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MORPHISMS, LINE BUNDLES AND MODULI SPACES IN REAL ALGEBRAIC GEOMETRY

by J. BOCHNAK, W. KUCHARZ *and* R. SILHOL*

1. Introduction

In the present article we investigate regular maps between real algebraic varieties and vector bundles on real algebraic varieties. The term *real algebraic variety* designates a locally ringed space isomorphic to a Zariski locally closed subset of $\mathbf{P}^n(\mathbf{R})$, for some n, endowed with the Zariski topology and the sheaf of $\mathbf{R}$-valued regular functions. Morphisms between real algebraic varieties are called *regular maps*. An equivalent description of real algebraic varieties can be obtained using reduced quasiprojective schemes over $\mathbf{R}$. Given such a scheme $\mathscr{X}$, let $\mathscr{X}(\mathbf{R})$ denote its set of $\mathbf{R}$-rational points. If $\mathscr{X}(\mathbf{R})$ is Zariski dense in $\mathscr{X}$, then we regard it as a real algebraic variety whose structure sheaf is the restriction of the structure sheaf of $\mathscr{X}$; up to isomorphism, each real algebraic variety is of this form. Note that $\mathscr{X}(\mathbf{R})$ is always contained in an affine open subset of $\mathscr{X}$, and hence each real algebraic variety is isomorphic to a Zariski closed subvariety of $\mathbf{R}^n$, for some n. Every real algebraic variety carries also the Euclidean topology, that is, the topology induced by the usual metric topology on $\mathbf{R}$. Unless explicitly stated otherwise, all topological notions related to real algebraic varieties will refer to the Euclidean topology.

Given two nonsingular real algebraic varieties X and Y, with X always assumed to be compact, we regard the set $\mathscr{R}(X, Y)$ of all regular maps from X into Y as a subset of the space $\mathscr{C}^\infty(X, Y)$ of all $\mathscr{C}^\infty$ maps from X into Y, endowed with the $\mathscr{C}^\infty$ topology, cf. [16]. The main object of our interest is the set

$$\mathscr{C}^\infty_{\mathscr{R}}(X, Y) = \text{the closure of } \mathscr{R}(X, Y) \text{ in } \mathscr{C}^\infty(X, Y).$$

In other words, we investigate which $\mathscr{C}^\infty$ maps from X into Y can be approximated by regular maps. Of course, a precursor of this problem is the classical Stone-Weierstrass

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approximation theorem, where $Y = \mathbf{R}$. The set $\mathscr{C}^\infty_{\mathscr{R}}(X, Y)$ has already been studied in [1, 2, 4, 5, 9], and related problems have been addressed in [3, 7, 10, 21]. The case where Y is the unit circle is completely elucidated in [5]. In this paper we deal almost exclusively with maps into the unit 2-sphere

$$S^2 = \{ (x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1 \}.$$

The approximation problem for maps into S^2 is closely tied to the theory of **C**-line bundles admitting an algebraic structure. Recall that a topological **C**-line bundle ξ on X is said to admit an *algebraic structure* if there exists an invertible $\mathscr{R}(X, \mathbf{C})$-module P, where **C** is identified with $\mathbf{R}^2$ and regarded as a real algebraic variety, such that the **C**-line bundle ξ_P on X associated with P in the usual way (Serre-Swan [27, 31]) is topologically isomorphic to ξ. Equivalently, ξ admits an algebraic structure if and only if it is topologically isomorphic to an algebraic **C**-vector subbundle of the trivial **C**-vector bundle on X with total space $X \times \mathbf{C}^k$ for some k [10]. We denote by $VB^1_{\mathbf{C}}(X)$ the group of isomorphism classes of topological **C**-line bundles on X, with group operation induced by tensor product of **C**-line bundles. Since X is compact, the subgroup $VB^1_{\mathbf{C}-\mathrm{alg}}(X)$ of $VB^1_{\mathbf{C}}(X)$ that consists of the isomorphism classes of topological **C**-line bundles on X admitting an algebraic structure is canonically isomorphic to the Picard group $\mathrm{Pic}(\mathscr{R}(X, \mathbf{C}))$ of isomorphism classes of invertible $\mathscr{R}(X, \mathbf{C})$-modules [1, Proposition 12.6.4]. We shall now explain how $\mathscr{C}^\infty_{\mathscr{R}}(X, S^2)$ and $VB^1_{\mathbf{C}-\mathrm{alg}}(X)$ are related to the Néron-Severi group.

Given an n-dimensional smooth projective scheme $\mathscr{V}$ over **C**, we regard its set of **C**-rational points $\mathscr{V}(\mathbf{C})$ as a complex manifold and denote by $H^2_{\mathrm{alg}}(\mathscr{V}(\mathbf{C}), \mathbf{Z})$ the subgroup of $H^2(\mathscr{V}(\mathbf{C}), \mathbf{Z})$ that consists of the cohomology classes Poincaré dual to the homology classes in $H_{2n-2}(\mathscr{V}(\mathbf{C}), \mathbf{Z})$ represented by divisors on $\mathscr{V}$. As usual, $H^2_{\mathrm{alg}}(\mathscr{V}(\mathbf{C}), \mathbf{Z})$ is identified with the Néron-Severi group of $\mathscr{V}$. If $\mathscr{X}$ is a smooth projective scheme over **R**, we put $\mathscr{X}_{\mathbf{C}} = \mathscr{X} \times_{\mathbf{R}} \mathbf{C}$ and do not distinguish between $\mathscr{X}_{\mathbf{C}}(\mathbf{C})$ and the set $\mathscr{X}(\mathbf{C})$ of **C**-rational points of $\mathscr{X}$. Thus, in particular, the group $H^2_{\mathrm{alg}}(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ is defined. Note that $\mathscr{X}(\mathbf{R})$ can be viewed as the set of fixed points of the action of the Galois group $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ on $\mathscr{X}(\mathbf{C})$. By the resolution of singularities theorem [15], there exists a smooth projective scheme $\mathscr{X}$ over **R** such that X and $\mathscr{X}(\mathbf{R})$ are isomorphic as real algebraic varieties. Identifying X with $\mathscr{X}(\mathbf{R})$, we set

$$H^2_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z}) = H^2(i)\,(H^2_{\mathrm{alg}}(\mathscr{X}(\mathbf{C}), \mathbf{Z})),$$

where $i : X \hookrightarrow \mathscr{X}(\mathbf{C})$ is the inclusion map. Throughout the paper, given a nonnegative integer k and a continuous map $\varphi : S \to T$ between topological spaces, we let $H^k(\varphi) : H^k(T, \mathbf{Z}) \to H^k(S, \mathbf{Z})$ denote the induced homomorphism. One easily sees that the subgroup $H^2_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z})$ of $H^2(X, \mathbf{Z})$ does not depend on the choice of $\mathscr{X}$ [10, p. 278]. The importance of the group $H^2_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z})$ stems from the following, already known result [10, Remark 5.4], [9, Proposition 2.2].

Theorem **1.0.** — *Let* X *be a compact nonsingular real algebraic variety. Then the canonical isomorphism*

$$c_1 : \mathrm{VB}^1_{\mathbf{C}}(\mathrm{X}) \to \mathrm{H}^2(\mathrm{X}, \mathbf{Z}),$$

induced by the first Chern class, maps $\mathrm{VB}^1_{\mathbf{C}-\mathrm{alg}}(\mathrm{X})$ *onto* $\mathrm{H}^2_{\mathbf{C}-\mathrm{alg}}(\mathrm{X}, \mathbf{Z})$. *Furthermore, given a* $\mathscr{C}^\infty$ *map* $f: \mathrm{X} \to \mathrm{S}^2$, *the following conditions are equivalent*:

a) f *is in* $\mathscr{C}^\infty_{\mathscr{R}}(\mathrm{X}, \mathrm{S}^2)$;

b) f *is homotopic to a regular map from* X *into* S^2;

c) $\mathrm{H}^2(f)(\kappa)$ *is in* $\mathrm{H}^2_{\mathbf{C}-\mathrm{alg}}(\mathrm{X}, \mathbf{Z})$, *where* κ *is a generator of* $\mathrm{H}^2(\mathrm{S}^2, \mathbf{Z}) \cong \mathbf{Z}$.

Let us examine more closely the case where X is a surface, which will play a special role in our considerations. Denote by $\pi^2(\mathrm{X})$ the set of homotopy classes $[f]$ of $\mathscr{C}^\infty$ maps $f: \mathrm{X} \to \mathrm{S}^2$. By Hopf's theorem, the map

$$h_{\mathrm{X}} : \pi^2(\mathrm{X}) \to \mathrm{H}^2(\mathrm{X}, \mathbf{Z}), \quad h_{\mathrm{X}}([f]) = \mathrm{H}^2(f)(\kappa)$$

is bijective and we endow $\pi^2(\mathrm{X})$ with group structure so that h_{X} becomes an isomorphism. The group $\pi^2(\mathrm{X})$ is known in topology as the second cohomotopy group of X, cf. [17]. It follows from Theorem 1.0 that the image of

$$\pi^2_{\mathscr{R}}(\mathrm{X}) = \{ [f] \in \pi^2(\mathrm{X}) \mid f \in \mathscr{R}(\mathrm{X}, \mathrm{S}^2) \}$$

under h_{X} is precisely $\mathrm{H}^2_{\mathbf{C}-\mathrm{alg}}(\mathrm{X}, \mathbf{Z})$. In particular, $\pi^2_{\mathscr{R}}(\mathrm{X})$ is a subgroup of $\pi^2(\mathrm{X})$ that determines completely $\mathscr{C}^\infty_{\mathscr{R}}(\mathrm{X}, \mathrm{S}^2)$. If X is connected and orientable, then $\pi^2(\mathrm{X})$ is isomorphic to $\mathbf{Z}$ and, in turn, the subgroup $\pi^2_{\mathscr{R}}(\mathrm{X})$ is determined completely by a single numerical invariant $b(\mathrm{X})$, which is a unique nonnegative integer satisfying

$$b(\mathrm{X})\, \pi^2(\mathrm{X}) = \pi^2_{\mathscr{R}}(\mathrm{X}).$$

Clearly, $b(\mathrm{X}) = 1$ if and only if the set $\mathscr{R}(\mathrm{X}, \mathrm{S}^2)$ is dense in $\mathscr{C}^\infty(\mathrm{X}, \mathrm{S}^2)$. Similarly, $b(\mathrm{X}) = 0$ if and only if every regular map from X into S^2 is null homotopic. More generally, a $\mathscr{C}^\infty$ map $f: \mathrm{X} \to \mathrm{S}^2$ belongs to $\mathscr{C}^\infty_{\mathscr{R}}(\mathrm{X}, \mathrm{S}^2)$ if and only if the topological degree $\deg(f)$ of f, computed with respect to some fixed orientations on X and S^2, is a multiple of $b(\mathrm{X})$.

Before stating new results of this paper, let us briefly review a few known facts concerning $\mathscr{C}^\infty_{\mathscr{R}}(\mathrm{X}, \mathrm{S}^2)$ and $\mathrm{VB}^1_{\mathbf{C}-\mathrm{alg}}(\mathrm{X})$ in order to give the reader an idea of diversity of occurring phenomena. Recall that a nonsingular real algebraic variety diffeomorphic to a $\mathscr{C}^\infty$ manifold M is called an *algebraic model* of M. Every closed $\mathscr{C}^\infty$ manifold admits uncountably many pairwise biregularly nonisomorphic algebraic models [6]. The following facts are known:

(i) Every closed $\mathscr{C}^\infty$ manifold M has an algebraic model X such that $\mathscr{R}(\mathrm{X}, \mathrm{S}^2)$ is dense in $\mathscr{C}^\infty(\mathrm{X}, \mathrm{S}^2)$ and $\mathrm{VB}^1_{\mathbf{C}-\mathrm{alg}}(\mathrm{X}) = \mathrm{VB}^1_{\mathbf{C}}(\mathrm{X})$ [8, Theorem 1.2].

(ii) A closed connected $\mathscr{C}^\infty$ surface M has the property that for every algebraic model X of M the set $\mathscr{R}(\mathrm{X}, \mathrm{S}^2)$ is dense in $\mathscr{C}^\infty(\mathrm{X}, \mathrm{S}^2)$ if and only if M is nonorientable and of odd genus [4, Theorem 2].

(iii) For every closed connected orientable $\mathscr{C}^\infty$ surface M and every nonnegative integer b there exists an algebraic model X of M with $b(X) = b$ [9, Theorem 1.1].

(iv) Orientable algebraic surfaces X in $\mathbf{P}^3(\mathbf{R})$ with $\pi^2_{\mathscr{R}}(X) = 0$ are generic in the following sense. Let $\mathscr{X}$ be a smooth subscheme of $\mathbf{P}^3_{\mathbf{R}}$ defined by a homogeneous polynomial of degree at least 4, whose all coefficients are algebraically independent over $\mathbf{Q}$. By [11], the Néron-Severi group $H^2_{alg}(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ of $\mathscr{X}_{\mathbf{C}}$ is generated by the cohomology class of a hyperplane section of $\mathscr{X}_{\mathbf{C}}$ in $\mathbf{P}^3_{\mathbf{C}}$. It follows that if $\mathscr{X}(\mathbf{R})$ is nonempty and orientable, then $H^2_{\mathbf{C}-alg}(\mathscr{X}(\mathbf{R}), \mathbf{Z}) = 0$ [10, Lemma 4.5], and hence $\pi^2_{\mathscr{R}}(\mathscr{X}(\mathbf{R})) = 0$.

(v) If S_{2n} is the Fermat 2-dimensional sphere of degree $2n$, that is,

$$S_{2n} = \{ (x, y, z) \in \mathbf{R}^3 \mid x^{2n} + y^{2n} + z^{2n} = 1 \},$$

then $\mathscr{R}(S_{2n}, S^2)$ is dense in $\mathscr{C}^\infty(S_{2n}, S^2)$ [4, Theorem 4.5]. In particular, " generic " cannot be omitted in (iv).

(vi) If F_k is the real Fermat curve in $\mathbf{P}^2(\mathbf{R})$ defined by $x^k + y^k = z^k$, then $b(F_1 \times F_1) = b(F_2 \times F_2) = 0$, $b(F_k \times F_k) = 1$ if k is odd and $k > 1$, and $1 \leqslant b(F_k \times F_k) \leqslant 2$ if k is even and $k > 2$ [9, Example 1.14]. We shall show in this paper that actually $b(F_4 \times F_4) = 2$, cf. Example 4.17.

(vii) Let $X = \mathscr{E}_1(\mathbf{R}) \times \ldots \times \mathscr{E}_n(\mathbf{R})$, where $\mathscr{E}_1, \ldots, \mathscr{E}_n$ are elliptic curves over $\mathbf{R}$, $n \geqslant 2$. Both $VB^1_{\mathbf{C}-alg}(X)$ and $\mathscr{C}^\infty_{\mathscr{R}}(X, S^2)$ are explicitly described in [7, 9]. For instance, $VB^1_{\mathbf{C}-alg}(X) = 0$ and $\mathscr{R}(X, S^2)$ contains only null homotopic maps if the elliptic curves $\mathscr{E}_{1\mathbf{C}}, \ldots, \mathscr{E}_{n\mathbf{C}}$ over $\mathbf{C}$ are pairwise nonisogenous. In particular, these conditions hold for a " generic " n-tuple $(\mathscr{E}_1, \ldots, \mathscr{E}_n)$. On the other hand, if one identifies isomorphic elliptic curves over $\mathbf{R}$, then $VB^1_{\mathbf{C}-alg}(X) = VB^1_{\mathbf{C}}(X)$, or equivalently, $\mathscr{R}(X, S^2)$ is dense in $\mathscr{C}^\infty(X, S^2)$ for countably many " exceptional " n-tuples $(\mathscr{E}_1, \ldots, \mathscr{E}_n)$, explicitly described by certain arithmetic conditions on periods of the $\mathscr{E}_i$.

In this paper we are especially interested in Abelian varieties over $\mathbf{R}$ and algebraic curves over $\mathbf{R}$.

Let $\mathscr{X}$ be a g-dimensional Abelian variety over $\mathbf{R}$. Then $X = \mathscr{X}(\mathbf{R})$ is a commutative real algebraic group with 2^r connected components for some integer r satisfying $0 \leqslant r \leqslant g$ (as $\mathscr{X}$ varies, all values of r with $0 \leqslant r \leqslant g$ do occur) [14, 26, 29]. Each connected component of X is diffeomorphic to the real torus $\mathbf{R}^g/\mathbf{Z}^g$ [14, 26, 29]. Given a point x in X, let $t_x : X \to X$ denote the translation by x, that is, $t_x(z) = x + z$ for z in X. Set

$$H^2(X, \mathbf{Z})^{inv} = \{ u \in H^2(X, \mathbf{Z}) \mid H^2(t_x)(u) = u \text{ for all } x \text{ in } X \},$$
$$VB^1_{\mathbf{C}}(X)^{inv} = \{ \alpha \in VB^1_{\mathbf{C}}(X) \mid t_x^*(\alpha) = \alpha \text{ for all } x \text{ in } X \},$$

where $t_x^* : VB^1_{\mathbf{C}}(X) \to VB^1_{\mathbf{C}}(X)$ is the isomorphism induced by pullback of $\mathbf{C}$-line bundles under t_x.

Proposition **1.1.** — *With the notation as above,* $VB^1_{\mathbf{C}}(X)^{inv}$ *and* $H^2(X, \mathbf{Z})^{inv}$ *are free Abelian groups of rank* $(g - 1)\, g/2$, *which satisfy*

$$c_1(VB^1_{\mathbf{C}}(X)^{inv}) = H^2(X, \mathbf{Z})^{inv},$$
$$VB^1_{\mathbf{C}-alg}(X) \subseteq VB^1_{\mathbf{C}}(X)^{inv}, \quad H^2_{\mathbf{C}-alg}(X, \mathbf{Z}) \subseteq H^2(X, \mathbf{Z})^{inv}.$$

Proposition 1.1, whose proof is quite simple, provides a natural " upper bound " for the size of the groups $VB^1_{C-alg}(X)$ and $H^2_{C-alg}(X, \mathbf{Z})$. Clearly, $H^2(X, \mathbf{Z})^{inv} = H^2(X, \mathbf{Z})$ is equivalent to the connectedness of X, and hence, in view of Proposition 1.1, X is connected if $VB^1_{C-alg}(X) = VB^1_C(X)$. Interjecting into this argument Theorem 1.0 and the fact that the group $H^2(X, \mathbf{Z})$ is generated by the elements of the form $H^2(f)(\kappa)$, where $f: X \to S^2$ is a $\mathscr{C}^\infty$ map, we also conclude that density of $\mathscr{R}(X, S^2)$ in $\mathscr{C}^\infty(X, S^2)$ implies connectedness of X.

Explicit computations that we intend to do will be in terms of a period matrix of $\mathscr{X}$. Let Ω be a complex $g \times 2g$ matrix such that the $\mathbf{Z}$-submodule $[\Omega]$ of $\mathbf{C}^g$ generated by the columns of Ω has rank $2g$ and is mapped onto itself by the complex conjugation of $\mathbf{C}^g$. The complex conjugation of $\mathbf{C}^g$ gives rise to group action of $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$ on the complex torus $\mathbf{C}^g/[\Omega]$. If there exists a $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-equivariant isomorphism between the complex Lie groups $\mathbf{C}^g/[\Omega]$ and $\mathscr{X}(\mathbf{C})$, then Ω is said to be a *period matrix* of $\mathscr{X}$. It is known that $\mathscr{X}$ admits a period matrix of the form (Z, I_g), where Z is a complex $g \times g$ matrix and I_g is the identity $g \times g$ matrix [26, 29]. Denote by Re Z and Im Z the real and the imaginary part of Z, respectively. One easily sees that then the matrix 2 Re Z has always integer entries, which justifies why below it is slightly more convenient to work with 2 Re Z instead of Re Z.

If A is a matrix, then ${}^tA$ will stand for its transpose. Denote by $\mathrm{Mat}_g(\mathbf{Z})$ the $\mathbf{Z}$-module of all $g \times g$ matrices with entries in $\mathbf{Z}$. Let $\mathrm{Alt}_g(\mathbf{Z})$ denote the $\mathbf{Z}$-submodule of $\mathrm{Mat}_g(\mathbf{Z})$ of all antisymmetric matrices; as usual, a matrix A is said to be antisymmetric if $A = -{}^tA$.

Given an arbitrary complex $g \times g$ matrix Z, define

$$C(Z)$$

to be the submodule of $\mathrm{Alt}_g(\mathbf{Z})$ that consists of all matrices C for which there exist matrices A in $\mathrm{Alt}_g(\mathbf{Z})$ and B in $\mathrm{Mat}_g(\mathbf{Z})$ such that for $M = 2\,\mathrm{Re}\,Z$ and $T = \mathrm{Im}\,Z$, the following conditions are satisfied:

$$\text{(1.2)}\qquad \begin{aligned} {}^tMCM - 4\,{}^tTCT &= 2(BM - {}^tM\,{}^tB) - 4A \\ {}^tMCT + {}^tTCM &= 2(BT - {}^tT\,{}^tB). \end{aligned}$$

Significance of C(Z) is explained by the following.

Theorem **1.3.** — *Let $\mathscr{X}$ be a g-dimensional Abelian variety over* $\mathbf{R}$ *and let* $X = \mathscr{X}(\mathbf{R})$. *If* $\Omega = (Z, I_g)$ *is a period matrix of* $\mathscr{X}$, *then every* $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$*-equivariant isomorphism of complex Lie groups* $\varphi : \mathbf{C}^g/[\Omega] \to \mathscr{X}(\mathbf{C})$ *gives rise to a group isomorphism*

$$\tau_\varphi : H^2(X, \mathbf{Z})^{inv} \to \mathrm{Alt}_g(\mathbf{Z})$$

satisfying

$$\tau_\varphi(H^2_{C-alg}(X, \mathbf{Z})) = C(Z).$$

Theorem 1.3 is a crucial ingredient in the proof of our next result, whose part (ii) is motivated by Proposition 1.1.

Theorem **1.4.** — *Let* $X = \mathscr{X}(\mathbf{R})$, *where* $\mathscr{X}$ *is a g-dimensional Abelian variety over* $\mathbf{R}$, *and let* (Z, I_g) *be a period matrix of* $\mathscr{X}$. *Then*:

(i) $VB^1_{C-alg}(X) = 0$ *if and only if the equation* ${}^t(\operatorname{Im} Z)\, C(\operatorname{Im} Z) = D$, *with* C *and* D *in* $\operatorname{Alt}_g(\mathbf{Z})$, *has only the trivial solution* $C = D = 0$.

(ii) $\operatorname{rank} VB^1_{C-alg}(X) = (g-1)\, g/2$ *if and only if every* 2×2 *minor determinant of the matrix* $\operatorname{Im} Z$ *is a rational number.*

An orientation of $X = \mathscr{X}(\mathbf{R})$, where $\mathscr{X}$ is an Abelian variety over $\mathbf{R}$, is said to be *invariant* if it is preserved by every translation $t_x : X \to X$ for x in X. Clearly, there are precisely two invariant orientations, regardless of the number of the connected components of X. This definition leads to a particularly simple interpretation of our results for Abelian surfaces.

Example **1.5.** — Let $X = \mathscr{X}(\mathbf{R})$, where $\mathscr{X}$ is an Abelian surface over $\mathbf{R}$. By Proposition 1.1, the groups $H^2(X, \mathbf{Z})^{inv}$ and $VB^1_C(X)^{inv}$ are isomorphic to $\mathbf{Z}$ and there exists a unique nonnegative integer $b(X)$ satisfying

$$H^2_{C-alg}(X, \mathbf{Z}) = b(X)\, H^2(X, \mathbf{Z})^{inv}, \qquad VB^1_{C-alg}(X) = b(X)\, VB^1_C(X)^{inv}.$$

We shall now give a characterization of the set $\mathscr{C}^\infty_{\mathscr{R}}(X, S^2)$ in terms of $b(X)$. To this end, endow X with an invariant orientation and fix an orientation on S^2. Let $X_1, \ldots, X_s$ be the connected components of X (recall that $s = 1, 2$ or 4, depending on the choice of $\mathscr{X}$). A $\mathscr{C}^\infty$ map $f : X \to S^2$ is in $\mathscr{C}^\infty_{\mathscr{R}}(X, S^2)$ if and only if

$$\deg(f \mid X_1) = \ldots = \deg(f \mid X_s) = k b(X)$$

for some integer k. Indeed this assertion follows from Theorem 1.0 and the first displayed equality in this example. In particular, $\operatorname{rank} \pi^2_{\mathscr{R}}(X) \leqslant 1$, whereas $\operatorname{rank} \pi^2(X) = s$. Of course, if X is connected, then the invariant $b(X)$ considered here coincides with the invariant $b(X)$ introduced subsequently to the definition of $\pi^2_{\mathscr{R}}(-)$.

If (Z, I_2) is a period matrix of $\mathscr{X}$, then Theorem 1.3 implies

$$C(Z) = b(X) \operatorname{Alt}_2(\mathbf{Z}).$$

Furthermore, in view of Theorem 1.4, $b(X) \neq 0$ if and only if $\det(\operatorname{Im} Z)$ is a rational number. The explicit computation of $b(X)$ is a difficult task due to the complexity of equations (1.2). This has been done in the case of $\mathscr{X}$ equal to the product of two elliptic curves over $\mathbf{R}$ [9]. The formulas for $b(X)$ strongly depend on the arithmetical properties of periods of the elliptic curves in question. As the elliptic curves vary, all nonnegative integers occur as values of $b(X)$.

Let us again consider Abelian varieties of arbitrary dimension. Theorem 1.4 (i) implies that $VB^1_{C-alg}(\mathscr{X}(\mathbf{R})) = 0$ for a " generic " Abelian variety $\mathscr{X}$ over $\mathbf{R}$. We shall make this vague remark precise for principally polarized Abelian varieties over $\mathbf{R}$.

Recall that the *moduli space* $\mathscr{A}_{\mathbf{R}}^{g}$ of g-dimensional principally polarized Abelian varieties over $\mathbf{R}$ is a topological space (even a stratified space with nonsingular real analytic strata [29, 30]), whose underlying set consists of the isomorphism classes $[\mathscr{Y}]$ of g-dimensional principally polarized Abelian varieties $\mathscr{Y}$ over $\mathbf{R}$. It follows from the local compactness of $\mathscr{A}_{\mathbf{R}}^{g}$ that it is a Baire space, that is, the intersection of any countable family of open and dense subset of $\mathscr{A}_{\mathbf{R}}^{g}$ is dense in $\mathscr{A}_{\mathbf{R}}^{g}$. Clearly, such an intersection is an uncountable set if $g \geqslant 1$.

Theorem **1.6.** — *Let g be a positive integer. Then:*

(i) *The set*

$$\{ [\mathscr{Y}] \in \mathscr{A}_{\mathbf{R}}^{g} \mid \mathrm{VB}^{1}_{\mathbf{C}-\mathrm{alg}}(\mathscr{Y}(\mathbf{R})) = 0 \}$$

is the intersection of a countable family of open and dense subsets of $\mathscr{A}_{\mathbf{R}}^{g}$.

(ii) *The set*

$$\{ [\mathscr{Y}] \in \mathscr{A}_{\mathbf{R}}^{g} \mid \mathrm{rank}\ \mathrm{VB}^{1}_{\mathbf{C}-\mathrm{alg}}(\mathscr{Y}(\mathbf{R})) = (g-1)\, g/2 \}$$

is uncountable and dense in $\mathscr{A}_{\mathbf{R}}^{g}$.

Recall that the number of the connected components of $\mathscr{A}_{\mathbf{R}}^{g}$ is equal to the integer part of $(3g+2)/2$, cf. [26, Theorem 6.1]. In particular, $\mathscr{A}_{\mathbf{R}}^{2}$ has 4 connected components and we have the following result, in which $b(-)$ stands for the invariant introduced in Example 1.5.

Theorem **1.7.** — *The intersection of the set*

$$\{ [\mathscr{Y}] \in \mathscr{A}_{\mathbf{R}}^{2} \mid b(\mathscr{Y}(\mathbf{R})) = 1 \}$$

with each connected component of $\mathscr{A}_{\mathbf{R}}^{2}$ is uncountable.

It is interesting to compare Theorem 1.7 with (vii) in our review of known facts above.

All results announced here dealing with Abelian varieties over $\mathbf{R}$ are proved in Section 2, which also contains some additional information on regular maps into S^2 and into unit spheres of dimension greater than 2.

We shall now describe our results concerning regular maps from X into S^2, where X is the product of compact nonsingular real algebraic curves $X_1, \ldots, X_n$ with $n \geqslant 2$. Since the group $H^2_{\mathbf{C}-\mathrm{alg}}(X_1 \times \ldots \times X_n, \mathbf{Z})$ is canonically isomorphic to the direct sum of the groups $H^2_{\mathbf{C}-\mathrm{alg}}(X_i \times X_j, \mathbf{Z})$ for $1 \leqslant i \leqslant j \leqslant n$ [9, Proposition 5 1], in view of Theorem 1.0, we restrict our attention to the case $n = 2$. Then the set $\mathscr{C}^{\infty}_{\mathscr{R}}(X_1 \times X_2, S^2)$ is completely determined by the group $\pi^2_{\mathscr{R}}(X_1 \times X_2)$, which is used as a main device in our presentation. The interested reader may himself recast the results in terms of $\mathbf{C}$-line bundles.

