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ALEXANDER S. MERKURJEV

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R-EQUIVALENCE AND RATIONALITY PROBLEM FOR SEMISIMPLE ADJOINT CLASSICAL ALGEBRAIC GROUPS

by A. S. MERKURJEV

Abstract. — The group of R-equivalence classes for all adjoint semisimple classical algebraic groups is computed. Examples of stably non-rational adjoint simple groups of type D_n, $n \geqslant 3$, are presented. The complete stable birational classification of adjoint simple groups of rank 3 is given.

Introduction

In [7, Cor. 2] Chevalley proved that over an algebraically closed field of characteristic zero the variety of any connected algebraic group is rational. He has also constructed examples of algebraic tori over local fields which are not rational.

Since all algebraic tori of dimension at most two are rational ([38, Th. 4.74]), and the variety of maximal tori in a connected algebraic group is rational ([7, Prop. 3] (in characteristic zero) [9, Exp. XIV, Th. 6.2] and [5, Th. 7.9]), it follows that all connected algebraic groups of rank at most 2 are rational. The special orthogonal group of a non-degenerate quadratic form over an arbitrary field is also an example of a rational variety (see Lemma 1 for a more general statement).

On the other hand, a semisimple group over a number field which is a counterexample to weak approximation (necessarily neither simply connected, nor adjoint [30, Th. 3.1, Cor. 5.4, Ex. 5.8]) provides an example of a non-rational group.

The first example of a simply connected semisimple not stably rational group was found by Platonov in [27] (the group $\mathbf{SL}_1(A)$ for a suitable simple algebra A). In [28] he has also constructed examples of quadratic forms q of a given dimension $d \geqslant 6$ such that the variety of the spinor group $\mathbf{Spin}(q)$ is not stably rational. For an arbitrary central simple algebra A with the index divisible by 4 the group $\mathbf{SL}_1(A)$ is not stably rational ([22]). On the other hand, the variety $\mathbf{Spin}(q)$ for a canonical quadratic form q (sum of squares) is always rational ([29]).

Key words and phrases. Adjoint simple algebraic groups, R-equivalence, rational varieties, algebras with involutions.

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For adjoint semisimple groups only a few results were known. Voskresenskiĭ and Klyachko proved in [39, Cor. of th. 8] that any adjoint simple group of type A_n with even n is rational. Černousov has shown ([6]) that the variety of the projective orthogonal group of a canonical quadratic form is rational.

As was noticed by Voskresenskiĭ (see [38]), the invariant (the group $SK_1(A)$) which Platonov has used in [27] to show that the group $G = \mathbf{SL}_1(A)$ is not stably rational, is nothing but the group of R-equivalence classes $G(F)/R$. The notion of R-equivalence was introduced by Manin in [20] and studied for linear algebraic groups in [8] by Colliot-Thélène and Sansuc. In particular, the group of R-equivalence classes for algebraic tori was computed in this paper.

The present paper is devoted to the computation of the group of R-equivalence classes for semisimple adjoint classical groups. As a consequence of this computation we give examples of stably non-rational simple adjoint groups.

In the preliminary section we remind basic definitions and facts in the theory of hermitian forms and algebras with involutions.

By a classical result of Weil ([40]), an adjoint simple group of classical series is the connected component of the automorphism group of some algebra with involution (except for some " non-classical " groups of type D_4). In the first section we compute the group of R-equivalence classes for such groups (Theorem 1). The result of the computation is given in terms of certain invariants of the corresponding algebra with involution. Note that in [12] Gille has studied the behavior of $G(F)/R$ under isogeny and in the special case of an even dimensional quadratic form came very close to Theorem 1 of the present paper (see Prop. 2.3 in [12]).

In the second section, following the classification of Weil, we consequently consider the classical types A, B, C and D of simple adjoint groups. In some cases we prove that adjoint groups are (stably) rational. For the type D we give a sufficient condition for an adjoint group to be not stably rational (Theorem 2).

In the next section, the complete stably rational classification and the classification of R-trivial simple adjoint classical groups of rank 3 is given (Theorem 3).

In the last section we give examples of not stably rational adjoint simple groups of type D_n for $n \geqslant 3$. The base field in these examples is an arbitrary number field for odd n and the field of rational functions in one variable over a number field if n is even.

In order to prove that the variety of an algebraic group G defined over a field F is not stably rational, we produce a field extension E/F such that $G(E)/R \neq 1$. We are forced to take E big enough, since it may happen that $G(E)/R$ is trivial for all algebraic extensions of F. (For example, if $G = \mathbf{SL}_1(A)$ for a central simple algebra A over a number field F.) A field E is obtained by passing over to iterated function fields of certain quadrics and Severi-Brauer varieties. The main ingredients in the proofs are the index reduction formula ([21, 24, 32]) and the results of [3] and [25].

The mail tool we use to prove the stable rationality of a certain adjoint simple group is the Proposition 3 with the corollaries. In order to apply them we need to know that the group of multipliers of the associated algebra with involution is " well rationally parameterized " (compare [6]).

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We use the following notation.

F is a field of characteristic different from 2.

All algebras considered in the paper are assumed to be finite-dimensional over the center. For a central simple algebra A over F we denote by $\mathrm{Nrd}: A^\times \to F^\times$ the reduced norm homomorphism. The image of Nrd we denote by Nrd(A). The index ind A is the square root of the dimension of the division algebra similar to A. For a field extension E/F the algebra $A \otimes_F E$ over E is denoted by A_E. We write $A \sim B$ for Brauer equivalent central simple algebras A and B. We use the notation $(a, b)_F$ for the *quaternion algebra* over F of dimension 4 given by generators i and j with the relations $i^2 = a \in F^\times$, $j^2 = b \in F^\times$, $ij = -ji$.

For a non-degenerate quadratic form q, we denote by $G(q)$ the group of multipliers of q: $G(q) = \{ x \in F^\times \text{ such that } x.q \simeq q \}$. The discriminant disc($q$) equals $(-1)^{\frac{n(n-1)}{2}} \det(q)$. For a field extension E/F the quadratic form $q \otimes_F E$ over E is denoted by q_E. For any $x_1, x_2, \ldots, x_n \in F^\times$ we denote the n-fold Pfister form $\bigotimes_{i=1}^n \langle 1, -x_i \rangle$ by $\langle\langle x_1, x_2, \ldots, x_n \rangle\rangle$.

For an algebraic variety X defined over a field F and any field extension E/F the set of E-points $\mathrm{Mor}_{\mathrm{Spec}\, F}(\mathrm{Spec}\, E, X)$ is denoted by X(E). If E/F is a finite separable extension and Y is an algebraic variety defined over a field E, then $R_{E/F}(Y)$ is the variety over F, obtained from Y by the restriction of scalars ([41]). The one-dimensional split torus Spec $F[t, t^{-1}]$ is denoted by $\mathbf{G}_{m,F}$.

0. Algebras with involutions

In this preliminary section we collect known facts in the theory of hermitian forms and algebras with involutions. The basic references are [14], [31].

Adjoint involution

Let P be a free right module over a ring D with an involution τ, $\varepsilon = \pm 1$. For a regular ε-hermitian form h on P over D with respect to τ one can define an involution $\sigma = \sigma_h$ on the ring $A = \mathrm{End}_D(P)$ by the equality

$$h(a(x), y) = h(x, \sigma(a)(y)), \quad x, y \in P;\ a \in A.$$

The involution σ_h is called *adjoint to h.*

Kind of an involution

Let F be any field of characteristic different from 2, Z be either a field F, or a quadratic etale extension of F (not necessarily a field). Consider an Azumaya algebra A over Z (if Z is a field, then A is a central simple algebra over Z) with involution σ such that F coincides with the subfield of σ-invariant elements in Z. The involution σ is of the *first kind* if it is trivial on Z (hence $Z = F$) and of the *second kind* otherwise.

