mathrm{Ext}^*_{Q'}(M, N) \arrow[r] & \dfrac{\mathrm{Ext}^*_{Q'}(M, N)}{(\chi_1', \ldots, \chi_{c-1}')\, \mathrm{Ext}^*_{Q'}(M, N)} \arrow[r, "\kappa'^*_{MN}"'] & \mathrm{Ext}^*_Q(M, N)
\end{tikzcd}$$

The composition in the lower row is ρ'^*_{MN}. As $\rho'^*_{MN}\,\rho''^*_{MN} = \rho^*_{MN}$ is surjective by assumption, it follows that the same holds for ρ'^*_{MN}. We now get from the induction hypothesis that the sequence $\chi_1', \ldots, \chi_{c-1}'$ is $\mathrm{Ext}^*_{Q'}(M, N)$-regular, and then conclude from the already established implication (i) $\Rightarrow$ (iii) that κ'^*_{MN} is an isomorphism. Thus, the middle vertical map in the diagram is surjective. It follows by Nakayama that so is ρ''^*_{MN}. The already established case $c = 1$ shows that χ_c is a non-zero-divisor on $\mathrm{Ext}^*_R(M, N)$, and that $\mathrm{Ext}^*_R(M, N)/\chi_c\, \mathrm{Ext}^*_R(M, N) \cong \mathrm{Ext}^*_{Q'}(M, N)$. We now know that the sequence $\chi_c, \chi_1, \ldots, \chi_{c-1}$ is $\mathrm{Ext}^*_R(M, N)$-regular, and then we conclude from Matsumura [29; (16.5)] that the sequence $\boldsymbol{\chi} = \chi_1, \ldots, \chi_c$ has the same property, as desired. □

(4.7) *Yoneda products.* — The graded R-module $\mathrm{Ext}^*_R(M, N)$ has a natural structure of *left* $\mathrm{Ext}^*_R(N, N)$- and *right* $\mathrm{Ext}^*_R(M, M)$-*bimodule*, given by Yoneda products, cf. [27; § III.5]. Mehta [30; (2.3)] shows that there are natural homomorphisms of graded R-algebras $\mathrm{Ext}^*_R(N, N) \leftarrow R[\chi] \to \mathrm{Ext}^*_R(M, M)$ whose images lie in the *centers* of the corresponding Yoneda algebras. Furthermore, the $\mathcal{S}^* = R[\chi]$-module structures on $\mathrm{Ext}^*_R(M, N)$ induced by either of these homomorphisms coincides with that described in (4.1), cf. [12].

(4.8) *Flat extensions.* — If $R \to R'$ is a local flat extension, then the canonical homomorphism $\mathrm{Ext}^*_R(M, M) \otimes_R R' \to \mathrm{Ext}^*_{R'}(M', M')$ is an isomorphism of graded R′-algebras, and $\mathrm{Ext}^*_R(M, N) \otimes_R R' \to \mathrm{Ext}^*_{R'}(M', N')$ is an equivariant homomorphism of graded right modules over it. We identify the objects connected by the canonical isomorphisms.

The following two finiteness results are crucial for this investigation.

(4.9) *Theorem.* — *If* M *is a finite module over a noetherian local ring* R, *and* $\mathrm{CI\text{-}dim}_R\, M$ *is finite, then for each finite* R*-module* N *the graded right module* $\mathrm{Ext}^*_R(M, N)$ *is finite over the* R*-subalgebra* $\mathscr{Z}^*$ *of* $\mathrm{Ext}^*_R(M, M)$ *generated by the central elements in* $\mathrm{Ext}^2_R(M, M)$.

If $R \to R' \leftarrow Q$ *is a quasi-deformation and* $\mathrm{pd}_Q\, M' < \infty$, *then* $\mathscr{Z}^* \otimes_R R'$ *is the graded* R′*-subalgebra* $\mathscr{Z}'^*$ *of* $\mathrm{Ext}^*_{R'}(M', M')$, *generated by the central elements of degree* 2.

Proof. — By the finiteness theorem [23; (2.3)] or [8; (2.1)], $\mathrm{Ext}^*_{R'}(M', N')$ and $\mathrm{Ext}^*_{R'}(M', M')$ are noetherian graded modules over the algebra $\mathscr{S}^*$ of cohomology operators of the deformation $R' \leftarrow Q$. In view of (4.7), we conclude that $\mathrm{Ext}^*_{R'}(M', M')$ and $\mathrm{Ext}^*_{R'}(M', N')$ are finite graded $\mathscr{Z}'^*$-modules. In particular, the R′-algebra $\mathrm{Ext}^*_{R'}(M', M')$ is finitely generated. By faithful flatness so is the R-algebra $\mathrm{Ext}^*_R(M, M)$. Let $\mu_1, \ldots, \mu_n$ be a set of homogeneous generators, and let $|\mu_j|$ denote the degree of μ_j. The map

$$\mathrm{Ext}^2_R(M, M) \xrightarrow{\alpha} \bigoplus_{j=1}^{n} \mathrm{Ext}^{|\mu_j|+2}_R(M, M), \qquad \alpha(\mu) = ([\mu, \mu_1], \ldots, [\mu, \mu_n]),$$

where $[\mu, \mu_j] = \mu\mu_j - \mu_j\,\mu$, is a homomorphism of R-modules with $\mathrm{Ker}\,\alpha = \mathscr{Z}^2$. As $\mu_1 \otimes 1, \ldots, \mu_n \otimes 1$ generate the R′-algebra $\mathrm{Ext}^*_R(M, M) \otimes_R R'$, we get

$$\mathscr{Z}^2 \otimes_R R' = (\mathrm{Ker}\,\alpha) \otimes_R R' = \mathrm{Ker}(\alpha \otimes_R R') = \mathscr{Z}'^2.$$

Thus, $\mathscr{Z}^* \otimes_R R' = \mathscr{Z}'^*$. Now $\mathrm{Ext}^*_R(M, N)$ is finite over $\mathscr{Z}^*$ by faithfully flat descent. □

When $(R, \mathfrak{m}, k)$ is a local ring, it is known from Milnor and Moore [31], André [3], and Sjödin [34], that $\mathrm{Ext}^*_R(k, k)$ is the universal enveloping algebra of a graded Lie algebra (also known as a Lie superalgebra), the *homotopy Lie algebra* $\pi^*(R)$. In this context, we make frequent use of the following result of [12].

(4.10) *Theorem.* — *If* $(R, \mathfrak{m}, k)$ *is noetherian local ring and* M *is a finite* R*-module such that* $\mathrm{CI\text{-}dim}_R\, M < \infty$, *then graded left module* $\mathrm{Ext}^*_R(M, k)$ *is finite over the* k*-subalgebra* $\mathscr{P}^*$ *of* $\mathrm{Ext}^*_R(k, k)$ *generated by the central elements in* $\pi^2(R)$. □

5. Complexity

In this section $(R, \mathfrak{m}, k)$ is a local ring, and M is a finite R-module.

One aspect of the asymptotic behavior of the minimal free resolution of M is captured by its complexity, a concept originating in the work of Alperin and Evens [1] on group representations and group cohomology, and adapted to local algebra in [8], [9]. We start this section by recalling some basic facts on this invariant, and proceed to establish a uniform bound on the complexities of all modules of finite CI-dimension over R. We finish by providing new and more direct proofs, based on the work in the preceding section, of some pivotal results of [8].

(5.1) *Complexity.* — The *complexity* of M over R is the number

$$\mathrm{cx}_R M = \inf\{ d \in \mathbf{N} \mid \text{there exists } \gamma \in \mathbf{R} \text{ such that } b_n^R(M) \leqslant \gamma n^{d-1} \text{ for } n \gg 0 \}.$$

We list some properties that hold quite generally, cf. [9; Appendix].

(5.2) *Proposition.* — *With* R *and* M *as above the following hold.*

(1) $\mathrm{cx}_R M = 0$ *if and only if* $\mathrm{pd}_R M < \infty$.

(2) *If a sequence* $\mathbf{y} \subset \mathfrak{m}$ *is both* R-*regular and* M-*regular, then* $\mathrm{cx}_R M/(\mathbf{y}) M = \mathrm{cx}_R M$.

(3) *If* $R \to R'$ *is a local flat extension and* $M' = M \otimes_R R'$, *then* $\mathrm{cx}_R M = \mathrm{cx}_{R'} M'$.

(4) *If* $R \leftarrow Q$ *is a codimension* c *deformation, then* $\mathrm{cx}_Q M \leqslant \mathrm{cx}_R M \leqslant \mathrm{cx}_Q M + c$.

Proof. — (1) is immediate from the definition.

(2) Let $\mathbf{K} = \mathbf{K}(\mathbf{y}; R)$ be the classical Koszul complex, and let $\mathbf{F}$ be a minimal free resolution of M over R. Since $H_n(\mathbf{F} \otimes_R \mathbf{K}) \cong H_n(M \otimes_R \mathbf{K}) = 0$ for $n \neq 0$ we see that $\mathbf{F} \otimes_R \mathbf{K}$ is a free resolution of $\overline{M} = M/(\mathbf{y}) M$ over R. It is clearly minimal, hence $P_{\overline{M}}^R(t) = (1 + t)^g P_M^R(t)$, with $g = \mathrm{card}\, \mathbf{y}$. The desired equality follows.

(3) and (4) follow easily from (1.5). □

A deeper understanding of complexity relies on a study of the Yoneda bimodule structure (4.7) on $\mathrm{Ext}_R^*(M, k)$. In the case of finite CI-dimension we obtain strong numerical bounds.

(5.3) *Theorem.* — *If* $\text{CI-dim}_R M < \infty$, *then* $\mathrm{cx}_R M$ *is finite, it is equal to the order of the pole at* $t = 1$ *of the Poincaré series* $P_M^R(t)$, *and*

$$(1) \qquad \mathrm{cx}_R M = \dim_{\mathscr{X}^*} \mathrm{Ext}_R^*(M, k) = \dim_{\mathscr{P}^*} \mathrm{Ext}_R^*(M, k).$$

Furthermore, for each finite R-*module* N *there is an inequality*

$$(2) \qquad \dim_{\mathscr{X}^*} \mathrm{Ext}_R^*(M, N) \leqslant \dim_R N + \mathrm{cx}_R M.$$

Proof. — (1) We know from (4.9) and (4.10) that $\mathrm{Ext}_R^*(M, k)$ is a finite module over the graded k-algebras $\mathscr{X}^*/\mathfrak{m}\mathscr{X}^*$ and $\mathscr{P}^*$, generated over k by a finite number of elements of degree 2. Thus, the desired equalities follow from basic properties of Hilbert functions.

(2) The argument is by induction on $n = \dim_R N$, and uses the fact that a short exact sequence of R-modules $0 \to N' \to N \to N'' \to 0$ induces an exact sequence

$$\mathrm{Ext}_R^*(M, N') \to \mathrm{Ext}_R^*(M, N) \to \mathrm{Ext}_R^*(M, N'') \to \Sigma^{-1} \mathrm{Ext}_R^*(M, N')$$

of degree zero homomorphisms of graded right $\mathrm{Ext}_R^*(M, M)$-modules, hence also one of $\mathscr{X}^*$-modules. (Note that as we use the homological suspension, $\Sigma^{-1} \mathrm{Ext}_R^*(M, N')$ is the graded module with degree i component equal to $\mathrm{Ext}_R^{i+1}(M, N')$.)

If $n = 0$, then $\dim_{\mathscr{X}^*} \operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N}) \leqslant \operatorname{cx}_{\mathrm{R}} \mathrm{M}$ follows from this exact sequence by induction on the length of M over R. So assume that $\dim_{\mathrm{R}} \mathrm{N} = n > 0$, and that the statement holds for modules of dimension less than n.