For convenience, a projective smooth scheme $\mathscr{X}$ over $\mathbf{R}$ of dimension 1 such that $\mathscr{X}_{\mathbf{C}}$ is irreducible will be called an *algebraic curve over* $\mathbf{R}$. We denote by $g(\mathscr{X})$ (resp. $s(\mathscr{X})$) the genus of $\mathscr{X}$ (resp. the number of the connected components of $\mathscr{X}(\mathbf{R})$). Recall that either $\mathscr{X}(\mathbf{C})\backslash\mathscr{X}(\mathbf{R})$ is connected, in which case $\mathscr{X}$ is said to be *nondividing*, or $\mathscr{X}(\mathbf{C})\backslash\mathscr{X}(\mathbf{R})$ has precisely 2 connected components and then $\mathscr{X}$ is said to be *dividing*. If $\mathscr{X}$ is nondividing (resp. dividing), then $0 \leqslant s(\mathscr{X}) \leqslant g(\mathscr{X})$ (resp. $1 \leqslant s(\mathscr{X}) \leqslant g(\mathscr{X}) + 1$ and $s(\mathscr{X}) \equiv g(\mathscr{X}) + 1 (\mathrm{mod}\ 2)$). In particular, $s(\mathscr{X}) = g(\mathscr{X})$ (resp. $s(\mathscr{X}) = g(\mathscr{X}) + 1$) implies that $\mathscr{X}$ is always nondividing (resp. dividing). These facts were essentially already known to F. Klein. For details the reader may refer to [14, 24, 26] and the literature cited there. For $\mathscr{X}$ nondividing with $s(\mathscr{X}) \geqslant 1$ (resp. $\mathscr{X}$ dividing or $\mathscr{X}$ with $s(\mathscr{X}) = 0$) we set $\varepsilon(\mathscr{X}) = 1$ (resp. $\varepsilon(\mathscr{X}) = 2$).

Let us recall that the moduli space $\mathscr{M}^g_{\mathbf{R}}$ of algebraic curves over $\mathbf{R}$ of genus g is a topological space (actually, even a stratified space with nonsingular real analytic strata [26, 30]), whose underlying set consists of the isomorphism classes $[\mathscr{X}]$ of algebraic curves $\mathscr{X}$ over $\mathbf{R}$ of genus g. It is well known that the family $\{\mathscr{M}^{(g,s,\varepsilon)}_{\mathbf{R}} \mid (s, \varepsilon) \in \Lambda_g \cup \{(0, 2)\}\}$, where

$$\Lambda_g = \Lambda^1_g \cup \Lambda^2_g,$$
$$\Lambda^1_g = \{(s, 1) \mid s \in \mathbf{Z},\ 1 \leqslant s \leqslant g\},$$
$$\Lambda^2_g = \{(s, 2) \mid s \in \mathbf{Z},\ 1 \leqslant s \leqslant g + 1,\ s \equiv g + 1 (\mathrm{mod}\ 2)\},$$
$$\mathscr{M}^{(g,s,\varepsilon)}_{\mathbf{R}} = \{[\mathscr{X}] \in \mathscr{M}^g_{\mathbf{R}} \mid (s(\mathscr{X}), \varepsilon(\mathscr{X})) = (s, \varepsilon)\},$$

is the set of connected components of $\mathscr{M}^g_{\mathbf{R}}$. Furthermore,

$$\dim \mathscr{M}^{(g,s,\varepsilon)}_{\mathbf{R}} = \begin{cases} g & \text{for } 0 \leqslant g \leqslant 1 \\ 3g - 3 & \text{for } g \geqslant 2 \end{cases}$$

for all (s, ε) in $\Lambda_g \cup \{(0, 2)\}$.

In most cases, algebraic curves $\mathscr{X}$ over $\mathbf{R}$ with $\mathscr{X}(\mathbf{R})$ empty (and hence the entire connected component $\mathscr{M}^{(g,0,2)}_{\mathbf{R}}$) will be of little interest to us.

Let $\mathscr{X}_k$ be an algebraic curve over $\mathbf{R}$ with $X_k = \mathscr{X}_k(\mathbf{R})$ nonempty for $k = 1, 2$. Obviously, $\pi^2(X_1 \times X_2)$ is a free Abelian group satisfying

$$\operatorname{rank} \pi^2(X_1 \times X_2) = s(\mathscr{X}_1)\, s(\mathscr{X}_2) \leqslant (g(\mathscr{X}_1) + 1)\, (g(\mathscr{X}_2) + 1).$$

Proposition **1.8.** — *With the notation as above,*

$$\operatorname{rank} \pi^2_{\mathscr{R}}(X_1 \times X_2) \leqslant (s(\mathscr{X}_1) - \varepsilon(\mathscr{X}_1) + 1)\, (s(\mathscr{X}_2) - \varepsilon(\mathscr{X}_2) + 1) \leqslant g(\mathscr{X}_1)\, g(\mathscr{X}_2).$$

The reader will find a slightly more detailed result in Proposition 3.8. An immediate consequence of Proposition 1.8 is that if $\mathscr{X}_k$ is dividing with X_k connected for $k = 1$ or $k = 2$, then $\pi^2_{\mathscr{R}}(X_1 \times X_2) = 0$. Other cases are much harder to handle. We investigate them by means of period matrices of $\mathscr{X}_1$ and $\mathscr{X}_2$.

It is useful to define first a certain $\mathbf{Z}$-module, which will play a similar role to the one defined by equations (1.2). Denote by $\mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$ the $\mathbf{Z}$-module of all $g_1 \times g_2$ matrices with entries in $\mathbf{Z}$. Let Z_k be a complex $g_k \times g_k$ matrix for $k = 1, 2$. We define

$$C(Z_1, Z_2)$$

to be the submodule of $\mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$ that consists of all matrices C for which there exist matrices G_1, G_2, H in $\mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$ such that for $M_k = 2 \operatorname{Re} Z_k$ and $T_k = \operatorname{Im} Z_k$, $k = 1, 2$, the following conditions are satisfied:

$$\text{(1.9)} \qquad \begin{aligned} {}^tM_1 CM_2 - 4\,{}^tT_1 CT_2 &= 2(G_1 M_2 - {}^tM_1 G_2) - 4H \\ {}^tM_1 CT_2 + {}^tT_1 CM_2 &= 2(G_1 T_2 - {}^tT_1 G_2). \end{aligned}$$

Theorem **1.10.** — *Let $\mathscr{X}_k$ be an algebraic curve over* $\mathbf{R}$ *of genus g_k with $X_k = \mathscr{X}_k(\mathbf{R})$ nonempty for $k = 1, 2$. Let (Z_k, I_{g_k}) be a period matrix of the Jacobian variety of $\mathscr{X}_k$. Then there exists a homomorphism*

$$\tau : \mathrm{Mat}(g_1 \times g_2, \mathbf{Z}) \to \pi^2(X_1 \times X_2)$$

satisfying

$$\tau(C(Z_1, Z_2)) = \pi^2_{\mathscr{R}}(X_1 \times X_2).$$

In particular, $\pi^2_{\mathscr{R}}(X_1 \times X_2)$ *is isomorphic to* $(C(Z_1, Z_2) + \operatorname{Ker} \tau)/\operatorname{Ker} \tau$. *Furthermore, if* $1 \leqslant g_k \leqslant 2$ *and if Z_k is suitably chosen for $k = 1, 2$, then* $\operatorname{Ker} \tau$ *and* $\tau(M(g_1 \times g_2, \mathbf{Z}))$ *can be explicitly described.*

We shall now elaborate on the last, vague statement in Theorem 1.10.

Let H_g be the space of all complex symmetric $g \times g$ matrices Z such that $2 \operatorname{Re} Z$ has integer entries and $\operatorname{Im} Z$ is positive definite; H_g is a subspace of the classical Siegel upper half space, suitable for the study of Abelian varieties over $\mathbf{R}$ and algebraic curves over $\mathbf{R}$, cf. [14, 26, 29, 30]. Every element Z in H_g determines in the usual way a principally polarized Abelian variety over $\mathbf{R}$, denoted by $\mathscr{Y}_Z$. By construction, $\mathscr{Y}_Z(\mathbf{C}) = \mathbf{C}^g/[\Omega]$, where $\Omega = (Z, I_g)$, and the principal polarization on $\mathscr{Y}_Z$ is determined by the alternating bilinear form $E_Z : [\Omega] \times [\Omega] \to \mathbf{Z}$ with matrix

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

with respect to the $\mathbf{Z}$-basis for $[\Omega]$ formed by the columns of Ω, cf. [14, 26, 28]. We say that Z is a *period matrix* of an algebraic curve $\mathscr{X}$ over $\mathbf{R}$ of genus g if $\mathscr{Y}_Z$ and the Jacobian variety of $\mathscr{X}$ are isomorphic as polarized Abelian varieties over $\mathbf{R}$.

It is well known that every algebraic curve over $\mathbf{R}$ of genus 1 has a unique period matrix in the subset

$$M^1 = \left\{ \frac{1}{2} + \sqrt{-1}t \mid t > 0 \right\} \cup \left\{ \sqrt{-1}t \mid t > 0 \right\}$$

of H_1, cf. [14, 26, 29] and Example 4.2. An analogous result for curves of genus 2 is new, interesting in its own right, and important in connection with Theorem 1.10.

Theorem **1.11.** — *There exists a subspace* M^2 *of* H_2, *described by a finite collection of explicitly known inequalities, such that every algebraic curve* $\mathscr{X}$ *over* $\mathbf{R}$ *of genus 2 has a unique period matrix* Z *in* M^2. *The correspondence* $\mathscr{X} \to Z$ *gives rise to a homeomorphism* $u : \mathscr{M}^2_{\mathbf{R}} \to M^2$.

Referring to the last statement in Theorem 1.10, we say that a period matrix of an algebraic curve over $\mathbf{R}$ of genus g, with $1 \leqslant g \leqslant 2$, is suitably chosen, provided that it belongs to M^g.

For example,

$$u(\mathscr{M}^{(2,1,1)}_{\mathbf{R}}) = \left\{ \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sqrt{-1}\begin{pmatrix}t_{11} & t_{12}\\ t_{12} & t_{22}\end{pmatrix} \in H_2 \mid 0 < t_{12} \leqslant t_{11} \leqslant t_{22} \right\},$$

and $u(\mathscr{M}^{(2,s,\varepsilon)}_{\mathbf{R}})$ can be described in a similar way for all (s, ε) in $\Lambda_2 \cup \{ (0, 2) \}$.

If $[\mathscr{X}_k]$ is in $\mathscr{M}^{(2,1,1)}_{\mathbf{R}}$, $X_k = \mathscr{X}_k(\mathbf{R})$, and $Z_k = u([\mathscr{X}_k])$ for $k = 1, 2$, then the homomorphism τ of Theorem 1.10 satisfies $\tau(M(2 \times 2, \mathbf{Z})) = \pi^2(X_1 \times X_2) \cong \mathbf{Z}$ and

$$\operatorname{Ker} \tau = \{ C = (c_{ij}) \in \operatorname{Mat}(2 \times 2, \mathbf{Z}) \mid c_{11} + c_{12} + c_{21} + c_{22} = 0 \}.$$

In all other cases, for curves of genus 1 or 2, a similar explicit description is known. Complete details related to Theorems 1.10 and 1.11 are given in Theorems 3.9, 4.3, 4.6, 4.7, and Examples 4.2, 4.5; Theorem 1.11 is included in Theorem 4.3.

We shall now describe a few results concerning the size of the subgroup $\pi^2_{\mathscr{R}}(-)$ of $\pi^2(-)$. Proposition 1.8 implies that the set

$$\{ ([\mathscr{X}_1], [\mathscr{X}_2]) \in \mathscr{M}^{(g_1, s_1, \varepsilon_1)}_{\mathbf{R}} \times \mathscr{M}^{(g_2, s_2, \varepsilon_2)}_{\mathbf{R}} \mid \operatorname{rank} \pi^2_{\mathscr{R}}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})) > (s_1 - \varepsilon_1 + 1)(s_2 - \varepsilon_2 + 1) \}$$

is empty for $((s_1, \varepsilon_1), (s_2, \varepsilon_2))$ in $\Lambda_{g_1} \times \Lambda_{g_2}$. The remaining possibilities for the rank of $\pi^2_{\mathscr{R}}(-)$ are examined below in the case $1 \leqslant g_k \leqslant 2$ for $k = 1, 2$.

Theorem **1.12.** — *Let* (s_k, ε_k) *be in* Λ_{g_k}, *where* $1 \leqslant g_k \leqslant 2$ *for* $k = 1, 2$, *and let* r *be an integer satisfying* $0 \leqslant r \leqslant (s_1 - \varepsilon_1 + 1)(s_2 - \varepsilon_2 + 1)$. *Then the set*

$$\{ ([\mathscr{X}_1], [\mathscr{X}_2]) \in \mathscr{M}^{(g_1, s_1, \varepsilon_1)}_{\mathbf{R}} \times \mathscr{M}^{(g_2, s_2, \varepsilon_2)}_{\mathbf{R}} \mid \operatorname{rank} \pi^2_{\mathscr{R}}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})) = r \}$$

is uncountable and dense in $\mathscr{M}^{(g_1, s_1, \varepsilon_1)}_{\mathbf{R}} \times \mathscr{M}^{(g_2, s_2, \varepsilon_2)}_{\mathbf{R}}$. *Furthermore, the set corresponding to* $r = 0$ *(that is,* $\pi^2_{\mathscr{R}}(-) = 0$*) is the intersection of a countable family of open and dense subsets of* $\mathscr{M}^{(g_1, s_1, \varepsilon_1)}_{\mathbf{R}} \times \mathscr{M}^{(g_2, s_2, \varepsilon_2)}_{\mathbf{R}}$.

Actually much more is known about the structure of the sets studied in Theorem 1.12. Each of them is described, in the sense made precise in Theorem 4.8, by inequalities involving a countable family of explicity known quadratic polynomials. Although each set is uncountable and dense, only the one corresponding to $r = 0$ is really " large ". The meaning of this last remark is clear since $\mathcal{M}_{\mathbf{R}}^{g_1} \times \mathcal{M}_{\mathbf{R}}^{g_2}$, due to its local compactness, is a Baire space. We conjecture that Theorem 1.12 is valid for curves of arbitrary genus.

Let us take a closer look at the case where $[\mathscr{X}_1] = [\mathscr{X}_2] = [\mathscr{X}]$ is in $\mathcal{M}_{\mathbf{R}}^{(g, s, \varepsilon)}$. Recall that then $\operatorname{rank} \pi^2_{\mathscr{R}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) \leqslant (s - \varepsilon + 1)^2 \leqslant g^2$, and $s - \varepsilon + 1 = g$ is equivalent to $s \geqslant g$.

Theorem **1.13.** — *For every* (s, ε) *in* Λ_g *with* $1 \leqslant g \leqslant 2$ *the set*

$$\{ [\mathscr{X}] \in \mathcal{M}_{\mathbf{R}}^{(g, s, \varepsilon)} \mid \pi^2_{\mathscr{R}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) = 0 \}$$

is the intersection of a countable family of open and dense subsets of $\mathcal{M}_{\mathbf{R}}^{(g, s, \varepsilon)}$, *whereas the set*

$$\{ [\mathscr{X}] \in \mathcal{M}_{\mathbf{R}}^{(g, s, \varepsilon)} \mid \operatorname{rank} \pi^2_{\mathscr{R}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) = (s - \varepsilon + 1)^2 \}$$

is dense in $\mathcal{M}_{\mathbf{R}}^{(g, s, \varepsilon)}$. *Furthermore, this last set is countable if and only if* $s \geqslant g$.

Let us mention that Theorem 1.13 is included in Theorem 4.9.

We have seen above that $\operatorname{rank} \pi^2_{\mathscr{R}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) \leqslant g^2$ for every algebraic curve $\mathscr{X}$ over $\mathbf{R}$ of genus g. The curves for which this maximum rank is attained are very special and it is not known whether they exist for $g \geqslant 4$, cf. Proposition 3.11 and the remark following its proof. On the other hand, according to Theorem 1.13, there are up to isomorphism precisely countably many such curves of genus g with $g = 1$ or $g = 2$.

As it transpires from our considerations above, the most common, " generic ", situation for a pair $(\mathscr{X}_1, \mathscr{X}_2)$ of algebraic curves over $\mathbf{R}$ of genus 1 or 2 is when $\mathscr{R}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R}), S^2)$ contains only null homotopic maps, that is,

$$\pi^2_{\mathscr{R}}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})) = 0.$$

We shall now consider the other extreme case, where $\mathscr{R}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R}), S^2)$ is dense in $\mathscr{C}^\infty(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R}), S^2)$ or, equivalently, where

$$\pi^2_{\mathscr{R}}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})) = \pi^2(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})).$$

The existence of such a pair of curves of given genera is far from obvious. It follows from Proposition 1.8 that both $\mathscr{X}_1$ and $\mathscr{X}_2$ must be nondividing, that is, $\varepsilon(\mathscr{X}_1) = \varepsilon(\mathscr{X}_2) = 1$. We shall see that this is the only restriction, at least for curves of genus 1 or 2. In fact our result is much stronger.

Theorem **1.14**. — (i) *For* $1 \leqslant g \leqslant 2$ *and* $1 \leqslant s \leqslant g$ *the set*

$$\{ [\mathscr{X}] \in \mathscr{M}_{\mathbf{R}}^{(g,s,1)} \mid \mathscr{C}^{\infty}_{\mathscr{R}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R}), \mathrm{S}^2) = \mathscr{C}^{\infty}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R}), \mathrm{S}^2) \}$$

is infinite. It is countable if and only if $s = g$.

(ii) *For* $1 \leqslant g_k \leqslant 2$ *and* $1 \leqslant s_k \leqslant g_k$, $k = 1, 2$, *the set*

$$\{ ([\mathscr{X}_1], [\mathscr{X}_2]) \in \mathscr{M}_{\mathbf{R}}^{(g_1,s_1,1)} \times \mathscr{M}_{\mathbf{R}}^{(g_2,s_2,1)} \mid \mathscr{C}^{\infty}_{\mathscr{R}}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R}), \mathrm{S}^2) = \mathscr{C}^{\infty}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R}), \mathrm{S}^2) \}$$

is infinite. Furthermore, it is countable if $(s_1, s_2) = (g_1, g_2)$, *and in all other cases, except perhaps when* $g_1 = g_2 = s_1 s_2 = 2$, *it is uncountable.*

Theorem 1.14 is equivalent to combined Theorems 4.10 and 4.11, whose proofs are quite long and use all main results of Sections 3 and 4.

2. Abelian varieties over R, line bundles, and regular maps

Let $\mathscr{X}$ be an Abelian variety over $\mathbf{R}$, let $\mathrm{X} = \mathscr{X}(\mathbf{R})$, and let X_0 be the connected component of X containing the identity element of the group X. Let $i : \mathrm{X} \hookrightarrow \mathscr{X}(\mathbf{C})$ and $j : \mathrm{X}_0 \hookrightarrow \mathrm{X}$ be the inclusion maps. For each nonnegative integer k, set

$$\mathrm{H}^k(\mathrm{X}, \mathbf{Z})^{\mathrm{inv}} = \{ u \in \mathrm{H}^k(\mathrm{X}, \mathbf{Z}) \mid \mathrm{H}^k(t_x)\,(u) = u \text{ for every } x \text{ in } \mathrm{X} \}$$

and define

$$\rho^k : \mathrm{H}^k(\mathrm{X}, \mathbf{Z})^{\mathrm{inv}} \to \mathrm{H}^k(\mathrm{X}_0, \mathbf{Z})$$

to be the restriction of the homomorphism $\mathrm{H}^k(j) : \mathrm{H}^k(\mathrm{X}, \mathbf{Z}) \to \mathrm{H}^k(\mathrm{X}_0, \mathbf{Z})$.

Proposition **2.1**. — *With the notation as above*:

(i) $\mathrm{H}^k(\mathrm{X}, \mathbf{Z})^{\mathrm{inv}} = \mathrm{H}^k(i)\,(\mathrm{H}^k(\mathscr{X}(\mathbf{C}), \mathbf{Z}))$,

(ii) ρ^k *is an isomorphism.*

Proof. — We shall first show that ρ^k is a monomorphism, that is, if $\mathrm{H}^k(j)\,(u) = 0$ for some u in $\mathrm{H}^k(\mathrm{X}, \mathbf{Z})^{\mathrm{inv}}$, then $u = 0$. To this end, let S denote a connected component of X, $j_{\mathrm{S}} : \mathrm{S} \hookrightarrow \mathrm{X}$ the inclusion map, x a point in S, and $q_x : \mathrm{X}_0 \to \mathrm{S}$ the restriction of the translation t_x. Since $t_x \circ j = j_{\mathrm{S}} \circ q_x$, we obtain

$$\mathrm{H}^k(q_x)\,(\mathrm{H}^k(j_{\mathrm{S}})\,(u)) = \mathrm{H}^k(j_{\mathrm{S}} \circ q_x)\,(u) = \mathrm{H}^k(t_x \circ j)\,(u) = \mathrm{H}^k(j)\,(\mathrm{H}^k(t_x)\,(u)) = \mathrm{H}^k(j)\,(u) = 0.$$

Clearly, q_x is a homeomorphism and hence $\mathrm{H}^*(j_{\mathrm{S}})\,(u) = 0$. This implies $u = 0$, the connected component S of X being arbitrary.

We now observe that $\mathrm{H}^k(i)\,(\mathrm{H}^k(\mathscr{X}(\mathbf{C}), \mathbf{Z})) \subseteq \mathrm{H}^k(\mathrm{X}, \mathbf{Z})^{\mathrm{inv}}$. Indeed let $i : \mathrm{X} \hookrightarrow \mathscr{X}(\mathbf{C})$ be the inclusion map. For each point y in $\mathscr{X}(\mathbf{C})$ the translation $\tau_y : \mathscr{X}(\mathbf{C}) \to \mathscr{X}(\mathbf{C})$ by y

is homotopic to the identity map of $\mathscr{X}(\mathbf{C})$. Since $\tau_y \circ i = i \circ t_y$ for y in X, the inclusion in question is satisfied.

To complete the proof of (i) and (ii) it suffices to show

$$H^k(j)\,(H^k(i)\,(H^k(\mathscr{X}(\mathbf{C}), \mathbf{Z}))) = H^k(X_0, \mathbf{Z}).$$

This however follows at once, provided that we can find a retraction $r : \mathscr{X}(\mathbf{C}) \to X_0$. Since $\mathscr{X}$ has a period matrix of the form $\Omega = (Z, I_g)$, the pairs $(\mathscr{X}(\mathbf{C}), X_0)$ and $(\mathbf{C}^g/[\Omega], \mathbf{R}^g/\mathbf{Z}^g)$ are homeomorphic, where $\mathbf{R}^g \subseteq \mathbf{C}^g$, $\mathbf{Z}^g = [\Omega] \cap \mathbf{R}^g$, and $\mathbf{R}^g/\mathbf{Z}^g$ is regarded as a subspace of $\mathbf{C}^g/[\Omega]$. Thus a retraction $r : \mathscr{X}(\mathbf{C}) \to X_0$ exists. □

Proof of Proposition 1.1. — It follows directly from the definitions of $VB^1_{\mathbf{C}}(X)^{\mathrm{inv}}$ and $H^2(X, \mathbf{Z})^{\mathrm{inv}}$ that $c_1(VB^1_{\mathbf{C}}(X)^{\mathrm{inv}}) = H^2(X, \mathbf{Z})^{\mathrm{inv}}$. Furthermore,

$$H^2_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z}) \subseteq H^2(X, \mathbf{Z})^{\mathrm{inv}}$$

is a consequence of the definition of $H^2_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z})$ and Proposition 2.1 (i), and therefore $VB^1_{\mathbf{C}-\mathrm{alg}}(X) \subseteq VB^1_{\mathbf{C}}(X)^{\mathrm{inv}}$.

Since X_0 is homeomorphic to $\mathbf{R}^g/\mathbf{Z}^g$, Proposition 2.1 (ii) implies

$$\operatorname{rank} H^2(X, \mathbf{Z})^{\mathrm{inv}} = \operatorname{rank} H^2(X_0, \mathbf{Z}) = (g-1)\,g/2.$$

Thus the proof is complete. □

Given a $\mathbf{Z}$-module Λ, we let $\mathrm{Alt}^2(\Lambda)$ denote the $\mathbf{Z}$-module of all alternating bilinear forms $\Lambda \times \Lambda \to \mathbf{Z}$.

Proof of Theorem 1.3. — Let $\Omega = (Z, I_g)$ be a period matrix of $\mathscr{X}$ and let $\varphi : \mathbf{C}^g/[\Omega] \to \mathscr{X}(\mathbf{C})$ be a $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-equivariant isomorphism of complex Lie groups. Note that $\mathbf{R}^g \cap [\Omega] = \mathbf{Z}^g$ is a lattice in $\mathbf{R}^g$ and regard the real torus $\mathbf{R}^g/\mathbf{Z}^g$ as embedded in the complex torus $\mathbf{C}^g/[\Omega]$. Since $\varphi(\mathbf{R}^g/\mathbf{Z}^g) = X_0$, where X_0 is the connected component of X containing the identity element of X, and the restriction $\varphi_0 : \mathbf{R}^g/\mathbf{Z}^g \to X_0$ of φ is a homeomorphism, it follows that

$$H^2(\varphi_0) : H^2(X_0, \mathbf{Z}) \to H^2(\mathbf{R}^g/\mathbf{Z}^g, \mathbf{Z})$$

is an isomorphism. Let

$$\varepsilon : H^2(\mathbf{R}^g/\mathbf{Z}^g, \mathbf{Z}) \to \mathrm{Alt}^2(\mathbf{Z}^g)$$

be the usual identification isomorphism, cf. [20], and let

$$\alpha : \mathrm{Alt}^2(\mathbf{Z}^g) \to \mathrm{Alt}_g(\mathbf{Z})$$

be the isomorphism which assigns to every alternating bilinear form in $\mathrm{Alt}^2(\mathbf{Z}^g)$ its matrix with respect to the canonical basis for $\mathbf{Z}^g$. Define

$$\tau_\varphi : H^2(X, \mathbf{Z})^{\mathrm{inv}} \to \mathrm{Alt}_g(\mathbf{Z})$$

to be the composition

$$\tau_\varphi = \alpha \circ \varepsilon \circ \mathrm{H}^2(\varphi_0) \circ \rho^2,$$

where $\rho^2 : \mathrm{H}^2(\mathrm{X}, \mathbf{Z})^{\mathrm{inv}} \to \mathrm{H}^2(\mathrm{X}_0, \mathbf{Z})$ is the isomorphism of Proposition 2.1 (ii) with $k = 2$. By construction, τ_φ is an isomorphism, and hence it remains to show that

$$\tau_\varphi(\mathrm{H}^2_{\mathbf{C}-\mathrm{alg}}(\mathrm{X}, \mathbf{Z})) = \mathrm{C}(\mathrm{Z}).$$

To this end we first recall how the Néron-Severi group $\mathrm{H}^2_{\mathrm{alg}}(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ of the Abelian variety $\mathscr{X}_{\mathbf{C}}$ over $\mathbf{C}$ can be described in terms of the period matrix $\Omega = (\mathrm{Z}, \mathrm{I}_g)$ of $\mathscr{X}$, which obviously is also a period matrix of $\mathscr{X}_{\mathbf{C}}$.

Let

$$\varphi^\# : \mathrm{H}^2(\mathscr{X}(\mathbf{C}), \mathbf{Z}) \to \mathrm{Alt}_{2g}(\mathbf{Z})$$

be the isomorphism obtained by composing the induced isomorphism

$$\mathrm{H}^2(\varphi) : \mathrm{H}^2(\mathscr{X}(\mathbf{C}), \mathbf{Z}) \to \mathrm{H}^2(\mathbf{C}^g/[\Omega], \mathbf{Z}),$$

the usual identification isomorphism (cf. [20])

$$\varepsilon_\Omega : \mathrm{H}^2(\mathbf{C}^g/[\Omega], \mathbf{Z}) \to \mathrm{Alt}^2([\Omega]),$$

and the isomorphism

$$\alpha_\Omega : \mathrm{Alt}^2([\Omega]) \to \mathrm{Alt}_{2g}(\mathbf{Z}),$$

which assigns to every alternating bilinear form in $\mathrm{Alt}^2([\Omega])$ its matrix with respect to the $\mathbf{Z}$-basis for $[\Omega]$ formed by the columns of Ω. Thus, explicitly,

$$\varphi^\# = \alpha_\Omega \circ \varepsilon_\Omega \circ \mathrm{H}^2(\varphi).$$

Denoting by $\mathrm{NS}(\Omega)$ the submodule of $\mathrm{Alt}_{2g}(\mathbf{Z})$ that consists of all matrices of the form

$$\begin{pmatrix} \mathrm{A} & \mathrm{B} \\ -{}^t\mathrm{B} & \mathrm{C} \end{pmatrix},$$

where A, C are in $\mathrm{Alt}_g(\mathbf{Z})$, B is in $\mathrm{Mat}_g(\mathbf{Z})$ and

$$\mathrm{A} - \mathrm{BZ} + {}^t\mathrm{Z}\,{}^t\mathrm{B} + {}^t\mathrm{ZCZ} = 0,$$

one obtains from [20, p. 43, Exercise 4]

$$\varphi^\#(\mathrm{H}^2_{\mathrm{alg}}(\mathscr{X}(\mathbf{C}), \mathbf{Z})) = \mathrm{NS}(\Omega).$$

It is now easy to complete the proof. A direct computation demonstrates $\tau_\varphi \circ \mathrm{H}^2(i) = r_g \circ \varphi^\#$ and $r_g(\mathrm{NS}(\Omega)) = \mathrm{C}(\mathrm{Z})$, where $i : \mathrm{X} \hookrightarrow \mathscr{X}(\mathbf{C})$ is the inclusion map and $r_g : \mathrm{Alt}_{2g}(\mathbf{Z}) \to \mathrm{Alt}_g(\mathbf{Z})$ is the epimorphism defined by

$$r_g\left(\begin{pmatrix} \mathrm{A} & \mathrm{B} \\ -{}^t\mathrm{B} & \mathrm{C} \end{pmatrix}\right) = \mathrm{C}$$

for A, C in $\mathrm{Alt}_g(\mathbf{Z})$ and B in $\mathrm{Mat}_g(\mathbf{Z})$. Hence, keeping in mind the definition of $\mathrm{H}^2_{\mathbf{C}-\mathrm{alg}}(\mathrm{X}, \mathbf{Z})$,

$$\begin{aligned}\tau_\varphi(\mathrm{H}^2_{\mathbf{C}-\mathrm{alg}}(\mathrm{X}, \mathbf{Z})) &= \tau_\varphi(\mathrm{H}^2(i)\ (\mathrm{H}^2_{\mathrm{alg}}(\mathscr{X}(\mathbf{C}), \mathbf{Z}))) \\ &= r_g(\varphi^\#(\mathrm{H}^2_{\mathrm{alg}}(\mathscr{X}(\mathbf{C}), \mathbf{Z}))) \\ &= r_g(\mathrm{NS}(\Omega)) \\ &= \mathrm{C}(\mathrm{Z})\end{aligned}$$

and the proof is finished. □

In connection with Theorem 1.3 a natural problem arises: given a complex $g \times g$ matrix Z compute C(Z) or, at least, rank C(Z). Below we shall give, in particular, an explicit characterization of these matrices Z for which rank C(Z) is the largest possible, that is, equal to $(g-1)\,g/2 = \mathrm{rank}\,\mathrm{Alt}_g(\mathbf{Z})$. It is useful to define first another $\mathbf{Z}$-submodule D(Z) of $\mathrm{Alt}_g(\mathbf{Z})$ by

$$\mathrm{D}(\mathrm{Z}) = \{\, \mathrm{C} \in \mathrm{Alt}_g(\mathbf{Z}) \mid {}^t(\mathrm{Im}\,\mathrm{Z})\,\mathrm{C}(\mathrm{Im}\,\mathrm{Z}) \in \mathrm{Alt}_g(\mathbf{Z})\, \}.$$

Lemma **2.2.** — *Let* Z *be a complex* $g \times g$ *matrix.*

(i) *If* $2\,\mathrm{Re}\,\mathrm{Z}$ *has integer entries* (*which is always satisfied if* $(\mathrm{Z}, \mathrm{I}_g)$ *is a period matrix of an Abelian variety over* **R**), *then* $4\,\mathrm{D}(\mathrm{Z}) \subseteq \mathrm{C}(\mathrm{Z})$ *and* $4\mathrm{C}(\mathrm{Z}) \subseteq \mathrm{D}(\mathrm{Z})$, *and hence* rank C(Z) = rank D(Z).