If Z is not a field, i.e. $Z = F \times F$, then there exists a central simple algebra B over F and Z-algebra isomorphism $A \simeq B \times B^{op}$ such that σ corresponds to the *switch involution* $(x, y^{op}) \mapsto (y, x^{op})$ on $B \times B^{op}$.

Type of an involution of the first kind

If σ is of the first kind (i.e. $Z = F$), then over a field extension splitting A, it is adjoint to some non-degenerate bilinear form h. The involution σ is called *orthogonal* (or, of *orthogonal type*) if h is symmetric and *symplectic* (or, of *symplectic type*) if h is skew-symmetric. For example, the canonical involution on a quaternion algebra is symplectic.

If σ is an adjoint involution to some ε-hermitian form with respect to an involution τ of the first kind, then σ and τ are of the same type if $\varepsilon = 1$ and of different types if $\varepsilon = -1$.

Hyperbolic involutions

Assume that Z is a field. By Wedderburn's theorem, $A = \mathrm{End}_D(V)$ for some central division Z-algebra D and a right vector D-space V. There is an involution τ on D of the same kind as σ (the restrictions of τ and σ on Z are equal). Then the involution σ is *adjoint* to some (unique up to an F-multiple) non-degenerate ε-hermitian form h on V over D with respect to τ (where $\varepsilon = \pm 1$ if σ is of the first kind and $\varepsilon = 1$ otherwise).

The involution σ is called *hyperbolic* (resp. *isotropic*) if h is a hyperbolic (resp. an isotropic) ε-hermitian form ([4]).

If Z is not a field, then the switch involution σ is also called hyperbolic.

Clifford algebra

Let A be a central simple algebra of degree $2n$ over F, and σ be an orthogonal involution on A. We denote by $C(A, \sigma)$ the (generalized, even) Clifford algebra of (A, σ) defined in [13, 36]. It is an Azumaya algebra over an etale quadratic extension L/F, called the *discriminant quadratic extension.* The *discriminant* $\mathrm{disc}(\sigma)$ of σ is the class $dF^{\times 2} \in F^{\times}/F^{\times 2}$ such that $L = F[t]/(t^2 - d)$. If the discriminant of σ is trivial (i.e. L splits), then $C(A, \sigma) = C^{+}(A, \sigma) \times C^{-}(A, \sigma)$ where $C^{\pm}(A, \sigma)$ are central simple F-algebras.

The Clifford algebra $C(A, \sigma)$ carries a canonical involution τ of the first kind if n is even and of the second kind if n is odd.

If $a \in A^{\times}$ is a skew-symmetric element, i.e. $\sigma(a) = -a$, then $\mathrm{Nrd}(a) \in (-1)^n \mathrm{disc}(\sigma)$ ([17]).

Example. — Consider an involution σ on $A = M_n(Q)$, where Q is a quaternion algebra, adjoint to a (-1)-hermitian form $h = \langle x_1, x_2, \ldots, x_n \rangle$ with respect to the canonical involution on Q (x_i are skew-symmetric elements in Q). Then the diagonal matrix $a = \operatorname{diag}(x_1, x_2, \ldots, x_n)$ is a skew-symmetric element in A with respect to σ and, therefore, the discriminant of σ equals $d\mathrm{F}^{\times 2}$ where

$$d = (-1)^n \operatorname{Nrd}_{\mathrm{A}}(a) = (-1)^n \prod_{i=1}^{n} \operatorname{Nrd}_{\mathrm{Q}}(x_i) = \prod_{i=1}^{n} (x_i^2).$$

The class $d\mathrm{F}^{\times 2}$ in $\mathrm{F}^{\times}/\mathrm{F}^{\times 2}$ we also call the *discriminant* of a (-1)-hermitian form h.

The discriminant of a hyperbolic involution is always trivial ([4, Cor. 2.3]).

When n is even, then $\operatorname{cor}_{\mathrm{L/F}}(\mathrm{C}(\mathrm{A}, \sigma))$ is Brauer equivalent to A over F and $\mathrm{C}(\mathrm{A}, \sigma) \otimes_{\mathrm{L}} \mathrm{C}(\mathrm{A}, \sigma) \sim 0$ over L. If n is odd, then $\operatorname{cor}_{\mathrm{L/F}}(\mathrm{C}(\mathrm{A}, \sigma)) \sim 0$ over F and $\mathrm{C}(\mathrm{A}, \sigma) \otimes_{\mathrm{L}} \mathrm{C}(\mathrm{A}, \sigma) \sim \mathrm{A}_{\mathrm{L}}$ over L ([33, Th. 3.3, Prop. 7]). In any case, the algebra $\mathrm{C}(\mathrm{A}, \sigma) \otimes_{\mathrm{F}} \mathrm{A}$ is equivalent to the conjugate of $\mathrm{C}(\mathrm{A}, \sigma)$ over L.

If A splits, $\mathrm{A} = \operatorname{End}_{\mathrm{F}}(\mathrm{V})$, then σ is adjoint to some symmetric bilinear form h and the Clifford algebra $\mathrm{C}(\mathrm{A}, \sigma)$ coincides with the even Clifford algebra $\mathrm{C}_0(\mathrm{V}, q)$ of the quadratic form q associated to h. The discriminant of q coincides with the one of (A, σ).

1. R-equivalence on adjoint simple classical groups

Let G be an algebraic group, defined over a field F. Any point $g \in \mathrm{G}(\mathrm{F}(t))$ of the group G over the rational function field $\mathrm{F}(t)$, in other words, an F-morphism $\operatorname{Spec} \mathrm{F}(t) \to \mathrm{G}$, can be considered as a rational map $g : \mathbf{A}^1_{\mathrm{F}} \to \mathrm{G}$, defined over F. Denote by RG(F) the normal subgroup in G(F) consisting of all elements $x \in \mathrm{G}(\mathrm{F})$ such that there exists a rational map $f : \mathbf{A}^1_{\mathrm{F}} \to \mathrm{G}$ over F (which can be considered as an element of the group $\mathrm{G}(\mathrm{F}(t))$), defined in the points 0 and 1 with $f(0) = 1$ and $f(1) = x$ ([12], Lemma 2.1). The factor-group G(F)/RG(F) we denote simply by G(F)/R. It is the group of R-equivalence classes introduced by Manin in [20] and studied for linear algebraic groups by Colliot-Thélène and Sansuc in [8].

An algebraic group G defined over F is called R-*trivial* if $\mathrm{G}(\mathrm{E})/\mathrm{R} = 1$ for any field extension E/F.

An irreducible algebraic variety X defined over F is called *stably rational* if the variety $\mathrm{X} \times_{\mathrm{F}} \mathbf{A}^n_{\mathrm{F}}$ is rational for some $n \in \mathbf{N}$. The relation between R-triviality and stable rationality of the variety of an algebraic group is given by the following

Proposition 1 ([8]). — *If the variety of a connected algebraic group* G, *defined over a field* F, *is stably rational, then the group* G *is* R-*trivial.*

Proof. — If the variety of G is stably rational over F, then it is stably rational over an arbitrary field extension. Hence, it is sufficient to show that $\mathrm{G}(\mathrm{F})/\mathrm{R} = 1$.

Assume first that the variety of G is rational. The case of a finite field F is considered in [8, Corollary 6]. Over an infinite field there is an open subset $\mathrm{U} \subset \mathrm{G}$ with $\mathrm{U}(\mathrm{F}) \neq \emptyset$ isomorphic to an open subset V in some affine space $\mathbf{A}^n_{\mathrm{F}}$. Let $x \in \mathrm{G}(\mathrm{F})$ be any rational point. Consider two translations U_1 and U_2 of U containing x and 1 respectively.