Let $\mathrm{N}' = \mathrm{H}^0_{\mathfrak{m}}(\mathrm{N})$ be the largest submodule of finite length in N. As the result is available in dimension zero, and $\dim_{\mathrm{R}} \mathrm{N}'' = \dim_{\mathrm{R}} \mathrm{N}$, the exact sequence shows it is sufficient to prove the inequality for N''. Changing notation, we assume that $\operatorname{depth}_{\mathrm{R}} \mathrm{N}$ is positive and choose an N-regular element $x \in \mathfrak{m}$. The cohomology exact sequence induced by the short exact sequence $0 \to \mathrm{N} \xrightarrow{x} \mathrm{N} \to \mathrm{N}/x\mathrm{N} \to 0$ shows that $\operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N})/x \operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N})$ is isomorphic to a $\mathscr{X}^*$-submodule of $\operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N}/x\mathrm{N})$. This gives the second inequality in the following computation, where the third one is provided by the induction hypothesis:

$$\begin{aligned}
\dim_{\mathscr{X}^*} \operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N}) &\leqslant \dim_{\mathscr{X}^*}(\operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N})/x \operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N})) + 1 \\
&\leqslant \dim_{\mathscr{X}^*} \operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{N}/x\mathrm{N}) + 1 \\
&\leqslant \dim_{\mathrm{R}}(\mathrm{N}/x\mathrm{N}) + \operatorname{cx}_{\mathrm{R}} \mathrm{M} + 1 \\
&= \dim_{\mathrm{R}} \mathrm{N} + \operatorname{cx}_{\mathrm{R}} \mathrm{M}. \quad \square
\end{aligned}$$

Ext algebras are almost never commutative and usually very complex, so we note:

(5.4) *Corollary. — When* $\text{CI-dim}_{\mathrm{R}} \mathrm{M} < \infty$ *the graded* R-*algebra* $\operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{M})$ *is module-finite over its center, that is a finitely generated graded* R-*algebra, and whose Krull dimension* m *satisfies* $\max\{\dim_{\mathrm{R}} \mathrm{M}, \operatorname{cx}_{\mathrm{R}} \mathrm{M}\} \leqslant m \leqslant \dim_{\mathrm{R}} \mathrm{M} + \operatorname{cx}_{\mathrm{R}} \mathrm{M}$.

Proof. — To see that $\dim_{\mathrm{R}} \mathrm{M}$ is a lower bound for the Krull dimension of the center of $\operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{M})$, note that $\operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{M})/\operatorname{Ext}^{\geqslant 1}_{\mathrm{R}}(\mathrm{M}, \mathrm{M}) \cong \operatorname{Hom}_{\mathrm{R}}(\mathrm{M}, \mathrm{M})$ is a finite module over this algebra, and its Krull dimension is $\dim_{\mathrm{R}} \mathrm{M}$. The lower bound $\operatorname{cx}_{\mathrm{R}} \mathrm{M}$ is obtained by remarking that $\operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}, k)$ is a finite module which has this Krull dimension. $\square$

Clearly, the upper and lower bounds on m coincide if $\dim_{\mathrm{R}} \mathrm{M}$ or $\operatorname{cx}_{\mathrm{R}} \mathrm{M}$ is zero, but

(5.5) *Example. — The inequality* $m \leqslant \dim_{\mathrm{R}} \mathrm{M} + \operatorname{cx}_{\mathrm{R}} \mathrm{M}$ *may be strict.*

Let $(\mathrm{S}, \mathfrak{p})$ be a local domain containing $\mathbf{Z}$ and having residual characteristic $p > 0$, and let $\mathrm{G} = \langle h \rangle$ be a cyclic group of order $q = p^e > 1$. The group algebra $\mathrm{R} = \mathrm{S}[\mathrm{G}]$ is then a local ring with maximal ideal $(\mathfrak{p}, h - 1)\,\mathrm{R}$. When S is considered as an R-module via the isomorphism $\mathrm{S} \cong \mathrm{R}/(h - 1)$, the deformation $\mathrm{R} \leftarrow \mathrm{S}[\mathrm{Y}]_{(\mathfrak{p}, \mathrm{Y}-1)}$, $h \leftarrow\!\shortmid \mathrm{Y}$, shows that $\text{CI-dim}_{\mathrm{R}} \mathrm{S}$ is finite. The sequence $\sigma : 0 \to \mathrm{S} \to \mathrm{R} \to \mathrm{R} \to \mathrm{S} \to 0$, whose middle map is multiplication by $h - 1$, is exact. Splicing it with itself we obtain a minimal R-free resolution of S which yields isomorphisms of R-algebras

$$\operatorname{Ext}^*_{\mathrm{R}}(\mathrm{S}, \mathrm{S}) = \mathscr{X}^* \cong \mathrm{S}[\mathrm{X}]/(q\mathrm{X}), \quad \text{with } \sigma \leftarrow\!\shortmid\!\rightarrow \mathrm{X}.$$

Thus, $\dim \mathscr{X}^* = \dim_{\mathrm{R}} \mathrm{S} < \dim_{\mathrm{R}} \mathrm{S} + 1 = \dim_{\mathrm{R}} \mathrm{S} + \operatorname{cx}_{\mathrm{R}} \mathrm{S}$.

The *embedding dimension* of R, edim R, is the minimal number of generators of $\mathfrak{m}$. Given the latitude for base change allowed by (1.2), it is quite remarkable that there exists an upper bound on the complexities of *all* R-modules of finite CI-dimension:

(5.6) *Theorem. — If* M *has finite* CI-*dimension, then* $\mathrm{cx}_R M \leqslant \mathrm{edim}\, R - \mathrm{depth}\, R$, *and the inequality is strict unless* R *is a complete intersection.*

Proof. — Let ζ^2 denote the (super) center of $\pi^*(R)$, and set $\mathrm{rank}_k \pi^i = e_i$. From (4.10) and (5.3.1) we get $\mathrm{cx}_R M \leqslant \mathrm{rank}_k \zeta^2$. It follows from [20; Theorem A] that $\mathrm{rank}_k \zeta^2 \leqslant \mathrm{edim}\, R - \mathrm{depth}\, R$, and when equality holds $e_i = 0$ for $i > 2$. In that case, Poincaré-Birkhoff-Witt yields $P_k^R(t) = (1+t)^{e_1}(1-t^2)^{-e_2}$, and we conclude that R is a complete intersection by the criterion of Assmus [4; (2.7)]. □

When $\mathrm{pd}_R M$ is finite, the Auslander-Buchsbaum Equality implies that $0 \leqslant \mathrm{pd}_R M \leqslant \mathrm{depth}\, R$, and exactness of the Koszul complexes on R-regular sequences shows that all intermediate values occur. Similarly, (1.4) shows that when $\mathrm{CI\text{-}dim}_R M$ is finite $0 \leqslant \mathrm{CI\text{-}dim}_R M \leqslant \mathrm{depth}\, R$, but we do not know whether these values appear in combination with each complexity allowed by (5.6). Nevertheless, the following example shows that many such combinations do occur: its list will be exhaustive if, as suggested by [9; (4.3)], $\mathrm{rank}_k \zeta^2$ is equal to the maximal codimension of an embedded deformation of $\hat{R}$.

(5.7) *Example. — If the local ring* R *has a codimension* c *deformation* Q *with* edim Q = edim R, *then for any pair of integers* (d, g) *with* $0 \leqslant d \leqslant c$ *and* $0 \leqslant g \leqslant \mathrm{depth}\, R$ *there exists an* R-*module* M *with* $\mathrm{cx}_R M = d$ *and* $\mathrm{CI\text{-}dim}_R M = g$.

Indeed, given d as above, the original deformation can be factored through an embedded deformation of codimension d, hence it suffices to construct the relevant examples when $d = c$. Fix an arbitrary integer r, and let M denote the module $M_r^{[c]}$ introduced in (3.1). That theorem yields $\mathrm{cx}_R M = c$ and $\mathrm{CI\text{-}dim}_R M = 0$, hence $\mathrm{depth}_R M = \mathrm{depth}\, R$ by (1.4). For g as above choose a length g sequence $\mathbf{y} \subset \mathfrak{m}$ which is regular on both R and M. Thus, $\mathrm{CI\text{-}dim}_R M/(\mathbf{y}) M = g$ by (1.12.1) and $\mathrm{cx}_R M/(\mathbf{y}) M = c$ by (5.2 2).

One of our main techniques for studying a module of finite CI-dimension is to construct a quasi-deformation that reflects a particular aspect of its minimal resolution.

(5.8) *Factorizations.* — Consider a deformation $R' \xleftarrow{\rho} Q$ with kernel $\mathfrak{c}$, and denote C the module $(\mathfrak{c}/\mathfrak{c}^2)^* = \mathrm{Hom}_R(\mathfrak{c}/\mathfrak{c}^2, R)$.

A factorization $R' \leftarrow Q' \leftarrow Q$ of ρ into a composition of deformations with kernels $\mathfrak{c}'$ and $\mathfrak{d} = \mathfrak{c}/\mathfrak{c}'$ yields an epimorphism of free R'-modules $\mathfrak{c}/\mathfrak{c}^2 \to \mathfrak{c}/(\mathfrak{c}' + \mathfrak{c}^2) = \mathfrak{d}/\mathfrak{d}^2$. Thus, $(\mathfrak{d}/\mathfrak{d}^2)^* = \mathrm{Hom}_R(\mathfrak{d}/\mathfrak{d}^2, R)$ is a direct summand of C. Conversely, given a decomposition $C = D \oplus E$, consider the direct summand $D^\perp = \bigcap_{\lambda \in D} \mathrm{Ker}\, \lambda$ of $\mathfrak{c}/\mathfrak{c}^2$. If a sequence

$\mathbf{y} \subset \mathfrak{c}$ lifts a basis of $D^\perp$, then it is Q-regular, and $\mathfrak{d} = \mathfrak{c}/(\mathbf{y})$ is generated by a $Q' = Q/(\mathbf{y})$-regular sequence. This provides a factorization $R' \leftarrow Q' \leftarrow Q$ of ρ such that $(\mathfrak{d}/\mathfrak{d}^2)^* = D$.

Let $x_1^*, \ldots, x_c^*$ be the basis of $(\Sigma^2(\mathfrak{c}/\mathfrak{c}^2))^*$ dual to the basis of $\Sigma^2(\mathfrak{c}/\mathfrak{c}^2)$ provided by the canonical images of $\Sigma^2(x_1), \ldots, \Sigma^2(x_c)$. Identify the symmetric algebra $\mathrm{Sym}^*_{R'}(\Sigma^{-2} C)$ with the algebra $\mathscr{S}^* = R[\chi]$ of cohomology operators defined by ρ by an isomorphism under which x_j^* corresponds to χ_j for $1 \leqslant j \leqslant c$. If $\boldsymbol{\eta} = \eta_1, \ldots, \eta_d$ is a basis for a direct summand D of C as above, then $R'[\eta]$ is identified with a subalgebra of $R[\chi]$, and thus $\mathrm{Ext}^*_R(M, N)$ becomes a graded $R'[\eta]$-module for arbitrary R-modules M and N. The functoriality of Eisenbud operators [18; (1.7)] shows that this module structure coincides with the one defined by the deformation $R' \leftarrow Q/(\mathbf{y}) = Q'$ through which ρ factors.

The next result can also be deduced from [8; (3.6)], but the proof of [*loc. cit.*] is considerably more complex.

(5.9) *Theorem.* — *If* $R \to R' \leftarrow Q$ *is a quasi-deformation with* $\mathrm{pd}_Q M' < \infty$, *then* $\mathrm{cx}_R M \leqslant \mathrm{pd}_Q R'$, *and the deformation* $R' \leftarrow Q$ *is embedded when equality holds.*

If furthermore the residue field ℓ *of* R' *is infinite, then* $R' \leftarrow Q$ *factors as a composition* $R' \leftarrow Q' \leftarrow Q$ *of deformations with* $\mathrm{cx}_R M = \mathrm{pd}_{Q'} R'$.

Proof. — The inequality $\mathrm{cx}_R M \leqslant \mathrm{pd}_Q R'$ is immediate from (5.2.4). Assume that equality holds, and that $\mathrm{edim}\, R' < \mathrm{edim}\, Q$. We can then generate $\mathfrak{c} = \mathrm{Ker}(Q \to R')$ by a regular sequence whose first element x is not in the square of the maximal ideal of the ring Q. The inequality above and [32; (27.5)] yield

$$\mathrm{cx}_R M \leqslant \mathrm{pd}_{Q/(x)} M' = \mathrm{pd}_Q M' - 1 = \mathrm{cx}_R M - 1,$$

which is absurd.

Set $\mathfrak{c} = \mathrm{Ker}(Q \to R)$, and note by (4.10) that $\mathrm{Ext}^*_{R'}(M', \ell)$ is a finite graded module over $\mathscr{S}^*$, and that its Krull dimension d is equal to $\mathrm{cx}_{R'} M' = \mathrm{cx}_R M$. As the residue field of R' is infinite, we may choose in $\mathscr{S}^2$ a homogeneous system of parameters η for $\mathrm{Ext}^*_{R'}(M', \ell)$, such that $R'\eta$ is a direct summand of $\mathscr{S}^2$. By (5.8) we identify $R'[\eta]$ with the algebra of cohomology operators defined of a codimension d deformation $R' \leftarrow Q'$ through which $R' \leftarrow Q$ factors. As $\mathrm{Ext}^*_{R'}(M', \ell)$ is finite over $R'[\eta]$, we get from (4.2) that $\mathrm{pd}_{Q'} M'$ is finite, hence Q' has the desired properties. □

As an immediate corollary, we obtain:

(5.10) *Theorem.* — *If* $\mathrm{CI\text{-}dim}_R M < \infty$, *then there exists a quasi-deformation* $R \to R' \leftarrow Q$ *such that* $\mathrm{cx}_R M = \mathrm{pd}_Q M'$. *Furthermore,* $\mathrm{edim}\, R' = \mathrm{edim}\, Q$, *and* Q *may be chosen complete with algebraically closed residue field.* □

In the context of CI-dimension it is natural to consider the following variant of the notion of the virtual projective dimension of [8 (3.3)]

$$\mathrm{qpd}_R M = \inf\{ \mathrm{pd}_Q(M \otimes_R R') \mid R \to R' \leftarrow Q \text{ is a quasi-deformation} \}.$$

Our last result contains the Auslander-Buchsbaum Equality, and its extension in [8; (3.5)].