(ii) C(Z) = D(Z), *provided that* Re Z *has integer entries.*

(iii) $\mathrm{rank}\,\mathrm{D}(\mathrm{Z}) = (g-1)\,g/2$ *if and only if every* 2×2 *minor determinant of the matrix* Im Z *is a rational number.*

Proof. — Let $\mathrm{M} = 2\,\mathrm{Re}\,\mathrm{Z}$ and $\mathrm{T} = \mathrm{Im}\,\mathrm{Z}$.

If C is in D(Z), then taking $\mathrm{A} = -\,{}^t\mathrm{MCM} + 4\,{}^t\mathrm{TCT}$ and $\mathrm{B} = 2\,{}^t\mathrm{MC}$ in (1.2), we see that 4C is in C(Z). Hence $4\,\mathrm{D}(\mathrm{Z}) \subseteq \mathrm{C}(\mathrm{Z})$, and the proof of (i) is finished since $4\mathrm{C}(\mathrm{Z}) \subseteq \mathrm{D}(\mathrm{Z})$ is obvious.

Assume that Re Z has integer entries, that is, M is in $2\,\mathrm{Mat}_g(\mathbf{Z})$. Then it is clear that $\mathrm{C}(\mathrm{Z}) \subseteq \mathrm{D}(\mathrm{Z})$. If C is in D(Z), take $\mathrm{A} = -\frac{1}{4}\,{}^t\mathrm{MCM} + {}^t\mathrm{TCM}$ and $\mathrm{B} = \frac{1}{2}\,{}^t\mathrm{MC}$ in (1.2), which shows that C belongs to C(Z). Thus (ii) is proved.

(iii) follows, by direct computation, from the equality $\mathrm{rank}\,\mathrm{Alt}_g(\mathbf{Z}) = (g-1)\,g/2$ and the fact that the $g \times g$ matrices A_{ij}, $1 \leqslant i < j \leqslant g$, with the (i,j)th (resp. (j,i)th) entry 1 (resp. -1) and all other entries 0, generate $\mathrm{Alt}_g(\mathbf{Z})$. □

We have not been able to find an explicit characterization of all complex $g \times g$ matrices Z such that $(\mathrm{Z}, \mathrm{I}_g)$ is a period matrix of some Abelian variety over **R** and $\mathrm{C}(\mathrm{Z}) = \mathrm{Alt}_g(\mathbf{Z})$. However, [9] which deals with products of elliptic curves over **R**, Lemma 2.2 (ii), and the proof of Theorem 1.7 given later on in this section are rich sources of examples with $\mathrm{C}(\mathrm{Z}) = \mathrm{Alt}_g(\mathbf{Z})$ satisfied. In connection with this remark, let us recall that if (Z, I_g) is a period matrix of an Abelian variety $\mathscr{X}$ over $\mathbf{R}$, then $\mathscr{X}(\mathbf{R})$ has 2^{g-r} connected components, where r is the rank of the matrix obtained by reducing modulo 2 all entries of $2\operatorname{Re} Z$ (as we already know, $2\operatorname{Re} Z$ has integer entries), cf. [26, 29]. Thus Lemma 2.2 (ii) corresponds to the case in which $\mathscr{X}(\mathbf{R})$ has 2^g connected components.

Corollary **2.3.** — *Let* $X = \mathscr{X}(\mathbf{R})$, *where* $\mathscr{X}$ *is a g-dimensional Abelian variety over* $\mathbf{R}$. *If* (Z, I_g) *is a period matrix of* $\mathscr{X}$, *then*

$$\operatorname{rank} VB^1_{C-\text{alg}}(X) = \operatorname{rank} D(Z).$$

In particular, $VB^1_{C-\text{alg}}(X) = 0$ *if and only if* $D(Z) = 0$.

Proof. — It suffices to apply Theorem 1.3 and Lemma 2.2 (i). □

Proof of Theorem 1.4. — The conclusion follows immediately from Corollary 2.3 and Lemma 2.2 (iii). □

At this point it would be possible to give proofs of Theorems 1.6 and 1.7. First, however, we wish to deduce from the results already proved some consequences concerning regular maps into the unit p-sphere

$$S^p = \{ (x_0, \ldots, x_p) \in \mathbf{R}^{p+1} \mid x_0^2 + \ldots + x_p^2 = 1 \}.$$

Fix once and for all an orientation of S^p and the corresponding generator κ_p of $H^p(S^p, \mathbf{Z}) \cong \mathbf{Z}$. For $X = \mathscr{X}(\mathbf{R})$, where $\mathscr{X}$ is an Abelian variety over $\mathbf{R}$, set

$$\mathscr{C}^\infty_*(X, S^p) = \{ f \in \mathscr{C}^\infty(X, S^p) \mid H^p(f)\,(\kappa_p) \in H^p(X, \mathbf{Z})^{\text{inv}} \}.$$

One easily generalizes some observations made in Section 1 for $p = 2$.

Lemma **2.4.** — *With the notation as above:*

(i) $H^p(X, \mathbf{Z})^{\text{inv}}$ *is generated by* $H^p(f)\,(\kappa_p)$ *as f runs through* $\mathscr{C}^\infty_*(X, S^2)$.

(ii) *If* $1 \leqslant p \leqslant \dim X$, *then the equality* $\mathscr{C}^\infty_*(X, S^p) = \mathscr{C}^\infty(X, S^p)$ *is equivalent to the connectedness of* X.

(iii) *If p is even, then* $\mathscr{R}(X, S^p) \subseteq \mathscr{C}^\infty_*(X, S^p)$.

Proof. — (i) is obvious since every connected component of X is diffeomorphic to a real torus.

In view of (i), $\mathscr{C}^\infty_*(X, S^p) = \mathscr{C}^\infty(X, S^p)$ is equivalent to $H^p(X, \mathbf{Z})^{\text{inv}} = H^p(X, \mathbf{Z})$. The last equality holds if and only if X is connected. Thus (ii) is proved.

It follows from [4, Proposition 1.2] that if p is even and f is in $\mathscr{R}(X, S^p)$, then $H^p(f)\,(\kappa_p)$ belongs to $H^p(i)\,(H^p(\mathscr{X}(\mathbf{C}), \mathbf{Z}))$, where $i : X \hookrightarrow \mathscr{X}(\mathbf{C})$ is the inclusion map. By Proposition 2.1 (i), the proof of (iii) is complete. □

Corollary **2.5.** — *If* X *is endowed with an invariant orientation,* $X_1, \ldots, X_s$ *are the connected components of* X, *and* $g = \dim X$ *is positive and even, then*

$$\deg(f \mid X_1) = \ldots = \deg(f \mid X_s)$$

for every regular map $f : X \to S^g$.

Proof. — The conclusion follows from Lemma 2.4 (iii). □

It should be mentioned that Corollary 2.5 is no longer valid if g is odd. Indeed, if $\dim X = g$ is odd and $d_1, \ldots, d_s$ are even integers, then arguing as in [3, Theorem 2.1] one can find a regular map $f: X \to S^p$ with $\deg(f \mid X_i) = d_i$ for $1 \leqslant i \leqslant s$.

For X as above and an integer n, define $n_X : X \to X$ by $n_X(x) = nx$ for all x in X. Obviously, n_X is a regular map.

Corollary **2.6.** — *Let* $X = \mathscr{X}(\mathbf{R})$, *where* $\mathscr{X}$ *is a g-dimensional Abelian variety over* $\mathbf{R}$, *and let* (Z, I_g) *be a period matrix of* $\mathscr{X}$. *Then:*

(i) *The equality* $D(Z) = 0$ *implies*

$$\mathscr{C}^\infty_{\mathscr{R}}(X, S^2) = \{f \in \mathscr{C}^\infty(X, S^2) \mid H^2(f)(\kappa_2) = 0\},$$

and the converse is true if $1 \leqslant g \leqslant 3$.

(ii) *Given a nonnegative integer n, the following conditions are equivalent:*

a) $f \circ n_X$ *is in* $\mathscr{C}^\infty_{\mathscr{R}}(X, S^2)$ *for every* f *in* $\mathscr{C}^\infty_*(X, S^2)$;
b) $H^2(n_X)(H^2(X, \mathbf{Z})^{\text{inv}}) \subseteq H^2_{\mathbf{C}-\text{alg}}(X, \mathbf{Z})$;
c) $n^2 \operatorname{Alt}_g(\mathbf{Z}) \subseteq C(Z)$.

Furthermore, the existence of a positive integer n for which a), b), c) are satisfied is equivalent to the fact that every 2×2 *minor determinant of* Im Z *is a rational number.*

Proof. — (i) It follows from Lemma 2.2 (i) that $D(Z) = 0$ is equivalent to $C(Z) = 0$. Therefore by Theorems 1.0 and 1.3, if $D(Z) = 0$ and h is in $\mathscr{C}^\infty_{\mathscr{R}}(X, S^2)$, then $H^2(h)(\kappa_2) = 0$, which proves the inclusion

$$\mathscr{C}^\infty_{\mathscr{R}}(X, S^2) \subseteq \{f \in \mathscr{C}^\infty(X, S^2) \mid H^2(f)(\kappa_2) = 0\}.$$

It follows from Theorem 1.0 that this inclusion is an equality. On the other hand, Theorem 1.3 also implies that there exists a nonzero element u in $H^2_{\mathbf{C}-\text{alg}}(X, \mathbf{Z})$, provided that $D(Z) \neq 0$. If $u = H^2(\varphi)(\kappa_2)$ for some $\mathscr{C}^\infty$ map $\varphi : X \to S^2$, then applying Theorem 1.0, we obtain that φ is in $\mathscr{C}^\infty_{\mathscr{R}}(X, S^2)$. Thus the last part of the conclusion follows since for $1 \leqslant g \leqslant 3$ every element of $H^2(X, \mathbf{Z})$ is of the form $H^2(\psi)(\kappa_2)$ for some $\mathscr{C}^\infty$ map $\psi : X \to S^2$.

(ii) We claim that $H^2(n_X)(v) = n^2 v$ for all v in $H^2(X, \mathbf{Z})^{\text{inv}}$. Indeed, define $n_{\mathscr{X}(\mathbf{C})} : \mathscr{X}(\mathbf{C}) \to \mathscr{X}(\mathbf{C})$ by $n_{\mathscr{X}(\mathbf{C})}(y) = ny$ for all y in $\mathscr{X}(\mathbf{C})$. Since $\mathscr{X}(\mathbf{C})$ is isomorphic to a complex torus, it follows that $H^2(n_{\mathscr{X}(\mathbf{C})})(w) = n^2 w$ for all w in $H^2(\mathscr{X}(\mathbf{C}), \mathbf{Z})$. Hence, by Proposition 2.1 (i), the claim holds.

The claim implies that $H^2(f \circ n_X) = n^2 H^2(f)$ on $H^2(S^2, \mathbf{Z})$ for every f in $\mathscr{C}^\infty_*(X, S^2)$, and hence *a)* is equivalent to *b)* in view of Lemma 2.4 (i). The equivalence of *b)* and *c)* follows from the claim and Theorem 1.3. The last assertion in (ii) is a consequence of Lemma 2.2 (iii) and the equality rank $\operatorname{Alt}_g(\mathbf{Z}) = (g-1)g/2$. □

One can deduce from Corollary 2.6 some results concerning regular maps into S^{2k} for $k \geqslant 2$. Here we confine ourselves to only the following.

Proposition **2.7.** — *Let* $X = \mathscr{X}(\mathbf{R})$, *where* $\mathscr{X}$ *is a g-dimensional Abelian variety over* $\mathbf{R}$. *Let* (Z, I_g) *be a period matrix of* $\mathscr{X}$. *Assume that every* 2×2 *minor determinant of* $\operatorname{Im} Z$ *is a rational number. Then for each integer* k, $2 \leqslant 2k \leqslant g$, *there exists a regular map* $f: X \to S^{2k}$ *such that the induced homomorphism* $H^{2k}(f): H^{2k}(S^{2k}, \mathbf{Z}) \to H^{2k}(X, \mathbf{Z})$ *is nonzero and hence, in particular,* f *is not null homotopic.*

Proof. — Denote by Y the k-fold product $S^2 \times \ldots \times S^2$. Since each connected component of X is diffeomorphic to $\mathbf{R}^g/\mathbf{Z}^g$, we can find a $\mathscr{C}^\infty$ map $F = (F_1, \ldots, F_k): X \to Y$ such that $H^{2k}(F): H^{2k}(Y, \mathbf{Z}) \to H^{2k}(X, \mathbf{Z})$ is a monomorphism and F_j belongs to $\mathscr{C}^\infty_*(X, S^2)$ for $1 \leqslant j \leqslant k$. The assumption and Corollary 2.6 (ii) imply that $F \circ n_X$ belongs to $\mathscr{C}^\infty_{\mathscr{R}}(X, Y)$ for some nonzero integer n. It is well known that there exists a regular map $G: Y \to S^{2k}$ with $H^{2k}(G) \neq 0$, cf. [21] or [1, Lemma 13.5.4]. By construction, $f = G \circ F \circ n_X : X \to S^{2k}$ is a regular map. Moreover,

$$H^{2k}(f) = H^{2k}(G \circ F \circ n_X) = H^{2k}(n_X) \circ H^{2k}(F) \circ H^{2k}(G)$$
$$= n^{2k}(H^{2k}(F) \circ H^{2k}(G))$$

and hence $H^{2k}(f) \neq 0$ (the reader should observe that $H^{2k}(n_X)(v) = n^{2k} v$ for all v in $H^{2k}(X, \mathbf{Z})^{\mathrm{inv}}$; cf. the proof of Corollary 2.6 (ii) for $k = 1$). □

We shall now deal with the moduli space $\mathscr{A}^g_{\mathbf{R}}$ of principally polarized g-dimensional Abelian varieties over $\mathbf{R}$. With the notation introduced between the statements of Theorems 1.10 and 1.11 in Section 1, the correspondence $Z \to \mathscr{Y}_Z$ gives rise to a surjective map

$$\pi_g : H_g \to \mathscr{A}^g_{\mathbf{R}}.$$

The topology on $\mathscr{A}^g_{\mathbf{R}}$ is induced via π_g from the topology on H_g. Actually, a more precise result is well known, cf. [29, 30]. Let Γ_g be the group of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix},$$

acting on H_g via

$$\begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix} . Z = AZ\,{}^tA + B\,{}^tA,$$

where A is in $\mathrm{Gl}_g(\mathbf{Z})$, B is in $\mathrm{Mat}_g(\mathbf{Z})$, and $B\,{}^tA$ is symmetric. The map π_g is constant on the orbits of Γ_g and induces a homeomorphism between the quotient $\Gamma_g \backslash H_g$ and $\mathscr{A}^g_{\mathbf{R}}$.

Proof of Theorem 1.6. — (i) One easily sees that the set

$$G = \{ Z \in H_g \mid D(Z) = 0 \}$$

is dense in H_g.

Choose a family $\{ K_n \mid n \in \mathbf{Z}^+ \}$ of compact subsets of H_g, whose union is equal to H_g. Given C and D in $\mathrm{Alt}_g(\mathbf{Z})$, $C \neq 0$, set

$$K_{n,C,D} = \{ Z \in K_n \mid (\mathrm{Im}\, Z)\, C(\mathrm{Im}\, Z) = D \}.$$

Then $K_{n,C,D}$ is a compact subset of H_g and

$$G = \bigcap_{(n,C,D)\in L} (H_g \setminus K_{n,C,D}),$$

where $L = \mathbf{Z}^+ \times (\mathrm{Alt}_g(\mathbf{Z})\setminus\{0\}) \times \mathrm{Alt}_g(\mathbf{Z})$. Since G is the union of orbits of Γ_g, we obtain

$$\pi_g(G) = \bigcap_{(n,C,D)\in L} (\mathscr{A}^g_{\mathbf{R}} \setminus \pi_g(K_{n,C,D})).$$

Obviously, each set $\mathscr{A}^g_{\mathbf{R}} \setminus \pi_g(K_{n,C,D})$ is open in $\mathscr{A}^g_{\mathbf{R}}$. Furthermore, since G is dense in H_g, it follows that $\pi_g(G)$ is dense in $\mathscr{A}^g_{\mathbf{R}}$, and therefore $\mathscr{A}^g_{\mathbf{R}} \setminus \pi_g(K_{n,C,D})$ is dense in $\mathscr{A}^g_{\mathbf{R}}$. In order to complete the proof it suffices to show

$$\pi_g(G) = \{ [\mathscr{Y}] \in \mathscr{A}^g_{\mathbf{R}} \mid \mathrm{VB}^1_{\mathbf{C}-\mathrm{alg}}(\mathscr{Y}(\mathbf{R})) = 0 \}.$$

This however readily follows from Theorems 1.0 and 1.3, and Lemma 2.2 (i).

(ii) By Lemma 2.2 (iii), the set

$$E = \{ Z \in H_g \mid \mathrm{rank}\, D(Z) = (g-1)\, g/2 \}$$

is uncountable and dense in H_g. Since Theorems 1.0 and 1.3, and Lemma 2.2 (i) imply

$$\pi_g(E) = \{ [\mathscr{Y}] \in \mathscr{A}^g_{\mathbf{R}} \mid \mathrm{rank}\, \mathrm{VB}^1_{\mathbf{C}-\mathrm{alg}}(\mathscr{Y}(\mathbf{R})) = (g-1)\, g/2 \},$$

the proof of (ii) is complete. □

Let us recall an explicit description of $\mathscr{A}^2_{\mathbf{R}}$ given in [29, 30]. The following topological subspaces of H_2 are connected:

$$A^{(2,2,1)} = \left\{ \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \mid t_{ij} \in \mathbf{R}, 0 \leqslant t_{12} \leqslant t_{11} \leqslant t_{22}, t_{11}t_{22} - t_{12}^2 > 0 \right\},$$

$$A^{(2,2,2)} = \left\{ \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \mid t_{ij} \in \mathbf{R}, 0 < t_{11} \leqslant t_{22}, 0 \leqslant 2t_{12} \leqslant t_{11} \right\},$$

$$A^{(2,1,1)} = \left\{ \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \mid t_{ij} \in \mathbf{R}, 0 \leqslant t_{12} \leqslant t_{11}, 0 \leqslant 2t_{12} \leqslant t_{22}, t_{11} > 0, t_{22} > 0 \right\},$$

$$A^{(2,0,2)} = \left\{ \sqrt{-1}\begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \mid t_{ij} \in \mathbf{R}, 0 < t_{11} \leqslant t_{22}, 0 \leqslant 2t_{22} \leqslant t_{11} \right\}.$$

Furthermore, if

$$A^2 = A^{(2,2,1)} \cup A^{(2,2,2)} \cup A^{(2,1,1)} \cup A^{(2,0,2)},$$

then $\pi_2 \mid A^2 : A^2 \to \mathscr{A}^2_{\mathbf{R}}$ is a homeomorphism [29, 30] (note that our notation is somewhat different than in [29, 30]). Hence the family $\{\mathscr{A}^{(2,\ell,\varepsilon)}_{\mathbf{R}}\}$, where $\mathscr{A}^{(2,\ell,\varepsilon)}_{\mathbf{R}} = \pi_2(A^{(2,\ell,\varepsilon)})$ and (ℓ, ε) belongs to $\{(2, 1), (2, 2), (1, 1), (0, 2)\}$, is the set of connected components of $\mathscr{A}^2_{\mathbf{R}}$. As mentioned above, if $[\mathscr{Y}]$ is in $\mathscr{A}^{(2,\ell,\varepsilon)}_{\mathbf{R}}$, then $\mathscr{Y}(\mathbf{R})$ has $2^{2-\ell}$ connected components.

Proof of Theorem 1.7. — Set

$$F^{(\ell,\varepsilon)} = \{ Z \in A^{(2,\ell,\varepsilon)} \mid C(Z) = \mathrm{Alt}_2(\mathbf{Z}) \}.$$

By Theorems 1.0 and 1.3, and Example 1.5,

$$\pi_2(F^{(\ell,\varepsilon)}) = \mathscr{A}^{(2,\ell,\varepsilon)}_{\mathbf{R}} \cap \{ [\mathscr{Y}] \in \mathscr{A}^2_{\mathbf{R}} \mid b(\mathscr{Y}(\mathbf{R})) = 1 \}.$$

Hence it suffices to show that each set $F^{(\ell,\varepsilon)}$ is uncountable.

Let t be a real number, $t > 1$. Set

$$Z^{(2,1)}_t = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} t & \sqrt{4t^2 - \frac{1}{4}} - t \\ \sqrt{4t^2 - \frac{1}{4}} - t & 5t - 2\sqrt{4t^2 - \frac{1}{4}} \end{pmatrix},$$

$$Z^{(2,2)}_t = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} \frac{1}{4t} & 0 \\ 0 & t \end{pmatrix},$$

$$Z^{(1,1)}_t = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} t & t - \sqrt{t^2 - 1} \\ t - \sqrt{t^2 - 1} & 2\left(t - \sqrt{t^2 - 1}\right) \end{pmatrix},$$

$$Z^{(0,2)}_t = \sqrt{-1}\begin{pmatrix} \frac{1}{t} & 0 \\ 0 & t \end{pmatrix}.$$

Note that $Z^{(\ell,\varepsilon)}_t$ belongs to $A^{(2,\ell,\varepsilon)}$. We shall now show that $Z^{(\ell,\varepsilon)}_t$ is in $F^{(\ell,\varepsilon)}$ for (ℓ, ε) in $\{(2, 1), (2, 2), (1, 1), (0, 2)\}$. Our argument is based on equations (1.2) and the obvious fact that the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

generates $\mathrm{Alt}_2(\mathbf{Z})$. Indeed, $Z_t^{(2,1)}$ is in $F^{(2,1)}$ since (1.2) holds for $M = 2\,\mathrm{Re}\,Z_t^{(2,1)}$, $T = \mathrm{Im}\,Z_t^{(2,1)}$, $A = 0$,

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Similarly, $Z_t^{(\ell,\varepsilon)}$ is in $F^{(\ell,\varepsilon)}$ for (ℓ, ε) in $\{(2, 2), (1, 1), (0, 2)\}$ since (1.2) holds for $M = 2\,\mathrm{Re}\,Z_t^{(\ell,\varepsilon)}$, $T = \mathrm{Im}\,Z_t^{(\ell,\varepsilon)}$,

$$A = C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence the proof is complete. □

3. Products of real algebraic varieties

Given topological spaces T_1 and T_2, we set

$$\Delta(T_1, T_2) = H^2(T_1 \times T_2, \mathbf{Z})/(H^2(pr_1)(H^2(T_1, \mathbf{Z})) + H^2(pr_2)(H^2(T_2, \mathbf{Z}))),$$

where $pr_k : T_1 \times T_2 \to T_k$ is the canonical projection, $k = 1, 2$. If $f_1 : T_1 \to S_1$ and $f_2 : T_2 \to S_2$ are continuous maps of topological spaces, then we define

$$\Delta(f_1, f_2) : \Delta(S_1, S_2) \to \Delta(T_1, T_2)$$

to be the homomorphism induced by

$$H^2(f_1 \times f_2) : H^2(S_1 \times S_2, \mathbf{Z}) \to H^2(T_1 \times T_2, \mathbf{Z}).$$

Let $\mathscr{V}_1$ and $\mathscr{V}_2$ be projective smooth irreducible schemes over $\mathbf{C}$. Denote by $\Delta_{\mathrm{alg}}(\mathscr{V}_1(\mathbf{C}), \mathscr{V}_2(\mathbf{C}))$ the image of $H^2_{\mathrm{alg}}(\mathscr{V}_1(\mathbf{C}) \times \mathscr{V}_2(\mathbf{C}), \mathbf{Z})$ under the canonical epimorphism

$$H^2(\mathscr{V}_1(\mathbf{C}) \times \mathscr{V}_2(\mathbf{C}), \mathbf{Z}) \to \Delta(\mathscr{V}_1(\mathbf{C}), \mathscr{V}_2(\mathbf{C})).$$

For future reference, let us observe that $\Delta_{\mathrm{alg}}(\mathscr{V}_1(\mathbf{C}), \mathscr{V}_2(\mathbf{C}))$ is canonically isomorphic to the group $\mathrm{Corr}(\mathscr{V}_1, \mathscr{V}_2)$ of divisorial correspondences on $\mathscr{V}_1 \times_{\mathbf{C}} \mathscr{V}_2$. Indeed, letting $q_k : \mathscr{V}_1 \times_{\mathbf{C}} \mathscr{V}_2 \to \mathscr{V}_k$ denote the canonical projection, we have

$$\mathrm{Corr}(\mathscr{V}_1, \mathscr{V}_2) = \mathrm{Pic}(\mathscr{V}_1 \times_{\mathbf{C}} \mathscr{V}_2)/(q_1^*(\mathrm{Pic}(\mathscr{V}_1)) + q_2^*(\mathrm{Pic}(\mathscr{V}_2))).$$

The homomorphism

$$c_1 : \mathrm{Pic}(\mathscr{V}_1 \times_{\mathbf{C}} \mathscr{V}_2) \to H^2(\mathscr{V}_1(\mathbf{C}) \times \mathscr{V}_2(\mathbf{C}), \mathbf{Z}),$$

determined by the first Chern class, gives rise to an epimorphism

$$\gamma : \mathrm{Corr}(\mathscr{V}_1, \mathscr{V}_2) \to \Delta_{\mathrm{alg}}(\mathscr{V}_1(\mathbf{C}), \mathscr{V}_2(\mathbf{C})).$$

It suffices to show that γ is an isomorphism or, equivalently, $\gamma^{-1}(0) = 0$. This, however, readily follows since $c_1^{-1}(0) = \mathrm{Pic}^0(\mathscr{V}_1 \times_{\mathbf{C}} \mathscr{V}_2)$ is a divisible group and $\mathrm{Corr}(\mathscr{V}_1, \mathscr{V}_2)$ is a finitely generated free Abelian group (cf. [18, p. 155]).