Choose any rational point y in the intersection $U_1 \cap U_2$. Translating the intersections with V of two appropriate straight lines from $\mathbf{A}^n_F$ to U_1 and U_2 we get elements $g(t), h(t) \in G(F(t))$ such that $g(0) = h(0) = y$, $g(1) = x$ and $h(1) = 1$. Then for $f(t) = g(t)\, h(t)^{-1}$ we have: $f(0) = 1$ and $f(1) = x$, hence $x \in RG(F)$.

If the variety of G is stably rational then $G \times_F \mathbf{A}^n_F$ is rational for some $n \in \mathbf{N}$. Since $\mathbf{A}^n_F$ is a rational algebraic group, it follows from the first part of the proof that $G(F)/R = G(F)/R \times \mathbf{A}^n_F(F)/R = (G \times_F \mathbf{A}^n_F)(F)/R = 1$. ■

Let F be any field of characteristic different from 2, Z be either a field F or a quadratic etale extension of F (not necessarily a field). Consider an Azumaya algebra A over Z with involution σ such that F coincides with the subfield of σ-invariant elements in Z. An element $a \in A^\times$ is called a *similitude* if $\sigma(a)\, a \in F^\times$. Denote by $\mathrm{Sim}(A, \sigma)$ the group of all similitudes. For any $a \in \mathrm{Sim}(A, \sigma)$ the element $\mu(a) = \sigma(a)\, a$ in $F^\times$ is called the *multiplier* of the similitude a. Moreover, we have a group homomorphism

$$\mu : \mathrm{Sim}(A, \sigma) \to F^\times.$$

The image of μ we denote by $G(A, \sigma)$.

If σ is adjoint to an ε-hermitian form h, then an element $x \in F^\times$ is the multiplier of a similitude if and only if $x.h \simeq h$. In particular, for a hyperbolic involution σ one has $G(A, \sigma) = F^\times$.

An element $a \in A^\times$ is called an *isometry* if $\sigma(a)\, a = 1$. The group of all isometries $\mathrm{Iso}(A, \sigma)$ coincides with the kernel of μ.

We consider the groups $\mathrm{Sim}(A, \sigma)$ and $\mathrm{Iso}(A, \sigma)$ as the groups of F-points of the corresponding algebraic groups $\mathbf{Sim}(A, \sigma)$ and $\mathbf{Iso}(A, \sigma)$. The latter algebraic group is the kernel of the algebraic group homomorphism induced by the map μ:

$$\mu : \mathbf{Sim}(A, \sigma) \to \mathbf{G}_{m, F}.$$

Since $Z^\times$ is in the center of $\mathrm{Sim}(A, \sigma)$, it follows that the torus $R_{Z/F}(\mathbf{G}_{m, Z})$ is a central subgroup in $\mathbf{Sim}(A, \sigma)$. The group of *projective similitudes* is the factor-group $\mathbf{Sim}(A, \sigma)/R_{Z/F}(\mathbf{G}_{m, Z})$ which we denote by $\mathbf{PSim}(A, \sigma)$. By Hilbert's theorem 90, the group of F-points of $\mathbf{PSim}(A, \sigma)$ equals

$$\mathbf{PSim}(A, \sigma)\,(F) = \mathrm{Sim}(A, \sigma)/Z^\times.$$

For any $a \in A^\times$ the inner automorphism $\mathrm{Int}(a)$ of A commutes with σ if and only if $a \in \mathrm{Sim}(A, \sigma)$. By Skolem-Noether's theorem, the correspondence $a \mapsto \mathrm{Int}(a)$ induces an isomorphism of the group $\mathrm{Sim}(A, \sigma)/Z^\times$ and the group $\mathrm{Aut}(A, \sigma)$ of F-automorphisms of A commuting with σ.

Denote by $\mathbf{Sim}_+(A, \sigma)$, $\mathbf{PSim}_+(A, \sigma)$, $\mathbf{Aut}_+(A, \sigma)$ and $\mathbf{Iso}_+(A, \sigma)$ the connected components of the identity in the corresponding algebraic groups. We have the canonical isomorphism

$$\mathbf{PSim}_+(A, \sigma) \simeq \mathbf{Aut}_+(A, \sigma).$$

By the fundamental work of Weil [40] any adjoint absolutely simple classical algebraic group defined over F is isomorphic to $\mathbf{PSim}_+(A, \sigma)$ for a suitable algebra with involution (A, σ) over F.

If $Z = F$ and $\deg A = 2n$, then the group of rational points $\mathrm{Sim}_+(A, \sigma)$ of $\mathbf{Sim}_+(A, \sigma)$ consists of all $a \in \mathrm{Sim}(A, \sigma)$ such that $\mathrm{Nrd}(a) = \mu(a)^n$ ([26, p. 14]).

Denote the subgroup $\mu(\mathrm{Sim}_+(A, \sigma)) \subset F^\times$ by $G_+(A, \sigma)$. In the split case, when σ is adjoint to a symmetric bilinear form (quadratic form) q one has $G_+(A, \sigma) = G(A, \sigma) = G(q)$.

Lemma 1 ([40]). — *The variety of the group* $\mathbf{Iso}_+(A, \sigma)$ *is rational.*

Proof. — The Cayley transformation $a \mapsto \dfrac{1-a}{1+a}$ establishes a birational isomorphism between $G = \mathbf{Iso}_+(A, \sigma)$ and the affine space of all skew-symmetric elements in A with respect to σ (the Lie algebra of G). ■

Remark. — In the case A splits, $A = \mathrm{End}(V)$, and σ is adjoint to a quadratic form q, the group $\mathbf{Iso}_+(A, \sigma)$ is isomorphic to the special orthogonal group $\mathbf{O}_+(q)$.

We will need the following generalization of a Cassels-Pfister theorem for algebras with involutions.

Lemma 2 ([35]). — *For any* $f(t)$ *in* $G(A_{F(t)}, \sigma_{F(t)}) \cap F[t]$ *there exists a polynomial* $a(t) \in A[t]$ *such that* $f(t) = \sigma(a(t)).a(t)$. ■

For an irreducible polynomial $p(t)$ over F we denote the field $F[t]/p(t)\,F[t]$ by $F(p)$. We will use the following statement which is a slight generalization of [35, Prop. 2.4].

Proposition 2. — *A rational function* $f(t) \in F(t)$ *such that* $f(0) = 1$ *belongs to* $G_+(A_{F(t)}, \sigma_{F(t)})$ *if and only if, for any irreducible polynomial* $p(t)$ *such that the* $p(t)$*-adic valuation of* $f(t)$ *is odd, the involution* σ *is hyperbolic over the field* $F(p)$.

Proof. — In the case Z splits the statement of the proposition easily holds since $G_+(A_{F(t)}, \sigma_{F(t)}) = F(t)^\times$ and σ is hyperbolic over any field extension. Thus, we may assume that Z is a field. Let $f \in G_+(A_{F(t)}, \sigma_{F(t)})$ and $p(t)$ be an irreducible polynomial such that the $p(t)$-adic valuation of $f(t)$ is odd; put $E = F(p)$. By Wedderburn's theorem, $A_E = \mathrm{End}_D(V)$ for some division algebra D and a right D-module V. The involution σ_E is adjoint to some ε-hermitian form h on V over D with respect to an involution τ on D. We have: $f(t).h_{E(t)} \simeq h_{E(t)}$. By induction on $\dim(h)$ we will prove that h is a hyperbolic form.

If $\varepsilon = -1$, $\sigma = id$ and $D = E$, then h is an alternating form over a field and, therefore, is hyperbolic. Otherwise we may assume that h is isomorphic to some diagonal

ε-hermitian form $\langle d_1, d_2, \ldots, d_m \rangle$, $d_i \in D^\times$ ([31], Chapter 7, Th. 6.3). Since $f(t).h_{E(t)} \simeq h_{E(t)}$, it follows that $d_1.f(t)$ is a value of h, i.e.

$$d_1.f(t).g(t)^2 = \sum_{i=1}^{m} \tau(x_i(t)).d_i.x_i(t)$$

for some polynomials $0 \neq g(t) \in E[t]$, $x_i(t) \in D[t]$.