(5.11) *Theorem.* — *For any* R-*module* M *there is an equality*

$$\operatorname{qpd}_R M = \operatorname{CI-dim}_R M + \operatorname{cx}_R M.$$

If $\operatorname{CI-dim}_R M$ *is finite, then* $\operatorname{qpd}_R M = \operatorname{depth} R - \operatorname{depth}_R M + \operatorname{cx}_R M < \infty$. *Furthermore,* $\operatorname{qpd}_R M \leqslant \operatorname{vpd}_R M$, *and equality holds when* $\operatorname{vpd}_R M$ *is finite.*

Proof. — By (5.3), a module of finite CI-dimension has finite complexity, hence $\operatorname{CI-dim}_R M + \operatorname{cx}_R M$ and $\operatorname{qpd}_R M$ are finite simultaneously. Assume this is the case, and choose a quasi-deformation $R \to R' \leftarrow Q$ with $\operatorname{qpd}_R M = \operatorname{pd}_Q M'$. By (1.14) and (5.2), it is harmless to assume that the residue field of R′ is infinite.

On the one hand, we then have

$$\begin{aligned}\operatorname{CI-dim}_R M + \operatorname{cx}_R M &\leqslant \operatorname{CI-dim}_R M + \operatorname{pd}_Q R' \\ &= (\operatorname{pd}_Q M' - \operatorname{pd}_Q R') + \operatorname{pd}_Q R' = \operatorname{qpd}_R M\end{aligned}$$

where the inequality comes from theorem (5.9), and the first equality from (1.7) and (1.3). On the other hand, replacing Q by a deformation Q′ constructed in (5.10), we get

$$\begin{aligned}\operatorname{qpd}_R M \leqslant \operatorname{pd}_{Q'} M' &= \operatorname{depth} Q' - \operatorname{depth}_{Q'} M' \\ &= \operatorname{pd}_{Q'} R' + \operatorname{depth}_{Q'} R' - \operatorname{depth}_{Q'} M' \\ &= \operatorname{cx}_R M + \operatorname{depth} R' - \operatorname{depth}_{R'} M' \\ &= \operatorname{cx}_R M + \operatorname{depth} R - \operatorname{depth}_R M \\ &= \operatorname{cx}_R M + \operatorname{CI-dim}_R M\end{aligned}$$

with equalities provided by Auslander-Buchsbaum (twice), (5.10), (1.7), and (1.3).

The inequality $\operatorname{qpd}_R M \leqslant \operatorname{vpd}_R M$ is clear from the definitions. If the virtual projective dimension is finite, then the preceding argument works for the flat extension $R \to R' = \widetilde{R}$, and shows that $\operatorname{vpd}_R M = \operatorname{cx}_R M + \operatorname{CI-dim}_R M$. □

6. Homological reductions

In this section $(R, \mathfrak{m}, k)$ is a local ring and M is finite R-module.

Much of the work in this paper revolves around the perception that for any given quasi-deformation $R \to R' \leftarrow Q$, the homological properties of the Q-module $M' = M \otimes_R R'$ provide an " upper bound " for those of the R-module M. A case in point is the coefficientwise inequality of formal power series

$$\textbf{(6.0)} \qquad P^R_M(t) \leqslant P^Q_{M'}(t)\,(1-t^2)^{-c}$$

recalled from (1.5). On a structural level, the standard resolution $\mathbf{E}\{\boldsymbol{\sigma}\}$ of Shamash and Eisenbud, recalled in (3.10), provides an approximation to the minimal resolution $\mathbf{F}'$ of M′. The latter is always a direct summand of $\mathbf{E}\{\boldsymbol{\sigma}\}$, and when these complexes coincide, then the minimal resolution of M over R is completely determined by data defined over Q. In this section we introduce and study the concept of *homological reducibility*, that describes how much structure can be induced on $\mathbf{F}'$.

Some early results provide handy conditions for equality to hold in (6.0).

(6.1) *Example.* — *If* $(Q, \mathfrak{n})$ *is a deformation of* R *and* M *is an* R*-module with* $\mathfrak{n}\, \mathrm{ann}_Q\, M \supseteq \mathrm{Ker}(Q \to R)$, *then equality holds in* (6.0) *by* [36; Theorem 5] *and* [33; § 3, Corollary 1].

First we explore what can be said in general in the extremal situation.

(6.2) *Proposition.* — *Let* ℓ *denote the residue field of* R′, *let* $\mathbf{E}\{\boldsymbol{\sigma}\}$ *be the standard* R′*-free resolution* (3.10) *of* $M' = M \otimes_R R'$ *constructed from a system of higher homotopies* $\boldsymbol{\sigma}$ *on a minimal* Q*-free resolution* $\mathbf{E}$ *of* M′, *and let* $R'[\chi] = R'[\chi_1, \ldots, \chi_c]$ *be the algebra of cohomology operators* (4.1) *defined by* $R' \leftarrow Q$. *The following conditions are equivalent.*

(i) $P_M^R(t) = P_{M'}^Q(t)\,(1 - t^2)^{-c}$.

(ii) $\sigma(\mathbf{E}) \subseteq \mathfrak{n}\mathbf{E}$ *for each* $\sigma \in \boldsymbol{\sigma}$.

(iii) $\mathbf{E}\{\boldsymbol{\sigma}\}$ *is a minimal* R*-free resolution.*

(iv) $\mathrm{Ext}_{R'}^*(M', \ell) \cong \ell[\chi] \otimes_\ell \mathrm{Ext}_Q^*(M', \ell)$ *as graded* $\ell[\chi]$*-modules.*

(v) $\boldsymbol{\chi} = \chi_1, \ldots, \chi_c$ *is an* $\mathrm{Ext}_{R'}^*(M', \ell)$*-regular sequence.*

In particular, the validity of (ii) *or* (iii) *does not depend on the choice of* $\boldsymbol{\sigma}$.

Proof. — The equivalence of the first three conditions follow from the expression (3.10.3) for the ranks of free modules in a standard R′-free resolution $\mathbf{E}\{\boldsymbol{\sigma}\}$, and from the observation that by (3.10.1) and (3.10.2) the resolution $\mathbf{E}\{\boldsymbol{\sigma}\}$ is minimal precisely when $\sigma(\mathbf{E}) \subseteq \mathfrak{n}\mathbf{E}$ for each $\sigma \in \boldsymbol{\sigma}$. The implication (iv) $\Rightarrow$ (v) is clear.

(iii) $\Rightarrow$ (iv). Consider the complexes $\mathrm{Hom}_{R'}(\mathbf{E}\{\boldsymbol{\sigma}\}, \ell)$ and $R'[\chi] \otimes_{R'} \mathrm{Hom}_Q(\mathbf{E}, \ell)$, whose differentials are zero in view of the minimality of the free resolutions $\mathbf{E}\{\boldsymbol{\sigma}\}$ and $\mathbf{E}$. By (4.5) they are isomorphic as DG $R'[\chi]$-modules, so we get an isomorphism in (iv).

(v) $\Rightarrow$ (i). Set $E^* = \mathrm{Ext}_{R'}^*(M', \ell)/(\boldsymbol{\chi})\, \mathrm{Ext}_{R'}^*(M', \ell)$. On the one hand, the regularity of the sequence $\chi_1, \ldots, \chi_c$ on $\mathrm{Ext}_{R'}^*(M', \ell)$ produces an equality of formal power series $P_{M'}^{R'}(t) = (\Sigma_{i \geqslant 0} \operatorname{rank}_\ell E^i\, t^i)\,(1 - t^2)^{-c}$. On the other hand, by (4.3) the same assumption yields an isomorphism $E^* \cong \mathrm{Ext}_Q^*(M', \ell)$, hence $\Sigma_{i \geqslant 0} \operatorname{rank}_\ell E^i\, t^i = P_{M'}^Q(t)$. $\square$

When the conditions above are fulfilled, the homological algebra of M over R is essentially determined by that of M′ over Q. This motivates the next definition.

(6.3) *Homological reduction.* — When the equivalent conditions of (6.2) hold, we say that a quasi-deformation $R \to R' \leftarrow Q$ is a *homological reduction* of the R-module M. The *homological redundancy* of M over R is defined to be the number

$$\mathrm{hom\ red}_R\, M = \sup\{\, c \in \mathbf{N} \mid M \text{ has a homological reduction of codimension } c\,\}.$$

A module M with $\mathrm{hom\ red}_R\, M = 0$ is said to be *homologically irreducible.* Note that such a module is necessarily non-zero, since clearly $\mathrm{hom\ red}\, 0 = \infty$.

To study homological redundancy, we introduce the *depth* of a graded module W over a commutative graded ring A as the supremum of the lengths of A-regular sequences of homogeneous elements of *non-zero* degree, and denote this number by $\operatorname{depth}_A W$. Note that $0 \leqslant \operatorname{depth}_A W \leqslant \infty$ when $W \neq 0$, and $\operatorname{depth}_A 0 = -\infty$. In the next theorem, that should be compared with (5.3.1), $\mathscr{X}^*$ and $\mathscr{P}^*$ are the algebras defined in (4.9) and (4.10).

(6.4) *Theorem.* — *If* $M \neq 0$ *is a finite* R-*module, then* $0 \leqslant \operatorname{hom\,red}_R M \leqslant \operatorname{cx}_R M$. *If furthermore* M *has finite* CI-*dimension, then*

$$\operatorname{hom\,red}_R M = \operatorname{depth}_{\mathscr{X}^\bullet} \operatorname{Ext}^*_R(M, k) = \operatorname{depth}_{\mathscr{P}^\bullet} \operatorname{Ext}^*_R(M, k).$$

A crucial part of the argument for (6.4) is provided by a factorization theorem.

(6.5) *Theorem.* — *In a quasi-deformation* $R \to R' \xleftarrow{\rho} Q$ *such that* $\operatorname{pd}_Q M' < \infty$ *and* R' *has an infinite residue field, the homomorphism* ρ *can be factored as a composition of deformations* $R' \leftarrow Q' \leftarrow Q$, *where* $\operatorname{hom\,red}_R M = \operatorname{pd}_{Q'} R'$ *and the* Q'-*module* M' *is homologically irreducible.*

After some preparation, the theorems are proved jointly at the end of the section.

(6.6) *Lemma.* — (1) *If* $R \to R' \leftarrow Q$ *and* $Q \to Q' \leftarrow P$ *are quasi-deformations, then the composition* $R \to R' \otimes_Q Q' \leftarrow P$ *defined in* (1.10) *is a homological reduction of* M *if and only if* $R \to R' \leftarrow Q$ *is one of* M *and* $Q \to Q' \leftarrow P$ *is one of* M'.

(2) *If* $R \to R' \leftarrow Q$ *is a homological reduction of* M, *then* $\operatorname{edim} R' = \operatorname{edim} Q$.

Proof. — (1) By multiplying each side of (6.0) with the corresponding side of the coefficientwise inequality $1 \leqslant (1-t^2)^{-\operatorname{depth} R}$, we obtain a symmetric relation

$$P^R_M(t)\,(1-t^2)^{-\operatorname{depth} R} \leqslant P^Q_{M'}(t)\,(1-t^2)^{-\operatorname{depth} Q}$$

that becomes an equality precisely when $R \to R' \leftarrow Q$ is a homological reduction of M. Thus, setting $M'' = M' \otimes_Q Q' = M \otimes_R (R' \otimes_Q Q')$, we get a concatenation of inequalities

$$P^R_M(t)\,(1-t^2)^{-\operatorname{depth} R} \leqslant P^Q_{M'}(t)\,(1-t^2)^{-\operatorname{depth} Q} \leqslant P^P_{M''}(t)\,(1-t^2)^{-\operatorname{depth} P}$$

in which the extreme two terms are equal if and only if all three are.