Let now X_1 and X_2 be compact nonsingular irreducible real algebraic varieties, and let $\Delta_{\mathbf{C}-\mathrm{alg}}(X_1, X_2)$ be the image of $H^2_{\mathbf{C}-\mathrm{alg}}(X_1 \times X_2, \mathbf{Z})$ under the canonical epimorphism

$$H^2(X_1 \times X_2, \mathbf{Z}) \to \Delta(X_1, X_2).$$

It is convenient to define in a canonical way a certain intermediate subgroup $\Delta_*(X_1, X_2)$ between $\Delta_{\mathbf{C}-\mathrm{alg}}(X_1, X_2)$ and $\Delta(X_1, X_2)$. This is done as follows. Let $\mathscr{X}_k$ be a projective smooth irreducible scheme over $\mathbf{R}$ with $\mathscr{X}_k(\mathbf{R})$ biregularly isomorphic to X_k for $k = 1, 2$. Let $h_k : X_k \to \mathscr{X}_k(\mathbf{R})$ be a biregular isomorphism and let $i_k : \mathscr{X}_k(\mathbf{R}) \hookrightarrow \mathscr{X}_k(\mathbf{C})$ be the inclusion map. We assert that the subgroup

$$\Delta_*(X_1, X_2) = \Delta(i_1 \circ h_1, i_2 \circ h_2)\,(\Delta(\mathscr{X}_1(\mathbf{C}), \mathscr{X}_2(\mathbf{C})))$$

of $\Delta(X_1, X_2)$ does not depend on $\mathscr{X}_k$ and h_k for $k = 1, 2$. Indeed, the subgroup $H^1(i_k \circ h_k)\,(H^1(\mathscr{X}_k(\mathbf{C}), \mathbf{Z}))$ of $H^1(X_k, \mathbf{Z})$ is independent of $\mathscr{X}_k$ and h_k since $H^1(\mathscr{X}_k(\mathbf{C}), \mathbf{Z})$ is a birational invariant of $\mathscr{X}_k$. Furthermore, our assumptions on $\mathscr{X}_k$ guarantee connectedness of $\mathscr{X}_k(\mathbf{C})$, and hence the Künneth formula implies that the cross product in cohomology induces a canonical isomorphism from $H^1(\mathscr{X}_1(\mathbf{C}), \mathbf{Z}) \otimes_{\mathbf{Z}} H^1(\mathscr{X}_2(\mathbf{C}), \mathbf{Z})$ onto $\Delta(\mathscr{X}_1(\mathbf{C}), \mathscr{X}_2(\mathbf{C}))$ (recall that the cohomology group $H^1(-, \mathbf{Z})$ is always free). The assertion follows. As in Section 1, we identify $\mathscr{X}_k(\mathbf{C})$ with $\mathscr{X}_{k\mathbf{C}}(\mathbf{C})$, where $\mathscr{X}_{k\mathbf{C}} = \mathscr{X}_k \times_{\mathbf{R}} \mathbf{C}$, and thus the group $\Delta_{\mathrm{alg}}(\mathscr{X}_1(\mathbf{C}), \mathscr{X}_2(\mathbf{C}))$ is defined. By definition of $H^2_{\mathbf{C}-\mathrm{alg}}(-, \mathbf{Z})$,

$$\Delta_{\mathbf{C}-\mathrm{alg}}(X_1, X_2) = \Delta(i_1 \circ h_1, i_2 \circ h_2)\,(\Delta_{\mathrm{alg}}(\mathscr{X}_1(\mathbf{C}), \mathscr{X}_2(\mathbf{C}))),$$

which yields

$$\Delta_{\mathbf{C}-\mathrm{alg}}(X_1, X_2) \subseteq \Delta_*(X_1, X_2) \subseteq \Delta(X_1, X_2).$$

It will also be convenient to define a canonical epimorphism

$$e : \Delta(X_1, X_2) \to \mathrm{Bil}(X_1, X_2),$$

where $\mathrm{Bil}(X_1, X_2)$ is the group of all $\mathbf{Z}$-bilinear maps $H_1(X_1, \mathbf{Z}) \times H_1(X_2, \mathbf{Z}) \to \mathbf{Z}$. To this end let η be an element of $\Delta(X_1, X_2)$ and let ξ be a cohomology class in $H^2(X_1 \times X_2, \mathbf{Z})$, whose residue class in $\Delta(X_1, X_2)$ is equal to η. Then for (u_1, u_2) in $H_1(X_1, \mathbf{Z}) \times H_1(X_2, \mathbf{Z})$, we set

$$e(\eta)\,(u_1, u_2) = \langle \xi, u_1 \times u_2 \rangle,$$

where $\langle\ ,\ \rangle$ is the Kronecker (that is, the scalar) product and $\times$ is the cross product in homology (cf. [12]). Clearly, $e(\eta)$ does not depend on the choice of ξ. Since $H^1(X_k, \mathbf{Z})$ is canonically isomorphic to $\mathrm{Hom}(H_1(X_k, \mathbf{Z}), \mathbf{Z})$, it follows from the Künneth formula that e is an epimorphism. One easily sees that, in general, e is not an isomorphism, unless X_1 and X_2 are connected. We claim however that the restriction of e to $\Delta_*(X_1, X_2)$ is injective. To see this it is enough to observe that every residue class ζ in $\Delta_*(X_1, X_2)$ can be represented by a linear combination of cohomology classes of the form $\xi_1 \times \xi_2$, where ξ_k is a cohomology class in $H^1(X_k, \mathbf{Z})$ for $k = 1, 2$, and $\times$ stands for the cross product in cohomology. Whenever convenient, we shall make no distinction between ζ in $\Delta_*(X_1, X_2)$ and the corresponding bilinear map $e(\zeta)$ (cf. Theorem 3.4 below).

The groups $\Delta_{\mathbf{C}-\mathrm{alg}}(X_1, X_2)$ and $\Delta_*(X_1, X_2)$ play a crucial role in our study of regular maps. Clearly, in all considerations involving $\Delta_{\mathbf{C}-\mathrm{alg}}(X_1, X_2)$ and $\Delta_*(X_1, X_2)$ we may assume without loss of generality that $X_k = \mathscr{X}_k(\mathbf{R})$ for $k = 1, 2$. Our goal in this section is to compute these groups in terms of period matrices of the Albanese varieties of $\mathscr{X}_1$ and $\mathscr{X}_2$. To this end we need some preparation.

Given nonnegative integers g_1 and g_2, we define

$$\pi_{(g_1, g_2)} : \mathrm{Alt}_{g_1+g_2}(\mathbf{Z}) \to \mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$$

by

$$\pi_{(g_1, g_2)}\left(\begin{pmatrix} B_1 & C \\ -{}^tC & B_2 \end{pmatrix}\right) = C,$$

where B_k belongs to $\mathrm{Alt}_{g_k}(\mathbf{Z})$ for $k = 1, 2$, and C belongs to $\mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$. Clearly, $\pi_{(g_1, g_2)}$ is an epimorphism.

Lemma **3.1.** — *Let* Z_k *be a complex* $g_k \times g_k$ *matrix for* $k = 1, 2$, *and let*

$$Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}.$$

Then $\pi_{(g_1, g_2)}(C(Z)) = C(Z_1, Z_2)$, *where* $C(Z)$ *and* $C(Z_1, Z_2)$ *are the* $\mathbf{Z}$*-modules defined in Section* 1 *by equations* (1.2) *and* (1.9), *respectively.*

Proof. — The conclusion follows from a direct calculation. □

Let $\mathscr{A}_k$ be a g_k-dimensional Abelian variety over $\mathbf{R}$, $k = 1, 2$. Then $\mathscr{A} = \mathscr{A}_1 \times_{\mathbf{R}} \mathscr{A}_2$ is an Abelian variety over $\mathbf{R}$ of dimension $g_1 + g_2$. We shall identify $\mathscr{A}(\mathbf{R})$ and $\mathscr{A}(\mathbf{C})$ with $\mathscr{A}_1(\mathbf{R}) \times \mathscr{A}_2(\mathbf{R})$ and $\mathscr{A}_1(\mathbf{C}) \times \mathscr{A}_2(\mathbf{C})$, respectively. Let $A_k = \mathscr{A}_k(\mathbf{R})$ and let $p_k : A_1 \times A_2 \to A_k$ be the canonical projection, $k = 1, 2$. By Proposition 2.1 (i), we have

$$\Delta_*(A_1, A_2) = H^2(A_1 \times A_2, \mathbf{Z})^{\mathrm{inv}}/(H^2(p_1)(H^2(A_1, \mathbf{Z})^{\mathrm{inv}}) + H^2(p_2)(H^2(A_2, \mathbf{Z})^{\mathrm{inv}})).$$

Let $\Omega_k = (Z_k, I_{g_k})$ be a period matrix of $\mathscr{A}_k$ and let $\varphi_k : \mathbf{C}^{g_k}/[\Omega_k] \to \mathscr{A}_k(\mathbf{C})$ be a $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-equivariant isomorphism of complex Lie groups. Note that $\Omega = (Z, I_{g_1+g_2})$, where

$$Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix},$$

is a period matrix of $\mathscr{A}$. Furthermore, $\varphi : \mathbf{C}^{g_1+g_2}/[\Omega] \to \mathscr{A}(\mathbf{C})$, defined by

$$\varphi((v_1, v_2) + [\Omega]) = (\varphi_1(v_1) + [\Omega_1], \varphi_2(v_2) + [\Omega_2])$$

for (v_1, v_2) in $\mathbf{C}^{g_1} \times \mathbf{C}^{g_2} = \mathbf{C}^{g_1+g_2}$, is a Gal($\mathbf{C}/\mathbf{R}$)-equivariant isomorphism of complex Lie groups. If

$$\tau_\varphi : \mathrm{H}^2(\mathrm{A}_1 \times \mathrm{A}_2, \mathbf{Z})^{\mathrm{inv}} \to \mathrm{Alt}_{g_1+g_2}(\mathbf{Z})$$

is the isomorphism of Theorem 1.3, then the kernel of the epimorphism

$$\pi_{(g_1, g_2)} \circ \tau_\varphi : \mathrm{H}^2(\mathrm{A}_1 \times \mathrm{A}_2, \mathbf{Z})^{\mathrm{inv}} \to \mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$$

can be easily computed, namely,

$$\mathrm{Ker}(\pi_{(g_1, g_2)} \circ \tau_\varphi) = \mathrm{H}^2(p_1)(\mathrm{H}^2(\mathrm{A}_1, \mathbf{Z})^{\mathrm{inv}}) + \mathrm{H}^2(p_2)(\mathrm{H}^2(\mathrm{A}_2, \mathbf{Z})^{\mathrm{inv}}).$$

Denote by

$$\rho_{(\varphi_1, \varphi_2)} : \Delta_*(\mathrm{A}_1, \mathrm{A}_2) \to \mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$$

the isomorphism induced by $\pi_{(g_1, g_2)} \circ \tau_\varphi$.

Proposition **3.2.** — *The isomorphism* $\rho_{(\varphi_1, \varphi_2)} : \Delta_*(\mathrm{A}_1, \mathrm{A}_2) \to \mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$ *satisfies*

$$\rho_{(\varphi_1, \varphi_2)}(\Delta_{\mathbf{C}-\mathrm{alg}}(\mathrm{A}_1, \mathrm{A}_2)) = \mathrm{C}(\mathrm{Z}_1, \mathrm{Z}_2).$$

Proof. — The conclusion follows from Theorem 1.3 and Lemma 3.1. □

We shall now prepare the setup for the main result of this section.

Let $\mathscr{X}_k$ be a projective smooth irreducible scheme over $\mathbf{R}$ with $\mathrm{X}_k = \mathscr{X}_k(\mathbf{R})$ nonempty. Let $\mathscr{A}_k$ be the Albanese variety of $\mathscr{X}_k$ and $\mathrm{A}_k = \mathscr{A}_k(\mathbf{R})$. Denote by $i_k : \mathrm{X}_k \hookrightarrow \mathscr{X}_k(\mathbf{C})$ and $j_k : \mathrm{A}_k \hookrightarrow \mathscr{A}_k(\mathbf{C})$ the inclusion maps. If $\alpha_k : \mathscr{X}_k \hookrightarrow \mathscr{A}_k$ is the Albanese morphism corresponding to some point x_k in X_k (that is, $\alpha_k(x_k) = 0$), then $\alpha_{k\mathbf{C}} \circ i_k = j_k \circ \alpha_{k\mathbf{R}}$, where $\alpha_{k\mathbf{C}} : \mathscr{X}_k(\mathbf{C}) \to \mathscr{A}_k(\mathbf{C})$ and $\alpha_{k\mathbf{R}} : \mathrm{X}_k \to \mathrm{A}_k$ are the maps determined by α_k. It follows that the diagram

$$\begin{array}{ccc} \Delta(\mathscr{A}_1(\mathbf{C}), \mathscr{A}_2(\mathbf{C})) & \xrightarrow{\Delta(\alpha_{1\mathbf{C}}, \alpha_{2\mathbf{C}})} & \Delta(\mathscr{X}_1(\mathbf{C}), \mathscr{X}_2(\mathbf{C})) \\ {\scriptstyle \Delta(j_1, j_2)}\downarrow & & \downarrow{\scriptstyle \Delta(i_1, i_2)} \\ \Delta(\mathrm{A}_1, \mathrm{A}_2) & \xrightarrow{\Delta(\alpha_{1\mathbf{R}}, \alpha_{2\mathbf{R}})} & \Delta(\mathrm{X}_1, \mathrm{X}_2) \end{array}$$

is commutative, and hence

$$\Delta(\alpha_{1\mathbf{R}}, \alpha_{2\mathbf{R}})(\Delta_*(\mathrm{A}_1, \mathrm{A}_2)) \subseteq \Delta_*(\mathrm{X}_1, \mathrm{X}_2).$$

We denote by

$$\delta_{(\mathscr{X}_1, \mathscr{X}_2)} : \Delta_*(\mathrm{A}_1, \mathrm{A}_2) \to \Delta_*(\mathrm{X}_1, \mathrm{X}_2)$$

the restriction of $\Delta(\alpha_{1\mathbf{R}}, \alpha_{2\mathbf{R}})$. We claim that $\delta_{(\mathscr{X}_1, \mathscr{X}_2)}$ does not depend on the choice of the point x_k in X_k to which α_k corresponds. Indeed, the induced homomorphism

$$H^\ell(\alpha_{k\mathbf{C}}) : H^\ell(\mathscr{A}_k(\mathbf{C}), \mathbf{Z}) \to H^\ell(\mathscr{X}_k(\mathbf{C}), \mathbf{Z})$$

for $\ell \geqslant 0$ is independent of x_k and hence commutativity of the diagram implies the claim.

Since $H^1(\alpha_{k\mathbf{C}})$ is an isomorphism for $k = 1, 2$, it follows from the Künneth formula that $\Delta(\alpha_{1\mathbf{C}}, \alpha_{2\mathbf{C}})$ is an isomorphism. Using again commutativity of the diagram, we obtain that $\delta_{(\mathscr{X}_1, \mathscr{X}_2)}$ is an epimorphism.

We know that the groups $\Delta_{\mathrm{alg}}(\mathscr{A}_1(\mathbf{C}), \mathscr{A}_2(\mathbf{C}))$ and $\mathrm{Corr}(\mathscr{A}_{1\mathbf{C}}, \mathscr{A}_{2\mathbf{C}})$ (resp. $\Delta_{\mathrm{alg}}(\mathscr{X}_1(\mathbf{C}), \mathscr{X}_2(\mathbf{C}))$ and $\mathrm{Corr}(\mathscr{X}_{1\mathbf{C}}, \mathscr{X}_{2\mathbf{C}})$) are canonically isomorphic. By the classical theorem on divisorial correspondences, $\mathrm{Corr}(\mathscr{A}_{1\mathbf{C}}, \mathscr{A}_{2\mathbf{C}})$ and $\mathrm{Corr}(\mathscr{X}_{1\mathbf{C}}, \mathscr{X}_{2\mathbf{C}})$ are also canonically isomorphic (cf. [18, p. 155]). By examining these isomorphisms, we obtain

$$\Delta(\alpha_{1\mathbf{C}}, \alpha_{2\mathbf{C}})\,(\Delta_{\mathrm{alg}}(\mathscr{A}_1(\mathbf{C}), \mathscr{A}_2(\mathbf{C}))) = \Delta_{\mathrm{alg}}(\mathscr{X}_1(\mathbf{C}), \mathscr{X}_2(\mathbf{C})).$$

This, together with the obvious equalities

$$\Delta(j_1, j_2)\,(\Delta_{\mathrm{alg}}(\mathscr{A}_1(\mathbf{C}), \mathscr{A}_2(\mathbf{C}))) = \Delta_{\mathbf{C}-\mathrm{alg}}(A_1, A_2),$$

$$\Delta(i_1, i_2)\,(\Delta_{\mathrm{alg}}(\mathscr{X}_1(\mathbf{C}), \mathscr{X}_2(\mathbf{C}))) = \Delta_{\mathbf{C}-\mathrm{alg}}(X_1, X_2),$$

implies $\delta_{(\mathscr{X}_1, \mathscr{X}_2)}(\Delta_{\mathbf{C}-\mathrm{alg}}(A_1, A_2)) = \Delta_{\mathbf{C}-\mathrm{alg}}(X_1, X_2)$ by virtue of commutativity of the diagram.

We summarize these observations in the following.

Proposition **3.3.** — *With the notation as above,*

$$\delta_{(\mathscr{X}_1, \mathscr{X}_2)} : \Delta_*(A_1, A_2) \to \Delta_*(X_1, X_2)$$

is an epimorphism and

$$\delta_{(\mathscr{X}_1, \mathscr{X}_2)}(\Delta_{\mathbf{C}-\mathrm{alg}}(A_1, A_2)) = \Delta_{\mathbf{C}-\mathrm{alg}}(X_1, X_2). \quad \square$$

Henceforth, given a nonnegative integer q and a continuous map $f: S \to T$ between topological spaces, we let $H_q(f) : H_q(S, \mathbf{Z}) \to H_q(T, \mathbf{Z})$ denote the induced homomorphism of homology groups.

We shall now describe $\delta_{(\mathscr{X}_1, \mathscr{X}_2)}$ in terms of period matrices of $\mathscr{A}_1$ and $\mathscr{A}_2$. Let $g_k = \dim \mathscr{A}_k$ and let $\Omega_k = (Z_k, I_{g_k})$ be a period matrix of $\mathscr{A}_k$ (as defined in Section 1). Denote by

$$\varepsilon_k : H_1(\mathbf{C}^{g_k}/[\Omega_k], \mathbf{Z}) \to [\Omega_k]$$

the usual identification isomorphism. Let $\varphi_k : \mathbf{C}^{g_k}/[\Omega_k] \to \mathscr{A}_k(\mathbf{C})$ be a $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-equivariant isomorphism of complex Lie groups and let

$$\hat{\varphi}_k : H_1(X_k, \mathbf{Z}) \to \mathbf{Z}^{g_k}$$

be the homomorphism defined by

$$\hat{\varphi}_k(u_k) = \varepsilon_k(\mathrm{H}_1(\varphi_k^{-1} \circ \alpha_{k\mathrm{C}} \circ i_k)(u_k))$$

for all u_k in $\mathrm{H}_1(\mathrm{X}_k, \mathbf{Z})$ (note that the element on the right-hand side of the equality belongs to $[\Omega_k] \cap \mathbf{R}^{g_k} = \mathbf{Z}^{g_k}$; elements of $\mathbf{Z}^{g_k}$ are viewed as $g_k \times 1$ matrices). Set

$$\mathrm{L}(\mathrm{X}_k, \varphi_k) = \hat{\varphi}_k(\mathrm{H}_1(\mathrm{X}_k, \mathbf{Z})).$$

We finally define a homomorphism

$$\sigma_{(\varphi_1, \varphi_2)} : \mathrm{Mat}(g_1 \times g_2, \mathbf{Z}) \to \Delta_*(\mathrm{X}_1, \mathrm{X}_2)$$

by setting

$$\sigma_{(\varphi_1, \varphi_2)} = \delta_{(\mathscr{X}_1, \mathscr{X}_2)} \circ (\rho_{(\varphi_1, \varphi_2)})^{-1},$$

where $\rho_{(\varphi_1, \varphi_2)} : \Delta_*(\mathrm{A}_1, \mathrm{A}_2) \to \mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$ is the isomorphism of Proposition 3.2. Since $\delta_{(\mathscr{X}_1, \mathscr{X}_2)}$ is an epimorphism, it follows that $\sigma_{(\varphi_1, \varphi_2)}$ is an epimorphism too.

Theorem **3.4.** — *The epimorphism* $\sigma_{(\varphi_1, \varphi_2)} : \mathrm{Mat}(g_1 \times g_2, \mathbf{Z}) \to \Delta_*(\mathrm{X}_1, \mathrm{X}_2)$ *satisfies*

$$\sigma_{(\varphi_1, \varphi_2)}(\mathrm{C}(\mathrm{Z}_1, \mathrm{Z}_2)) = \Delta_{\mathrm{C-alg}}(\mathrm{X}_1, \mathrm{X}_2).$$

Furthermore, for every matrix C *in* $\mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$ *the element* $\sigma_{(\varphi_1, \varphi_2)}(\mathrm{C})$ *of* $\Delta_*(\mathrm{X}_1, \mathrm{X}_2)$, *viewed as a bilinear map* $\sigma_{(\varphi_1, \varphi_2)}(\mathrm{C}) : \mathrm{H}_1(\mathrm{X}_1, \mathbf{Z}) \times \mathrm{H}_1(\mathrm{X}_2, \mathbf{Z}) \to \mathbf{Z}$, *is given by*

$$(\sigma_{(\varphi_1, \varphi_2)}(\mathrm{C}))(u_1, u_2) = {}^t\hat{\varphi}_1(u_1)\,\mathrm{C}\,\hat{\varphi}_2(u_2)$$

for all (u_1, u_2) *in* $\mathrm{H}_1(\mathrm{X}_1, \mathbf{Z}) \times \mathrm{H}_1(\mathrm{X}_2, \mathbf{Z})$. *In particular,*

$$\mathrm{Ker}\ \sigma_{(\varphi_1, \varphi_2)} = \{\, \mathrm{C} \in \mathrm{Mat}(g_1 \times g_2, \mathbf{Z}) \mid {}^t\lambda_1 \mathrm{C} \lambda_2 = 0 \ \textit{for all}\ \lambda_k \in \mathrm{L}(\mathrm{X}_k, \varphi_k),\ k = 1, 2 \,\}.$$

Proof. — The first assertion,

$$\sigma_{(\varphi_1, \varphi_2)}(\mathrm{C}(\mathrm{Z}_1, \mathrm{Z}_2)) = \Delta_{\mathrm{C-alg}}(\mathrm{X}_1, \mathrm{X}_2),$$

follows at once from Propositions 3.2 and 3.3. We shall now prove the second part of the theorem.

Let C be a matrix in $\mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$ and let η be an element in $\mathrm{H}^2(\mathscr{A}_1(\mathbf{C}) \times \mathscr{A}_2(\mathbf{C}), \mathbf{Z})$ such that $\Delta(j_1, j_2)([\eta]) = (\rho_{(\varphi_1, \varphi_2)})^{-1}(\mathrm{C})$, where $[\eta]$ is the residue class of η in $\Delta(\mathscr{A}_1(\mathbf{C}), \mathscr{A}_2(\mathbf{C}))$. Then, using commutativity of the diagram above, we have

$$\begin{aligned} \sigma_{(\varphi_1, \varphi_2)}(\mathrm{C}) &= \Delta(\alpha_{1\mathbf{R}}, \alpha_{2\mathbf{R}})(\Delta(j_1, j_2)([\eta])) \\ &= \Delta(i_1, i_2)(\Delta(\alpha_{1\mathrm{C}}, \alpha_{2\mathrm{C}})([\eta])) \qquad \textbf{(3.4.1)} \\ &= [\mathrm{H}^2(i_1 \times i_2)(\mathrm{H}^2(\alpha_{1\mathrm{C}} \times \alpha_{2\mathrm{C}})(\eta))], \end{aligned}$$

where the last element on the right-hand side is the residue class in $\Delta_*(\mathrm{X}_1, \mathrm{X}_2)$.

Let Z be the matrix in $\mathrm{Mat}_{g_1+g_2}(\mathbf{C})$ defined by

$$Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix}.$$

Then $\Omega = (Z, I_{g_1+g_2})$ is a period matrix of the Abelian variety $\mathscr{A}_1 \times_{\mathbf{R}} \mathscr{A}_2$ over $\mathbf{R}$, and $\varphi : \mathbf{C}^{g_1+g_2}/[\Omega] \to \mathscr{A}_1(\mathbf{C}) \times \mathscr{A}_2(\mathbf{C})$, given by

$$\varphi((v_1, v_2) + [\Omega]) = (\varphi_1(v_1 + [\Omega_1]), \varphi_2(v_2 + [\Omega_2]))$$

for (v_1, v_2) in $\mathbf{C}^{g_1} \times \mathbf{C}^{g_2} = \mathbf{C}^{g_1+g_2}$, is a $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-equivariant isomorphism of complex Lie groups. Let $E : [\Omega] \times [\Omega] \to \mathbf{Z}$ be the alternating bilinear form corresponding to the cohomology class $H^2(\varphi)(\eta)$ in $H^2(\mathbf{C}^{g_1+g_2}/[\Omega], \mathbf{Z})$ under the usual identification of $H^2(\mathbf{C}^{g_1+g_2}/[\Omega], \mathbf{Z})$ with the group of alternating bilinear forms $[\Omega] \times [\Omega] \to \mathbf{Z}$.

Let u_k be a homology class in $H_1(X_k, \mathbf{Z})$ and let $\lambda_k = \hat{\varphi}_k(u_k)$ for $k = 1, 2$. Then λ_k is in $[\Omega_k] \cap \mathbf{R}^{g_k} = \mathbf{Z}^{g_k}$, and

$$\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}$$

belong to $[\Omega] \cap \mathbf{R}^{g_1+g_2} = \mathbf{Z}^{g_1+g_2}$. Regarding $\sigma_{(\varphi_1, \varphi_2)}(C)$ as a bilinear map,

$$\sigma_{(\varphi_1, \varphi_2)}(C) : H_1(X_1, \mathbf{Z}) \times H_1(X_2, \mathbf{Z}) \to \mathbf{Z},$$

and applying (3.4.1), we obtain

$$\begin{aligned}(\sigma_{(\varphi_1, \varphi_2)}(C))(u_1, u_2) &= \langle H^2(i_1 \times i_2)(H^2(\alpha_{1C} \times \alpha_{2C})(\eta)), u_1 \times u_2 \rangle \\ &= \langle \eta, H_2(\alpha_{1C} \times \alpha_{2C})(H_2(i_1 \times i_2)(u_1 \times u_2)) \rangle \\ &= \langle \eta, (H_1(\alpha_{1C} \circ i_1)(u_1)) \times (H_1(\alpha_{2C} \circ i_2)(u_2)) \rangle \\ &= E\left(\begin{pmatrix} \lambda_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}\right).\end{aligned} \tag{3.4.2}$$

Note that the matrix of E with respect to the $\mathbf{Z}$-basis for $[\Omega]$ formed by the columns of Ω is of the form

$$\begin{pmatrix} P_1 & Q \\ -{}^tQ & P_2 \end{pmatrix},$$

where P_1, P_2 are in $\mathrm{Alt}_{g_1+g_2}(\mathbf{Z})$ and Q is in $\mathrm{Mat}_{g_1+g_2}(\mathbf{Z})$. Furthermore, since $\rho_{(\varphi_1, \varphi_2)}(\Delta(j_1, j_2)([\eta])) = C$, it follows from the definition of $\rho_{(\varphi_1, \varphi_2)}$ that

$$P_2 = \begin{pmatrix} B_1 & C \\ -{}^tC & B_2 \end{pmatrix}$$

for some B_k in $\mathrm{Alt}_{g_k}(\mathbf{Z})$, $k = 1, 2$. Writing

$$\begin{pmatrix}\lambda_1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\\lambda_2\end{pmatrix}$$

as $\mathbf{Z}$-linear combinations of the columns of Ω and using the matrix representation of E described above, one readily shows

$$E\left(\begin{pmatrix}\lambda_1\\0\end{pmatrix}, \begin{pmatrix}0\\\lambda_2\end{pmatrix}\right) = {}^t\lambda_1\, C\lambda_2.$$

Hence, by (3.4.2), we obtain

$$(\sigma_{(\varphi_1,\varphi_2)}(C))\,(u_1, u_2) = {}^t\lambda_1\, C\lambda_2 = {}^t\hat{\varphi}_1(u_1)\, C\hat{\varphi}_2(u_2)$$

as desired.

The formula for $\mathrm{Ker}\ \sigma_{(\varphi_1,\varphi_2)}$ is now obvious. □

As a straightforward application, which does not require the full strength of the above results, we obtain the following.

Corollary **3.5.** — *Let $\mathscr{X}_k$ be a projective smooth irreducible scheme over* $\mathbf{R}$ *with* $X_k = \mathscr{X}_k(\mathbf{R})$ *nonempty for* $k = 1, 2$. *If*

$$b_1(\mathscr{X}_1(\mathbf{C}))\, b_1(\mathscr{X}_2(\mathbf{C})) < 4 b_1(X_1)\, b_1(X_2),$$

where $b_1(M)$ *denotes the first Betti number of* M, *then* $\mathscr{C}^\infty_{\mathscr{R}}(X_1 \times X_2, S^2) \neq \mathscr{C}^\infty(X_1 \times X_2, S^2)$.

Proof. — Let g_k be the dimension of the Albanese variety of $\mathscr{X}_k$. It follows from Theorem 3.4 that

$$\mathrm{rank}\ \Delta_{\mathbf{C}-\mathrm{alg}}(X_1, X_2) \leqslant \mathrm{rank}\ \Delta_*(X_1, X_2) \leqslant g_1 g_2.$$

Since $b_1(\mathscr{X}_k(\mathbf{C})) = 2g_k$, by assumption we have

$$g_1 g_2 < b_1(X_1)\, b_1(X_2) = (\mathrm{rank}\ H^1(X_1, \mathbf{Z}))\,(\mathrm{rank}\ H^1(X_2, \mathbf{Z})),$$

and hence

$$\mathrm{rank}\ \Delta_{\mathbf{C}-\mathrm{alg}}(X_1, X_2) < (\mathrm{rank}\ H^1(X_1, \mathbf{Z}))\,(\mathrm{rank}\ H^1(X_2, \mathbf{Z})).$$

Note that the cohomology cross product induces a monomorphism from

$$H^1(X_1, \mathbf{Z}) \otimes_{\mathbf{Z}} H^1(X_2, \mathbf{Z})$$

into $\Delta(X_1, X_2)$ and therefore, in view of the last inequality, one can find v_k in $H^1(X_k, \mathbf{Z})$ for $k = 1, 2$ such that $v_1 \times v_2$ does not belong to $H^2_{\mathbf{C}-\mathrm{alg}}(X_1 \times X_2, \mathbf{Z})$. Let κ (resp. μ) be a generator of $H^2(S^2, \mathbf{Z})$ (resp. $H^1(S^1, \mathbf{Z})$). Pick $\mathscr{C}^\infty$ maps $h : S^1 \times S^1 \to S^2$ and

$f_k : X_k \to S^1$ such that $H^2(h)(\kappa) = \mu \times \mu$ in $H^2(S^1 \times S^1, \mathbf{Z})$, and $H^1(f_k)(\mu) = v_k$ for $k = 1, 2$. Setting $f = h \circ (f_1 \times f_2)$, we obtain $H^2(f)(\kappa) = v_1 \times v_2$ and therefore, by Theorem 1.0, f does not belong to $\mathscr{C}^\infty_\mathscr{R}(X_1 \times X_2, S^2)$. □

Example **3.6.** — (i) With the notation as in Corollary 3.5, if $b_1(\mathscr{X}_1(\mathbf{C}))\, b_1(\mathscr{X}_2(\mathbf{C})) = 0$ and $b_1(X_1)\, b_1(X_2) \neq 0$, then $\mathscr{C}^\infty_\mathscr{R}(X_1 \times X_2, S^2) \neq \mathscr{C}^\infty(X_1 \times X_2, S^2)$.

(ii) Let X be a compact nonsingular real algebraic variety. If $b_1(X) \neq 0$, then $\mathscr{C}^\infty_\mathscr{R}(X \times S^1, S^2) \neq \mathscr{C}^\infty(X \times S^1, S^2)$; the assertion follows from (i) and the fact that S^1 is biregularly isomorphic to $\mathbf{P}^1(\mathbf{R})$.