Let θ be a root of $p(t)$ in E. The $(t-\theta)$-adic valuation of $f(t)$ is odd. Hence, dividing both sides of the equality by an appropriate even power of $t-\theta$, we may assume that $x_i(\theta) \neq 0$ for some i and the left side is divisible by $t-\theta$. Substituting $t=\theta$ one gets

$$0 = \sum_{i=1}^{m} \tau(x_i(\theta)).d_i.x_i(\theta),$$

i.e. h is isotropic. Hence, $h = h' \perp \mathbf{H}$, where $\mathbf{H}$ is the hyperbolic plane, and $f(t).h' \simeq h'$. By the induction hypothesis the forms h' and, therefore, h are hyperbolic.

For the proof of the converse statement we may assume that $f(t) = p(t)$ is irreducible, $p(0) = 1$ and the involution σ_E is hyperbolic over the field $E = F(p)$. Denote by h an ε-hermitian form over a division F-algebra D with involution τ associated to the involution σ. Since $\sigma_{E(t)}$ is also hyperbolic, $G(A_{E(t)}, \sigma_{E(t)}) = E(t)^\times$; in particular, $\lambda(t).h_{E(t)} \simeq h_{E(t)}$ where $\lambda(t) = \dfrac{\theta - t}{\theta}$.

By [31, Chapter 2, Lemma 5.8] there exists an $F(t)$-linear function $s: E(t) \to F(t)$ such that

$$s_*(\langle 1 \rangle - \langle \lambda(t) \rangle) = \langle 1 \rangle - \langle N_{E(t)/F(t)}(\lambda(t)) \rangle = \langle 1 \rangle - \langle p(t) \rangle$$

in the Witt ring of $F(t)$, where s_* is a transfer map.

The linear map s defines also a transfer homomorphism of Witt groups of the classes of ε-hermitian forms ([14, p. 62])

$$s_*: W^\varepsilon(D_{E(t)}, \tau_{E(t)}) \to W^\varepsilon(D_{F(t)}, \tau_{F(t)}).$$

We have

$$\begin{aligned} h_{F(t)} - p(t).h_{F(t)} &= (\langle 1 \rangle - \langle p(t) \rangle).h_{F(t)} = s_*(\langle 1 \rangle - \langle \lambda(t) \rangle).h_{F(t)} \\ &= s_*(h_{E(t)} - \lambda(t).h_{E(t)}) = s_*(0) = 0 \end{aligned}$$

in the Witt group $W^\varepsilon(D_{F(t)}, \tau_{F(t)})$. The Witt cancellation implies that $p(t).h_{F(t)} \simeq h_{F(t)}$, hence, $p(t) \in G(A_{F(t)}, \sigma_{F(t)})$. By Lemma 2, $p(t) = \sigma(a(t)).a(t)$ for some $a(t) \in A[t]$. Since $1 = p(0) = \sigma(a(0)).a(0)$, it follows that $a(0) \in \mathrm{Iso}(A, \sigma)$. Replacing $a(t)$ by $a(t).a(0)^{-1}$ we may assume $a(0) = 1$. In order to prove that $p(t) \in G_+(A_{F(t)}, \sigma_{F(t)})$ we need to consider only the case of an orthogonal involution σ on a central simple algebra A of even degree $2n$ since otherwise the group $\mathbf{Sim}(A, \sigma)$ is connected. Substituting $t = 0$ to the equality $\mathrm{Nrd}(a(t)) = \pm p(t)^n$ one gets the sign " + ". ■

Denote by $NZ^\times$ the subgroup in $F^\times$ consisting of all elements of the type $\sigma z . z$, $z \in Z^\times$. If σ is an involution of the first kind (i.e. $Z = F$), then $NZ^\times = F^{\times 2}$. If σ is of the second kind, then $NZ^\times$ is the group of norms $N_{Z/F}(Z^\times)$ of the quadratic extension Z/F.

Denote by $\mathrm{Hyp}(A, \sigma)$ the subgroup in $F^\times$ generated by the norms of all finite extensions E/F such that σ_E is a hyperbolic involution.

The following Theorem describes the group $\mathbf{PSim}_+(A, \sigma)$ modulo R-equivalence.

Theorem 1. — There is a natural isomorphism

$$\mathbf{PSim}_+(A, \sigma)\ (F)/R \simeq G_+(A, \sigma)/NZ^\times . \mathrm{Hyp}(A, \sigma).$$

Proof. — It follows from the exact sequence

$$\mathrm{Iso}_+(A, \sigma) \to \mathrm{PSim}_+(A, \sigma) \to G_+(A, \sigma)/NZ^\times \to 1$$

that the following sequence is also exact:

$$\mathbf{Iso}_+(A, \sigma)\ (F)/R \to \mathbf{PSim}_+(A, \sigma)\ (F)/R \to G_+(A, \sigma)/NZ^\times . U \to 1$$

where U is the subgroup in $F^\times$ consisting of the elements $\mu(a(1))$ for all $a(t) \in \mathrm{Sim}_+(A_{F(t)}, \sigma_{F(t)})$, defined in the points 0 and 1 with $a(0) = 1$.

Lemma 1 shows that the group $\mathbf{Iso}_+(A, \sigma)$ is rational. Hence by Proposition 1 we obtain the isomorphism

$$\mathbf{PSim}_+(A, \sigma)\ (F)/R \simeq G_+(A, \sigma)/NZ^\times . U.$$

It remains to show that $NZ^\times . U = NZ^\times . \mathrm{Hyp}(A, \sigma)$. Let $f(t) = \mu(a(t))$ where $a(t) \in \mathrm{Sim}_+(A_{F(t)}, \sigma_{F(t)})$ defined in the points 0 and 1 with $a(0) = 1$. Then $f(t) \in G_+(A_{F(t)}, \sigma_{F(t)})$ and $f(0) = 1$. By Proposition 2, $f(1)$ is the product of a square and elements of the type $p(1)$ where $p(t)$ is an irreducible polynomial such that $p(0) = 1$ and the involution $\sigma_{F(p)}$ is hyperbolic. Since $p(1) = p(1)\, p(0)^{-1}$ is a norm of the extension $F(p)/F$, it follows that $f(1) \in F^{\times 2} . \mathrm{Hyp}(A, \sigma) \subset NZ^\times . \mathrm{Hyp}(A, \sigma)$, hence $U \subset NZ^\times . \mathrm{Hyp}(A, \sigma)$.

Conversely, let E/F be a finite extension such that the involution σ_E is hyperbolic, $x \in E^\times$, $y = N_{E/F}(x)$, $E_0 = F(x)$. Then $y = z^k$ where $z = N_{E_0/F}(x)$ and $k = [E : E_0]$. If k is even then $y \in F^{\times 2}$. If k is odd then the involution σ_{E_0} is hyperbolic ([4, Prop. 4.1]). Denote by $p(t)$ the minimal polynomial of $s = \dfrac{1}{1-x}$ such that $p(0) = 1$. It follows from Proposition 2 that $p(t) \in G_+(A_{F(t)}, \sigma_{F(t)})$. Lemma 2 implies that $p(t) = \mu(a(t))$ for some polynomial $a(t) \in \mathrm{Sim}(A_{F(t)}, \sigma_{F(t)}) \cap A[t]$. Replacing $a(t)$ by $a(t) . a(0)^{-1}$ we assume that $a(0) = 1$. As at the end of the proof of Proposition 2 we check that $a(t) \in \mathrm{Sim}_+(A_{F(t)}, \sigma_{F(t)})$, hence $p(1) = \mu(a(1)) \in U$. Finally,

$$z = N_{E_0/F}(x) = N_{E_0/F}\left(\frac{s-1}{s}\right) = p(1) . p(0)^{-1} = p(1)$$

and

$$y = z^k = z^{k-1} . p(1) \in F^{\times 2} . U \subset NZ^\times . U. \quad \blacksquare$$

Two irreducible algebraic varieties X and Y defined over a field F are called *stably birationally isomorphic* if the varieties $X \times \mathbf{A}_F^n$ and $Y \times \mathbf{A}_F^m$ are birationally isomorphic for some n and m.