(2) Suppose that $\operatorname{edim} R' < \operatorname{edim} Q$, and choose a minimal generating set for $\operatorname{Ker}(Q \to R')$ whose first element x is not in the square of the maximal ideal of Q. From (1) we see that $R \to R' \leftarrow Q' = Q/(x)$ is a homological reduction of M, hence $P^R_M(t) = P^{Q'}_{M'}(t)\,(1-t^2)^{-1}$. On the other hand, $P^R_M(t) = P^{R'}_{M'}(t) = P^{Q'}_M(t)\,(1+t)^{-1}$ due respectively to (1.5), and to [32; (27.3)] in view of the choice of x. The two expressions for $P^R_M(t)$ being incompatible, we get a contradiction, whence $\operatorname{edim} R' = \operatorname{edim} Q$. □

The essence of the following remark is that properties invariant under finite descent are also invariant under the more general operation of replacing the original ring by another ring of operators, that acts in a *compatible* way, and over which the module is *noetherian.*

(6.7) *Bimodules.* — Let R be a commutative ring, let A and B be graded commutative R-algebras, and let W be a graded A-B-bimodule (over commutative algebras we drop the distinction between left and right actions). Equivalently, W is a graded module over the graded R-algebra $T = A \otimes_R B$, and thus it has a canonical structure of faithful module over the graded R-algebra $C = T/\mathrm{ann}_T W$, such that the natural homomorphisms of graded R-algebras $\alpha : A \to C$ and $C \leftarrow B : \beta$ induce the original actions of A and B on W.

Let R be noetherian, the R-algebras A and B finitely generated, and the module W finite over *each one* of them. In this case C is finitely generated, W is a finite C-module, and the maps α and β are both finite. Indeed, $\gamma : T \to \mathrm{End}_A(W)$, $\gamma(a \otimes b)(m) = (-1)^{|m||b|} amb$, induces an A-linear embedding of C into the noetherian A-module $\mathrm{Hom}_A(W, W)$. The finiteness of α follows. That of β results by symmetry.

Proof of Theorem (6.4) *and Theorem* (6.5). — The inequality $\mathrm{hom\,red}_R\, M \geqslant 0$ is simply the observation that the constant deformation (1.10) is a homological reduction of any M. For the other inequality, note that if a quasi-deformation $R \to R' \leftarrow Q$ is a codimension c homological reduction of an R-module $M \neq 0$, and $M' = M \otimes_R R'$, then there are coefficientwise inequalities $P^R_M(t) = P^Q_{M'}(t)(1-t^2)^{-c} \geqslant (1-t^2)^{-c}$, that is, have $b^R_{2n}(M) \geqslant \binom{2n+c-1}{c-1}$ for $n \geqslant 0$. Thus, $\mathrm{cx}_R\, M \geqslant c$, and hence $\mathrm{cx}_R\, M \geqslant \mathrm{hom\,red}_R\, M$.

For the rest of the argument we assume that $\text{CI-dim}_R\, M$ is finite, pick a quasi-deformation $R \to R' \leftarrow Q$ such that $\mathrm{pd}_Q\, M' < \infty$ and the residue field ℓ of R′ is infinite, and set $M' = M \otimes_R R'$. Also, we denote by $\mathscr{X}^*$ and $\mathscr{X}'^*$ the subalgebras generated by the degree 2 central elements of $\mathrm{Ext}^2_R(M, M)$ and $\mathrm{Ext}^2_{R'}(M', M')$, respectively, and note that $\mathscr{X}^* \otimes_R R' = \mathscr{X}'^*$ by (4.9). Finally, we set $\mathrm{depth}_{\mathscr{X}^*}(\mathrm{Ext}^*_R(M, k)) = g$.

If $\mathbf{F}$ is a minimal free resolution of M over R, then

$$\mathrm{Ext}^*_R(M, k) \otimes_k \ell = H_* \mathrm{Hom}_R(\mathbf{F}, k) \otimes_k \ell \cong H_* \mathrm{Hom}_{R'}(\mathbf{F} \otimes_R R', \ell) = \mathrm{Ext}^*_{R'}(M', \ell).$$

This provides an identification of $(\mathscr{X}^* \otimes_R k) \otimes_k \ell$ with $\mathscr{X}'^* \otimes_{R'} \ell$, that is compatible with the isomorphism $\mathrm{Ext}^*_R(M, k) \otimes_k \ell \cong \mathrm{Ext}^*_{R'}(M', \ell)$ of graded right modules over these algebras. As by (4.9) the $\mathscr{X}^* \otimes_R k$-module $\mathrm{Ext}^*_R(M, k)$ is finite, and the depth of a finite graded module over a finitely generated graded k-algebra is invariant under base change by field extensions of k, we see that $\mathrm{depth}_{(\mathscr{X}'^* \otimes_{R'} \ell)}(\mathrm{Ext}^*_R(M, k) \otimes_k \ell) = g$. This implies that the depth of $\mathrm{Ext}^*_{R'}(M', \ell)$ over $\mathscr{X}'^*$ is equal to g, and hence, in view of (4.7), so is its depth over the algebra $\mathscr{S}^*$ of cohomology operators of the deformation $R' \leftarrow Q$.

If the R-module M has a homological reduction of codimension c, then (6.2) shows that $c \leqslant \mathrm{depth}_{\mathscr{S}^*}(\mathrm{Ext}^*_R(M, k) \otimes_k \ell)$, that is, $c \leqslant g$. As ℓ is infinite, then we can choose in $\mathscr{S}^2$ a regular sequence $\boldsymbol{\eta}$ of length g, that generates an R′-direct summand of that free R′-module. By (5.8), there is a factorization $R' \leftarrow Q' \leftarrow Q$ such that the algebra of cohomology operators defined by $R' \leftarrow Q'$ is equal to $R'[\eta]$. From (6.6.1) we see that $R \to R' \leftarrow Q'$ is a homological reduction of M, hence $\mathrm{hom\,red}_R\, M \geqslant g$.

We now know that hom $\mathrm{red}_{\mathrm{R}}\, \mathrm{M} = g$. Assuming the Q′-module M′ has a homological reduction $\mathrm{Q}' \to \mathrm{Q}'' \leftarrow \mathrm{P}$ of positive codimension b, we see from (6.6.1) that $\mathrm{R} \to \mathrm{R}'' \leftarrow \mathrm{P}$ is a homological reduction of the R-module M. But its codimension $g + b$ is strictly greater than hom $\mathrm{red}_{\mathrm{R}}\, \mathrm{M}$, which is absurd. This proves M′ is homologically irreducible over Q′.

Finally, recall that $\mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, k)$ is a bimodule over $\mathscr{X}^*$ and $\mathscr{P}^*$, that by (4.9) and (4.10) is finite over each one of these algebras. As depth is invariant by finite descent, we conclude from (6.7) that $\mathrm{depth}_{\mathscr{X}^*} \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, k) = \mathrm{depth}_{\mathscr{P}^*} \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, k)$.

All the assertions of (6.4) and (6.5) have now been established. □

7. Critical degree and growth of Betti numbers

We consider finite modules over a local ring $(\mathrm{R}, \mathfrak{m}, k)$.

We prove there is a critical degree in the minimal resolution of each module M of finite CI-dimension, after which asymptotically stable patterns develop. In particular, we prove that beyond this degree the Betti sequence is non-decreasing, answering a basic question of [7] in the case of finite CI-dimension—and thus for all modules over complete intersections.

We provide examples to illustrate unstable phenomena at the start of Betti sequences. Furthermore, we show that the syzygy following the critical degree is the simplest one in the entire syzygy sequence: it admits a homological reduction of highest codimension, and thus its properties are determined by a module of lowest possible complexity.

(7.1) *Critical degree.* — An R-module M is said to have critical degree at most s, denoted by $\mathrm{cr\,deg}_{\mathrm{R}}\, \mathrm{M} \leqslant s$, if its minimal resolution $\mathbf{F}$ has a chain endomorphism μ of degree $q < 0$, such that $\mu_{n+q} : \mathrm{F}_{n+q} \to \mathrm{F}_n$ is surjective for all $n > s$; if no such s exists, we set $\mathrm{cr\,deg}_{\mathrm{R}}\, \mathrm{M} = \infty$. Clearly, $\mathrm{cr\,deg}\, 0 = -\infty$, and $-1 \leqslant \mathrm{cr\,deg}_{\mathrm{R}}\, \mathrm{M} \leqslant \infty$ when $\mathrm{M} \neq 0$.

If $\mathrm{M} \neq 0$ and $\mathrm{pd}_{\mathrm{R}}\, \mathrm{M}$ is finite, then $\mathrm{cr\,deg}_{\mathrm{R}}\, \mathrm{M} = \mathrm{pd}_{\mathrm{R}}\, \mathrm{M}$, so the critical degree sharpens the measure of inhomogeneity of free resolutions expressed by the notion of projective dimension. More generally, if M *has period q after s steps*, in the sense that $\mathrm{Syz}^{\mathrm{R}}_n(\mathrm{M}) \cong \mathrm{Syz}^{\mathrm{R}}_{n+q}(\mathrm{M})$ for $n > s$, then $\mathrm{cr\,deg}_{\mathrm{R}}\, \mathrm{M} \leqslant s$. For modules of finite CI-dimension we have a cohomological characterization, using the graded algebras $\mathscr{X}^*$ and $\mathscr{P}^*$ introduced in (4.9) and (4.10).

(7.2) *Proposition.* — *If* $\mathrm{M} \neq 0$ *is a finite* R*-module with* $\mathrm{CI\text{-}dim}_{\mathrm{R}}\, \mathrm{M} < \infty$, *then the critical degree of* M *is finite, say* $\mathrm{cr\,deg}_{\mathrm{R}}\, \mathrm{M} = s$, *and the following hold.*

(1) *There are equalities*

$$\begin{aligned}\mathrm{cr\,deg}_{\mathrm{R}}\, \mathrm{M} &= \sup\{ r \in \mathbf{N} \mid \mathrm{depth}_{\mathscr{P}^*} \mathrm{Ext}^{\geqslant r}_{\mathrm{R}}(\mathrm{M}, k) = 0 \} \\ &= \sup\{ r \in \mathbf{N} \mid \mathrm{depth}_{\mathscr{X}^*} \mathrm{Ext}^{\geqslant r}_{\mathrm{R}}(\mathrm{M}, k) = 0 \}.\end{aligned}$$

(2) *There is a codimension* 1 *quasi-deformation* $\mathrm{R} \to \mathrm{R}' \leftarrow \mathrm{Q}$, *such that* $\mathrm{cr\,deg}_{\mathrm{R}'}\, \mathrm{M}' = s$ *and the Eisenbud operator on the minimal resolution of* M′ *is surjective in degrees* $n > s$.

Part (0) of the next theorem is well known and is included only for comparison. Part (1), that can also be obtained from [8; (4.4)] and extends [18; (7.2)], shows that modules of finite CI-dimension satisfy a conjecture of Eisenbud [18], that fails in general [21].

(7.3) *Theorem. — Let* $\mathrm{M} \neq 0$ *be a finite* R*-module with* $\operatorname{depth} \mathrm{R} - \operatorname{depth}_{\mathrm{R}} \mathrm{M} = g$ *and* $\operatorname{cr}\deg_{\mathrm{R}} \mathrm{M} = s$. *When* $\mathrm{CI}\text{-}\dim_{\mathrm{R}} \mathrm{M}$ *is finite and* $n > s$, *the following three cases occur.*

(0) *If* $\operatorname{cx}_{\mathrm{R}} \mathrm{M} = 0$ *then* $s = g$ *and* $b_n^{\mathrm{R}}(\mathrm{M}) = 0$.

(1) *If* $\operatorname{cx}_{\mathrm{R}} \mathrm{M} = 1$ *then* $s \leqslant g$ *and* $b_n^{\mathrm{R}}(\mathrm{M}) = b$ *for some integer* $b > 0$; *furthermore,* M *has period* 2 *after* $s + 1$ *steps.*

(2) *If* $\operatorname{cx}_{\mathrm{R}} \mathrm{M} \geqslant 2$ *then* $s < \infty$ *and* $b_n^{\mathrm{R}}(\mathrm{M}) < b_{n+1}^{\mathrm{R}}(\mathrm{M})$.

The next result shows that $\operatorname{cr}\deg_{\mathrm{R}} \mathrm{M}$ determines the homological reducibility of all the syzygies M_n. It contains the result of Eisenbud [18; (8.2)] that high syzygies have a homological reduction of codimension 1; we also prove that, surprisingly, they cannot be deformed to any higher codimension when $n > \operatorname{cr}\deg_{\mathrm{R}} \mathrm{M} + 1$.