The assumption $b_1(X) \neq 0$ cannot be omitted as the example of the Fermat 2-sphere

$$S^2_{2n} = \{ (x, y, z) \in \mathbf{R}^3 \mid x^{2n} + y^{2n} + z^{2n} = 1 \}$$

shows. Indeed, by [10, Proposition 4.8], $H^2_{\mathbf{C}-\mathrm{alg}}(S^2_{2n}, \mathbf{Z}) = H^2(S^2_{2n}, \mathbf{Z})$, and hence $H^2_{\mathbf{C}-\mathrm{alg}}(S^2_{2n} \times S^1, \mathbf{Z}) = H^2(S^2_{2n} \times S^1, \mathbf{Z})$ since $H^1(S^2_{2n}, \mathbf{Z}) = 0$. It follows from Theorem 1.0 that $\mathscr{C}^\infty_\mathscr{R}(S^2_{2n} \times S^1, S^2) = \mathscr{C}^\infty(S^2_{2n} \times S^1, S^2)$. □

The most interesting applications of Theorem 3.4 concern the case $\dim \mathscr{X}_k = 1$ for $k = 1, 2$ (cf. also Section 4). Of course, then the Albanese variety of $\mathscr{X}_k$ is just the Jacobian variety of the curve $\mathscr{X}_k$. First, we need some preparation.

Recall that the term algebraic curve over $\mathbf{R}$ designates a projective smooth scheme $\mathscr{X}$ over $\mathbf{R}$ of dimension 1 such that $\mathscr{X} \times_\mathbf{R} \mathbf{C}$ is irreducible (cf. Section 1). If $\mathscr{X}(\mathbf{R})$ is nonempty, then the above definition simply means that $\mathscr{X}$ is a projective smooth irreducible scheme over $\mathbf{R}$ of dimension 1. We shall freely use terminology and notations related to algebraic curves over $\mathbf{R}$ introduced in Section 1. In particular, $g(\mathscr{X})$, $s(\mathscr{X})$, and $\varepsilon(\mathscr{X})$ will be used. Obviously, $H_1(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ and $H^1(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ are free Abelian groups of rank $2g(\mathscr{X})$. We shall now record a well known fact concerning the topology of the pair $(\mathscr{X}(\mathbf{C}), \mathscr{X}(\mathbf{R}))$ (a proof is given for the convenience of the reader).

Lemma **3.7.** — *Let $\mathscr{X}$ be an algebraic curve over $\mathbf{R}$ with $\mathscr{X}(\mathbf{R})$ nonempty, and let $i : \mathscr{X}(\mathbf{R}) \to \mathscr{X}(\mathbf{C})$ be the inclusion map. Then:*

(i) $H_1(i)(H_1(\mathscr{X}(\mathbf{R}), \mathbf{Z}))$ *is a free direct summand in* $H_1(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ *of rank* $s(\mathscr{X}) - \varepsilon(\mathscr{X}) + 1$.

(ii) $H^1(\mathscr{X}(\mathbf{R}), \mathbf{Z})/H^1(i)(H^1(\mathscr{X}(\mathbf{C}), \mathbf{Z}))$ *is a free Abelian group of rank* $\varepsilon(\mathscr{X}) - 1$.

Proof. — Let $s = s(\mathscr{X})$ and $r = s(\mathscr{X}) - \varepsilon(\mathscr{X}) + 1$. Let $C_1, \ldots, C_s$ be the connected components of $\mathscr{X}(\mathbf{R})$. Fix an orientation on C_i and denote by $[C_i]$ the homology class in $H_1(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ represented by C_i. By construction, $H_1(i)(H_1(\mathscr{X}(\mathbf{R}), \mathbf{Z}))$ is generated by $[C_1], \ldots, [C_r]$. It follows that $\mathscr{X}(\mathbf{C}) \backslash (C_1 \cup \ldots \cup C_r)$ is connected (cf. [28, p. 339]) and hence there exist $\mathscr{C}^\infty$ compact oriented curves $D_1, \ldots, D_r$ in $\mathscr{X}(\mathbf{C})$ such that the intersection number $C_i.D_j$ is the Kronecker delta, that is, $C_i.D_j = \delta_{ij}$. This implies that the subgroup of $H_1(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ generated by $[C_1], \ldots, [C_r]$ is a free direct summand in $H_1(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ of rank r. Thus (i) is proved.

A standard topological argument shows that (ii) is a consequence of (i). □

Recall from Section 1 that if X is a compact nonsingular real algebraic surface, then the isomorphism

$$h_X : \pi^2(X) \to H^2(X, \mathbf{Z})$$

satisfies

$$h_X(\pi^2_{\mathscr{R}}(X)) = H^2_{\mathbf{C}-\mathrm{alg}}(X, \mathbf{Z}),$$

where $h_X([f]) = H^2(f)(\kappa)$ for every $\mathscr{C}^\infty$ map $f: X \to S^2$, and κ is a fixed generator of the group $H^2(S^2, \mathbf{Z}) \cong \mathbf{Z}$.

Assuming that X_1 and X_2 are compact nonsingular real algebraic curves, we observe that

$$\Delta(X_1, X_2) = H^2(X_1, \times X_2, \mathbf{Z})$$

and define the subgroup $\pi^2_*(X_1 \times X_2)$ of $\pi^2(X_1 \times X_2)$ by

$$\pi^2_*(X_1 \times X_2) = \{ [f] \in \pi^2(X_1 \times X_2) \mid h_{X_1 \times X_2}([f]) \in \Delta_*(X_1, X_2) \}.$$

Since

$$\Delta_{\mathbf{C}-\mathrm{alg}}(X_1, X_2) = H^2_{\mathbf{C}-\mathrm{alg}}(X_1 \times X_2, \mathbf{Z}),$$

we obtain

$$\pi^2_{\mathscr{R}}(X_1 \times X_2) \subseteq \pi^2_*(X_1 \times X_2) \subseteq \pi^2(X_1 \times X_2).$$

The following is a simple but useful consequence of Lemma 3.7.

Proposition **3.8.** — *Let* $\mathscr{X}_k$ *be an algebraic curve over* $\mathbf{R}$ *with* $X_k = \mathscr{X}_k(\mathbf{R})$ *nonempty for* $k = 1, 2$. *Then*

$$\operatorname{rank} \pi^2_*(X_1 \times X_2) = (s(\mathscr{X}_1) - \varepsilon(\mathscr{X}_1) + 1)(s(\mathscr{X}_2) - \varepsilon(\mathscr{X}_2) + 1) \leqslant g(\mathscr{X}_1)\, g(\mathscr{X}_2).$$

In particular,

$$\operatorname{rank} \pi^2_{\mathscr{R}}(X_1 \times X_2) \leqslant (s(\mathscr{X}_1) - \varepsilon(\mathscr{X}_1) + 1)(s(\mathscr{X}_2) - \varepsilon(\mathscr{X}_2) + 1) \leqslant g(\mathscr{X}_1)\, g(\mathscr{X}_2).$$

Furthermore, $\pi^2(X_1 \times X_2)/\pi^2_*(X_1 \times X_2)$ *is a free Abelian group with*

$$\operatorname{rank}(\pi^2(X_1 \times X_2)/\pi^2_*(X_1 \times X_2)) = s(\mathscr{X}_1)\, s(\mathscr{X}_2) - (s(\mathscr{X}_1) - \varepsilon(\mathscr{X}_1) + 1)(s(\mathscr{X}_2) - \varepsilon(\mathscr{X}_2) + 1).$$

Proof. — Obviously, $H^1(X_k, \mathbf{Z})$ is a free Abelian group of rank $s(\mathscr{X}_k)$ for $k = 1, 2$. Thus, by the Künneth formula and Lemma 3.7, $\Delta(X_1, X_2)/\Delta_*(X_1, X_2)$ is a free Abelian group of rank

$$s(\mathscr{X}_1)\, s(\mathscr{X}_2) - (s(\mathscr{X}_1) - \varepsilon(\mathscr{X}_1) + 1)\,(s(\mathscr{X}_2) - \varepsilon(\mathscr{X}_2) + 1).$$

The isomorphism $h_{X_1 \times X_2} : \pi^2(X_1 \times X_2) \to \Delta(X_1, X_2)$ satisfies

$$h_{X_1 \times X_2}(\pi^2_*(X_1 \times X_2)) = \Delta_*(X_1, X_2),$$

and hence the last assertion in the proposition is proved.

The formula for rank $\pi^2_*(X_1 \times X_2)$ follows now at once, while the upper bound on rank $\pi^2_*(X_1 \times X_2)$ is obvious since $s(\mathscr{X}_k) - \varepsilon(\mathscr{X}_k) + 1 \leqslant g(\mathscr{X}_k)$ for $k = 1, 2$ (cf. Section 1). In view of $\pi^2_{\mathscr{R}}(X_1 \times X_2) \subseteq \pi^2_*(X_1 \times X_2)$, the upper bound on rank $\pi^2_{\mathscr{R}}(X_1 \times X_2)$ also follows. □

We shall now give a version of Theorem 3.4 which is more convenient for the study of algebraic curves over $\mathbf{R}$.

Let $\mathscr{X}_k$ be an algebraic curve over $\mathbf{R}$ of genus g_k (that is, $g_k = g(\mathscr{X}_k)$) with $X_k = \mathscr{X}_k(\mathbf{R})$ nonempty for $k = 1, 2$. Let $\mathscr{J}_k$ be the Jacobian variety of $\mathscr{X}_k$ and let $\Omega_k = (Z_k, I_{g_k})$ be a period matrix of $\mathscr{J}_k$. Let $\varphi_k : \mathbf{C}^{g_k}/[\Omega_k] \to \mathscr{J}_k(\mathbf{C})$ be a $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-equivariant isomorphism of complex Lie groups. We have the subgroup $L(X_k, \varphi_k)$ of $\mathbf{Z}^{g_k}$ for $k = 1, 2$, and the epimorphism

$$\sigma_{(\varphi_1, \varphi_2)} : \mathrm{Mat}(g_1 \times g_2, \mathbf{Z}) \to \Delta_*(X_1, X_2) \subseteq H^2(X_1 \times X_2, \mathbf{Z})$$

(cf. Theorem 3.4). Define

$$\tau_{(\varphi_1, \varphi_2)} : \mathrm{Mat}(g_1 \times g_2, \mathbf{Z}) \to \pi^2_*(X_1 \times X_2)$$

by setting

$$\tau_{(\varphi_1, \varphi_2)}(C) = (h_{X_1, X_2})^{-1}(\sigma_{(\varphi_1, \varphi_2)}(C))$$

for C in $\mathrm{Mat}(g_1 \times g_2), \mathbf{Z}$. By construction, $\tau_{(\varphi_1, \varphi_2)}$ is a group epimorphism.

Theorem **3.9.** — *The epimorphism*

$$\tau_{(\varphi_1, \varphi_2)} : \mathrm{Mat}(g_1 \times g_2, \mathbf{Z}) \to \pi^2_*(X_1 \times X_2)$$

and the subgroup $L(X_k, \varphi_k)$ *of* $\mathbf{Z}^{g_k}$ *have the following properties*:

(i) $\tau_{(\varphi_1, \varphi_2)}(C(Z_1, Z_2)) = \pi^2_{\mathscr{R}}(X_1 \times X_2)$,

(ii) $\mathrm{Ker}\, \tau_{(\varphi_1, \varphi_2)} = \{ C \in \mathrm{Mat}(g_1 \times g_2, \mathbf{Z}) \mid {}^t\lambda_1 C \lambda_2 = 0$ *for all* $\lambda_k \in L(X_k, \varphi_k), k = 1, 2 \}$,

(iii) $L(X_k, \varphi_k)$ *is a free direct summand in* $\mathbf{Z}^{g_k}$ *of rank* $s(\mathscr{X}_k) - \varepsilon(\mathscr{X}_k) + 1$,

(iv) $\tau_{(\varphi_1, \varphi_2)}$ *is an isomorphism if and only if* $s(\mathscr{X}_k) \geqslant g_k$ *for* $k = 1, 2$.

Proof. — Properties (i) and (ii) follow directly from Theorem 3.4.

We shall now prove (iii). Note that if $\alpha_k : \mathscr{X}_k \to \mathscr{J}_k$ is the canonical morphism (that is, the Albanese morphism) corresponding to some point x_k in X_k, and $\alpha_{k\mathbf{C}} : \mathscr{X}_k(\mathbf{C}) \to \mathscr{J}_k(\mathbf{C})$ is the embedding determined by α_k, then the induced homomorphism $H_1(\alpha_{k\mathbf{C}}) : H_1(\mathscr{X}_k(\mathbf{C}), \mathbf{Z}) \to H_1(\mathscr{J}_k(\mathbf{C}), \mathbf{Z})$ is an isomorphism. By examining the definition of $L(X_k, \varphi_k)$ (cf. the paragraph preceding Theorem 3.4) and applying Lemma 3.7 (i), one readily obtains (iii).

It follows from (ii) and (iii) that $\tau_{(\varphi_1, \varphi_2)}$ is an isomorphism if and only if $s(\mathscr{X}_k) - \varepsilon(\mathscr{X}_k) + 1 = g(\mathscr{X}_k)$ for $k = 1, 2$. The last condition is satisfied if and only if $s(\mathscr{X}_k) \geqslant g(\mathscr{X}_k) = g_k$ (cf. Section 1 for the relations between $g(\mathscr{X}_k)$, $s(\mathscr{X}_k)$, and $\varepsilon(\mathscr{X}_k)$). Hence (iv) holds. □

For many applications of Theorem 3.9 a certain technical result, Lemma 3.10 below, is very useful.

Let Z_k be a complex $g_k \times g_k$ matrix for $k = 1, 2$. Clearly,

$$D(Z_1, Z_2) = \{ C \in \mathrm{Mat}(g_1 \times g_2, \mathbf{Z}) \mid {}^t(\mathrm{Im}\, Z_1)\, C(\mathrm{Im}\, Z_2) \in \mathrm{Mat}(g_1 \times g_2, \mathbf{Z}) \}$$

is a $\mathbf{Z}$-submodule of $\mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$. We shall give, in particular, an explicit characterization of these matrices Z_1 and Z_2 for which $\mathrm{rank}\, C(Z_1, Z_2) = g_1 g_2 = \mathrm{rank}\, \mathrm{Mat}(g_1 \times g_2, \mathbf{Z})$.

Lemma **3.10.** — *With the notation as above:*

(i) *If* $2\,\mathrm{Re}\, Z_k$ *has integer entries for* $k = 1, 2$, *then* $4\, D(Z_1, Z_2) \subseteq C(Z_1, Z_2)$ *and* $4C(Z_1, Z_2) \subseteq D(Z_1, Z_2)$, *and hence* $\mathrm{rank}\, C(Z_1, Z_2) = \mathrm{rank}\, D(Z_1, Z_2)$.

(ii) $C(Z_1, Z_2) = D(Z_1, Z_2)$, *provided that* $\mathrm{Re}\, Z_k$ *has integer entries for* $k = 1, 2$.

(iii) $\mathrm{rank}\, D(Z_1, Z_2) = g_1 g_2$ *if and only if* $t^1_{\alpha\beta}\, t^2_{\gamma\delta}$ *is a rational number for all* $1 \leqslant \alpha \leqslant g_1$, $1 \leqslant \beta \leqslant g_1$, $1 \leqslant \gamma \leqslant g_2$, $1 \leqslant \delta \leqslant g_2$, *where* $\mathrm{Im}\, Z_1 = (t^1_{\alpha\beta})$, $\mathrm{Im}\, Z_2 = (t^2_{\gamma\delta})$.

Proof. — The argument is straightforward and we leave it for the reader (cf. the proof of Lemma 2.2). □

By Proposition 3.8, $\mathrm{rank}\, \pi^2_{\mathscr{B}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) \leqslant g^2$ for every algebraic curve $\mathscr{X}$ over $\mathbf{R}$ of genus g with $\mathscr{X}(\mathbf{R})$ nonempty. We shall now give a characterization of the exceptional curves for which this maximum rank is attained.

Proposition **3.11.** — *Let* $\mathscr{X}$ *be an algebraic curve over* $\mathbf{R}$ *of positive genus* g *such that* $\mathscr{X}(\mathbf{R})$ *has* s *connected components,* $s \geqslant 1$. *Then* $\mathrm{rank}\, \pi^2_{\mathscr{B}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) = g^2$ *if and only if* $s \geqslant g$ *and the Jacobian variety of the curve* $\mathscr{X} \times_{\mathbf{R}} \mathbf{C}$ *over* $\mathbf{C}$ *is isomorphic over* $\mathbf{C}$ *to the product of* g *pairwise isogenous elliptic curves over* $\mathbf{C}$ *with complex multiplication. Furthermore, the set*

$$\{ [\mathscr{X}] \in \mathscr{M}^g_{\mathbf{R}} \mid \mathscr{X}(\mathbf{R}) \neq \emptyset \text{ and } \mathrm{rank}\, \pi^2_{\mathscr{B}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) = g^2 \}$$

is at most countable.

Proof. — Recall that an Abelian variety $\mathscr{V}$ over $\mathbf{C}$ of positive dimension g is isomorphic to the product of g pairwise isogenous elliptic curves over $\mathbf{C}$ with complex multiplication if and only if $\mathscr{V}$ admits a period matrix whose all entries belong to the imaginary quadratic extension $\mathbf{Q}(\sqrt{-d})$ of $\mathbf{Q}$ for some positive integer d [19]. Obviously, if all entries of W are in $\mathbf{Q}(\sqrt{-d})$ for some period matrix $(\mathrm{W}, \mathrm{I}_g)$ of $\mathscr{V}$, then all entries of W′ are in $\mathbf{Q}(\sqrt{-d})$ for any period matrix $(\mathrm{W}', \mathrm{I}_g)$ of $\mathscr{V}$.

Let $\mathscr{J}$ be the Jacobian variety of $\mathscr{X}$ and let Z be a period matrix of $\mathscr{X}$. The Abelian variety $\mathscr{J} \times_{\mathbf{R}} \mathbf{C}$ over $\mathbf{C}$ is the Jacobian variety of $\mathscr{X} \times_{\mathbf{R}} \mathbf{C}$, and $(\mathrm{Z}, \mathrm{I}_g)$ is a period matrix of $\mathscr{J}$ and of $\mathscr{J} \times_{\mathbf{R}} \mathbf{C}$.

It follows from Proposition 3.8, Theorem 3.9 (iv), and Lemma 3.10 (i), (iii) that the condition rank $\pi^2_{\mathscr{R}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) = g^2$ holds if and only if $s \geqslant g$ and all entries of Z belong to $\mathbf{Q}(\sqrt{-d})$ for some positive integer d.

The first assertion of the proposition is a consequence of the facts listed above. Since $\mathscr{X}(\mathbf{R})$ is nonempty, Torelli's theorem for algebraic curves over $\mathbf{R}$ [14, 22, 26] implies that $\mathscr{X}$ is determined up to isomorphism by the isomorphism class of its polarized Jacobian variety. Thus the second assertion of the proposition also follows. □

Since no algebraic curve over $\mathbf{C}$ with genus greater than 3 and Jacobian variety isomorphic to the product of elliptic curves over $\mathbf{C}$ is known [13], the question of existence of an algebraic curve $\mathscr{X}$ over $\mathbf{R}$ of genus g greater than 3 with rank $\pi^2_{\mathscr{R}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) = g^2$ is, a fortiori, wide open. Curves of genus 1 or 2 are discussed in this context in the remark following Theorem 1.13 in Section 1.

4. Algebraic curves over R of small genus

We shall show that the results of Section 3 take a very appealing and concrete form for algebraic curves over $\mathbf{R}$ of genus 1 or 2 (the reader may consult [7, 9] for the genus 1 case). Towards the end of this section we shall also discuss concrete examples of curves of higher genus.

We already considered the moduli space $\mathscr{M}^g_{\mathbf{R}}$ (resp. $\mathscr{A}^g_{\mathbf{R}}$) of algebraic curves over $\mathbf{R}$ of genus g (resp. principally polarized Abelian varieties over $\mathbf{R}$ of dimension g). Let

$$t_g : \mathscr{M}^g_{\mathbf{R}} \to \mathscr{A}^g_{\mathbf{R}}$$

be the Torelli map, that is, $t_g([\mathscr{X}]) = [\mathscr{J}]$ for all $[\mathscr{X}]$ in $\mathscr{M}^g_{\mathbf{R}}$, where $\mathscr{J}$ is the Jacobian variety of $\mathscr{X}$ endowed with the canonical polarization. If $g \geqslant 2$, then t_g is injective, while for $g = 0$ or 1 the restrictions of t_g to

$$\{ [\mathscr{X}] \in \mathscr{M}^g_{\mathbf{R}} \mid \mathscr{X}(\mathbf{R}) \neq \emptyset \} \quad \text{and} \quad \{ [\mathscr{X}] \in \mathscr{M}^g_{\mathbf{R}} \mid \mathscr{X}(\mathbf{R}) = \emptyset \}$$

are injective [14, 22]. In particular, for every $g \geqslant 0$ the restriction of t_g to each connected component $\mathscr{M}^{(g,s,\varepsilon)}_{\mathbf{R}}$ of $\mathscr{M}^g_{\mathbf{R}}$ is injective (cf. Section 1 for the definition of $\mathscr{M}^{(g,s,\varepsilon)}_{\mathbf{R}}$).

Before we state the next general property of t_g, recall that a topological embedding is a continuous, injective map $f : \mathrm{S} \to \mathrm{T}$ between topological spaces such that f maps homeomorphically S onto $f(\mathrm{S})$ endowed with the topology induced from T.

Proposition **4.1.** — *If* $g \geqslant 2$, *then* $t_g : \mathscr{M}^g_{\mathbf{R}} \to \mathscr{A}^g_{\mathbf{R}}$ *is a topological embedding.*

Proof. — Since t_g is continuous (cf. [26]) and injective, it suffices to prove that if $\{ \mathcal{X}_n \}$ is a sequence in $\mathcal{M}^g_{\mathbf{R}}$ (we identify curves and their isomorphism classes) and $\{ t_g(\mathcal{X}_n) \}$ converges in $\mathcal{A}^g_{\mathbf{R}}$, then $\{ \mathcal{X}_n \}$ converges in $\mathcal{M}^g_{\mathbf{R}}$.

Let $\bar{\mathcal{M}}^g_{\mathbf{R}}$ be the moduli space of stable curves over $\mathbf{R}$ (cf. [25]). In particular, $\mathcal{M}^g_{\mathbf{R}} \subset \bar{\mathcal{M}}^g_{\mathbf{R}}$ and $\bar{\mathcal{M}}^g_{\mathbf{R}}$ is a compactification of $\mathcal{M}^g_{\mathbf{R}}$. We claim that if a subsequence $\{ \mathcal{X}_{n_k} \}$ of $\{ \mathcal{X}_n \}$ converges to a curve $\mathcal{X}$ in $\bar{\mathcal{M}}^g_{\mathbf{R}}$, then $\mathcal{X}$ belongs to $\mathcal{M}^g_{\mathbf{R}}$, that is, $\mathcal{X}$ is smooth and $\mathcal{X} \times_{\mathbf{R}} \mathbf{C}$ is irreducible. To this end, let $\mathcal{M}^g_{\mathbf{C}}$, $\bar{\mathcal{M}}^g_{\mathbf{C}}$ and $\mathcal{A}^g_{\mathbf{C}}$ denote the complex counterparts of $\mathcal{M}^g_{\mathbf{R}}$, $\bar{\mathcal{M}}^g_{\mathbf{R}}$ and $\mathcal{A}^g_{\mathbf{R}}$, respectively. The map $\bar{\mathcal{M}}^g_{\mathbf{R}} \to \bar{\mathcal{M}}^g_{\mathbf{C}}$, $\mathcal{Y} \to \mathcal{Y} \times_{\mathbf{R}} \mathbf{C}$ is continuous and hence the sequence $\{ \mathcal{X}_{n_k} \times_{\mathbf{R}} \mathbf{C} \}$ converges to $\mathcal{X} \times_{\mathbf{R}} \mathbf{C}$ in $\bar{\mathcal{M}}^g_{\mathbf{C}}$. Let $\mathrm{T}_g : \mathcal{M}^g_{\mathbf{C}} \to \mathcal{A}^g_{\mathbf{C}}$ be the Torelli map. Since the map $\mathcal{A}^g_{\mathbf{R}} \to \mathcal{A}^g_{\mathbf{C}}$, $\mathcal{A} \to \mathcal{A} \times_{\mathbf{R}} \mathbf{C}$ is continuous and $\mathrm{T}_g(\mathcal{X}_n \times_{\mathbf{R}} \mathbf{C}) = t_g(\mathcal{X}_n) \times_{\mathbf{R}} \mathbf{C}$, we conclude that the sequence $\{ \mathrm{T}_g(\mathcal{X}_n \times_{\mathbf{R}} \mathbf{C}) \}$ converges in $\mathcal{A}^g_{\mathbf{C}}$. By [23, p. 111, 112], T_g is a topological embedding, and hence $\{ \mathcal{X}_n \times_{\mathbf{R}} \mathbf{C} \}$ converges in $\mathcal{M}^g_{\mathbf{C}}$. Since $\bar{\mathcal{M}}^g_{\mathbf{C}}$ is a Hausdorff space, it follows that $\mathcal{X}$ is smooth and $\mathcal{X} \times_{\mathbf{R}} \mathbf{C}$ is irreducible; thus the claim is proved.

The claim implies that $\{ \mathcal{X}_n \}$ converges in $\mathcal{M}^g_{\mathbf{R}}$, and hence the proof of the proposition is finished. □

We proceed to give an explicit description of $t_g(\mathcal{M}^g_{\mathbf{R}})$ for $g \leqslant 2$. This is trivial if $g = 0$ since $\mathcal{A}^0_{\mathbf{R}}$ consists of one point. To deal with the case $g > 0$, recall from Section 2 (see the text following the proof of Proposition 2.7) that the map

$$\pi_g : \mathrm{H}_g \to \mathcal{A}^g_{\mathbf{R}}, \qquad \pi_g(\mathrm{Z}) = [\mathcal{Y}_{\mathrm{Z}}]$$

is continuous and surjective. Obviously, an element Z of H_g is a period matrix of an algebraic curve $\mathcal{X}$ over $\mathbf{R}$ of genus g if and only if $\pi_g(\mathrm{Z}) = t_g([\mathcal{X}])$.

Example **4.2.** — Set

$$\mathrm{A}^{(1,1,1)} = \left\{ \frac{1}{2} + \sqrt{-1}\, t \mid t \in \mathbf{R}, t > 0 \right\},$$

$$\mathrm{A}^{(1,0,2)} = \left\{ \sqrt{-1}\, t \mid t \in \mathbf{R}, t > 0 \right\},$$

$$\mathrm{A}^1 = \mathrm{A}^{(1,1,1)} \cup \mathrm{A}^{(1,0,2)}.$$

It is well known that $\pi_1 \mid \mathrm{A}^1 : \mathrm{A}^1 \to \mathcal{A}^1_{\mathbf{R}}$ is a homeomorphism (even a real analytic isomorphism [14, 29]), and hence

$$\mathcal{A}^{(1,1,1)}_{\mathbf{R}} = (\pi_1 \mid \mathrm{A}^1)\,(\mathrm{A}^{(1,1,1)}), \qquad \mathcal{A}^{(1,0,2)}_{\mathbf{R}} = (\pi_1 \mid \mathrm{A}^1)\,(\mathrm{A}^{(1,0,2)})$$

are the connected components of $\mathcal{A}^1_{\mathbf{R}}$. Furthermore,

$$t_1(\mathcal{M}^{(1,1,1)}_{\mathbf{R}}) = \mathcal{A}^{(1,1,1)}_{\mathbf{R}},$$

$$t_1(\mathcal{M}^{(1,2,2)}_{\mathbf{R}}) = \mathcal{A}^{(1,0,2)}_{\mathbf{R}}, \qquad t_1(\mathcal{M}^{(1,0,2)}_{\mathbf{R}}) = \mathcal{A}^{(1,0,2)}_{\mathbf{R}},$$

and the restriction of t_1 to each connected component of $\mathcal{M}^1_{\mathbf{R}}$ is a topological embedding of this component in $\mathcal{A}^1_{\mathbf{R}}$. Later on in this section we shall make use of the map

$$u_1 = (\pi_1 \mid \mathrm{A}^1)^{-1} \circ t_1 : \mathcal{M}^1_{\mathbf{R}} \to \mathrm{A}^1.$$

Every algebraic curve $\mathscr{X}$ over $\mathbf{R}$ of genus 1 has a unique period (period matrix, if we want to conform to general terminology) Z in A^1, namely $Z = u_1([\mathscr{X}])$. It will be also convenient to set

$$M^1 = A^1, \quad M^{(1,1,1)} = A^{(1,1,1)}, \quad \text{and} \quad M^{(1,2,2)} = A^{(1,0,2)}. \quad \square$$

Some preparation is still required to describe $t_2(\mathscr{M}^2_{\mathbf{R}})$. Recall (cf. Section 2, the paragraph preceding the proof of Theorem 1.7) that $\pi_2 \mid A^2 : A^2 \to \mathscr{A}^2_{\mathbf{R}}$ is a homeomorphism, where

$$A^2 = A^{(2,2,1)} \cup A^{(2,1,1)} \cup A^{(2,0,2)} \cup A^{(2,2,2)}$$

and the $A^{(2,\ell,\varepsilon)}$ are the connected components of A^2, explicitly described in H_2 by simple inequalities. The family $\{ \mathscr{A}^{(2,\ell,\varepsilon)}_{\mathbf{R}} = \pi_2(A^{(2,\ell,\varepsilon)}) \}$ is the set of connected components of $\mathscr{A}^2_{\mathbf{R}}$; thus $\mathscr{A}^2_{\mathbf{R}}$ has 4 connected components.