We will use the following Proposition in the next section in order to derive stably rational groups.

Proposition 3. — Let $\alpha : X \to T$ *and* $\beta : Y \to T$ *be two morphisms of irreducible algebraic varieties defined over a field* F. *Assume that*

(1) *The fiber of* α *and* β *over any field-valued point of* T *is an irreducible stably rational variety provided it has a rational point.*

(2) *For a field extension* E/F *the images of* $\alpha(E)$ *and* $\beta(E)$ *in* $T(E)$ *coincide.*

Then X *and* Y *are stably birationally isomorphic.*

Proof. — Let $V = X \times_T Y$ and $W = \operatorname{Spec} E \times_X V = \operatorname{Spec} E \times_T Y$ be the fiber of β over the generic point $\theta : \operatorname{Spec} E \to X$ where $E = F(X)$ is the function field of X. Since by assumption $\operatorname{im} \alpha(E) = \operatorname{im} \beta(E)$, it follows that there is a morphism $\gamma : \operatorname{Spec} E \to Y$ such that $\beta \circ \gamma = \alpha \circ \theta$. By the universal property of the fiber product, the variety W has a rational point over E and by assumption is an irreducible stably rational variety. Hence V is an irreducible variety and since $F(V) = E(W)$, it follows that X and V are stably birationally isomorphic. Analogously, Y and V and, therefore, X and Y are stably birationally isomorphic. ■

Corollary 1. — Let A *be an algebra with involution* σ *over* F *as above,* X *be an irreducible stably rational algebraic variety defined over* F *and* $\alpha : X \to \mathbf{G}_{m,F}$ *be a morphism defined over* F. *Assume that*

(1) *The fiber of* α *over any field valued-point of* $\mathbf{G}_{m,F}$ *is an irreducible stably rational variety provided it has a rational point.*

(2) *For any field extension* E/F *the image of* $\alpha(E)$ *in* $E^\times$ *equals* $G_+(A_E, \sigma_E)$.

Then the variety of the group $\mathbf{PSim}_+(A, \sigma)$ *is stably rational.*

Proof. — We can take $Y = \mathbf{Sim}_+(A, \sigma)$, $T = \mathbf{G}_{m,F}$ and the multiplier map $\beta = \mu : Y \to T$. If the fiber of μ over a field E has a rational point, then as an E-variety, it is isomorphic to $\mathbf{Iso}_+(A, \sigma)_E$ and hence is an E-rational variety by Lemma 1. By Proposition 3, the variety of the group $\mathbf{Sim}_+(A, \sigma)$ and X are stably birationally isomorphic, hence $\mathbf{Sim}_+(A, \sigma)$ is stably rational.

The variety of $\mathbf{Sim}_+(A, \sigma)$ is a $R_{Z/F}(\mathbf{G}_{m,F})$-torsor over $\mathbf{PSim}_+(A, \sigma)$, hence, by Hilbert's theorem 90, the stably rational variety $\mathbf{Sim}_+(A, \sigma)$ is birationally isomorphic to the product $\mathbf{PSim}_+(A, \sigma) \times \mathbf{A}_F^i$ where $i = (Z : F)$ ([38, Th. 4.15]) and, therefore, the variety of $\mathbf{PSim}_+(A, \sigma)$ is stably rational. ■

The following statement is a particular case of Corollary 1.

Corollary 2. — *Let* G *be a connected stably rational algebraic group over a field* F *and* $\alpha : G \to \mathbf{G}_{m,F}$ *be a homomorphism defined over* F. *Assume that*

(1) *The kernel of* α *is a connected stably rational algebraic group defined over* F.
(2) *For any field extension* E/F *the image of* $\alpha(E)$ *in* $E^\times$ *equals* $G_+(A_E, \sigma_E)$.

Then the variety of the group $\mathbf{PSim}_+(A, \sigma)$ *is stably rational.* ■

2. Classification and examples

An arbitrary semisimple adjoint algebraic group over a field F is isomorphic to a direct product of several groups of type $G_1 = R_{E/F}(G)$, where E/F is a finite separable field extension and G is an absolutely simple adjoint algebraic group over E ([37, 3.1.2]). Since the R-equivalence commutes with direct products and $G_1(F)/R = G(E)/R$, the computation of the group of R-equivalence classes for a semisimple adjoint algebraic group reduces to the case of absolutely simple adjoint algebraic groups.

Below we give the list of all classical absolutely simple adjoint groups following [40]. In some cases we prove that these groups are stably rational and hence R-trivial. The main statement (Theorem 2) gives a condition which is sufficient for an adjoint group of type D_n not to be R-trivial and hence not to be stably rational.

Type A_{n-1}

An arbitrary absolutely simple adjoint algebraic group of type A_{n-1} is isomorphic to a connected group $\mathbf{PSim}(A, \sigma)$, where A is an Azumaya algebra of degree n over an etale quadratic extension Z of F and σ is an involution of the second kind trivial on F.

Consider first the case when Z splits, $Z = F \times F$. Then A is isomorphic to $B \times B^{op}$ with the switch involution, where B is a central simple algebra of degree n over F. It is easy to see that the map $b \mapsto (b, (b^{-1})^{op})$ gives rise to an isomorphism of $\mathbf{PGL}_1(B)$ and $\mathbf{PSim}(A, \sigma)$. The group $\mathbf{PGL}_1(B)$ embeds as an open subset in the projective space $\mathbf{P}(B)$, hence, the variety of this group is rational.

Consider now the case when Z is a field. The involution σ_Z is hyperbolic since $Z \otimes_F Z$ splits, hence $NZ^\times \subset \mathrm{Hyp}(A, \sigma)$. It follows from Theorem 1 that

$$\mathbf{PSim}(A, \sigma)\ (F)/R = G(A, \sigma)/\mathrm{Hyp}(A, \sigma).$$

Consider the following particular cases:

Case 1: *n is odd.*

We claim that $G(A, \sigma) = \mathrm{Hyp}(A, \sigma) = NZ^\times$. Indeed, $NZ^\times \subset \mathrm{Hyp}(A, \sigma) \subset G(A, \sigma)$. Conversely, if $x = \sigma(a)\, a \in F^\times$, then taking the reduced norms of both sides one gets: $x^n = N_{Z/F}(\mathrm{Nrd}(a))$, hence $x \in NZ^\times$ since n is odd. The equality we proved implies that $\mathbf{PSim}(A, \sigma)\ (F)/R = 1$ in this case. It reflects the fact that the algebraic group $\mathbf{PSim}(A, \sigma)$ is rational if n is odd ([39, Cor. of Th. 8]).

Case 2: $n = 2$.

In this case A is a quaternion algebra over Z. Denote by $(a \mapsto \bar{a})$ the canonical involution on A and consider the quaternion F-subalgebra

$$Q = \{ a \in A \text{ such that } \sigma a = \bar{a} \}$$

in A. We claim that $\mathrm{Sim}(A, \sigma) = Q^\times . Z^\times$. The inclusion " $\supset$ " is clear. Let $a \in \mathrm{Sim}(A, \sigma)$, i.e. $\sigma(a)\, a \in F^\times$. Since $\sigma a . \bar{a}^{-1} = (\sigma(a)\, a) . (\bar{a}a)^{-1} \in Z^\times$ and $N_{Z/F}(\sigma a . \bar{a}^{-1}) = 1$, it follows from Hilbert's theorem 90 that $\sigma a . \bar{a}^{-1} = \sigma z . z^{-1}$ for some $z \in Z^\times$. Hence, $\sigma(az^{-1}) = \overline{az^{-1}}$, $az^{-1} \in Q^\times$ and $a \in Q^\times . Z^\times$.