(7.4) *Theorem. — Let* M *be a finite* R*-module of finite* CI*-dimension, with* $s = \operatorname{cr}\deg_{\mathrm{R}} \mathrm{M}$. *The homological reducibility of the n-th syzygy* M_n *of* M *is then described as follows:*

(1) M_n *is homologically irreducible for* $0 \leqslant n \leqslant s$;

(2) M_{s+1} *has a homological reduction of codimension* $h \geqslant 1$;

(3) M_n *has a reduction of codimension* 1 *and none of higher codimension, if* $n > s + 1$.

If $h \geqslant 2$, *then* $\operatorname{Ext}_{\mathrm{R}}^{\leqslant s}(\mathrm{M}, k)$ *is the largest submodule of finite length of* $\operatorname{Ext}_{\mathrm{R}}^{*}(\mathrm{M}, k)$, *both as a left* $\operatorname{Ext}_{\mathrm{R}}^{*}(k, k)$*-module and as a right* $\operatorname{Ext}_{\mathrm{R}}^{*}(\mathrm{M}, \mathrm{M})$*-module, and the decomposition*

$$\operatorname{Ext}_{\mathrm{R}}^{*}(\mathrm{M}, k) = \operatorname{Ext}_{\mathrm{R}}^{\leqslant s}(\mathrm{M}, k) \oplus \operatorname{Ext}_{\mathrm{R}}^{\geqslant s}(\mathrm{M}, k)$$

is compatible with both structures.

Before giving arguments, we illustrate " strange " patterns of growth of Betti numbers of modules of finite CI-dimension, based on the modules $\mathrm{M}_i = \mathrm{M}_i^{[c]}$ from (3.1), with $(\mathrm{Q}, \mathfrak{n})$ is a local ring with $\operatorname{edim} \mathrm{Q} = m$, $\mathbf{x}$ is a length $c \geqslant 2$ regular sequence in $\mathfrak{n}^2$, and $\mathrm{R} = \mathrm{Q}/(\mathbf{x})$.

The critical degree of a module of complexity $d \leqslant 1$ is bounded by depth R, but

(7.5) *Example. — There is no bound on the critical degree, valid for all modules of complexity equal to some integer* $d \geqslant 2$.

Indeed, fix $s \geqslant 0$ and note that M_0 is the $s + 1$-st syzygy of M_{-s-1}. As the graded $\mathscr{P}^*$-module $\operatorname{Ext}_{\mathrm{R}}^{*}(\mathrm{M}_{-s-1}, k)$ is isomorphic to

$$\operatorname{Ext}_{\mathrm{R}}^{\leqslant s}(\mathrm{M}_{-s-1}, k) \oplus k[\chi] \otimes_k \Sigma^{-s-1} \operatorname{Ext}_{\mathrm{Q}}^{*}(\mathrm{M}_0, k)$$

by (6.2) and (7.4), it follows from (7.2.1) that $\operatorname{cr}\deg_{\mathrm{R}} \mathrm{M}_{-s-1} = s$.

The first (depth R + 1) Betti numbers contain the information that $\mathrm{cx}_R\, M = 0$, but

(7.6) *Example.* — *No finite interval of constant Betti numbers implies that* $\mathrm{cx}_R\, M = 1$.

Indeed, for $c = 2$, $s \geqslant 0$, and $0 \leqslant n \leqslant s$ we have $b_n^R(M_0 \oplus M_{-s-1}) = 4^m(s+2)$ by (3.1).

The Betti sequence of any module is monotonic after the critical degree, but

(7.7) *Example.* — *Strict growth of the Betti sequence does not signal the critical degree.*

Take $c = 2$ and $s \geqslant 0$ and note that $M_0 \oplus M_0 \oplus M_{-s-1}$ has a strictly increasing Betti sequence by (3.1), and critical degree s by (7.5).

Among the consequences of the finiteness of $\mathrm{cr\,deg}_R\, M$ is the validity of a weak form of a conjecture of Eisenbud [18; p. 37] (boundedness implies periodicity), as well as a partial answer to a question raised in [7; p. 34] (is the Betti sequence of each finite R-module eventually non-decreasing?). Note also that the next result implies that the non-periodic modules with constant Betti numbers constructed in [21; § 3] have infinite critical degree.

(7.8) *Theorem.* — *If* M *is a finite* R*-module such that* $\mathrm{cr\,deg}_R\, M = s < \infty$ *and* $\mu : \mathbf{F} \to \mathbf{F}$ *is a chain endomorphism that satisfies* (7.1), *then either* $\mathrm{cx}_R\, M \leqslant 1$ *and* M *has period* q *after* $\operatorname{depth} R - \operatorname{depth}_R M$ *steps, or* $\mathrm{cx}_R\, M > 1$ *and* $b_n^R(M) < b_{n+q}^R(M)$ *for* $n > s$.

If furthermore $q \leqslant 2$, *then* $b_n^R(M) \leqslant b_{n+1}^R(M)$ *for* $n > s$, *with equality when* $\mathrm{cx}_R\, M \leqslant 1$.

Proof. — Let $\mathbf{F}$ and μ be as in (7.1), set $M_n = \partial(F_n)$ for $n \in \mathbf{Z}$, let $\tilde{\mu} : F_q \to M$ be the composition of $\mu_q : F_q \to F_0$ with the augmentation $F_0 \to M$, and identify $\mathrm{Ext}_R^*(M, k)$ with $H^* \operatorname{Hom}_R(\mathbf{F}, k)$. Consider the pushout diagram with exact rows

$$\begin{tikzcd}
F_{q+1} \arrow[r, "\partial_{q+1}"] & F_q \arrow[r, "\partial_q"] \arrow[d, "\tilde{\mu}"'] & F_{q-1} \arrow[r, "\partial_{q-1}"] \arrow[d] & F_{q-2} \arrow[r, "\partial_{q-2}"] \arrow[d, equal] & \cdots \arrow[r, "\partial_1"] & F_0 \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & M(\mu) \arrow[r] & F_{q-2} \arrow[r, "\partial_{q-2}"] & \cdots \arrow[r, "\partial_1"] & F_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

It follows from [27; § III.9] that $\mu^n = \operatorname{Hom}_R(\mu_{n+q}, k)$ is the iterated connecting homomorphism of the bottom row, which itself is the Yoneda splice of the exact sequences

(7.8.1) $\qquad 0 \to M \to M(\mu) \to M_{q-1} \to 0,$

(7.8.2) $\qquad 0 \to M_{q-1} \to F_{q-2} \xrightarrow{\partial_{q-2}} \cdots \xrightarrow{\partial_1} F_0 \to M \to 0.$

Thus, $\mu^n = \pm\, \beta^{n+1} \alpha^n$, with α^n the connecting homomorphism of (7.8.1) and β^{n+1} the iterated connecting homomorphism of (7.8.2). As β^{n+1} is bijective for all n, the cohomology exact sequence of (7.8.1) takes the form

$$\ldots \to \mathrm{Ext}_R^n(M(\mu), k) \to \mathrm{Ext}_R^n(M, k) \xrightarrow{\mu^n} \mathrm{Ext}_R^{n+q}(M, k) \\ \to \mathrm{Ext}_R^{n+1}(M(\mu), k) \to \ldots .$$

Since μ^n is the k-dual of $\mu_{n+q}: F_{n+q} \otimes_R k \to F_n \otimes_R k$, that by assumption is surjective for $n > s$, we see that μ^n is injective for $n > s$, the exact sequence splits, and

$$\textbf{(7.8.3)} \qquad b^R_{n+q}(M) = b^R_n(M) + b^R_{n+1}(M(\mu)).$$

We see that M has complexity $\leqslant 1$ precisely when $r = \mathrm{pd}_R\, M(\mu) < \infty$. Set $b_n = b^R_n(M)$.

If $\mathrm{cx}_R\, M > 1$, then (7.8.3) yields $b_n < b_{n+q}$ for $n > s$. On the other hand, if $\mathrm{cx}_R\, M \leqslant 1$, then μ^n is an isomorphism for $n > r$, hence by Nakayama μ_{n+q} is a surjective homomorphism of free R-modules of the same rank, and thus an isomorphism.

As $r = \mathrm{depth}\, R - \mathrm{depth}_R\, M(\mu) \geqslant 0$, to finish the proof of the first part we show that $\mathrm{depth}_R\, M \leqslant \mathrm{depth}_R\, M(\mu)$. Assuming $\mathrm{depth}_R\, M > \mathrm{depth}\, R$, we use (7.8.2) to get $\mathrm{depth}_R\, M_{q-1} \geqslant \mathrm{depth}\, R$, and then (7.8.1) implies $\mathrm{depth}_R\, M(\mu) \geqslant \mathrm{depth}\, R$. As $M(\mu)$ has finite projective dimension over R, it is free, so M is itself periodic, and thus an infinite syzygy. This forces $\mathrm{depth}_R\, M = \mathrm{depth}\, R$, a contradiction. Assuming $\mathrm{depth}_R\, M(\mu) < \mathrm{depth}_R\, M \leqslant \mathrm{depth}\, R$, we get $\mathrm{depth}_R\, M_{q-1} = \mathrm{depth}_R\, M(\mu) < \mathrm{depth}_R\, M$ from (7.8.1) and $\mathrm{depth}_R\, M_{q-1} \geqslant \mathrm{depth}_R\, M$ from (7.8.2). This is a new contradiction, so we are done.

In the proof of the last assertion we may assume that $q = 2$. The map μ_{n+2} is surjective for $n > s$, and thus it induces a surjection $M_{n+2} \to M_n$. Choosing a minimal prime ideal $\mathfrak{p}$ of R and counting lengths over $R_\mathfrak{p}$ in the localizations of (7.8.2) and (7.8.1), we get

$$(b_{n+1} - b_n)\, \mathrm{length}(R_\mathfrak{p}) = \mathrm{length}(M_{n+2})_\mathfrak{p} - \mathrm{length}(M_n)_\mathfrak{p} \geqslant 0,$$

hence $b_{n+1} \geqslant b_n$ for $n > s$. If $\mathrm{cx}_R \leqslant 1$, then M_{r+1} is periodic of period 2 by (1), hence for $n > r$ we have $M_{n+2} \cong M_n$, and thus $b_{n+1} = b_n$. □

Proof of Proposition (7.2). — The finiteness of $\mathrm{cr\,deg}_R\, M = s$ follows from the equalities in (1), in view of (4.9). As both (1) and (2) are trivial when $\mathrm{pd}_R\, M$ is finite, we assume $\mathrm{cx}_R\, M > 0$. We use the notation of the preceding proof.

(1) By [27; § III.9], the homomorphism μ^n is, up to sign, given by Yoneda multiplication of $\mathrm{Ext}^n_R(M, k)$ with $\mathrm{cls}(\widetilde{\mu}) \in \mathrm{Ext}^q_R(M, M)$. Thus, the splitting of the cohomology exact sequence established above shows that $\mathrm{cls}(\widetilde{\mu})$ is a non-zero-divisor on $\mathrm{Ext}^{\geqslant s}_R(M, k)$. The graded subalgebra $\mathscr{X}^*[\mathrm{cls}(\widetilde{\mu})]$ of $\mathrm{Ext}^*_R(M, M)$ is commutative since $\mathscr{X}^*$ is central, and by (4.9) the graded $\mathscr{X}^*[\mathrm{cls}(\widetilde{\mu})]$-module $\mathrm{Ext}^{\geqslant s}_R(M, k)$ is finite. Thus, its depth over $\mathscr{X}^*$ coincides with its depth over $\mathscr{X}^*[\mathrm{cls}(\widetilde{\mu})]$, that we have just seen to be positive.

This implies $s = \mathrm{cr\,deg}_R\, M \geqslant \max\{ r \in \mathbf{N} \mid \mathrm{depth}_{\mathscr{X}^*} \mathrm{Ext}^{\geqslant r}_R(M, k) = 0 \}$. As each element of $\mathrm{Ext}^*_R(M, M)$, in particular of $\mathscr{X}^*$, comes from a chain endomorphism of $\mathbf{F}$, the opposite inequality is clear, and thus $\mathrm{cr\,deg}_R\, M \geqslant \max\{ r \in \mathbf{N} \mid \mathrm{depth}_{\mathscr{X}^*} \mathrm{Ext}^{\geqslant r}_R(M, k) = 0 \}$. On the other hand, by (4.9) and (4.10) the right $\mathscr{X}^*$- and left $\mathscr{P}^*$-bimodule $\mathrm{Ext}^{\geqslant r}_R(M, k)$ is finite on each side. It follows from the construction in (6.7) that its depth is the same over either algebra.