We remember from Section 1 that $\mathscr{M}^2_{\mathbf{R}}$ has 5 connected components $\mathscr{M}^{(2,s,\varepsilon)}_{\mathbf{R}}$, where (s, ε) belongs to $\{ (1, 1), (2, 1), (3, 2), (1, 2), (0, 2) \}$. It is well known that

$$t_2(\mathscr{M}^{(2,1,1)}_{\mathbf{R}}) \subseteq \mathscr{A}^{(2,2,1)}_{\mathbf{R}}, \quad t_2(\mathscr{M}^{(2,2,1)}_{\mathbf{R}}) \subseteq \mathscr{A}^{(2,1,1)}_{\mathbf{R}}, \quad t_2(\mathscr{M}^{(2,3,2)}_{\mathbf{R}}) \subseteq \mathscr{A}^{(2,0,2)}_{\mathbf{R}},$$

$$t_2(\mathscr{M}^{(2,1,2)}_{\mathbf{R}}) \cup t_2(\mathscr{M}^{(2,0,2)}_{\mathbf{R}}) \subseteq \mathscr{A}^{(2,2,2)}_{\mathbf{R}}$$

(cf. [14, 26]). Define

$$u_2 : \mathscr{M}^2_{\mathbf{R}} \to A^2$$

by $u_2 = (\pi_2 \mid A^2)^{-1} \circ t_2$. By construction, every algebraic curve $\mathscr{X}$ over $\mathbf{R}$ of genus 2 has a unique period matrix Z in A^2, namely $Z = u_2([\mathscr{X}])$. It turns out that $u_2(\mathscr{M}^{(2,s,\varepsilon)}_{\mathbf{R}})$ can be explicitly described. To this end, set

$$M^{(2,1,1)} = \{ Z \in A^{(2,2,1)} \mid \operatorname{Im} Z = (t_{ij}),\ t_{12} > 0 \},$$

$$M^{(2,2,1)} = \{ Z \in A^{(2,1,1)} \mid \operatorname{Im} Z = (t_{ij}),\ t_{12} > 0 \},$$

$$M^{(2,3,2)} = \{ Z \in A^{(2,0,2)} \mid \operatorname{Im} Z = (t_{ij}),\ t_{12} > 0 \},$$

$$M^{(2,1,2)} = \left\{ Z \in A^{(2,2,2)} \mid \det(\operatorname{Im} Z) < \frac{1}{4} \right\},$$

$$M^{(2,0,2)} = \left\{ Z \in A^{(2,2,2)} \mid \det(\operatorname{Im} Z) > \frac{1}{4} \right\},$$

$$M^2 = M^{(2,1,1)} \cup M^{(2,2,1)} \cup M^{(2,3,2)} \cup M^{(2,1,2)} \cup M^{(2,0,2)}.$$

Theorem **4.3.** — *The map* $u_2 : \mathscr{M}^2_{\mathbf{R}} \to A^2$ *is a topological embedding. Moreover,*

$$u_2(\mathscr{M}^2_{\mathbf{R}}) = M^2 \quad \textit{and} \quad u_2(\mathscr{M}^{(2,s,\varepsilon)}_{\mathbf{R}}) = M^{(2,s,\varepsilon)}$$

for all (s, ε) *in* $\{ (1, 1), (2, 1), (3, 2), (1, 2), (0, 2) \}$.

Proof. — Let $\mathscr{Y}$ be a principally polarized Abelian surface over $\mathbf{R}$. We shall view $\mathscr{Y}_{\mathbf{C}} = \mathscr{Y} \times_{\mathbf{R}} \mathbf{C}$ as a principally polarized Abelian surface over $\mathbf{C}$ and identify, as usual, $\mathscr{Y}(\mathbf{C})$ and $\mathscr{Y}_{\mathbf{C}}(\mathbf{C})$. Since every algebraic curve of genus 2 is hyperelliptic, it follows from Torelli's theorem (cf. the version in [22, Theorem 12.1]) that $\mathscr{Y}$ is the Jacobian variety of an algebraic curve over $\mathbf{R}$ if and only if $\mathscr{Y}_{\mathbf{C}}$ is the Jacobian variety of an algebraic curve (projective and smooth) over $\mathbf{C}$. Hence, by [20, p. 348, (8.2)], $\mathscr{Y}$ is not the Jacobian variety of an algebraic curve over $\mathbf{R}$ if and only if $\mathscr{Y}_{\mathbf{C}}$ is isomorphic as a polarized Abelian variety to $\mathscr{E}_1 \times_{\mathbf{C}} \mathscr{E}_2$, where $\mathscr{E}_1$ and $\mathscr{E}_2$ are elliptic curves over $\mathbf{C}$ endowed with the canonical polarizations. We identify $\mathscr{Y}(\mathbf{C})$ with $\mathscr{E}_1(\mathbf{C}) \times \mathscr{E}_2(\mathbf{C})$ and regard $\mathscr{E}_k(\mathbf{C})$ as a subset of $\mathscr{Y}(\mathbf{C})$, $k = 1, 2$. If $\sigma : \mathscr{Y}(\mathbf{C}) \to \mathscr{Y}(\mathbf{C})$ is the complex conjugation, then either

$$\textbf{(4.3.1)} \qquad \sigma(\mathscr{E}_k(\mathbf{C})) = \mathscr{E}_k(\mathbf{C}) \quad \text{for } k = 1, 2,$$

or

$$\textbf{(4.3.2)} \qquad \sigma(\mathscr{E}_\ell(\mathbf{C})) \neq \mathscr{E}_\ell(\mathbf{C}), \quad \text{where } \ell = 1 \text{ or } \ell = 2.$$

If (4.3.1) holds, then there exists an elliptic curve $\mathscr{D}_k$ over $\mathbf{R}$ such that $\mathscr{E}_k = \mathscr{D}_k \times_{\mathbf{R}} \mathbf{C}$ for $k = 1, 2$. It follows that $[\mathscr{Y}] = [\mathscr{D}_1 \times_{\mathbf{R}} \mathscr{D}_2]$ in $\mathscr{A}^2_{\mathbf{R}}$. By example 4.2, we have $\pi_1(Z_k) = [\mathscr{D}_k]$ for some Z_k in A^1, $k = 1, 2$. Hence $(\pi_2 \mid A^2)^{-1}([\mathscr{Y}])$ belongs to

$$P = (A^{(2,2,1)} \backslash M^{(2,1,1)}) \cup (A^{(2,1,1)} \backslash M^{(2,2,1)}) \cup (A^{(2,0,2)} \backslash M^{(2,3,2)}).$$

Conversely, if $(\pi_2 \mid A^2)^{-1}([\mathscr{Y}])$ belongs to P, then there exist elliptic curves $\mathscr{E}_1$ and $\mathscr{E}_2$ over $\mathbf{C}$ such that $\mathscr{Y}_{\mathbf{C}}$ and $\mathscr{E}_1 \times_{\mathbf{C}} \mathscr{E}_2$ are isomorphic as polarized Abelian varieties, and (4.3.1) is satisfied.

If (4.3.2) holds, then by [20, p. 348, (8.1)], there exists an isomorphism of complex Lie groups $h : \mathscr{Y}_{\mathbf{C}}(\mathbf{C}) \to \mathscr{E}_\ell(\mathbf{C}) \times \sigma(\mathscr{E}_\ell(\mathbf{C}))$. Moreover, if h is constructed as in the proof of [20, p. 348, (8.1)], and $h(v) = (v_1, v_2)$, where v is in $\mathscr{Y}_{\mathbf{C}}(\mathbf{C})$, v_1 is in $\mathscr{E}_\ell(\mathbf{C})$, v_2 is in $\sigma(\mathscr{E}_\ell(\mathbf{C}))$, then $h(\sigma(v)) = (\sigma(v_2), \sigma(v_1))$. Note that if $\bar{\mathscr{E}}_\ell$ is the conjugate of $\mathscr{E}_\ell$, then $\bar{\mathscr{E}}_\ell(\mathbf{C}) = \sigma(\mathscr{E}_\ell(\mathbf{C}))$ and h is induced by an isomorphism $\mathscr{Y}_{\mathbf{C}} \to \mathscr{E}_\ell \times_{\mathbf{C}} \bar{\mathscr{E}}_\ell$ of polarized Abelian varieties. By [30, Lemma 10.10 and its proof], $(\pi_2 \mid A^2)^{-1}([\mathscr{Y}])$ belongs to

$$Q = \left\{ Z \in A^{(2,2,2)} \mid \det(\operatorname{Im} Z) = \frac{1}{4} \right\} = A^{(2,2,2)} \backslash (M^{(2,1,2)} \cup M^{(2,0,2)}).$$

Conversely, if $(\pi_2 \mid A^2)^{-1}([\mathscr{Y}])$ belongs to Q, using [30, Lemma 10.10], one readily shows that (4.3.2) is satisfied.

Summarizing, we have

$$u_2(\mathscr{M}^2_{\mathbf{R}}) = M^2,$$

$$u_2(\mathscr{M}^{(2,s,\varepsilon)}_{\mathbf{R}}) = M^{(2,s,\varepsilon)} \quad \text{for } (s, \varepsilon) \in \{ (1,1), (2,1), (3,2) \},$$

$$u_2(\mathscr{M}^{(2,1,2)}_{\mathbf{R}}) \cup u_2(\mathscr{M}^{(2,0,2)}_{\mathbf{R}}) = M^{(2,1,2)} \cup M^{(2,0,2)}.$$

Applying [30, Lemma 10.10] once again, we also get

$$u_2(\mathcal{M}_{\mathbf{R}}^{(2,1,2)}) = \mathrm{M}^{(2,1,2)}, \quad u_2(\mathcal{M}_{\mathbf{R}}^{(2,0,2)}) = \mathrm{M}^{(2,0,2)}.$$

Hence the proof of the theorem is complete. □

Let $\mathscr{X}$ be an algebraic curve over $\mathbf{R}$ of genus g with $\mathrm{X} = \mathscr{X}(\mathbf{R})$ nonempty, that is, $g(\mathscr{X}) = g$ and $s(\mathscr{X}) \geqslant 1$. Let $\mathscr{J}$ be the Jacobian variety of $\mathscr{X}$. If Z is a period matrix of $\mathscr{X}$, then one can find an isomorphism $\Phi : \mathscr{Y}_{\mathrm{Z}} \to \mathscr{J}$ of polarized Abelian varieties over $\mathbf{R}$. Let $\Phi_{\mathbf{C}} : \mathscr{Y}_{\mathrm{Z}}(\mathbf{C}) \to \mathscr{J}(\mathbf{C})$ be the map determined by Φ. Since $\mathscr{Y}_{\mathrm{Z}}(\mathbf{C}) = \mathbf{C}^g/[\Omega]$, where $\Omega = (\mathrm{Z}, \mathrm{I}_g)$, and $\Phi_{\mathbf{C}}$ is a $\mathrm{Gal}(\mathbf{C}/\mathbf{R})$-equivariant isomorphism of complex Lie groups, we have the homomorphism

$$\hat{\Phi}_{\mathbf{C}} : \mathrm{H}_1(\mathrm{X}, \mathbf{Z}) \to \mathbf{Z}^g$$

constructed in the paragraph following Proposition 3.3 (recall that $\mathbf{Z}^g = [\Omega] \cap \mathbf{R}^g$, and elements of $\mathbf{Z}^g$ are viewed as $g \times 1$ matrices) and the subgroup

$$\mathrm{L}(\mathrm{X}, \Phi_{\mathbf{C}}) = \hat{\Phi}_{\mathbf{C}}(\mathrm{H}_1(\mathrm{X}, \mathbf{Z}))$$

of $\mathbf{Z}^g$. Explicit computation of $\mathrm{L}(\mathrm{X}, \Phi_{\mathbf{C}})$ is crucial for effective applications of Theorem 3.9. Directly from Theorem 3.9 we obtain the following facts for $g = 1$ or 2. If $g = 1$, then $\mathrm{L}(\mathrm{X}, \Phi_{\mathbf{C}}) = \mathbf{Z}$. If $g = 2$ and $s(\mathscr{X}) \geqslant 2$ (resp. $g = 2$ and $(s(\mathscr{X}), \varepsilon(\mathscr{X})) = (1, 2)$) then $\mathrm{L}(\mathrm{X}, \Phi_{\mathbf{C}}) = \mathbf{Z}^2$ (resp. $\mathrm{L}(\mathrm{X}, \Phi_{\mathbf{C}}) = 0$). The only remaining case for $g = 2$, namely $(s(\mathscr{X}), \varepsilon(\mathscr{X})) = (1, 1)$, is much harder and is dealt with below.

Proposition **4.4.** — *Let $\mathscr{X}$ be an algebraic curve over* $\mathbf{R}$ *of genus* 2. *Assume that $s(\mathscr{X}) = 1$ and $\varepsilon(\mathscr{X}) = 1$, that is, $[\mathscr{X}]$ belongs to $\mathcal{M}_{\mathbf{R}}^{(2,1,1)}$. If $\mathrm{Z} = u_2([\mathscr{X}])$ and $\mathscr{J}$ is the Jacobian variety of $\mathscr{X}$, then there exists an isomorphism $\Phi : \mathscr{Y}_{\mathrm{Z}} \to \mathscr{J}$ of polarized Abelian varieties over* $\mathbf{R}$ *such that*

$$\mathrm{L}(\mathrm{X}, \Phi_{\mathbf{C}}) = \mathbf{Z}\begin{pmatrix}1\\1\end{pmatrix}.$$

Proof. — Let $\mathscr{P}$ be the space of all polynomials Q of degree 6 with real coefficients and the leading coefficient 1 such that the complex roots of Q are distinct, and -1, 1 are the only real roots of Q. It is well known that $\mathscr{X}$ can be given by an affine equation

$$y^2 = \mathrm{P}(x),$$

for some P in $\mathscr{P}$ (cf. [14, p. 170]). We regard $\mathscr{X}(\mathbf{C})$ as the 2-sheeted branched covering of $\mathbf{P}^1(\mathbf{C})$,

$$\pi : \mathscr{X}(\mathbf{C}) \to \mathbf{P}^1(\mathbf{C}),$$

ramified over the roots of P. As usual, we identify $\mathbf{P}^1(\mathbf{C}) \setminus \{[1:0]\}$ with $\mathbf{C}$, and $\mathscr{X}(\mathbf{C}) \setminus \pi^{-1}([1:0])$ with

$$\mathrm{C}_{\mathrm{P}} = \{ (x, y) \in \mathbf{C}^2 \mid y^2 = \mathrm{P}(x) \}.$$

Then π, viewed as a map of C_P into $\mathbf{C}$, is given by

$$\pi(x,y) = x.$$

Let $\sigma : \mathscr{X}(\mathbf{C}) \to \mathscr{X}(\mathbf{C})$ be the complex conjugation.

We shall now construct a symplectic basis $(\beta_1, \beta_2, \alpha_1, \alpha_2)$ for $H_1(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ (α_k and β_k will be represented by cycles passing through the ramification points, which is a convenient, for our purposes, modification of the usual construction, cf. for example [20, p. 345, 346]) such that

(4.4.1) $\quad H_1(\sigma)(\alpha_k) = \beta_k - \alpha_k, \quad H_1(\sigma)(\beta_k) = \beta_k \quad$ for $k = 1, 2$,

(4.4.2) $\quad \gamma = \beta_1 + \beta_2$,

where γ is the homology class in $H_1(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ represented by $\mathscr{X}(\mathbf{R})$ endowed with a suitable orientation.

To this end, let $p_0, \ldots, p_5$ be the roots of P ordered in such a way that $p_0 = 1$, $p_3 = -1$, $\bar{p}_1 = p_5$, $\bar{p}_2 = p_4$, $\operatorname{Im} p_1 > 0$, $\operatorname{Im} p_2 > 0$, $\operatorname{Re} p_1 \geqslant \operatorname{Re} p_2$. Let a_1 and a_2 be the oriented segments from p_1 to p_0, and from p_4 to p_3, respectively. Let b_1 and b_2 be simple oriented arcs in $\mathbf{C}$ from p_1 to p_5, and from p_4 to p_2, respectively. We choose b_k so that $\tau(b_k) = b_k$ as sets for $k = 1, 2$, where $\tau : \mathbf{C} \to \mathbf{C}$ is the complex conjugation. Denote by c the union $(-\infty, -1] \cup [1, \infty)$ oriented in such a way that the preferred direction on $(-\infty, -1]$ (resp. $[1, \infty)$) is from $-\infty$ to -1 (resp. ∞ to 1). Let r_j be a ray with beginning point p_j, $j = 0, \ldots, 5$.

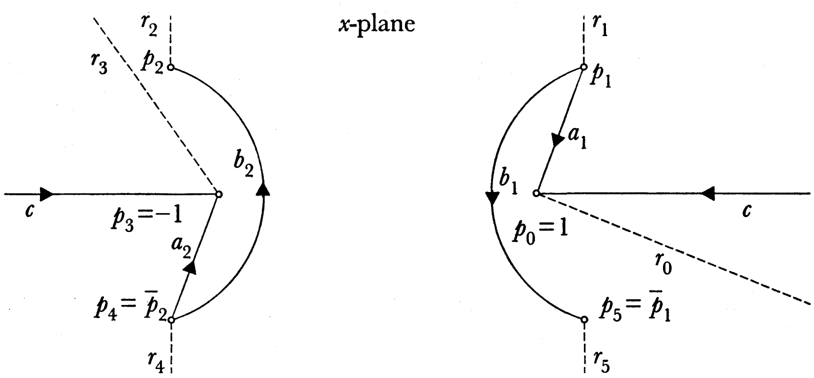

FIG. 1

We choose b_1, b_2, and $r_0, \ldots, r_5$ as on Figure 1. In particular, the sets we consider intersect only at the points indicated on Figure 1, and a_1 is transverse to b_1 at p_1 (resp. a_2 is transverse to b_2 at p_4).

Note that $\pi^{-1}(a_k)$, $\pi^{-1}(b_k)$ are $\mathscr{C}^\infty$ curves in $\mathscr{X}(\mathbf{C})$ for $k = 1, 2$, and $\pi^{-1}(c) = \mathscr{X}(\mathbf{R})$. Choose the determination of $\sqrt{P(x)}$ on $\mathbf{C}\backslash(r_0 \cup \ldots \cup r_5)$ such that $\operatorname{Im}\sqrt{P(0)} > 0$. Clearly, the set

$$C_P^+ = \left\{ (x,y) \in \mathbf{C}^2 \mid x \in \mathbf{C}\backslash(r_0 \cup \ldots \cup r_5),\ y = \sqrt{P(x)} \right\}$$

is contained in C_P. We pick the orientations on $\pi^{-1}(a_k)$, $\pi^{-1}(b_k)$, $\mathscr{X}(\mathbf{R})$ in such a way that the embeddings

$$\pi : \pi^{-1}(a_k) \cap C_P^+ \to a_k, \qquad \pi : \pi^{-1}(b_k) \cap C_P^+ \to b_k,$$
$$\pi : \mathscr{X}(\mathbf{R}) \cap C_P^+ \to c$$

preserve the orientations (obviously, we can do this for the first two embeddings, and the choice of the orientation on c implies that this can be done also for the third embedding).

Let α_k, β_k, and γ be the homology classes in $H_1(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ represented by $\pi^{-1}(a_k)$, $\pi^{-1}(b_k)$, and $\mathscr{X}(\mathbf{R})$, respectively. One readily verifies that $(\beta_1, \beta_2, \alpha_1, \alpha_2)$ is a symplectic basis for $H_1(\mathscr{X}(\mathbf{C}), \mathbf{Z})$ and (4.4.1) holds. Furthermore, the intersection number of γ and α_k is 1 for $k = 1, 2$, and hence (4.4.2) is also satisfied.

It is now easy to obtain a period matrix of $\mathscr{X}$. Indeed, regard

$$\omega = \frac{dx}{y} \quad \text{and} \quad \eta = \frac{x\,dx}{y}$$

as holomorphic forms on $\mathscr{X}(\mathbf{C})$ (they are linearly independent), and set

$$A = \begin{pmatrix} \int_{\alpha_1} \omega & \int_{\alpha_2} \omega \\ \int_{\alpha_1} \eta & \int_{\alpha_2} \eta \end{pmatrix}, \qquad B = \begin{pmatrix} \int_{\beta_1} \omega & \int_{\beta_2} \omega \\ \int_{\beta_1} \eta & \int_{\beta_2} \eta \end{pmatrix}.$$

Since ω and η are defined over $\mathbf{R}$, it follows from (4.4.1) that the lattice generated by the columns of (A, B) is mapped onto itself by the complex conjugation. Thus

$$Z_P = B^{-1} A$$

belongs to H_2 and is a period matrix of $\mathscr{X}$. Furthermore (4.4.2) implies that one can find an isomorphism $\Psi : \mathscr{Y}_{Z_P} \to \mathscr{J}$ of polarized Abelian varieties over $\mathbf{R}$ such that

(4.4.3) $$L(X, \Psi_{\mathbf{C}}) = \mathbf{Z} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The proof is not finished, however, because Z_P need not be equal to $Z = u_2([\mathscr{X}])$.

In order to complete the proof, we first observe that Z_P can be computed by integrating suitable forms on the x-plane. Let

$$\omega_P = \frac{dx}{\sqrt{P(x)}}, \qquad \eta_P = \frac{x\,dx}{\sqrt{P(x)}}$$

on $\mathbf{C} \backslash (r_0 \cup \ldots \cup r_5)$, and

$$A_P = \begin{pmatrix} \int_{a_1} \omega_P & \int_{a_2} \omega_P \\ \int_{a_1} \eta_P & \int_{a_2} \eta_P \end{pmatrix}, \qquad B_P = \begin{pmatrix} \int_{b_1} \omega_P & \int_{b_2} \omega_P \\ \int_{b_1} \eta_P & \int_{b_2} \eta_P \end{pmatrix}.$$

Then $A = 2A_P$, $B = 2B_P$, and

(4.4.4) $$Z_P = B_P^{-1} A_P.$$

We shall now consider a particular algebraic curve $\mathscr{X}_0$ over $\mathbf{R}$ given by the equation $y^2 = P_0(x)$, where

$$P_0(x) = x^6 - 1.$$

We claim that

(4.4.5) $$Z_{P_0} = u_2([\mathscr{X}_0]).$$

Indeed, we can use the construction described above for the polynomial P_0. If $\zeta = (1/2) + \sqrt{-1}(\sqrt{3}/2)$, then $p_j = \zeta^j$, $j = 0, \ldots, 5$, are the roots of P_0. As b_1 (resp. b_2) we can take the oriented segment from p_1 to p_5 (resp. from p_4 to p_2).

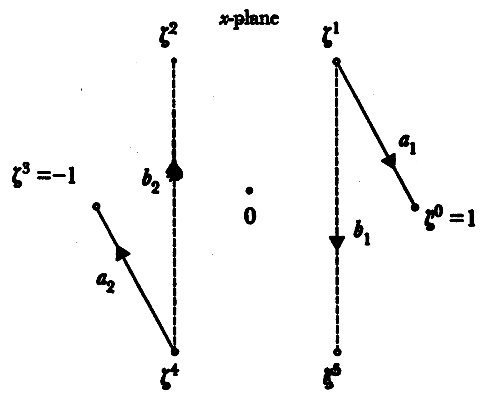

FIG. 2

All the integrals below are taken along the line segments. If

$$r = \int_0^1 \omega_{P_0} \quad \text{and} \quad s = \int_0^1 \eta_{P_0},$$

then an obvious change of variables yields

(4.4.6) $$\int_0^{\zeta^j} \omega_{P_0} = \zeta^j r \quad \text{and} \quad \int_0^{\zeta^j} \eta_{P_0} = \zeta^{2j} s$$

for $j = 0, \ldots, 5$. Using (4.4.6), we can easily express the integrals of ω_{P_0} and η_{P_0} along a_k and b_k in terms of ζ, r, s. For example,

$$\int_{a_1} \omega_{P_0} = \int_{\zeta^1}^{\zeta^0} \omega_{P_0} = \int_0^1 \omega_{P_0} - \int_0^{\zeta} \omega_{P_0} = r(1 - \zeta^1).$$

For the other integrals one gets analogous formulas, which leads to

$$A_{P_0} = \begin{pmatrix} r(1-\zeta^1) & -r(1+\zeta^4) \\ s(1-\zeta^2) & s(1-\zeta^2) \end{pmatrix},$$

$$B_{P_0} = \begin{pmatrix} r(\zeta^5-\zeta^1) & r(\zeta^2-\zeta^4) \\ s(\zeta^4-\zeta^2) & s(\zeta^4-\zeta^2) \end{pmatrix} = -2\sqrt{-1}\sqrt{3}\begin{pmatrix} r & -r \\ s & s \end{pmatrix}.$$

Hence, by (4.4.4),

$$Z_{P_0} = B_{P_0}^{-1} A_{P_0} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

Clearly, Z_{P_0} is in $M^{(2,1,1)}$ and therefore (4.4.5) is proved. In view of (4.4.3), the proposition is proved for $\mathscr{X}_0$.

We shall now consider the general case. Let $S_2(\mathbf{R})$ be the vector space of all real symmetric 2×2 matrices and let $S_2^+(\mathbf{R})$ be the cone in $S_2(\mathbf{R})$ of the positive definite matrices,

$$S_2^+(\mathbf{R}) = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \in S_2(\mathbf{R}) \mid t_{11} > 0,\ t_{11}t_{22} - t_{12}^2 > 0 \right\}.$$

Observe that the space $\mathscr{P}$ is connected, the map

$$F : \mathscr{P} \to H_2, \quad F(Q) = Z_Q$$

is continuous, and $2\,\mathrm{Re}\,F(P_0) = 2\,\mathrm{Re}\,Z_{P_0} = I_2$. It follows that

$$F(\mathscr{P}) \subseteq \frac{1}{2} I_2 + \sqrt{-1}\, S_2^+(\mathbf{R}).$$

Set

$$R = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{12} & t_{22} \end{pmatrix} \in S_2^+(\mathbf{R}) \mid 0 < t_{12} \leqslant t_{11} \leqslant t_{22} \right\},$$

$$C_n = \begin{pmatrix} n & 1-n \\ 1+n & -n \end{pmatrix},$$

$$S_0 = R \cup C_0 R\, {}^tC_0 \quad \left(\text{of course, } C_0 = {}^tC_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

$$S_n = C_n S_0\, {}^tC_n, \quad S'_n = C_0 S_n\, {}^tC_0.$$

Clearly, $S_0 = S'_0$. Moreover, S_n is the convex subset of $S_2^+(\mathbf{R})$ bounded by the planes in $S_2(\mathbf{R})$ passing through the pairs of the lines

$$\mathbf{R}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \quad \mathbf{R}\begin{pmatrix}n^2 & n(n+1)\\ n(n+1) & (n+1)^2\end{pmatrix}, \quad \mathbf{R}\begin{pmatrix}(n-1)^2 & n(n-1)\\ n(n-1) & n^2\end{pmatrix}.$$

Note that the plane V_n passing through the last two lines is of the form

$$V_n = C_n V_0 \, {}^tC_n,$$

where

$$V_0 = \left\{ \begin{pmatrix} t_{11} & t_{12}\\ t_{12} & t_{22}\end{pmatrix} \in S_2(\mathbf{R}) \mid t_{12} = 0 \right\}.$$

Set

$$V'_n = C_0 V_n \, {}^tC_0.$$

If L is the plane in $S_2(\mathbf{R})$ defined by

$$L = \left\{ \begin{pmatrix} t_{11} & t_{12}\\ t_{12} & t_{22}\end{pmatrix} \in S_2(\mathbf{R}) \mid t_{22} = -t_{11} + 1 \right\},$$

then the intersection $S_2^+(\mathbf{R}) \cap L$ is an open disc. The sets $S_n \cap L$, $S'_n \cap L$, $V_n \cap L$, $V'_n \cap L$ are shown on Figure 3 for $n = 0, 1, 2$. One can verify that

$$S = \bigcup_{n \geqslant 0} S_n \cup \bigcup_{n \geqslant 0} S'_n$$

is a convex open subset of $S_2^+(\mathbf{R})$ with boundary ∂S contained in

$$V = \bigcup_{n \geqslant 0} V_n \cup \bigcup_{n \geqslant 0} V'_n.$$

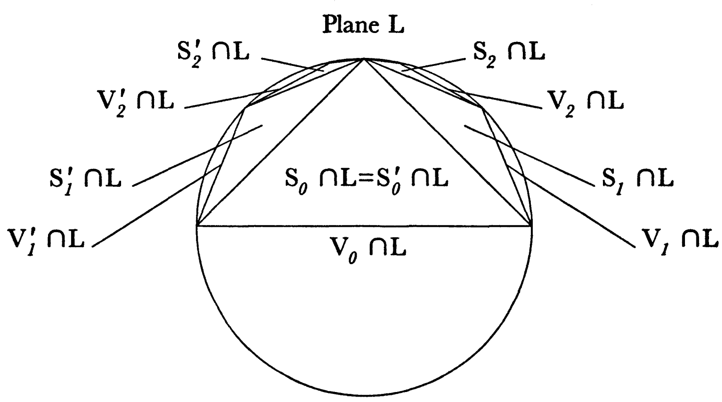

FIG. 3

We assert that

$$\text{(4.4.7)} \qquad F(\mathscr{P}) \subseteq \frac{1}{2} I_2 + \sqrt{-1} S.$$

Indeed, let us consider the action of the group Γ_2 on H_2 (cf. Section 2, the paragraph preceding the proof of Theorem 1.6 for the definition of Γ_2 and its action on H_2). Observe that if C belongs to $Gl_2(\mathbf{Z})$, then

$$K = \begin{pmatrix} C & \frac{1}{2}({}^tC^{-1} - C) \\ 0 & {}^tC^{-1} \end{pmatrix}$$

belongs to Γ_2, and the action of K on $W = \frac{1}{2} I_2 + \sqrt{-1}T$ in H_2 is given by

$$(K, W) \to K.W = \frac{1}{2} I_2 + \sqrt{-1} CT\,{}^tC.$$

Recall that $\mathscr{Y}_W$ and $\mathscr{Y}_{K.W}$ are isomorphic as polarized Abelian varieties over $\mathbf{R}$. It is obvious that no matrix in $\frac{1}{2} I_2 + \sqrt{-1} V_0$ is a period matrix of an algebraic curve over $\mathbf{R}$ (cf. the proof of Theorem 4.3), and hence no matrix in V is a period matrix of an algebraic curve over $\mathbf{R}$. Since $\partial S \subseteq V$, $\mathscr{P}$ is connected, F is continuous, $F(P_0)$ belongs to $\frac{1}{2} I_2 + \sqrt{-1} S$, and $F(Q) = Z_Q$ is a period matrix of an algebraic curve over $\mathbf{R}$ for all Q in $\mathscr{P}$, it follows that (4.4.7) is satisfied.