It follows from the formula above that $\mathrm{Sim}(A, \sigma)/Z^\times \simeq Q^\times/F^\times$, hence $\mathbf{PSim}(A, \sigma) \simeq \mathbf{PGL}_1(Q)$ is a rational algebraic group, and also $G(A, \sigma) = \mathrm{Nrd}(Q) . NZ^\times$. Since $NZ^\times \subset \mathrm{Hyp}(A, \sigma)$, it follows that $\mathrm{Hyp}(A, \sigma) = G(A, \sigma) = \mathrm{Nrd}(Q) . NZ^\times$.

Type B_n

An arbitrary absolutely simple adjoint algebraic group of the type B_n is isomorphic to a connected group $\mathbf{PSim}(A, \sigma)$ where A is a central simple algebra of degree $2n + 1$ over a field F with involution σ of the first kind.

The algebra A necessarily splits, hence the involution σ is adjoint to some quadratic form q (uniquely determined up to a scalar) of dimension $2n + 1$. The algebraic group $\mathbf{PSim}(A, \sigma)$ is equal to the projective orthogonal group $\mathbf{PGO}(q)$ of the form q which is naturally isomorphic to the special orthogonal group $\mathbf{O}_+(q) = \mathbf{Iso}_+(A, \sigma)$. This group is known to be rational by Lemma 1.

Since q is of odd dimension, it is not hyperbolic over any field extension of F, hence $\mathrm{Hyp}(A, \sigma) = 1$. If $x \in G(A, \sigma) = G(q)$, i.e. $x . q \simeq q$, then, taking the determinant of the both sides, one sees that $x \in F^{\times 2}$ and $G(A, \sigma) = F^{\times 2}$.

Type C_n

An arbitrary absolutely simple adjoint algebraic group of type C_n is isomorphic to a connected group $\mathbf{PSim}(A, \sigma)$ where A is a central simple algebra of degree $2n$ over a field F with a symplectic involution σ.

Consider the following particular cases:

Case 1: $n = 1$.

In this case A is a quaternion algebra and σ is the canonical involution. Hence, $\mathrm{Sim}(A, \sigma) = A^\times$ and $\mathbf{PSim}(A, \sigma) \simeq \mathbf{PGL}_1(A)$ are rational algebraic groups of type $C_1 = A_1$. It is clear that $G(A, \sigma) = \mathrm{Hyp}(A, \sigma) = \mathrm{Nrd}(A)$.

Case 2: $n = 2$.

The space of skew-symmetric elements of trivial reduced trace

$$V = \{ a \in A \text{ such that } \sigma a = -a \text{ and } \mathrm{Trd}(a) = 0 \}$$

carries the 5-dimensional quadratic form $q(a) = a^2 \in F$. The map $\mathrm{Sim}(A, \sigma) \to O_+(q)$, $a \mapsto \mathrm{Int}(a)|_V$ induces an isomorphism of simple adjoint algebraic groups

$\mathbf{PSim}(A, \sigma) \simeq \mathbf{O}_+(q)$ of type $C_2 = B_2$ [15; 23, Prop. 5.4]. In particular, the group $\mathbf{PSim}(A, \sigma)$ is rational.

Case 3: *n is odd.*

Lemma 3. — *If n is odd, then* $G(A, \sigma) = \mathrm{Hyp}(A, \sigma) = \mathrm{Nrd}(A)$.

Proof. — Since n is odd, the algebra A is similar to a quaternion algebra $Q = (a\ b)_F$ and the involution σ is adjoint to a hermitian form h on some space V over Q with the canonical involution on Q. Any element x in Nrd(A) is a norm of some quadratic extension E/F which splits the algebra A. Since A_E splits, the involution σ_E is adjoint to some skew-symmetric form over E and, therefore, is hyperbolic, hence, $x \in \mathrm{Hyp}(A, \sigma)$. Conversely, assume that $x \in G(A, \sigma)$, i.e. $x.h \simeq h$. The hermitian form h gives a $4n$-dimensional quadratic form q on V considered as an F-space. If $h = \langle a_1, a_2, \ldots, a_n \rangle$ with $a_i \in F^\times$, then $q = \langle\langle a, b \rangle\rangle \otimes f$ where f is a quadratic form of dimension n over F with the same diagonalization as h. Thus,

$$x \langle\langle a, b \rangle\rangle \otimes f \simeq \langle\langle a, b \rangle\rangle \otimes f,$$

hence the form $\langle\langle a, b, x \rangle\rangle \otimes f$ is hyperbolic. Since f is of odd dimension, it follows that $\langle\langle a, b, x \rangle\rangle$ is a hyperbolic form ([18, Ch. 8, Cor. 6.7]), i.e. $x \in \mathrm{Nrd}(Q) = \mathrm{Nrd}(A)$. ∎

In particular, we have proved that the group $\mathbf{PSim}(A, \sigma)$ is R-trivial. This statement is also a consequence of the following

Proposition 4. — *Any absolutely simple adjoint group of the type* C_n *with odd n is stably rational.*

Proof. — We apply Corollary 2 of Proposition 3 to the rational algebraic group $G = \mathbf{GL}_1(Q)$ and to the reduced norm homomorphism $\mathrm{Nrd} = \alpha : G \to \mathbf{G}_{m,F}$ (we use the notation of the proof of Lemma 3). The kernel of α is the rational algebraic group $\mathbf{SL}_1(Q)$ (affine quadric with a rational point). Finally, by Lemma 3, $G(A_E, \sigma_E) = \mathrm{Nrd}(A_E) = \mathrm{im}\ \alpha(E)$ for any field extension E/F. ∎

Type D_n

An arbitrary adjoint algebraic group of the type D_n (except for some non-classical groups of the type D_4) is isomorphic to a group $\mathbf{PSim}_+(A, \sigma)$ where A is a central simple algebra of degree $2n$ over F with an orthogonal involution σ. The group of F-points of $\mathbf{PSim}_+(A, \sigma)$ equals $\mathrm{Sim}_+(A, \sigma)/F^\times$ where $\mathrm{Sim}_+(A, \sigma)$ consists of *proper similitudes*, i.e. similitudes $a \in \mathrm{Sim}(A, \sigma)$, such that $\mathrm{Nrd}(a) = \mu(a)^n$.

Denote by $C = C(A, \sigma)$ the Clifford algebra of algebra with involution (A, σ) with the center L/F being an etale quadratic extension over F (section 0). By functoriality there is a natural homomorphism

$$\rho : \mathrm{PSim}_+(A, \sigma) = \mathrm{Aut}_+(A, \sigma) \to \mathrm{Aut}_L(C) = \mathrm{PGL}_1(C) = C^\times/L^\times.$$

Lemma 4. — $\mathrm{Hyp}(A, \sigma) \subset N_{L/F}(\mathrm{Nrd}\, C(A, \sigma))$.

Proof. — Let E be a finite extension of the field F such that the involution σ_E is hyperbolic. Since the discriminant of σ_E is trivial, we may assume that $L \subset E$, therefore, $L \otimes_F E = E \times E$ and one of the two components of the Clifford algebra $C(A_E, \sigma_E)$, say $C^+ = C^+(A_E, \sigma_E)$, splits ([2, Th. 3]). Hence,

$$N_{E/F}(E^\times) = N_{E/F}(N_{L\otimes E/E}(E^\times \times 1)) \subset N_{E/F}(N_{L\otimes E/E}(\mathrm{Nrd}(C^+ \times C^-)) = N_{L\otimes E/F}(\mathrm{Nrd}\, C(A_E, \sigma_E)) \subset N_{L/F}(\mathrm{Nrd}\, C(A, \sigma)). \quad \blacksquare$$

Consider the following particular cases:

Case 1: $n = 2$.