(2) By (1.14), choose a quasi-deformation $R \to R' \leftarrow Q$ such that $\mathfrak{m}R'$ is the maximal ideal of R', the residue field ℓ of $(Q, \mathfrak{n})$ is infinite, and $\mathrm{pd}_Q M'$ is finite. Note that $\mathrm{depth}_{\mathscr{S}} \mathrm{Ext}_{R'}^{\geqslant r}(M', \ell) = \mathrm{depth}_{\mathscr{T}} \mathrm{Ext}_{R'}^{\geqslant r}(M', \ell)$ for each r, so (1) yields $\mathrm{cr\,deg}_{R'} M' = \mathrm{cr\,deg}_R M$. In view of (4.7), it also shows that

$$\mathrm{depth}_{\mathscr{S}} \mathrm{Ext}_{R'}^{\geqslant s}(M', \ell) = \mathrm{depth}_{\mathscr{T}} \mathrm{Ext}_{R'}^{\geqslant s}(M', \ell) > 0.$$

As ℓ is infinite, there is an $\mathrm{Ext}_{R'}^{\geqslant s}(M', \ell)$-regular element η in $\mathscr{S}^2$. Note that $\eta \notin \mathfrak{n}\mathscr{S}^2$, for otherwise μ would annihilate $\mathrm{Ext}_{R'}^*(M', \ell)$. Thus, $R'\eta$ is a direct summand of $\mathscr{S}^2$. By (5.8) there is a factorization $R' \leftarrow Q' \leftarrow Q$ whose Eisenbud operator t' induces on $\mathrm{Ext}_{R'}^*(M', \ell)$ the same action as η. It follows that $\mathrm{Hom}_R(t'_{n+2}, k)$ is injective for $n > s$, hence t'_{n+2} is surjective by Nakayama. □

Proof of Theorem (7.3). — Let $R \to R' \leftarrow Q'$ be a quasi-deformation given by (7.2.2), in particular, $\mathrm{cr\,deg}_{R'} M' = \mathrm{cr\,deg}_R M = s$. As $R \to R'$ is a flat extension and $\mathrm{Syz}_n^{R'}(M') \cong \mathrm{Syz}_n^R(M) \otimes_R R'$ by (1.8.2), $\mathrm{Syz}_n^{R'}(M') \cong \mathrm{Syz}_{n+2}^{R'}(M')$ implies $\mathrm{Syz}_n^R(M) \cong \mathrm{Syz}_{n+2}^R(M)$, cf. [22; (2.5.8)]. Thus, neither the hypotheses nor the conclusions of the theorem change when we replace the R-module M by the R'-module M'. Changing notation, we assume $R = Q/(x)$ for a local ring Q and a non-zero-divisor x, and $t = t(Q, x, \mathbf{F})$ is an Eisenbud operator on a minimal resolution $\mathbf{F}$ of M over R, with t_{n+2} surjective when $n > s$.

By (4.1), there is a lifting of the complex $(\mathbf{F}, \partial)$ to a sequence of homomorphisms of free Q-modules $(\widetilde{\mathbf{F}}, \widetilde{\partial})$, and a degree -2 endomorphism $\widetilde{t}$ of $\widetilde{\mathbf{F}}$ such that $t = \widetilde{t} \otimes_Q R$. As $t_{n+2} = \widetilde{t}_{n+2} \otimes_Q Q/(x)$ is surjective, so is $\widetilde{t}_{n+2}$ by Nakayama. Thus, $\widetilde{F}_{n+2} = E \oplus G$ with $E = \mathrm{Ker}\, \widetilde{t}_{n+2}$, and the restriction θ of $\widetilde{t}_{n+2}$ to G is an isomorphism with $\widetilde{F}_n$.

Next we assume that $b_n^R(M) = b_{n+1}^R(M) = b \neq 0$ for some $n > s$, and show that $b_{n+2}^R(M) = b$. Let $\gamma : G \to \widetilde{F}_{n+1}$ be the restriction of $\widetilde{\partial}_{n+2}$. As $\widetilde{\partial}_{n+1}\gamma$ is the restriction to G of $\widetilde{\partial}_{n+1}\widetilde{\partial}_{n+2} = x\widetilde{t}_{n+2}$, we have $\widetilde{\partial}_{n+1}\gamma = x\theta$, and hence

$$\overset{b}{\wedge} \widetilde{\partial}_{n+1} \overset{b}{\wedge} \gamma = \overset{b}{\wedge} (\widetilde{\partial}_{n+1}\gamma) = \overset{b}{\wedge} (x\theta) = x^b \overset{b}{\wedge} \theta.$$

Note that G, $\widetilde{F}_{n+1}$, and $\widetilde{F}_n$ have rank b and fix isomorphisms of Q with $\wedge^b(G)$, $\wedge^b(\widetilde{F}_{n+1})$, and $\wedge^b(\widetilde{F}_n)$. The maps $\wedge^b \widetilde{\partial}_{n+1}$, $\wedge^b \gamma$, and $\wedge^b \theta$ are then given by multiplication with elements of Q, say y, z, and u, respectively. The equality above becomes $yz = x^b u$. As θ is bijective so is $\wedge^b(\theta)$, hence u is a unit in Q. Since x is a non-zero-divisor in Q, so is y, hence $\widetilde{\partial}_{n+1}$ is injective. From $\widetilde{\partial}_{n+1}\widetilde{\partial}_{n+2}(E) = x\widetilde{t}_{n+2}(E) = 0$ we now see that $E \subseteq \mathrm{Ker}\, \widetilde{\partial}_{n+2}$, so $\mathrm{Im}\, \widetilde{\partial}_{n+2}$ is a homomorphic image of $\widetilde{F}_{n+2}/E \cong G$. Remarking that

$$(\mathrm{Coker}\, \widetilde{\partial}_{n+3}) \otimes_Q R \cong \mathrm{Coker}\, \partial_{n+3} = M_{n+2}$$

we conclude that M_{n+2} is a homomorphic image of the free R-module $G \otimes_Q R \cong R^b$. It follows that $b_{n+2}^R(M) \leqslant b = b_n^R(M)$. On the other hand, we know from (7.8) that $b_{n+2}^R(M) \geqslant b_{n+1}^R(M) \geqslant b_n^R(M)$, hence all three numbers are equal to b.

Thus, the sequence $\{ b^{\mathrm{R}}_n(\mathrm{M}) \}_{n>s}$ is either strictly increasing or constant. In the second case the equality $\mathrm{rank}_{\mathrm{R}}\, \mathrm{F}_{n+2} = \mathrm{rank}_{\mathrm{R}}\, \mathrm{F}_n$ implies $\mathrm{E} = 0$, so the surjective homomorphism t_{n+2} is bijective, and hence the induced homomorphism $\mathrm{M}_{n+2} \to \mathrm{M}_n$ is bijective. □

Proof of Theorem (7.4). — By (6.4) and (7.2.1),

$$\mathrm{hom\ red}_{\mathrm{R}}\, \mathrm{M}_n = \mathrm{depth}_{\mathscr{P}^\bullet}\, \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}_n, k) = \mathrm{depth}_{\mathscr{P}^\bullet}\, \mathrm{Ext}^{\geqslant n}_{\mathrm{R}}(\mathrm{M}, k).$$

This number is 0 precisely for $n \leqslant s$. To see it is 1 for $n > s+1$, use the homological characterization of depth and the exact sequence of $\mathscr{P}^*$-modules

$$0 \to \mathrm{Ext}^{\geqslant n+1}_{\mathrm{R}}(\mathrm{M}, k) \to \mathrm{Ext}^{\geqslant n}_{\mathrm{R}}(\mathrm{M}, k) \to \Sigma^{-n} k^b \to 0,$$

where $b = b^{\mathrm{R}}_n(\mathrm{M})$. Set next $h = \mathrm{hom\ red}_{\mathrm{R}}\, \mathrm{M}_{s+1}$, and consider the exact sequence

$$0 \to \mathrm{Ext}^{>s}_{\mathrm{R}}(\mathrm{M}, k) \to \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, k) \to \mathrm{Ext}^{\leqslant s}_{\mathrm{R}}(\mathrm{M}, k) \to 0$$

of left $\mathrm{Ext}^*_{\mathrm{R}}(k, k) \otimes_{\mathrm{R}} \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}, \mathrm{M})^{\mathrm{op}}$-modules. The center of the tensor product contains the image of $\mathscr{P}^* \otimes_{\mathrm{R}} \mathscr{X}^*$, over which the graded module $\mathrm{Ext}^{>s}_{\mathrm{R}}(\mathrm{M}, k)$ is finite. Using again the construction of (6.7) and the invariance of depth under finite homomorphisms, we get

$$h = \mathrm{depth}_{\mathscr{P}^\bullet}\, \mathrm{Ext}^{>s}_{\mathrm{R}}(\mathrm{M}, k) = \mathrm{depth}_{(\mathscr{P}^\bullet \otimes \mathscr{X}^\bullet)}\, \mathrm{Ext}^{>s}_{\mathrm{R}}(\mathrm{M}, k) = \mathrm{depth}_{\mathscr{X}^\bullet}\, \mathrm{Ext}^{>s}_{\mathrm{R}}(\mathrm{M}, k).$$

Thus, an inequality $h \geqslant 2$ implies both that $\mathrm{Ext}^{>s}_{\mathrm{R}}(\mathrm{M}, k)$ has no non-trivial submodule of finite length, and that the exact sequence above splits. □

8. Asymptotes of Betti sequences

In this section R is a local ring, and M is a finite R-module of finite CI-dimension.

We show that the rational function representing the Poincaré series of M satisfies non-trivial arithmetical properties. As a consequence, we determine the first two terms of the asymptote of the Betti sequence of modules of higher complexity, and show that for big n the *gaps* $\{ b^{\mathrm{R}}_{n+1}(\mathrm{M}) - b^{\mathrm{R}}_n(\mathrm{M}) \}$ between consecutive Betti numbers grow essentially like a polynomial of degree $d-2$. This may be viewed as a quantitative sharpening of the assertion of (7.3.2), that if $\mathrm{cx}_{\mathrm{R}}\, \mathrm{M} \geqslant 2$, then the Betti numbers of M are eventually non-decreasing. An application of these results yields a description of all Betti sequences over several classes of local rings that are " close " to complete intersections.

(8.1) *Theorem. — Let* M *be a finite module over a local ring* R. *If* $\mathrm{CI\text{-}dim}_{\mathrm{R}}\, \mathrm{M} < \infty$ *(for instance, if* R *is a complete intersection) and* $\mathrm{cx}_{\mathrm{R}}\, \mathrm{M} = d \geqslant 2$, *then for* $n \gg 0$

$$b^{\mathrm{R}}_n(\mathrm{M}) = \begin{cases} b_+(n) & \text{when } n \text{ is even;} \\ b_-(n) & \text{when } n \text{ is odd,} \end{cases}$$

where $b_\pm \in \mathbf{Q}[t]$ *are polynomials of the form*

$$b_\pm(t) = \frac{b}{2^e(d-1)!}\, t^{d-1} + \frac{c_\pm}{2^d(d-2)!}\, t^{d-2} + \textit{lower order terms}$$

with integers b, $c_\pm$, e *such that either* $0 \leqslant e \leqslant d-2$, $c_+ = c_-$ *and* $b > 0$, *or* $e = d-1$ *and* $b > |\, c_+ - c_- \,|$. *In particular, both difference polynomials* $b_\pm(t+1) - b_\mp(t)$ *have degree* $d-2$ *and positive leading coefficient.*

(8.2) *Corollary.* — *When* $d \geqslant 2$ *ther exist periodic of period* 2 *functions* $\delta_i : \mathbf{Z} \to \mathbf{Q}$, $0 \leqslant i \leqslant d-2$, *such that* $\delta_0(0) > 0$, $\delta_0(1) > 0$, *and for* $n \gg 0$

$$b_n^{\mathrm{R}}(\mathrm{M}) - b_{n-1}^{\mathrm{R}}(\mathrm{M}) = \delta_0(n)\, n^{d-2} + \ldots + \delta_{d-2}(n). \quad \square$$

From (8.1), or from [8; (4.2)], one has $\lim_{n\to\infty} b_n^{\mathrm{R}}(\mathrm{M})/n^{d-1} \in (2^e(d-1)!)^{-1}\,\mathbf{Z}$, but

(8.3) *Example.* — *When* $c_+ = c_-$ *the sequence* $(b_n^{\mathrm{R}}(\mathrm{M}) - b_{n-1}^{\mathrm{R}}(\mathrm{M}))/n^{d-2}$ *converges to a number in* $(2^{d-1}(d-2)!)^{-1}\,\mathbf{Z}$; *when* $c_+ \neq c_-$ *it diverges*; *furthermore, both cases occur.*

Indeed, let $(\mathrm{Q}, \mathfrak{n})$ be a two-dimensional regular local ring, let $\mathbf{x} = x_1, x_2$ be a Q-regular sequence contained in $\mathfrak{n}^3$, set $(\mathrm{R}, \mathfrak{m}) = (\mathrm{Q}/(\mathbf{x}), \mathfrak{n}/(\mathbf{x}))$, and note that

$$b_n^{\mathrm{R}}(\mathrm{R}/\mathfrak{m}) = n+1 \quad \text{for each } n \geqslant 0$$

by the result of Tate recalled in (6.2). On the other hand, by [10; (2.1)]

$$b_n^{\mathrm{R}}(\mathrm{R}/\mathfrak{m}^2) = \frac{3}{2}n + 1 \ \text{ for even } n \geqslant 0$$

and

$$b_n^{\mathrm{R}}(\mathrm{R}/\mathfrak{m}^2) = \frac{3}{2}n + \frac{3}{2} \ \text{ for odd } n \geqslant 1.$$

The theorem has an interesting application to local rings " close " to complete intersections. Indeed, if R satisfies one of conditions *a)* through *d)* of the next theorem, then by [11] all finite R-modules have rational Poincaré series with a common denominator. Furthermore, [9] determines the possible denominators and shows that if $\mathrm{P}_{\mathrm{M}}^{\mathrm{R}}(t)$ has radius of convergence $\geqslant 1$, then M has finite virtual projective dimension, hence finite CI-dimension. For the rings in *e)* the corresponding analysis is carried out in [26]. Thus, we get

(8.4) *Theorem.* — *Let* R *be a local ring that satisfies one of the following conditions:*

a) R *is one link from a complete intersection*;
b) R *is two links from a complete intersection and* R *is Gorenstein*;
c) edim R — depth R $\leqslant 3$;
d) edim R — depth R $= 4$ *and* R *is Gorenstein*;
e) edim R — depth R $= 4$ *and* R *is a Cohen-Macaulay almost complete intersection of residual characteristic* $\neq 2$.