By (4.4.7), the period matrix Z_P of $\mathscr{X}$ is of the form

$$\text{(4.4.8)} \qquad Z_P = \begin{pmatrix} C & \frac{1}{2}({}^tC^{-1} - C) \\ 0 & {}^tC^{-1} \end{pmatrix} . Z',$$

where Z′ belongs to $\frac{1}{2} I_2 + \sqrt{-1} R = M^{(2,1,1)}$ and either $C = C_n$ or $C = C_0 C_n$ for some $n \geqslant 0$. Since Z′ is a period matrix of $\mathscr{X}$ and Z is in $M^{(2,1,1)}$, we must have $Z' = Z = u_2([\mathscr{X}])$. It follows from (4.4.8) that there exists an isomorphism

$$G : \mathscr{Y}_{Z_P} \to \mathscr{Y}_{Z'} = \mathscr{Y}_Z$$

of polarized Abelian varieties over $\mathbf{R}$ such that the map

$$G_{\mathbf{C}} : \mathscr{Y}_{Z_P}(\mathbf{C}) = \mathbf{C}^2/[(Z_P, I_2)] \to \mathscr{Y}_Z(\mathbf{C}) = \mathbf{C}^2/[(Z, I_2)]$$

determined by G is induced by the linear isomorphism

$$\mathbf{C}^2 \to \mathbf{C}^2, \quad v \to C^{-1} v.$$

Recall that we already constructed the isomorphism $\Psi : \mathscr{Y}_{Z_P} \to \mathscr{J}$ of polarized Abelian varieties over $\mathbf{R}$ such that (4.4.3) is satisfied. Note that

$$\Phi : \Psi \circ G^{-1} : \mathscr{Y}_Z \to \mathscr{J}$$

is an isomorphism of polarized Abelian varieties over $\mathbf{R}$, and since

$$C\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix},$$

we obtain

$$L(X, \Phi_C) = \mathbf{Z}\begin{pmatrix}1\\1\end{pmatrix}.$$

Thus the proof of the proposition is finished. □

We are now in a position to give a very explicit description of $\pi^2_{\mathscr{R}}(X_1 \times X_2)$, where $X_k = \mathscr{X}_k(\mathbf{R})$ and $\mathscr{X}_k$ is an algebraic curve over $\mathbf{R}$ of genus 1 or 2 for $k = 1, 2$. In particular, the map $u_g : \mathscr{M}^g_{\mathbf{R}} \to A^g$ for $g = 1, 2$ (cf. Example 4.2 and the paragraph preceding Theorem 4.3) will be used. We begin with the simplest case, $g(\mathscr{X}_k) = 1$ for $k = 1, 2$.

Example **4.5.** — Let $\mathscr{X}_k$ be an algebraic curve over $\mathbf{R}$ of genus 1 with $X_k = \mathscr{X}_k(\mathbf{R})$ nonempty, $k = 1, 2$. Let $Z_k = u_1([\mathscr{X}_k])$ for $k = 1, 2$. Then there exists an isomorphism

$$\tau_{11} : \mathrm{Mat}(1 \times 1, \mathbf{Z}) \to \pi^2_*(X_1 \times X_2)$$

such that

$$\tau_{11}(C(Z_1, Z_2)) = \pi^2_{\mathscr{R}}(X_1 \times X_2).$$

This follows immediately from Theorem 3.9. Of course, $\mathrm{Mat}(1 \times 1, \mathbf{Z}) = \mathbf{Z}$, and we should mention that the subgroup $C(Z_1, Z_2)$ of $\mathbf{Z}$ is explicitly computed in [9] (note that the notation in [9] is somewhat different than here). □

The case $g(\mathscr{X}_k) = k$ for $k = 1, 2$ is considerably more difficult and requires, among other things, Proposition 4.4.

Theorem **4.6.** — *Let* $\mathscr{X}_k$ *be an algebraic curve over* $\mathbf{R}$ *of genus* k *with* $X_k = \mathscr{X}_k(\mathbf{R})$ *nonempty,* $k = 1, 2$. *Let* $Z_k = u_k([\mathscr{X}_k])$ *for* $k = 1, 2$. *Then there exists an epimorphism*

$$\tau_{12} : \mathrm{Mat}(1 \times 2, \mathbf{Z}) \to \pi^2_*(X_1 \times X_2)$$

such that

(i) $\tau_{12}(C(Z_1, Z_2)) = \pi^2_{\mathscr{R}}(X_1 \times X_2)$,

(ii) τ_{12} *is an isomorphism if* $s(\mathscr{X}_2) \geqslant 2$,

(iii) $\mathrm{Ker}\, \tau_{12} = \{ (c_1, c_2) \in \mathrm{Mat}(1 \times 2, \mathbf{Z}) \mid c_1 + c_2 = 0 \}$ *if* $(s(\mathscr{X}_2), \varepsilon(\mathscr{X}_2)) = (1, 1)$,

(iv) $\mathrm{Ker}\, \tau_{12} = \mathrm{Mat}(1 \times 2, \mathbf{Z})$, *that is,* $\pi^2_*(X_1 \times X_2) = 0$ *if* $(s(\mathscr{X}_2), \varepsilon(\mathscr{X}_2)) = (1, 2)$.

Proof. — The existence of τ_{12} satisfying (i) and (ii) follows from Theorem 3.9, while in order to obtain (iii) one applies also Proposition 4.4. Condition (iv) is a consequence of Proposition 3.8. □

We shall now consider the last case, $g(\mathscr{X}_k) = 2$ for $k = 1, 2$. Of course, without loss of generality we may assume that $s(\mathscr{X}_1) \leqslant s(\mathscr{X}_2)$.

Theorem **4.7.** — *Let $\mathscr{X}_k$ be an algebraic curve over* $\mathbf{R}$ *of genus 2 with* $X_k = \mathscr{X}_k(\mathbf{R})$ *nonempty,* $k = 1, 2$. *Let* $Z_k = u_2([\mathscr{X}_k])$ *for* $k = 1, 2$. *Then there exists an epimorphism*

$$\tau_{22} : \mathrm{Mat}(2 \times 2, \mathbf{Z}) \to \pi_*^2(X_1 \times X_2)$$

such that

(i) $\tau_{22}(C(Z_1, Z_2)) = \pi^2_{\mathscr{R}}(X_1 \times X_2)$,

(ii) τ_{22} *is an isomorphism if* $s(\mathscr{X}_k) \geqslant 2$ *for* $k = 1, 2$,

(iii) $\mathrm{Ker}\,\tau_{22} = \{ (c_{ij}) \in \mathrm{Mat}(2 \times 2, \mathbf{Z}) \mid c_{11} + c_{21} = 0,\ c_{12} + c_{22} = 0 \}$ *if* $(s(\mathscr{X}_1), \varepsilon(\mathscr{X}_1)) = (1, 1)$ *and* $s(\mathscr{X}_2) \geqslant 2$,

(iv) $\mathrm{Ker}\,\tau_{22} = \{ (c_{ij}) \in \mathrm{Mat}(2 \times 2, \mathbf{Z}) \mid c_{11} + c_{12} + c_{21} + c_{22} = 0 \}$ *if* $(s(\mathscr{X}_k), \varepsilon(\mathscr{X}_k)) = (1, 1)$ *for* $k = 1, 2$,

(v) $\mathrm{Ker}\,\tau_{22} = \mathrm{Mat}(2 \times 2, \mathbf{Z})$, *that is,* $\pi_*^2(X_1 \times X_2) = 0$ *if* $(s(\mathscr{X}_\ell), \varepsilon(\mathscr{X}_\ell)) = (1, 2)$ *for* $\ell = 1$ *or* $\ell = 2$.

Proof. — As in the proof of Theorem 4.6, the existence of τ_{22} satisfying (i) and (ii) follows from Theorem 3.9. To obtain (iii) and (iv), one applies in addition Proposition 4.4. Condition (v) follows from Proposition 3.8. □

Before stating our next result, let us recall from Section 1 that

$$\{ \mathscr{M}_{\mathbf{R}}^{(g,s,\varepsilon)} \mid (s, \varepsilon) \in \Lambda_g \cup \{ (0, 2) \} \}$$

is the set of connected components of $\mathscr{M}_{\mathbf{R}}^g$.

By Proposition 3.8, if $([\mathscr{X}_1], [\mathscr{X}_2])$ is in $\mathscr{M}_{\mathbf{R}}^{(g_1, s_1, \varepsilon_1)} \times \mathscr{M}_{\mathbf{R}}^{(g_2, s_2, \varepsilon_2)}$ with (s_k, ε_k) in Λ_{g_k} for $k = 1, 2$, then

$$\mathrm{rank}\,\pi^2_{\mathscr{R}}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})) \leqslant \mathrm{rank}\,\pi_*^2(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})) \leqslant (s_1 - \varepsilon_1 + 1)\,(s_2 - \varepsilon_2 + 1).$$

Given a nonnegative integer r, we put

$$\mathscr{R}_r(g_1, s_1, \varepsilon_1; g_2, s_2, \varepsilon_2) = \{ ([\mathscr{X}_1], [\mathscr{X}_2]) \in \mathscr{M}_{\mathbf{R}}^{(g_1, s_1, \varepsilon_1)} \times \mathscr{M}_{\mathbf{R}}^{(g_2, s_2, \varepsilon_2)} \mid \mathrm{rank}\,\pi^2_{\mathscr{R}}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})) = r \}.$$

In particular, for $r = 0$ we have

$$\mathscr{R}_0(g_1, s_1, \varepsilon_1; g_2, s_2, \varepsilon_2) = \{ ([\mathscr{X}_1], [\mathscr{X}_2]) \in \mathscr{M}_{\mathbf{R}}^{(g_1, s_1, \varepsilon_1)} \times \mathscr{M}_{\mathbf{R}}^{(g_2, s_2, \varepsilon_2)} \mid \pi^2_{\mathscr{R}}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})) = 0 \}.$$

In order to describe the sets $\mathscr{R}_r(g_1, s_1, \varepsilon_1; g_2, s_2, \varepsilon_2)$, it is convenient to denote by $S_n(\mathbf{R})$ the $\mathbf{R}$-vector space of all real, symmetric $n \times n$ matrices. Our next result is a more detailed version of Theorem 1.12.

Theorem **4.8.** — *Let* (s_k, ε_k) *be in* Λ_{g_k}, *where* $1 \leqslant g_k \leqslant 2$ *for* $k = 1, 2$. *Let* $p = (s_1 - \varepsilon_1 + 1)(s_2 - \varepsilon_2 + 1)$. *Then there exists a chain of sets*

$$S_{g_1}(\mathbf{R}) \times S_{g_2}(\mathbf{R}) = V_0 \supset V_1 \supset \ldots \supset V_p \supset V_{p+1} = \emptyset$$

such that

(i) *for each integer* ℓ *satisfying* $1 \leqslant \ell \leqslant p$, *the set* V_ℓ *is the union of a countable family of algebraic subsets of* V_0,

(ii) $V_r \backslash V_{r+1}$ *is dense in* V_0 *(in the metric topology) for all* $0 \leqslant r \leqslant p$,

(iii) *the set* $U \cap (V_r \backslash V_{r+1})$ *is uncountable for each nonempty open subset* U *of* V_0 *and all* $0 \leqslant r \leqslant p$,

(iv) *given* $[\mathscr{X}_k]$ *in* $\mathscr{M}_{\mathbf{R}}^{(g_k, s_k, \varepsilon_k)}$ *with* $Z_k = u_{g_k}([\mathscr{X}_k])$ *for* $k = 1, 2$, *and an integer* r *satisfying* $0 \leqslant r \leqslant p$, *one has*

$$\operatorname{rank} \pi^2_{\mathscr{R}}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})) = r$$

if and only if $(\operatorname{Im} Z_1, \operatorname{Im} Z_2)$ *belongs to* $V_r \backslash V_{r+1}$.

Furthermore, $\mathscr{R}_r(g_1, s_1, \varepsilon_1; g_2, s_2, \varepsilon_2)$ *is an uncountable and dense subset of* $\mathscr{M}_{\mathbf{R}}^{(g_1, s_1, \varepsilon_1)} \times \mathscr{M}_{\mathbf{R}}^{(g_2, s_2, \varepsilon_2)}$ *for all* $0 \leqslant r \leqslant p$, *and the set* $\mathscr{R}_0(g_1, s_1, \varepsilon_1; g_2, s_2, \varepsilon_2)$ *is the intersection of a countable family of open and dense subsets of* $\mathscr{M}_{\mathbf{R}}^{(g_1, s_1, \varepsilon_1)} \times \mathscr{M}_{\mathbf{R}}^{(g_2, s_2, \varepsilon_2)}$.

Proof. — Let $q = g_1 g_2$, $W_0 = S_{g_1}(\mathbf{R}) \times S_{g_2}(\mathbf{R})$, and $W_{q+1} = \emptyset$. Given an integer n, $1 \leqslant n \leqslant q$, denote by Ω_n the set of all n-tuples $(C_1, \ldots, C_n)$ of linearly independent elements of $\operatorname{Mat}(g_1 \times g_2, \mathbf{Z})$. Obviously, for each $(C_1, \ldots, C_n)$ in Ω_n the set

$$W_{(C_1, \ldots, C_n)} = \{ (T_1, T_2) \in W_0 \mid T_1 C_j T_2 \in \operatorname{Mat}(g_1 \times g_2, \mathbf{Z}) \text{ for } 1 \leqslant j \leqslant n \}$$

is the union of a countable family of algebraic subsets of W_0. Setting

$$W_n = \bigcup_{(C_1, \ldots, C_n) \in \Omega_n} W_{(C_1, \ldots, C_n)},$$

we obtain a chain of sets

$$S_{g_1}(\mathbf{R}) \times S_{g_2}(\mathbf{R}) = W_0 \supset W_1 \supset \ldots \supset W_q \supset W_{q+1} = \emptyset.$$

In particular,

(4.8.1) $W_0 \backslash W_1$ is dense in W_0.

Furthermore, one easily verifies that if $(E_1, \ldots, E_n)$ is in Ω_n and each matrix E_j has precisely one entry equal to 1 and all other entries 0, then the following conditions are satisfied:

(4.8.2) $W_{(E_1, \ldots, E_n)} \backslash W_{n+1}$ is dense in W_0;

(4.8.3) $(W_{(E_1, \ldots, E_n)} \backslash W_{n+1}) \cap U$ is uncountable for each nonempty open subset U of W_0.

In order to define the sets $V_1, \ldots, V_p$ (of course, $V_0 = W_0$, $V_{p+1} = \emptyset$), we have to consider several cases.

Case 1. — Suppose $s_k \geqslant g_k$ for $k = 1, 2$. Then $p = q$ and we set $V_n = W_n$ for $1 \leqslant n \leqslant p$.

Case 2. — If $p = 0$, that is, $(g_k, s_k, \varepsilon_k) = (2, 1, 2)$ for $k = 1$ or $k = 2$, then no new set has to be defined.

Listing other cases we may assume without loss of generality that $g_1 \leqslant g_2$. Moreover, if $g_1 = g_2$, then we may assume $s_1 \leqslant s_2$. Thus it remains to consider three additional cases.

Case 3. — Suppose $g_1 = 1$ and $(g_2, s_2, \varepsilon_2) = (2, 1, 1)$. Then $p = 1$ and we set $V_1 = \bigcup_{C \in K_1} W_{(C)}$, where

$$K_1 = \{ C = (c_1, c_2) \in \mathrm{Mat}(1 \times 2, \mathbf{Z}) \mid c_1 + c_2 \neq 0 \}.$$

Case 4. — Suppose $(g_k, s_k, \varepsilon_k) = (2, 1, 1)$ for $k = 1, 2$. Then $p = 1$ and we set $V_1 = \bigcup_{C \in K_2} W_{(C)}$, where

$$K_2 = \{ C = (c_{ij}) \in \mathrm{Mat}(2 \times 2, \mathbf{Z}) \mid c_{11} + c_{12} + c_{21} + c_{22} \neq 0 \}.$$

Case 5. — Suppose $(g_1, s_1, \varepsilon_1) = (2, 1, 1)$ and $s_2 \geqslant g_2 = 2$. Then $p = 2$ and we set $V_1 = \bigcup_{C \in L_1} W_{(C)}$, $V_2 = \bigcup_{(C_1, C_2) \in L_2} W_{(C_1, C_2)}$, where

$$L_1 = \mathrm{Mat}(2 \times 2, \mathbf{Z}) \setminus L,$$
$$L_2 = \{ (C_1, C_2) \in \Omega_2 \mid (\mathbf{Z}C_1 + \mathbf{Z}C_2) \cap L = \{ 0 \} \},$$
$$L = \{ C = (c_{ij}) \in \mathrm{Mat}(2 \times 2, \mathbf{Z}) \mid c_{11} + c_{21} = 0,\ c_{12} + c_{22} = 0 \}.$$

Now, (i) is obvious, while (ii) and (iii) follow from (4 8.1), (4.8.2), and (4.8.3). In order to establish (iv) it suffices to recall that $\mathrm{rank}\, C(Z_1, Z_2) = \mathrm{rank}\, D(Z_1, Z_2)$ (cf. Lemma 3.10 (i)) and apply Example 4.5 and Theorems 4.6, 4.7. Indeed, for $[\mathscr{X}_k]$ and Z_k as in (iv), one has

$$\mathrm{rank}\, \pi^2_{\mathscr{R}}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})) = \mathrm{rank}\, \tau_{g_1 g_2}(D(Z_1, Z_2)).$$

Furthermore, $(\mathrm{Im}\, Z_1, \mathrm{Im}\, Z_2)$ belongs to $W_{(C_1, \ldots, C_n)}$ if and only if C_i belongs to $D(Z_1, Z_2)$ for all $1 \leqslant i \leqslant n$.

By Example 4.2 and Theorem 4.3, the map

$$u_{g_1} \times u_{g_2} : \mathscr{M}_{\mathbf{R}}^{(g_1, s_1, \varepsilon_1)} \times \mathscr{M}_{\mathbf{R}}^{(g_2, s_2, \varepsilon_2)} \to M^{(g_1, s_1, \varepsilon_1)} \times M^{(g_2, s_2, \varepsilon_2)}$$

is a homeomorphism which, in view of (iv), transforms $\mathscr{R}_r(g_1, s_1, \varepsilon_1; g_2, s_2, \varepsilon_2)$ onto

$$\{ (Z_1, Z_2) \in M^{(g_1, s_1, \varepsilon_1)} \times M^{(g_2, s_2, \varepsilon_2)} \mid (\mathrm{Im}\, Z_1, \mathrm{Im}\, Z_2) \in V_r \setminus V_{r+1} \}.$$

Since M^{g_k} is contained in the closure of the interior of M^{g_k} in H_{g_k} for $k = 1, 2$, the last assertion of the theorem follows from (ii) and (iii). □

If $[\mathscr{X}]$ is in $\mathscr{M}_{\mathbf{R}}^{(g, s, \varepsilon)}$ with (s, ε) in Λ_g, then Proposition 3.8 implies

$$\operatorname{rank} \pi^2_{\mathscr{R}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) \leqslant \operatorname{rank} \pi^2_*(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) = (s - \varepsilon + 1)^2.$$

We put

$$\mathscr{R}_0(g, s, \varepsilon) = \{ [\mathscr{X}] \in \mathscr{M}_{\mathbf{R}}^{(g, s, \varepsilon)} \mid \pi^2_{\mathscr{R}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) = 0 \},$$

$$\mathscr{R}(g, s, \varepsilon) = \{ [\mathscr{X}] \in \mathscr{M}_{\mathbf{R}}^{(g, s, \varepsilon)} \mid \operatorname{rank} \pi^2_{\mathscr{R}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) = (s - \varepsilon + 1)^2 \}.$$

The following result includes Theorem 1.13.

Theorem **4.9.** — *Let (s, ε) be in Λ_g, where $g = 1$ or $g = 2$. Then $\mathscr{R}_0(g, s, \varepsilon)$ is the intersection of a countable family of open and dense subsets of $\mathscr{M}_{\mathbf{R}}^{(g, s, \varepsilon)}$, whereas the set $\mathscr{R}(g, s, \varepsilon)$ is dense in $\mathscr{M}_{\mathbf{R}}^{(g, s, \varepsilon)}$. Furthermore, the following conditions are satisfied:*

(i) $\mathscr{R}_0(g, s, \varepsilon) = \mathscr{R}(g, s, \varepsilon) = \mathscr{M}_{\mathbf{R}}^{(g, s, \varepsilon)}$ *if* $(g, s, \varepsilon) = (2, 1, 2)$,
(ii) $\mathscr{R}(g, s, \varepsilon)$ *is uncountable if* $(g, s, \varepsilon) = (2, 1, 1)$,
(iii) $\mathscr{R}(g, s, \varepsilon)$ *is countable in all other cases.*

Proof. — Define

$$G_g = \bigcap_{(C, D) \in L} (S_g(\mathbf{R}) \setminus \{ T \in S_g(\mathbf{R}) \mid TCT = D \}),$$

where $L = (\operatorname{Mat}(g \times g, \mathbf{Z}) \setminus \{ 0 \}) \times \operatorname{Mat}(g \times g, \mathbf{Z})$. Clearly, G_g is the intersection of a countable family of open and dense subsets of $S_g(\mathbf{R})$. Moreover, an element Z of H_g satisfies $D(Z, Z) = 0$ if and only if $\operatorname{Im} Z$ belongs to G_g. It follows from Example 4.5, Theorem 4.7, and Lemma 3.10 (i) that

(4.9.1) $$\mathscr{R}_0(g, s, \varepsilon) = \{ [\mathscr{X}] \in \mathscr{M}_{\mathbf{R}}^{(g, s, \varepsilon)} \mid \operatorname{Im} Z \in G_g, \text{ where } Z = u_g([\mathscr{X}]) \}.$$

Let now

$$E = \{ T \in S_2(\mathbf{R}) \mid \exists\, C = (c_{ij}) \in \operatorname{Mat}(2 \times 2, \mathbf{Z}) \text{ such that } c_{11} + c_{12} + c_{21} + c_{22} \neq 0 \text{ and } TCT \in \operatorname{Mat}(2 \times 2, \mathbf{Z}) \},$$

$$F_g = \{ T \in S_g(\mathbf{R}) \mid TCT \text{ has rational entries for all } C \in \operatorname{Mat}(g \times g, \mathbf{Z}) \},$$

$$A(g, s, \varepsilon) = \begin{cases} S_2(\mathbf{R}) & \text{if } (g, s, \varepsilon) = (2, 1, 2) \\ E & \text{if } (g, s, \varepsilon) = (2, 1, 1) \\ F_g & \text{in all other cases.} \end{cases}$$

One easily sees that E is dense in $S_2(\mathbf{R})$, and $E \cap U$ is uncountable for each nonempty, open subset U of $S_2(\mathbf{R})$. On the other hand, F_g is dense in $S_g(\mathbf{R})$ and countable for

$g = 1, 2$. By Lemma 3.10 (i), we have rank $D(Z, Z) = \text{rank } C(Z, Z)$ for Z in H_g and hence, in view of Example 4.5 and Theorem 4.7,

(4.9.2) $$\mathscr{R}(g, s, \varepsilon) = \{ [\mathscr{X}] \in \mathscr{M}_{\mathbf{R}}^{(g, s, \varepsilon)} \mid \operatorname{Im} Z \in A(g, s, \varepsilon), \text{ where } Z = u_g([\mathscr{X}]) \}.$$

By (4.9.1), (4.9.2), Example 4.2, and Theorem 4.3, the homeomorphism $u_g : \mathscr{M}_{\mathbf{R}}^{(g, s, \varepsilon)} \to M^{(g, s, \varepsilon)}$ transforms $\mathscr{R}_0(g, s, \varepsilon)$ onto $\{ Z \in M^{(g, s, \varepsilon)} \mid \operatorname{Im} Z \in G_g \}$ and $\mathscr{R}(g, s, \varepsilon)$ onto $\{ Z \in M^{(g, s, \varepsilon)} \mid \operatorname{Im} Z \in A(g, s, \varepsilon) \}$. Since $M^{(g, s, \varepsilon)}$ is contained in the closure of the interior of $M^{(g, s, \varepsilon)}$ in H_g, and since (i) is a direct consequence of Proposition 3.8, all the assertions of Theorem 4.9 follow. □

Let $([\mathscr{X}_1], [\mathscr{X}_2])$ be in $\mathscr{M}_{\mathbf{R}}^{(g_1, s_1, \varepsilon_1)} \times \mathscr{M}_{\mathbf{R}}^{(g_2, s_2, \varepsilon_2)}$, where (s_k, ε_k) belongs to Λ_{g_k} for $k = 1, 2$. We know from Section 1 that $\mathscr{R}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R}), S^2)$ is dense in $\mathscr{C}^\infty(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R}), S^2)$ if and only if $\pi^2_{\mathscr{R}}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R}), S^2) = \pi^2(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R}), S^2)$. Moreover, by Proposition 3.8, if these equivalent conditions are satisfied then $\varepsilon_1 = \varepsilon_2 = 1$. We shall now study the sets

$$\mathscr{D}(g_1, s_1; g_2, s_2) =$$
$$\{ ([\mathscr{X}_1], [\mathscr{X}_2]) \in \mathscr{M}_{\mathbf{R}}^{(g_1, s_1, 1)} \times \mathscr{M}_{\mathbf{R}}^{(g_2, s_2, 1)} \mid \pi^2_{\mathscr{R}}(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})) = \pi^2(\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})) \}$$

with $(s_k, 1)$ in Λ_{g_k} for $k = 1, 2$, and

$$\mathscr{D}(g, s) = \{ [\mathscr{X}] \in \mathscr{M}_{\mathbf{R}}^{(g, s, 1)} \mid \pi^2_{\mathscr{R}}(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) = \pi^2(\mathscr{X}(\mathbf{R}) \times \mathscr{X}(\mathbf{R})) \}$$

with $(s, 1)$ in Λ_g. The proofs of our next two theorems apply the full strength of the results described in Examples 4.2 and 4.5, and Theorems 4.3, 4.6, and 4.7. We shall freely use the notation introduced in these statements. We shall also frequently refer to equalities (1.9) of Section 1. The reader will notice that Theorems 4.10 and 4.11 combined together are equivalent to Theorem 1.14.

Theorem **4.10.** — *The sets* $\mathscr{D}(1, 1)$ *and* $\mathscr{D}(2, 2)$ *are infinite countable, while the set* $\mathscr{D}(2, 1)$ *is uncountable.*

Proof. — *Case* $\mathscr{D}(1, 1)$. The conclusion concerning $\mathscr{D}(1, 1)$ is proved in [9, Corollary 1.6].

Case $\mathscr{D}(2, 2)$. — We shall first exhibit an infinite sequence Z^j of distinct elements of $M^{(2, 2, 1)}$ such that $C(Z^j, Z^j) = \operatorname{Mat}(2 \times 2, \mathbf{Z})$ for $j = 0, 1, 2, \ldots$ Define Z^j by

$$Z^j = \frac{1}{2} M^j + \sqrt{-1}\, T^j,$$

$$M^j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T^j = \sqrt{12j + 3} \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & 3 \end{pmatrix}$$

Obviously, Z^j belongs to $M^{(2,2,1)}$. We shall study Z^j with the help of the following table:

C	G_1	G_2	H
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 11 & -3 \\ 7 & -2 \end{pmatrix}$	$\begin{pmatrix} 8 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 8 & 4 \\ 8 & 3 \end{pmatrix} + 3j \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 5 & 14 \\ 4 & 9 \end{pmatrix} + 3j \begin{pmatrix} 6 & 18 \\ 4 & 12 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 6 & -1 \\ 5 & -1 \end{pmatrix}$	$\begin{pmatrix} 5 & 2 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 5 & 2 \\ 16 & 9 \end{pmatrix} + 3j \begin{pmatrix} 6 & 4 \\ 18 & 12 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 3 & 9 \\ 9 & 27 \end{pmatrix} + 3j \begin{pmatrix} 4 & 12 \\ 12 & 36 \end{pmatrix}$

The first column of the table contains the canonical generators of the module $\mathrm{Mat}(2 \times 2, \mathbf{Z})$. We claim that each of these generators belongs to $C(Z^j, Z^j)$; in other words, $C(Z^j, Z^j) = \mathrm{Mat}(2 \times 2, \mathbf{Z})$. Indeed, equalities (1.9) are satisfied for $M_1 = M_2 = M^j$, $T_1 = T_2 = T^j$, and C, G_1, G_2, H in each row of the table. Thus the claim is proved. We thank Joost van Hamel for helping us in the computation of G_1, G_2, and H in the above table.

We can now complete the proof of the case under consideration. Applying Theorem 4.3, choose a curve $\mathscr{X}^j$ such that $[\mathscr{X}^j]$ is in $\mathscr{M}_{\mathbf{R}}^{(2,2,1)}$ and $u_2([\mathscr{X}^j]) = Z^j$. Let $X^j = \mathscr{X}^j(\mathbf{R})$. It follows from the claim and Theorem 4.7 that

$$\pi^2_{\mathscr{R}}(X^j \times X^j) = \pi^2(X^j \times X^j) \qquad \text{for } j = 0, 1, 2, \ldots$$

This shows that the set $\mathscr{D}(2, 2)$ is infinite. Obviously, $\mathscr{D}(2, 2) \subseteq \mathscr{R}(2, 2, 1)$ and, by Theorem 4.9, $\mathscr{R}(2, 2, 1)$ is a countable set. Hence $\mathscr{D}(2, 2)$ is infinite countable and the proof is finished.