Proposition 5 (compare [26, Prop. 1.15]). —

(1) *Any adjoint group of the type* $D_2 = A_2$ *is isomorphic to*

$$\mathbf{PSim}_+(A, \sigma) \simeq R_{L/F}(\mathbf{PGL}_1(C))$$

and hence is rational.

(2) $\mathrm{Hyp}(A, \sigma) = N_{L/F}\, \mathrm{Nrd}\, C$ *and* $G_+(A, \sigma) = F^{\times 2}.N_{L/F}\, \mathrm{Nrd}\, C$.

Proof. — 1. The homomorphism ρ induces the desired isomorphism ([23], Prop. 5.3]).

2. Let $x \in N_{L/F}\, \mathrm{Nrd}(C)$. We will prove that $x \in \mathrm{Hyp}(A, \sigma)$.

Claim: we may assume that x is a norm in a field extension E/F such that the discriminant of σ_E is trivial and one of the components of the Clifford algebra $C^\pm = C^\pm(A_E, \sigma_E)$ splits. For the proof of the claim assume first that L is a field. Then there is a field extension E/L such that C splits over E and x is a norm in E/F. Clearly, the split algebra $C \otimes_L E$ is one of the components $C^\pm$. Consider now the case $L = F \times F$. Then $N_{L/F}\, \mathrm{Nrd}\, C = \mathrm{Nrd}(C^+).\mathrm{Nrd}(C^-)$ and we may assume that $x \in \mathrm{Nrd}(C^\pm)$. Hence, there exists a field extension E/F such that $C^\pm$ splits and x is a norm in E/F.

Now in order to prove that $x \in \mathrm{Hyp}(A, \sigma)$ it suffices to show that σ is hyperbolic provided the discriminant is trivial and one of the components $C^\pm = C(A, \sigma)$ splits. The algebra A is isomorphic to the tensor product of two quaternion algebras $C^+ \times C^-$ and the involution σ is isomorphic to the tensor product of two canonical involutions ([33, Prop. 4.5]). Since one of the components splits it follows that the corresponding canonical involution, and hence σ, is hyperbolic. We have proved that $x \in \mathrm{Hyp}(A, \sigma)$, i.e. $N_{L/F}\, \mathrm{Nrd}\, C \subset \mathrm{Hyp}(A, \sigma)$. The inverse inclusion is given by Lemma 4. The second equality follows from Theorem 1 and Proposition 1. ■

Corollary ([26, Cor. 1.16]). — *If* $\mathrm{disc}(\sigma)$ *is trivial, then*

$$\mathbf{PSim}_+(A, \sigma) \simeq \mathbf{PGL}_1(C^+) \times \mathbf{PGL}_1(C^-)$$

where $C^\pm = C^\pm(A, \sigma)$ *and* $G_+(A, \sigma) = \mathrm{Hyp}(A, \sigma) = \mathrm{Nrd}(C^+).\mathrm{Nrd}(C^-)$. ■

Case 2: $n = 3$.

Lemma 5. — *Let* E/F *be a field extension. For a central simple algebra* A *of degree* 6 *over* F *with an orthogonal involution* σ *the following conditions are equivalent*:

(1) σ *is hyperbolic over* E;

(2) A *is split over* E *and the involution* σ *is adjoint to a* 6-*dimensional hyperbolic quadratic form*;

(3) L *can be embedded into* E *over* F *and the algebra* $C \otimes_L E$ *is split.*

Proof. — (1) $\Rightarrow$ (2). Since the dimension of the hyperbolic ε-hermitian form associated to σ is even and the index of A is a 2-power, it follows that the algebra A must be split.

(2) $\Rightarrow$ (1) is trivial.

(2) $\Rightarrow$ (3). The discriminant of a hyperbolic involution is trivial, hence $dE^{\times 2} = \mathrm{disc}(\sigma_E) = E^{\times 2}$, i.e. L can be embedded into E. The algebra $C \otimes_L E$ is similar to the Clifford algebra of a hyperbolic 6-dimensional quadratic form over E and, therefore, is split.

(3) $\Rightarrow$ (2). Since $A_L \sim C \otimes_L C$ (see section 0) and $C \otimes_L E$ is split, it follows that A is split over E. Hence, the involution σ_E is adjoint to some quadratic form q over E of dimension 6 and trivial discriminant. The Clifford algebra $C(q) \sim C \otimes_L E$ is split, hence q is a hyperbolic form ([1]). ■

Corollary. — *We have* $\mathrm{Hyp}(A, \sigma) = N_{L/F}\,\mathrm{Nrd}\,C$ *and*

$$\mathbf{PSim}_+(A, \sigma)\,(F)/R = G_+(A, \sigma)/F^{\times 2}.N_{L/F}\,\mathrm{Nrd}\,C. \quad ■$$

Assume that the discriminant of the involution σ is trivial (i.e. L splits, $L \simeq F \times F$). In this case $C \simeq C^+ \times C^{+op}$ for some central simple algebra C^+ of degree 4 over F. The following statement is the consequence of the Corollary, Theorem 1 [23, Prop. 5.5] and [26, Cor. 1.19].

Proposition 6. — *Let* $\mathrm{disc}(\sigma) \in F^{\times 2}$. *Then*:

1. *The homomorphism* ρ *gives the isomorphism of algebraic groups*

$$\mathbf{PSim}_+(A, \sigma) \simeq \mathbf{PGL}_1(C^+)$$

of the types $D_3 = A_3$. *In particular, the former group is rational.*

2. $\mathrm{Hyp}(A, \sigma) = \mathrm{Nrd}(C^+)$ *and* $G_+(A, \sigma) = F^{\times 2}.\mathrm{Nrd}(C^+)$. ■

Case 3: *The algebra* A *splits*: $A = \mathrm{End}_F(V)$.

In this case the involution σ is adjoint to some non-degenerate quadratic form q on the space V. The groups $\mathbf{Iso}_+(A, \sigma)$, $\mathbf{Sim}_+(A, \sigma)$ and $\mathbf{PSim}_+(A, \sigma)$ are respectively equal to the special orthogonal group $\mathbf{O}_+(q)$, the group of proper similitudes $\mathbf{GO}_+(q)$ and the projective special orthogonal group $\mathbf{PGO}_+(q)$ of the quadratic form q. The group $\mathrm{Hyp}(A, \sigma)$ equals the subgroup $\mathrm{Hyp}(q)$ of $F^\times$ generated by the norms in all finite extensions E/F such that the quadratic form q_E is hyperbolic. The groups $G(A, \sigma)$ and $G_+(A, \sigma)$ are equal to $G(q)$. The algebra $C(A, \sigma)$ equals the even Clifford algebra $C_0(q)$ with the center L.

The following statement gives examples of stably rational groups $\mathbf{PGO}_+(q)$.