If M *is a finite* R*-module whose Poincaré series* $P^R_M(t)$ *has radius of convergence* $\rho \geqslant 1$, *then* $\mathrm{cx}_R\, M = d \leqslant \mathrm{edim}\, R - \mathrm{depth}\, R$ *and there exist polynomials* Δ_1 *and* Δ_2, *both of degree* $d - 2$ *and with positive leading coefficients, such that*

$$\Delta_1(n) \leqslant b^R_n(M) - b^R_{n-1}(M) \leqslant \Delta_2(n) \quad \textit{for } n \gg 0.$$

In particular, $\{ b^R_n(M) \}_{n \geqslant 0}$ *is eventually either constant or strictly increasing.* □

Remark. — Using the information on the denominators of Poincaré series from [9] and [26], Sun [35] has shown that if $\rho < 1$, then $\lim_{n\to\infty} b^R_{n+1}(M)/b^R_n(M) = 1/\rho > 1$. Thus, the asymptotic pattern—in particular, the eventual growth—of Betti sequences is now known for all modules over complete intersections, and over the rings described in (8.4).

We establish Theorem (8.1) in an equivalent form, stated in terms of Poincaré series.

A consequence of the finiteness result in (4.10) is that the Poincaré series of an R-module M of finite CI-dimension can be written uniquely in the form

(8.5) $$P^R_M(t) = \frac{p_M(t)}{(1-t)^d(1+t)^e} \quad \text{with } p_M \in \mathbf{Z}[t] \text{ such that } p_M(\pm 1) \neq 0.$$

The next theorem shows that it satisfies non-trivial arithmetical relations.

(8.6) *Theorem.* — *If* M *is a finite* R*-module with* $\mathrm{CI\text{-}dim}_R\, M < \infty$ *and Poincaré series given by* (8.5), *then* $\mathrm{cx}_R\, M = d$, $p_M(1) > 0$, *and one of the following cases occurs.*

(0) $d = 0 : e < 0$, *or* $e = 0$ *with* $p_M(-1) > 0$; *also,*

$$\deg p_M = \mathrm{depth}\, R - \mathrm{depth}_R\, M + e.$$

(1) $d = 1 : e \leqslant 0$, *and* $\deg p_M \leqslant \mathrm{depth}\, R - \mathrm{depth}_R\, M + e$.

(2) $d \geqslant 2 : e < d - 1$, *or* $e = d - 1$ *with* $p_M(1) > | p_M(-1) |$.

Proof. — Basic dimension theory and (5.3.1) yield $d = \mathrm{cx}_R\, M$. Set $s = \mathrm{cr\,deg}_R\, M$.

(0) Here $\mathrm{pd}_R\, M < \infty$, so $P^R_M(t)$ is a polynomial of degree $\mathrm{depth}\, R - \mathrm{depth}_R\, M$, hence $e \leqslant 0$ and $\deg p_M = \mathrm{depth}\, R - \mathrm{depth}_R\, M + e$. As $P^R_M(t)$ has positive coefficients, we get $p_M(1) = 2^e P^R_M(1) > 0$. If $e = 0$, then $p_M(-1) = P^R_M(-1)$ is the Euler characteristic of M, which is non-negative. As -1 is not a root of p_M, this yields $p_M(-1) > 0$.

(1) By (7.3.1), we have

$$s \leqslant \mathrm{depth}\, R - \mathrm{depth}_R\, M \quad \text{and} \quad \Sigma_{i > s}\, b^R_i(M)\, t^i = bt^{s+1}(1-t)^{-1} \neq 0.$$

(2) In view of (1.5) and (7.2.2), after performing a local flat extension of R we may assume that it has a deformation $R \leftarrow Q$ such that $\mathrm{pd}_Q\, M < \infty$, and that there exists a degree -2 chain endomorphism μ of the minimal free resolution of M, that is surjective in degrees $n > s$. Thus, we can switch to the notation of the proof of (7.8), with $q = 2$.

We prove that $e \leqslant d-1$ by induction on d. For $d=1$ this is contained in (1), so let $d \geqslant 2$. The exact sequence (7.8.2) shows that $\mathrm{pd}_{\mathrm{Q}}\, \mathrm{M}_1$ is finite, and then the exact sequence (7.8.1) yields $\mathrm{pd}_{\mathrm{Q}}\, \mathrm{M}(\mu) < \infty$, hence $\mathrm{CI}\text{-}\dim_{\mathrm{R}} \mathrm{M}(\mu) < \infty$. Write the Poincaré series

$$\text{(8.7.1)} \qquad \mathrm{P}^{\mathrm{R}}_{\mathrm{M}(\mu)}(t) = \frac{p_{\mathrm{M}(\mu)}(t)}{(1-t)^{d'}(1+t)^{e'}}$$

in irreducible form (8.5). The equalities (7.8.3) translate into an equality of power series $\mathrm{P}^{\mathrm{R}}_{\mathrm{M}}(t) = t^2\, \mathrm{P}^{\mathrm{R}}_{\mathrm{M}}(t) + t\mathrm{P}^{\mathrm{R}}_{\mathrm{M}(\mu)}(t) + r(t)$, for some polynomial $r \in \mathbf{Z}[t]$. Thus, we get

$$\text{(8.7.2)} \qquad \mathrm{P}^{\mathrm{R}}_{\mathrm{M}(\mu)}(t) = \frac{p_{\mathrm{M}}(t) - (1-t)^{d-1}(1+t)^{e-1}\, r(t)}{t(1-t)^{d-1}(1+t)^{e-1}}.$$

At $t=1$ the numerator above is equal to $p_{\mathrm{M}}(1) \neq 0$, so no factor $(1-t)$ cancels. Comparing (8.7.1) with (8.7.2), we see that $d' = \mathrm{cx}_{\mathrm{R}}\, \mathrm{M}(\mu) = d-1$, hence the induction hypothesis applies to $\mathrm{M}(\mu)$. For $e \leqslant 1$ the desired inequality $e \leqslant d-1$ is clear, so let $e > 1$. The numerator of (8.7.2) at $t=1$ is then $p_{\mathrm{M}}(-1) \neq 0$; by another comparison with (8.7.1):

$$\text{(8.7.3)} \qquad e = e' + 1 \leqslant d' = d-1.$$

Note that $u_0 = p_{\mathrm{M}}(1)/2^e$ and $v_0 = p_{\mathrm{M}}(-1)/2^d$ in the prime fractions decomposition

$$\frac{p_{\mathrm{M}}(t)}{(1-t)^d(1+t)^e} = \sum_{i=1}^{d} \frac{u_{d-i}}{(1-t)^i} + \sum_{j=1}^{e} \frac{v_{e-j}}{(1+t)^j} + g(t).$$

Thus, there are h_+ and h_- in $\mathbf{Q}[t]$, of degree at most $d-2$ and $e-2$, respectively, and

$$\text{(8.7.4)} \qquad b^{\mathrm{R}}_n(\mathrm{M}) = \frac{p_{\mathrm{M}}(1)}{2^e}\binom{n+d-1}{d-1} + (-1)^n \frac{p_{\mathrm{M}}(-1)}{2^d}\binom{n+e-1}{e-1} + h_+(n) + (-1)^n h_-(n)$$

for $n \gg 0$. As $d > e$ and the Betti sequence is positive, we see that $p_{\mathrm{M}}(1) > 0$.

To finish the proof of (8.6) we assume that $e = d-1 \geqslant 1$ and show by another induction on d that $p_{\mathrm{M}}(1) > |\, p_{\mathrm{M}}(-1)\,|$. If $d=2$, then for $n \gg 0$ in (8.7.4) $h_+(n) = h$, a constant, and $h_-(n) = 0$. The formula for the Betti numbers simplifies to

$$b^{\mathrm{R}}_n(\mathrm{M}) = \frac{p_{\mathrm{M}}(1)}{2}\, n + (-1)^n \frac{p_{\mathrm{M}}(-1)}{4} + h \quad \text{for } n \gg 0.$$

By (7.3.2) the Betti sequence of M is eventually strictly increasing. This means that the function $b^{\mathrm{R}}_n(\mathrm{M}) - b^{\mathrm{R}}_{n-1}(\mathrm{M}) = \frac{1}{2} p_{\mathrm{M}}(1) + (-1)^n \frac{1}{2} p_{\mathrm{M}}(-1)$ is positive for $n \gg 0$. In other words, $p_{\mathrm{M}}(1) > |\, p_{\mathrm{M}}(-1)\,|$, hence we have a basis for our induction. To perform the induction step, choose $d \geqslant 3$ and note from (8.7.3) that the module $M(\mu)$ has $e' = e - 1 = d - 2 = d' - 1 \geqslant 2$, hence the induction assumption applies to it. Finally, note the equality

$$p_M(t) = tp_{M(\mu)}(t) + (1-t)^{d-1}(1+t)^{e-1}r(t)$$

that results from (8.7.1), (8.7.2) and (8.7.3). It shows that $p_M(1) = p_{M(\mu)}(1)$ and $p_M(-1) = -p_{M(\mu)}(-1)$, hence $|p_M(1)| > |p_M(-1)|$ by induction. □

Proof of Theorem (8.1). — The polynomials $b_\pm$ whose existence is asserted by the theorem are obtained by splitting (8.7.4) into two expressions, one for each parity of n. By (8.7.3) we have $e \leqslant d - 1$. If the inequality is strict, then both polynomials b_+ and b_- have the same coefficient of t^{d-2}, hence both difference polynomials have degree $d - 2$ and leading coefficient $b/2^e(d-2)! > 0$. If $e = d - 1$, then the coefficients of t^{d-2} in the difference polynomials are $(b + c_\pm - c_\mp)/2^{d-1}(d-2)!$. As the numerator of this fraction is equal to $p_M(1) \pm p_M(-1)$, it is positive by (8.6.2). □

9. Obstructions to embeddings into standard resolutions

Consider a local ring $(R, \mathfrak{m}, k)$ with a deformation $\rho : Q \to R$ with kernel generated by a regular sequence $\mathbf{x} = x_1, \ldots, x_c$, and a finite R-module M. For the standard R-free resolution (3.10) of M, Eisenbud [18; p. 37] makes the following

Conjecture. — If M is a finite R-module M with $\mathrm{pd}_Q M < \infty$, then its minimal R-free resolution is a subcomplex of a standard resolution in such a way that the maps t_j may be chosen to be induced by the standard χ_j. In particular, the t_j may be chosen to commute.

We produce obstructions to its validity, and show that any ring R with an embedded deformation of codimension 2 has modules with non-vanishing obstruction.

(9.1) *Theorem. — If there exist a family* $\{ t_j = t_j(Q, \mathbf{x}, \mathbf{F}) \}_{1 \leqslant j \leqslant c}$ *of Eisenbud operators on the minimal* R*-free resolution* $\mathbf{F}$ *of* M, *a system* $\boldsymbol{\sigma}$ *of higher homotopies on* $\mathbf{E}$, *and a comparison* $\alpha : \mathbf{F} \to \mathbf{E}\{\boldsymbol{\sigma}\}$ *of* R*-free resolutions of* M *such that* $\alpha t_j = \chi_j \alpha$ *for* $1 \leqslant j \leqslant c$, *then the spectral sequence* (4.4.1) *stops at* ${}^2\mathrm{E}$.