Case $\mathscr{D}(2, 1)$. — Let

$$Z_\gamma = \frac{1}{2} M_\gamma + \sqrt{-1}\, T_\gamma,$$

where

$$M_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_\gamma = \sqrt{3} \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & \gamma \end{pmatrix}, \quad \gamma \in \mathbf{R}, \quad \gamma \geqslant \frac{3}{2}.$$

Then Z_γ belongs to $M^{(2,1,1)}$ and, in view of Theorem 4.3, there exists an algebraic curve $\mathscr{X}_\gamma$ over $\mathbf{R}$ such that $[\mathscr{X}_\gamma]$ is in $\mathscr{M}_{\mathbf{R}}^{(2,1,1)}$ and $u_2([\mathscr{X}_\gamma]) = Z_\gamma$. Let $X_\gamma = \mathscr{X}_\gamma(\mathbf{R})$

and let $\tau_{22} : \mathrm{Mat}(2 \times 2, \mathbf{Z}) \to \pi^2(X_Y \times X_Y)$ be the epimorphism of Theorem 4.7. By Theorem 4.7 (iv), one has $\mathbf{Z}C + \mathrm{Ker}\,\tau_{22} = \mathrm{Mat}(2 \times 2, \mathbf{Z})$, where

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, by Theorem 4.7 (i), in order to prove that $\pi^2_{\mathscr{R}}(X_Y \times X_Y) = \pi^2(X_Y \times X_Y)$ it suffices to show that C belongs to $C(Z_Y, Z_Y)$. The latter holds since equalities (1.9) are satisfied for $M_1 = M_2 = M_Y$, $T_1 = T_2 = T_Y$,

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad G_1 = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}.$$

It follows that $\mathscr{D}(2, 1)$ is uncountable. □

We shall now deal with the sets $\mathscr{D}(g_1, s_1; g_2, s_2)$ for $1 \leqslant g_1 \leqslant 2$, $1 \leqslant g_2 \leqslant 2$. Without loss of generality we may assume, and always do so below, that $g_1 \leqslant g_2$, and if $g_1 = g_2$, then $s_1 \leqslant s_2$. Thus we have precisely 6 distinct sets to consider; they are listed in our next result.

Theorem **4.11.** — (i) *The sets* $\mathscr{D}(1, 1; 1, 1)$, $\mathscr{D}(1, 1; 2, 2)$, *and* $\mathscr{D}(2, 2; 2, 2)$ *are infinite countable.*

(ii) *The sets* $\mathscr{D}(1, 1; 2, 1)$ *and* $\mathscr{D}(2, 1; 2, 1)$ *are uncountable.*

(iii) *The set* $\mathscr{D}(2, 1; 2, 2)$ *is infinite.*

Proof. — *Case* $\mathscr{D}(1, 1; 1, 1)$. A full description of $\mathscr{D}(1, 1; 1, 1)$ is given in [9, Corollary 1.8]. In particular, this set is infinite countable.

Case $\mathscr{D}(1, 1; 2, 2)$. — For each $j = 0, 1, 2, \ldots$ define Z_1^j and Z_2^j by

$$Z_1^j = \frac{1}{2} M_1^j + \sqrt{-1}\, T_1^j, \quad Z_2^j = \frac{1}{2} M_2^j + \sqrt{-1}\, T_2^j,$$

where

$$M_1^j = 1, \quad T_1^j = \frac{1}{2}\sqrt{12j + 3},$$

$$M_2^j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_2^j = \sqrt{12j + 3}\begin{pmatrix} \frac{3}{2} & 1 \\ 1 & 2 \end{pmatrix}.$$

By construction, Z_1^j belongs to $M^{(1,1,1)}$ and Z_2^j belongs to $M^{(2,2,1)}$. We claim that $C(Z_1^j, Z_2^j) = \mathrm{Mat}(1 \times 2, \mathbf{Z})$. Indeed, consider the following table:

C	G_1	G_2	H
(1, 0)	(0, 1)	(0, 3)	$(2, 0) + 3j(3, 2)$
(0, 1)	(0, 0)	(− 1, − 2)	$(2, 4) + 3j(2, 4)$

The first column of the table contains the canonical generators of the module $\mathrm{Mat}(1 \times 2, \mathbf{Z})$. Equalities (1.9) are satisfied for $M_1 = M_1^j$, $T_1 = T_1^j$, $M_2 = M_2^j$, $T_2 = T_2^j$, and C, G_1, G_2, H in each row of the table. Hence the claim is proved.

Using Example 4.2 and Theorem 4.3, choose curves $\mathscr{X}_1^j$ and $\mathscr{X}_2^j$ such that $[\mathscr{X}_1^j]$ is in $\mathscr{M}_{\mathbf{R}}^{(1,1,1)}$, $u_1([\mathscr{X}_1^j]) = Z_1^j$, $[\mathscr{X}_2^j]$ is in $\mathscr{M}_{\mathbf{R}}^{(2,2,1)}$, and $u_2([\mathscr{X}_2^j]) = Z_2^j$. It follows from the claim and Theorem 4.6 that $\pi^2_{\mathscr{R}}(\mathscr{X}_1^j(\mathbf{R}) \times \mathscr{X}_2^j(\mathbf{R})) = \pi^2(\mathscr{X}_1^j(\mathbf{R}) \times \mathscr{X}_2^j(\mathbf{R}))$ for $j = 0, 1, 2, \ldots$ This shows that the set $\mathscr{D}(1, 1; 2, 2)$ is infinite. Countability of $\mathscr{D}(1, 1; 2, 2)$ follows from Proposition 4.12 below.

Case $\mathscr{D}(2, 2; 2, 2)$. — Obviously $\mathscr{D}(2, 2)$ is contained in $\mathscr{D}(2, 2; 2, 2)$. Since, by Theorem 4.10, $\mathscr{D}(2, 2)$ is an infinite set, $\mathscr{D}(2, 2; 2, 2)$ is infinite as well. Countability of $\mathscr{D}(2, 2; 2, 2)$ follows from Proposition 4.12 below.

Case $\mathscr{D}(1, 1; 2, 1)$. — Let

$$Z = \frac{1}{2} M + \sqrt{-1}\, T, \quad Z_\gamma = \frac{1}{2} M_\gamma + \sqrt{-1}\, T_\gamma,$$

where

$$M = 1, \quad T = \frac{\sqrt{3}}{2}, \quad M_\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$T_\gamma = \sqrt{3} \begin{pmatrix} \frac{3}{2} & 1 \\ 1 & \gamma \end{pmatrix}, \quad \gamma \in \mathbf{R}, \quad \gamma \geqslant \frac{3}{2}.$$

Then Z belongs to $M^{(1,1,1)}$ and Z_γ belongs to $M^{(2,1,1)}$. By Example 4.2 and Theorem 4.3, there exist algebraic curves $\mathscr{X}$ and $\mathscr{X}_\gamma$ over $\mathbf{R}$ such that $[\mathscr{X}]$ is in $\mathscr{M}_{\mathbf{R}}^{(1,1,1)}$, $u_1([\mathscr{X}]) = Z$, $[\mathscr{X}_\gamma]$ is in $\mathscr{M}_{\mathbf{R}}^{(2,1,1)}$, and $u_2([\mathscr{X}_\gamma]) = Z_\gamma$. Let $X = \mathscr{X}(\mathbf{R})$, and $X_\gamma = \mathscr{X}_\gamma(\mathbf{R})$, and let $\tau_{12} : \mathrm{Mat}(1 \times 2, \mathbf{Z}) \to \pi^2(X \times X_\gamma)$ be the epimorphism of Theorem 4.6. By Theorem 4.6 (iii), we have $\mathbf{Z}C + \mathrm{Ker}\, \tau_{12} = \mathrm{Mat}(1 \times 2, \mathbf{Z})$, where $C = (1, 0)$. Thus, by Theorem 4.6 (i), in order to prove that $\pi^2_{\mathscr{R}}(X \times X_\gamma) = \pi^2(X \times X_\gamma)$ it suffices to show that C belongs to $C(Z, Z_\gamma)$. The last condition is satisfied since equalities (1.9) hold for $M_1 = M$, $T_1 = T$, $M_2 = M_\gamma$, $T_2 = T_\gamma$, $C = (1, 0)$, $G_1 = (0, 0)$, $G_2 = (-2, -1)$, $H = (3, 2)$. We conclude that the set $\mathscr{D}(1, 1; 2, 1)$ is uncountable.

Case $\mathscr{D}(2, 1; 2, 1)$. — Since $\mathscr{D}(2, 1; 2, 1)$ contains the uncountable set $\mathscr{D}(2, 1)$, it is itself uncountable.

Case $\mathscr{D}(2, 1; 2, 2)$. — We conjecture that this set is uncountable, but are only able to prove that it is infinite. Let

$$Z_1^j = \frac{1}{2} M_1^j + \sqrt{-1}\, T_1^j, \quad Z_2^j = \frac{1}{2} M_2^j + \sqrt{-1}\, T_2^j,$$

where

$$M_1^j = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_2^j = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$T_1^j = \sqrt{12j+3}\begin{pmatrix} \frac{3}{2} & 1 \\ 1 & 4 \end{pmatrix}, \quad T_2^j = \sqrt{12j+3}\begin{pmatrix} \frac{3}{2} & 1 \\ 1 & 2 \end{pmatrix}$$

for $j = 0, 1, 2, \ldots$ Then Z_1^j belongs to $M^{(2,1,1)}$ and Z_2^j belongs to $M^{(2,2,1)}$. By Theorem 4.3, we can find algebraic curves $\mathscr{X}_1^j$ and $\mathscr{X}_2^j$ over $\mathbf{R}$ such that $[\mathscr{X}_1^j]$ is in $\mathscr{M}_{\mathbf{R}}^{(2,1,1)}$ $u_2([\mathscr{X}_1^j]) = Z_1^j$, $[\mathscr{X}_2^j]$ is in $\mathscr{M}_{\mathbf{R}}^{(2,2,1)}$ and $u_2([\mathscr{X}_2^j]) = Z_2^j$. Let $X_1^j = \mathscr{X}_1^j(\mathbf{R})$ and $X_2^j = \mathscr{X}_2^j(\mathbf{R})$, and let $\tau_{22} : \mathrm{Mat}(2 \times 2, \mathbf{Z}) \to \pi^2(X_1^j \times X_2^j)$ be the epimorphism of Theorem 4.7. By Theorem 4.7 (iii), we have $\mathbf{Z}C_1 + \mathbf{Z}C_2 + \mathrm{Ker}\, \tau_{22} = \mathrm{Mat}(2 \times 2, \mathbf{Z})$, where

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

By Theorem 4.7 (i), in order to prove $\pi^2_{\mathscr{R}}(X_1^j \times X_2^j) = \pi^2(X_1^j \times X_2^j)$ it suffices to show that C_1 and C_2 are in $C(Z_1^j, Z_2^j)$. To this end, let us consider the following table:

C	G_1	G_2	H
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix} + 3j\begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ 4 & -6 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}$	$\begin{pmatrix} 5 & 9 \\ 5 & 7 \end{pmatrix} + 3j\begin{pmatrix} 6 & 12 \\ 4 & 8 \end{pmatrix}$

Since equalities (1.9) are satisfied for $M_1 = M_1^j$, $T_1 = T_1^j$, $M_2 = M_2^j$, $T_2 = T_2^j$, and C, G_1, G_2, H in each row of the table, it follows that C_1 and C_2 belong to $C(Z_1^j, Z_2^j)$. We conclude that the set $\mathscr{D}(2, 1; 2, 2)$ is infinite.

The proof of the theorem is now complete, except that in two cases above we referred to Proposition 4.12. □

Given two algebraic curves $\mathscr{X}_1$ and $\mathscr{X}_2$ over $\mathbf{R}$ with $X_1 = \mathscr{X}_1(\mathbf{R})$ and $X_2 = \mathscr{X}_2(\mathbf{R})$ nonempty, let us set

$$\Gamma(\mathscr{X}_1, \mathscr{X}_2) = \pi^2(X_1 \times X_2)/\pi^2_{\mathscr{R}}(X_1 \times X_2).$$

Clearly, $\Gamma(\mathscr{X}_1, \mathscr{X}_2)$ is a finite (resp. trivial) group if and only if

$$\mathrm{rank}\, \pi^2_{\mathscr{R}}(X_1 \times X_2) = \mathrm{rank}\, \pi^2(X_1 \times X_2)$$

(resp. $\pi^2_{\mathscr{R}}(X_1 \times X_2) = \pi^2(X_1 \times X_2)$).

Proposition **4.12.** — *Let* (g_k, s_k) *be in* $\{(1, 1), (2, 2)\}$ *for* $k = 1, 2$. *Then the set* $\mathscr{E}(g_1, s_1; g_2, s_2)$ *defined as*

$$\{([\mathscr{X}_1], [\mathscr{X}_2]) \in \mathscr{M}_{\mathbf{R}}^{(g_1, s_1, 1)} \times \mathscr{M}_{\mathbf{R}}^{(g_2, s_2, 1)} \mid \Gamma(\mathscr{X}_1, \mathscr{X}_2) \textit{ is a finite group of odd order}\}$$

is infinite countable.

Proof. — Clearly, the set $\mathscr{E}(g_1, s_1; g_2, s_2)$ contains $\mathscr{D}(g_1, s_1; g_2, s_2)$ and therefore is infinite by the proof of Theorem 4.11 (note that infinity of $\mathscr{D}(g_1, s_1; g_2, s_2)$ was established without any reference to Proposition 4.12). We shall show countability of $\mathscr{E}(g_1, s_1; g_2, s_2)$ by proving the following stronger assertion.

Assertion. — If $([\mathscr{X}_1], [\mathscr{X}_2])$ is in $\mathscr{E}(g_1, s_1; g_2, s_2)$ and $Z_k = u_{g_k}([\mathscr{X}_k])$ for $k = 1, 2$, then the entries of the matrix $\operatorname{Im} Z_k$ belong to some quadratic extension of $\mathbf{Q}$.

Without loss of generality we may assume that $g_1 \leqslant g_2$. This leads to three cases, which are considered below.

Case $\mathscr{E}(1, 1; 1, 1)$. — The assertion is proved in [9, Corollary 1.8].

Case $\mathscr{E}(1, 1; 2, 2)$. — The group $\operatorname{Mat}(1 \times 2, \mathbf{Z})/C(Z_1, Z_2)$ is isomorphic to $\Gamma(\mathscr{X}_1, \mathscr{X}_2)$ (cf. Theorem 4.6 (i), (ii)). If $\Gamma(\mathscr{X}_1, \mathscr{X}_2)$ is of odd order, then $C(Z_1, Z_2)$ contains elements $(p, 0)$ and $(0, q)$ for some odd integers p and q. Let

$$Z_1 = \frac{1}{2} + \alpha\sqrt{-1}, \quad Z_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

Then α, a, b, c are strictly positive real numbers. Substituting $C = (p, 0)$, $M_i = 2 \operatorname{Re} Z_i$, $T_i = \operatorname{Im} Z_i$, $i = 1, 2$, into equations (1.9) defining $C(Z_1, Z_2)$, one gets

(4.12.1) $\quad (p, 0) - 4p\alpha(a, b) \in 2\mathbf{Z} \times 2\mathbf{Z}$

(4.12.2) $\quad p(a, b) + p\alpha(1, 0) = 2(ag_1 + bg_2 - \alpha g_1', bg_1 + cg_2 - \alpha g_2')$

for some g_i, g_i' in $\mathbf{Z}$, $i = 1, 2$. From (4.12.1) one deduces that $\alpha = \frac{r}{a}$, $b = sa$ for some r, s in $\mathbf{Q}\backslash\{0\}$. Substituting this into (4.12.2) one gets

$$a^2(p - 2g_1 - 2sg_2) + r(p + 2g_1') = 0.$$

Therefore a^2 is in $\mathbf{Q}$ (otherwise $p + 2g_1' = 0$ which is impossible, p being odd). It follows that a, α and b are in some quadratic extension of $\mathbf{Q}$. Substituting $C = (0, q)$ into the first equation in (1.9), one obtains that $2q\alpha c$ is in $\mathbf{Z}$, which implies that c is in the same quadratic extension of $\mathbf{Q}$ as α.

Case $\mathscr{E}(2, 2; 2, 2)$. — Again, the groups $\operatorname{Mat}(2 \times 2, \mathbf{Z})/C(Z_1, Z_2)$ and $\Gamma(\mathscr{X}_1, \mathscr{X}_2)$ are isomorphic (cf. Theorem 4.7 (i), (ii)). If the latter is of odd order, then $C(Z_1, Z_2)$ must contain matrices

$$\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$$

for some odd integers p and q. Let

$$Z_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} \alpha_1 & \beta_1 \\ \beta_1 & \gamma_1 \end{pmatrix}, \quad Z_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} \alpha_2 & \beta_2 \\ \beta_2 & \gamma_2 \end{pmatrix},$$

where the entries α_k, β_k, γ_k are strictly positive real numbers for $k = 1, 2$. Substituting $M_k = 2 \operatorname{Re} Z_k$, $T_k = \operatorname{Im} Z_k$,

$$C = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$$

into (1.9), one gets

$$\textbf{(4.12.3)} \qquad p\begin{pmatrix} 4\alpha_1\alpha_2 - 1 & 4\alpha_1\beta_2 \\ 4\beta_1\alpha_2 & 4\beta_1\beta_2 \end{pmatrix} \in 2\,\mathrm{Mat}(2 \times 2, \mathbf{Z})$$

$$\textbf{(4.12.4)} \qquad p\begin{pmatrix} \alpha_1 + \alpha_2 & \beta_1 \\ \beta_1 & 0 \end{pmatrix} = 2\begin{pmatrix} m\alpha_2 + n\beta_2 + \ell\alpha_1 + s\beta_1 & * \\ * & * \end{pmatrix},$$

where m, n, ℓ, s are in $\mathbf{Z}$ and $*$ stands for the entries which are of no interest to us. It follows from (4.12.3) that $\alpha_2 = d/(4p\alpha_1)$, $\beta_1 = 2t\alpha_1/d$, $\beta_2 = r/\alpha_1$, where d, t are in $\mathbf{Z}$, d is odd, and r is in $\mathbf{Q}\backslash\{0\}$. Substituting these into (4.12.4), one gets

$$\alpha_1^2\left(p - 2\ell - \frac{4ts}{d}\right) \in \mathbf{Q}.$$

This implies that α_1^2 is in $\mathbf{Q}$, p and d being odd integers. Hence α_1, and therefore α_2, β_1, β_2 are in some quadratic extension of $\mathbf{Q}$.

Finally, substituting

$$C = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$$

into the first equation in (1.9), one obtains

$$2q\begin{pmatrix} \beta_1\beta_2 & \beta_1\gamma_2 \\ \gamma_1\beta_2 & \gamma_1\gamma_2 \end{pmatrix} \in \mathrm{Mat}(2 \times 2, \mathbf{Z}),$$

which implies that γ_1 and γ_2 are also in the same quadratic extension of $\mathbf{Q}$ as α. This completes the proof.

Let us mention that a more careful computation demonstrates that the matrices Z_1 and Z_2 considered above have all entries in the quadratic field $\mathbf{Q}(\sqrt{-d})$, where d is a positive integer satisfying $d \equiv 3 \pmod 4$. □

Remark **4.13.** — Combining Theorem 4.8 and Proposition 4.12, one sees immediately that for (g_k, s_k) in $\{(1, 1), (2, 2)\}$, $k = 1, 2$, the set

$$\{([\mathscr{X}_1], [\mathscr{X}_2]) \in \mathscr{M}_{\mathbf{R}}^{(g_1, s_1, 1)} \times \mathscr{M}_{\mathbf{R}}^{(g_2, s_2, 1)} \mid \Gamma(\mathscr{X}_1, \mathscr{X}_2) \text{ is finite of even order}\}$$

is uncountable. □

Example **4.14.** — Let

$$Z_1 = \frac{1}{2}(1 + \sqrt{-1}) \quad \text{and} \quad Z_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sqrt{-1}\begin{pmatrix} \frac{5}{2} & 1 \\ 1 & 2 \end{pmatrix}.$$

Then Z_k belongs to $M^{(k, k, 1)}$ for $k = 1, 2$. We claim that

$$C(Z_1, Z_2) = \{(2n, m) \in \mathrm{Mat}(1 \times 2, \mathbf{Z}) \mid n, m \in \mathbf{Z}\}.$$

Indeed, $C = (1, 0)$ (resp. $C = (2, 0)$) is in $C(Z_1, Z_2)$ since equalities (1.9) are satisfied for $C = (0, 1)$, $G_1 = (0, 0)$, $G_2 = (-1, -2)$, $H = (1, 2)$ (resp. $C = (2, 0)$, $G_1 = (1, 0)$, $G_2 = (-1, 0)$, $H = (3, 1)$). On the other hand, $(1, 0)$ is not in $C(Z_1, Z_2)$ as can be seen by a direct and simple computation. Thus the claim is proved. By Example 4.2 and Theorem 4.3, there exists an algebraic curve $\mathscr{X}_k$ over $\mathbf{R}$ such that $[\mathscr{X}_k]$ is in $\mathscr{M}_{\mathbf{R}}^{(k, k, 1)}$ and $u_k([\mathscr{X}_k]) = Z_k$ for $k = 1, 2$. Applying Theorem 4.6, one obtains $\Gamma(\mathscr{X}_1, \mathscr{X}_2) \cong \mathbf{Z}/2$. In other words, there exists a connected component Y of $\mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R})$ such that a continuous map $f \colon \mathscr{X}_1(\mathbf{R}) \times \mathscr{X}_2(\mathbf{R}) \to S^2$ is homotopic to a regular map if and only if $\deg(f \mid Y)$ is even. □

We shall conclude this section by considering the group $\pi^2_{\mathscr{R}}(X \times X)$ for algebraic curves $\mathscr{X}$ over $\mathbf{R}$ (given by specific equations) with $X = \mathscr{X}(\mathbf{R})$ connected and $\varepsilon(\mathscr{X}) = 1$. By Proposition 3.8, we have $\pi^2_*(X \times X) = \pi^2(X \times X) \cong \mathbf{Z}$, and hence

$$\pi^2_{\mathscr{R}}(X \times X) = \beta(X)\, \pi^2(X \times X)$$

for some uniquely determined nonnegative integer $\beta(X)$. In other words, a continuous map $f \colon X \times X \to S^2$ is homotopic to a regular map if and only if $\deg(f)$ is a multiple of $\beta(X)$. Note that $\beta(X) = b(X \times X)$, where $b(-)$ is the invariant introduced in Section 1 in the paragraph containing the definition of $\pi^2(-)$. We have three examples: the first one deals with a curve of genus 2, while the remaining two are concerned with curves of genus 3.

Example **4.15.** — Let $\mathscr{X}$ be the algebraic curve over $\mathbf{R}$ given by the affine equation $y^2 = x^6 - 1$. We already know that $[\mathscr{X}]$ belongs to $\mathscr{M}_{\mathbf{R}}^{(2, 1, 1)}$ and $u_2([\mathscr{X}]) = Z$, where $Z = \frac{1}{2} M + \sqrt{-1}\, T$,

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \frac{1}{2\sqrt{3}}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(cf. the proof of Proposition 4.4). Let $X = \mathscr{X}(\mathbf{R})$ and let $\tau_{22} : \mathrm{Mat}(2 \times 2, \mathbf{Z}) \to \pi^2(X \times X)$ be the epimorphism of Theorem 4.7. Identifying $\pi^2(X \times X)$ with $\mathbf{Z}$, we obtain from Theorem 4.7 (iv) that τ_{22} is necessarily of the form

$$\tau_{22}(C) = \pm \sum_{i,j=1}^{2} c_{ij}$$

for $C = (c_{ij})$ in $\mathrm{Mat}(2 \times 2, \mathbf{Z})$. Moreover, by Theorem 4.7 (i), we have

$$\tau_{22}(C(Z, Z)) = \beta(X)\, \mathbf{Z}.$$

For

$$C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $\mathrm{Mat}(2 \times 2, \mathbf{Z})$, one has

$$MCM - 4TCT = C - 4TCT = -\frac{1}{3}\begin{pmatrix} a + 2b + 2c + d & 2a + b + c + 2d \\ 2a + b + c + 2d & a + 2b + 2c + d \end{pmatrix}.$$

If C is in $C(Z, Z)$, then $MCM - 4TCT$ is in $2\,\mathrm{Mat}(2 \times 2, \mathbf{Z})$ and hence $a + d \equiv 0 \pmod 2$, $b + c \equiv 0 \pmod 2$; in particular, $\tau_{22}(C) \equiv 0 \pmod 2$. Therefore $\beta(X)$ must be even. On the other hand, $MC_0M - 4TC_0T$ is in $2\,\mathrm{Mat}(2 \times 2, \mathbf{Z})$ for

$$C_0 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

which easily implies that $2C_0$ belongs to $C(Z, Z)$. Since $\tau_{22}(2C_0) = \pm 4$, we conclude that $\beta(X) = 2$ or $\beta(X) = 4$. □

Example **4.16.** — Let $\mathscr{K}$ be the Klein quartic curve given by the equation $x^3y + y^3z + z^3x = 0$. Here we regard $\mathscr{K}$ as an algebraic curve over $\mathbf{R}$. It can be shown by a rather lengthy computation that $Z = \frac{1}{2}M + \sqrt{-1}\,T$, where

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \frac{1}{2\sqrt{7}}\begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix},$$

is a period matrix of $\mathscr{K}$. Furthermore, if $\mathscr{J}$ is the Jacobian variety of $\mathscr{K}$, then one can find an isomorphism $\Phi : \mathscr{Y}_Z \to \mathscr{J}$ of principally polarized Abelian varieties over $\mathbf{R}$ such that

$$L(\mathscr{K}, \Phi_C) = \mathbf{Z}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

with $K = \mathscr{K}(\mathbf{R})$. In particular, $[\mathscr{K}]$ is in $\mathscr{M}_{\mathbf{R}}^{(3,1,1)}$ (cf. [14] or [26, Lemma 7.1]), and if $\tau : \mathrm{Mat}(3 \times 3, \mathbf{Z}) \to \pi^2(K \times K)$ is the epimorphism of Theorem 3.9 and if $\pi^2(K \times K)$ is identified with $\mathbf{Z}$, then Theorem 3.9 (ii) implies

$$\tau(C) = \pm \sum_{i,j=1}^{3} c_{ij}$$

for $C = (c_{ij})$ in $\mathrm{Mat}(3 \times 3, \mathbf{Z})$. Therefore, by Theorem 3.9 (i),

$$\tau(C(Z, Z)) = \beta(K)\, \mathbf{Z}.$$

If C is in C(Z, Z), then $MCM - 4TCT = C - 4TCT$ is in $2\,\mathrm{Mat}(3 \times 3, \mathbf{Z})$, and a straightforward calculation demonstrates that $\tau(C) = -\tau(C - 4TCT)$. Thus $\tau(C)$ is even, which implies that $\beta(K)$ is even. On the other hand, $MC_0M - 4TC_0T$ is in $2\,\mathrm{Mat}(3 \times 3, \mathbf{Z})$ for

$$C_0 = \begin{pmatrix} 10 & 0 & 0 \\ -4 & 0 & 0 \\ -4 & 0 & 0 \end{pmatrix},$$

which implies that $2C_0$ is in C(Z, Z). Clearly, $\tau(2C_0) = \pm 4$ and therefore $\beta(K) = 2$ or $\beta(K) = 4$. □

Example **4.17.** — Let $\mathscr{F}_k$ be the Fermat curve $x^k + y^k = z^k$, viewed as an algebraic curve over $\mathbf{R}$. Clearly, $F_k = \mathscr{F}_k(\mathbf{R})$ is connected. It is proved in [9, Example 1.14] that $\beta(F_1) = \beta(F_2) = 0$, $\beta(F_k) = 1$ for k odd satisfying $k \geqslant 3$, and $1 \leqslant \beta(F_k) \leqslant 2$ for k even, $k \geqslant 4$. We assert that $\beta(F_4) = 2$.

Let $\mathscr{J}_4$ be the Jacobian variety of $\mathscr{F}_4$. It can be shown (we do not include the necessary calculations here) that $Z = \frac{1}{2} M + \sqrt{-1}\, T$, where

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad T = \frac{1}{2} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

is a period matrix of $\mathscr{F}_4$, and there is an isomorphism $\Phi : \mathscr{Y}_Z \to \mathscr{J}_4$ of polarized Abelian varieties over $\mathbf{R}$ such that

$$L(F_4, \Phi_C) = \mathbf{Z} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Arguing as in Example 4.16, we see that if $\tau : \mathrm{Mat}(3 \times 3, \mathbf{Z}) \to \mathbf{Z}$ is defined by

$$\tau(C) = \sum_{i, j=1}^{3} c_{ij}$$

for $C = (c_{ij})$ in $\mathrm{Mat}(3 \times 3, \mathbf{Z})$, then

$$\tau(C(Z, Z)) = \beta(F_4)\,\mathbf{Z}.$$

In order to prove our assertion it suffices to show that $\beta(F_4) \neq 1$. Equivalently, we shall demonstrate that there is no matrix

$$C = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

in $\mathrm{Mat}(3 \times 3, \mathbf{Z})$ satisfying the following conditions:

(4.17.1) $\tau(C) = a + b + c + d + e + f + g + h + i = 1,$

(4.17.2) $C - 4TCT \in 2\,\mathrm{Mat}(3 \times 3, \mathbf{Z}),$

(4.17.3) $CT + TC = 2(G_1 T - TG_2)$
for some $G_1 = (g_{ij})$ and $G_2 = (g'_{ij})$ in $\mathrm{Mat}(3 \times 3, \mathbf{Z})$.

Suppose that C satisfying (4.17.1), (4.17.2) and (4.17.3) exists. If $(a_{ij}) = C - 4TCT$, then from (4.17.2) one gets

$$a_{11} = 3a + 2(b + c + d + g) + e + f + h + i \equiv 0 \pmod 2.$$

In particular,

(4.17.4) $a + e + f + h + i \equiv 0 \pmod 2.$

Let $(b_{ij}) = CT + TC$ and $(d_{ij}) = 2(G_1 T - TG_2)$. Then

$$b_{11} = 2a + \frac{1}{2}(b + c + d + g),$$

$$d_{11} = 2\left(g_{11} - g'_{11} + \frac{1}{2}(g_{12} + g_{13} - g'_{21} - g'_{31})\right).$$

Since, by (4.17.3), $b_{11} = d_{11}$, it follows that

(4.17.5) $b + c + d + g \equiv 0 \pmod 2.$

Adding (4.17.4) and (4.17.5), one gets

$$a + b + c + d + e + f + g + h + i \equiv 0 \pmod 2,$$

which contradicts (4.17.1). Thus the assertion is proved. □

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