In particular, the canonical homomorphism

$$\kappa^*_{Mk} : \frac{\mathrm{Ext}^*_R(M, k)}{(\boldsymbol{\chi})\,\mathrm{Ext}^*_R(M, k)} \to \mathrm{Ext}^*_Q(M, k)$$

from (4.3.2) *is injective, hence* $\mathrm{Ext}^*_R(M, k)$ *is generated over* $k[\chi]$ *in degrees* $\leqslant \mathrm{pd}_Q M$.

The theorem suggests a more precise version of an earlier conjecture, cf. [6; (5.3.1)]:

(9.2) *Conjecture.* — If $\mathbf{F}$ has *any* structure of DG $R[\chi]$-module, then κ^*_{Mk} is injective.

(9.3) *Example.* — *When* $s \geqslant 0$ *and* $c \geqslant 2$ *no embedding of a minimal resolution of the* R*-module* $\mathrm{M}_{-s-1} = \mathrm{M}^{[c]}_{-s-1}$ *into a standard resolution coming from a codimension* c *deformation is compatible with the action of the Eisenbud operators.*

Indeed, the direct sum decomposition of $\operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}_{-s-1}, k)$ described in (7.5) is also one of graded $k[\chi]$-modules, cf. (4.7), hence any system of generators contains elements of degree $\operatorname{pd}_{\mathrm{Q}} \mathrm{M}_0 + s + 1$. As $\operatorname{pd}_{\mathrm{Q}} \mathrm{M}_{-s-1} = \operatorname{pd}_{\mathrm{Q}} \mathrm{M}_0 = c$ by (3.1), we conclude by (9.1).

Proof of Theorem (9.1). — The hypotheses imply that by setting $\chi_j f = t_j(\mathrm{Q}, \mathbf{x}, \mathbf{F})(f)$ for $f \in \mathbf{F}$ and $1 \leqslant j \leqslant c$ one obtains on $\mathbf{F}$ a structure of DG $\mathrm{R}[\chi]$-module such that $\alpha : \mathbf{F} \to \mathbf{E}\{\boldsymbol{\sigma}\}$ becomes a morphism of DG modules over $\mathrm{R}[\chi]$. This produces a morphism

$$\operatorname{Hom}_{\mathrm{R}}(\alpha, k) : \operatorname{Hom}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, k) \to \operatorname{Hom}_{\mathrm{R}}(\mathbf{F}, k)$$

of DG modules. Filtering the Koszul DG module $\mathbf{K}$ by wedge degree, we get a morphism

$$\operatorname{Hom}_{\mathrm{R}}(\alpha, k) \otimes_{\mathrm{R}[\chi]} \mathbf{K} : \operatorname{Hom}_{\mathrm{R}}(\mathbf{E}\{\boldsymbol{\sigma}\}, k) \otimes_{\mathrm{R}[\chi]} \mathbf{K} \to \operatorname{Hom}_{\mathrm{R}}(\mathbf{F}, k) \otimes_{\mathrm{R}[\chi]} \mathbf{K}$$

of filtered DG modules. The first page of the induced morphism of spectral sequences is the map $\mathbf{K}(\boldsymbol{\chi}; \mathrm{H}^* \operatorname{Hom}_{\mathrm{R}}(\alpha, k))$ of Koszul complexes. As α is a comparison of R-free resolutions, $\mathrm{H}^* \operatorname{Hom}_{\mathrm{R}}(\alpha, k)$ is bijective, so the sequences are isomorphic from the page ${}^1\mathrm{E}$ onwards. Due to the minimality of the resolution $\mathbf{F}$, the differential of $\operatorname{Hom}_{\mathrm{R}}(\mathbf{F}, k)$ is trivial, hence the second spectral sequence stops at ${}^2\mathrm{E}$.

This implies the first spectral sequence, that by construction is the one in (4.4.1) with $\mathrm{N} = k$, also collapses at ${}^2\mathrm{E}$, and thus its vertical edge homomorphism ${}^2v^*_{\mathrm{M}k}$ is injective. By (4.4.3), this map is equal to the reduced change of rings homomorphism $\kappa^*_{\mathrm{M}k}$. □

Next we prove that the cohomology over R of the high R-syzygies of each finite R-module M completely determines its cohomology over Q. Thus, the obstructions of (9.1) vanish for such syzygies, leaving open the asymptotic form of Eisenbud's conjecture stated in [18; p. 37] as follows: " In the spirit of this paper, it would be interesting to prove this conjecture just for some truncation of the minimal free resolution. "

(9.4) *Theorem.* — *Let* $\boldsymbol{\chi}$ *be a system of cohomology operators defined by a codimension* c *deformation* $\rho : \mathrm{Q} \to \mathrm{R}$, *and let* M *be a finite* R*-module such that* $\operatorname{pd}_{\mathrm{Q}} \mathrm{M}$ *is finite. When* $n \gg 0$ *the spectral sequence* (4.4.1) *for the modules* $\mathrm{M}_n = \operatorname{Syz}^{\mathrm{R}}_n(\mathrm{M})$ *and* k *stops at the page* ${}^2\mathrm{E}$, *and produces for each* $p \in \mathbf{Z}$ *a natural exact sequence*

$$0 \to \mathrm{H}_{p-1}(\boldsymbol{\chi}; \operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}_n, k))^{2p-1} \to \operatorname{Ext}^p_{\mathrm{Q}}(\mathrm{M}_n, k) \xrightarrow{\delta^p} \mathrm{H}_p(\boldsymbol{\chi}; \operatorname{Ext}^*_{\mathrm{R}}(\mathrm{M}_n, k))^{2p} \to 0$$

of k*-linear homomorphisms, in which* δ^p *is the edge homomorphism from* (4.4.4).

Proof. — The graded module $\mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}_n, k) = \Sigma^n \mathrm{Ext}^{\geqslant n}_{\mathrm{R}}(\mathrm{M}, k)$ over the polynomial ring $k[\chi]$ splits as a direct sum of its even and odd parts. For $n \gg 0$ the minimal resolution of the even (respectively, odd) submodules has its p-th free module generated purely in degree $-2p$ (respectively, $-2p-1$): By simple regrading this can be deduced from [19; (1.1)], which shows that if $\mathscr{M}$ is a finite graded module over a polynomial ring generated by elements of upper degree 1 then for $n \gg 0$ the module $\mathscr{M}^{\geqslant n}$ has a (finite) linear resolution. As

$$\mathrm{Tor}^{k[\chi]}_p(k, \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}_n, k))^q \cong \mathrm{H}_p(\boldsymbol{\chi}; \mathrm{Ext}^*_{\mathrm{R}}(\mathrm{M}_n, k))^q,$$

we see that the sequence (4.4.1) for M_n and k has ${}^2\mathrm{E}^q_p = 0$, unless $-1 \leqslant 2p + q \leqslant 0$.

This leaves no space for non-trivial differentials, hence the sequence stops at ${}^2\mathrm{E}$ and produces a two-tier filtration of the limit term $\mathrm{Ext}^{q-p}_{\mathrm{Q}}(\mathrm{M}_n, k)$. The desired short exact sequences represent precisely this information. □

REFERENCES

[1] J. Alperin, L. Evens, Representations, resolutions, and Quillen's dimension theorem, *J. Pure Appl. Algebra* **22** (1981), 1-9.

[2] D. Anick, A counterexample to a conjecture of Serre, *Ann. of Math.* **115** (1982), 1-33.

[3] M. André, Hopf algebras with divided powers, *J. Algebra* **18** (1971), 19-50.

[4] E. F. Assmus, Jr., On the homology of local rings, *Ill. J. Math.* **3** (1959), 187-199.

[5] M. Auslander, M. Bridger, *Stable module theory*, *Mem. Amer. Math. Soc.* **94** (1969).

[6] L. L. Avramov, Obstructions to the existence of multiplicative structures on minimal free resolutions, *Amer. J. Math.* **103** (1981), 1-31.

[7] L. L. Avramov, Local algebra and rational homotopy, Homotopie algébrique et algèbre locale (J.-M. Lemaire, J.-C. Thomas, eds.), *Astérisque*, vol. 113-114, Soc. Math. France, Paris, 1984, p. 15-43.

[8] L. L. Avramov, Modules of finite virtual projective dimension, *Invent. math.* **96** (1989), 71-101.

[9] L. L. Avramov, Homological asymptotics of modules over local rings, *Commutative algebra* (M. Hochster, C. Huneke, J. Sally, eds.), *MSRI Publ.*, vol. 15, Springer, New York, 1989, p. 33-62.

[10] L. L. Avramov, Local rings over which all modules have rational Poincaré series, *J. Pure Appl. Algebra* **91** (1994), 29-48.

[11] L. L. Avramov, A. R. Kustin, M. Miller, Poincaré series of modules over local rings of small embedding codepth or small linking number, *J. Algebra* **118** (1988), 162-204.

[12] L. L. Avramov, L.-C. Sun, Cohomology operators defined by a deformation, *J. Algebra*, to appear.

[13] D. J. Benson, J. F. Carlson, Projective resolutions and Poincaré duality complexes, *Trans. Amer. Math. Soc.* **342** (1994), 447-488.

[14] N. Bourbaki, *Algèbre. III*, Nouvelle édition, Paris, Hermann, 1970.

[15] N. Bourbaki, *Algèbre commutative. IX*, Paris, Masson, 1983.

[16] R.-O. Buchweitz, G.-M. Greuel, F. Schreyer, Cohen-Macaulay modules on hypersurface singularities. II, *Invent. math.* **88** (1987), 165-182.

[17] J. A. Eagon, M. Hochster, R-sequences and indeterminates, *Quart. J. Math. Oxford Ser.* (2) **25** (1974), 61-71.

[18] D. Eisenbud, Homological algebra on a complete intersection, with an application to group representations, *Trans. Amer. Math. Soc.* **260** (1980), 35-64.

[19] D. Eisenbud, S. Goto, Linear free resolutions and minimal multiplicity, *J. Algebra* **88** (1984), 89-133.

[20] Y. Félix, S. Halperin, C. Jacobsson, C. Löfwall, J.-C. Thomas, The radical of the homotopy Lie algebra, *Amer. J. Math.*, **110** (1988), 301-322.

[21] V. N. Gasharov, I. V. Peeva, Boundedness versus periodicity over commutative local rings, *Trans. Amer. Math. Soc.* **320** (1990), 569-580.

[22] A. Grothendieck, Éléments de géométrie algébrique. IV_2, *Publ. Math. IHES* **24** (1965).

[23] T. H. Gulliksen, A change of rings theorem, with applications to Poincaré series and intersection multiplicity, *Math. Scand.* **34** (1974), 167-183.
[24] T. H. Gulliksen, On the deviations of a local ring, *Math. Scand.* **47** (1980), 5-20.
[25] J. Herzog, B. Ulrich, J. Backelin, Linear maximal Cohen-Macaulay modules over strict complete intersections, *J. Pure Appl. Algebra* **71** (1991), 187-202.
[26] A. R. Kustin, S. M. Palmer, The Poincaré series of every finitely generated module over a codimension 4 almost complete intersection is a rational function, *J. Pure Appl. Algebra* **95** (1994), 271-295.
[27] S. MacLane, *Homology*, Grundlehren Math. Wiss., vol. 114, Springer, Berlin, 1963.
[28] Yu. I. Manin, Some remarks on Koszul algebras and quantum groups, *Ann. Inst. Fourier* (Grenoble) **37** (1987), 191-205.
[29] H. Matsumura, *Commutative ring theory*, *Stud. Adv. Math.*, vol. 8, Cambridge, Univ. Press, 1986.
[30] V. B. Mehta, *Endomorphisms of complexes and modules over Golod rings*, Ph. D. Thesis, Univ. of California, Berkeley, 1976.
[31] J. W. Milnor, J. C. Moore, On the structure of Hopf algebras, *Ann. of Math.* (2) **81** (1965), 211-264.
[32] M. Nagata, *Local rings*, New York, Wiley, 1962.
[33] J. Shamash, The Poincaré series of a local rings, *J. Algebra* **12** (1969), 453-470.
[34] G. Sjödin, Hopf algebras and derivations, *J. Algebra* **64** (1980), 218-229.
[35] L.-C. Sun, Growth of Betti numbers over local rings of small embedding codepth or small linking number, *J. Pure Appl. Algebra* **96** (1994), 57-71.
[36] J. Tate, Homology of Noetherian rings and of local rings, *Ill. J. Math.* **1** (1957), 14-27.

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