Proposition 7. — If $q = f \otimes_F g$ *is the tensor product of a Pfister form* f *and a form* g *of odd dimension over* F *then the group* $\mathbf{PGO}_+(q)$ *is stably rational.*

Proof. — We apply Corollary 1 of Proposition 3 to the rational variety X of anisotropic vectors in the space of definition of f and to the morphism $\alpha : X \to \mathbf{G}_{m,F}$ defined by the equality $\alpha(v) = f(v)$. Any fiber of this map with a rational point is an affine quadric with a rational point and hence is a rational variety. Finally, the image of $\alpha(E)$ for any field extension E/F, i.e. the set of non-zero values of f_E is equal to the group of multipliers $G(f_E) = G(q_E)$ since f is a Pfister form and the dimension of g is odd ([18, Cor. VIII.6.7, Cor. X.1.7]). ■

The main result of the section is the following

Theorem 2. — Let A *be a central simple algebra of even degree over a field* F *with an orthogonal involution* σ. *If* $\mathrm{disc}(\sigma)$ *is not trivial and* $\mathrm{ind}\, C(A, \sigma) \geqslant 4$, *then the group* $\mathbf{PSim}_+(A, \sigma)$ *is not* R*-trivial and hence is not stably rational.*

We start with some preliminary results. Let D be a central simple algebra over a field F.

Lemma 6 ([19, Prop. 7; 42]). — *A monic rational function* $f(t) \in F(t)^\times$ *belongs to* $\mathrm{Nrd}(D_{F(t)})$ *if and only if, for any irreducible polynomial* $p(t)$, *the* $p(t)$*-adic valuation of* $f(t)$ *is divisible by* $\mathrm{ind}\, D_{F(p)}$. ■

Let $L = F(\sqrt{d})$, $d \in F^\times$, be a quadratic extension. Denote by X the affine conic curve given in $\mathbf{A}^2_F$ by the equation $u^2 - dv^2 = a$, $a \in F^\times$. We assume that X is not split or, equivalently, $X(F) = \emptyset$, or $a \notin N_{L/F}(L^\times)$. The degree of any closed point in X is even. The conic curve X is split over L, $X_L \simeq \mathbf{A}^1_L - \mathrm{pt}$.

Lemma 7. — If $\mathrm{ind}(D_L)$ *is divisible by* 4 *then*

$$a \notin F(X)^{\times 2} . N_{L(X)/F(X)}(\mathrm{Nrd}\, D_{L(X)}).$$

Proof. — Assume that

$$u^2 - dv^2 = a = f^2 . N_{L(X)/F(X)}(g)$$

where $f \in F(X)^\times$ and $g \in \mathrm{Nrd}(D_{L(X)})$. Hilbert's theorem 90 then yields $u + v\sqrt{d} = fgl\bar{l}^{-1}$ for some $l \in L(X)^\times$. Replacing f by $fl\bar{l}$, we have $u + v\sqrt{d} = fgh^2$ with $f \in F(X)^\times$, $g \in \mathrm{Nrd}(D_{L(X)})$ and $h \in L(X)^\times$.

Let x' and x'' be the two L-points at infinity on the projective conic $\overline{X}_L$, which are defined by $\operatorname{div}\left(u + v\sqrt{d}\right) = x' - x''$. Then

$$1 = v_{x'}\left(u + v\sqrt{d}\right) = v_{x'}(f) + v_{x'}(g) + 2v_{x'}(h).$$

By Lemma 6, $v_{x'}(g)$ is even, so that $v_{x'}(f)$ is odd. Let x be the closed point (of degree 2) at infinity of the projective conic $\overline{X}$. From $f \in F(X)^\times$ we conclude that $v_x(f)$ is an odd integer. Since the degree of a closed point in $\overline{X}$ is even, it follows that there exists a closed point $y \in X$ such that $v_y(f)$ is odd and the degree of y is twice an odd number. Let y' be a closed point in X_L over y. The degree of y' is either an odd number or twice an odd number. Since D_L has an index divisible by 4, the index of $D_{L(y')}$ is even in all cases. By Lemma 6, this implies that $v_{y'}(g)$ is even. Now $v_{y'}(f) = v_y(f)$ is odd. From the equality $0 = v_{y'}\left(u + v\sqrt{d}\right) = v_{y'}(fgh^2) = v_{y'}(f) + v_{y'}(g) + 2v_{y'}(h)$ we get a contradiction. ■

Lemma 8. — Let $L = F\left(\sqrt{d}\right)$ *be a quadratic extension and* D *be a central simple algebra over* F. *Then for any quadratic form* f *of even dimension and trivial discriminant there exists a field extension* E/F, *linearly disjoint from* L/F, *such that the group* $G(f_E)$ *is not contained in the norm group of the extension* EL/E *and* $\operatorname{ind}(D_{EL}) = \operatorname{ind}(D_L)$.

Proof. — We proceed by induction on $2n = \dim f$. If $n = 1$, then f is hyperbolic and we can take $E = F(t)$ since $t \in E^\times = G(f_E)$ and t is not a norm in the extension EL/E. In the general case we may assume that $n \geqslant 2$ and write f in the form $h \perp \langle u, v, w \rangle$ for a quadratic form h of dimension $2n - 3$ and $u, v, w \in F^\times$. Hence in the Witt ring of F, $f = g + x.\langle\langle a, b \rangle\rangle$ for the form $g = h \perp \langle -uvw \rangle$ of dimension $(2n - 2)$, trivial discriminant and $a = -uv$, $b = -uw$, $x = u$. By the induction hypothesis, replacing F by some extension, we may assume that the group $G(g)$ is not contained in the norm group of the extension L/F. Choose any $c \in G(g)$ that is not a norm in the extension L/F. In particular, the quaternion algebra $Q = (c, d)_F$ does not split.

Consider the function field E of the projective quadric given by the form $q = \langle 1, -a, -b, ab, -c \rangle$. Since $c \in G(\langle\langle a, b \rangle\rangle_E)$ and $c \in G(g_E)$, it follows that $c \in G(f_E)$. On the other hand, the function field of the quadric given by q does not change the index of any central simple algebra ([21, Cor. 3 of Th. 1]). In particular, $\operatorname{ind}(D_{EL}) = \operatorname{ind}(D_L)$ and $\operatorname{ind}(Q_E) = \operatorname{ind} Q$, hence Q_E does not split, i.e. c is not a norm in the extension EL/E. ■

Proof of Theorem 2. — Assume first that A is split. Then the involution σ is adjoint to some quadratic form q. Denote by $dF^{\times 2}$ the discriminant of q and by D the Clifford algebra $C(q)$. Then $C_0(q)$ is similar to D_L where $L = F\left(\sqrt{d}\right)$. By assumption $\operatorname{ind} D_L \geqslant 4$. Consider the quadratic form $f = q \perp \langle 1, -d \rangle$ of trivial discriminant. By Lemma 8, we may assume (extending the ground field if necessary) that there exists $a \in G(f)$ such that $a \notin N_{L/F}(L^\times)$.

Consider the affine conic curve X given by the equation $u^2 - dv^2 = a$. It is not split since $a \notin N_{L/F}(L^\times)$. By Lemma 7, $a \notin F(X)^{\times 2}.N_{L(X)/F(X)}(\mathrm{Nrd}\, D_{L(X)})$. Replacing F by F(X) we may assume that $a \notin F^{\times 2}.N_{L/F}(\mathrm{Nrd}\, D_L)$ but $a \in N_{L/F}(L^\times)$. Lemma 4 shows that $a \notin F^{\times 2}.\mathrm{Hyp}(q)$. It follows from the equality $N_{L/F}(L^\times) = G(\langle 1, -d \rangle)$ that $a \in G(f) \cap N_{L/F}(L^\times) \subset G(q)$, hence a represents a nontrivial element in $\mathbf{PGO}_+(q)\,(F)/R = G(q)/F^{\times 2}.\mathrm{Hyp}(q)$.

Now consider the general case. Let Y be the Severi-Brauer variety corresponding to the algebra A, E = F(Y) be the function field of Y. By the index reduction formula ([24, § 5; 32, Th. 1.3, Th. 1.6]),

$$\mathrm{ind}(C(A, \sigma) \otimes_F E) = \min(\mathrm{ind}\, C(A, \sigma), \mathrm{ind}(C(A, \sigma) \otimes_F A)) \geqslant 4$$

since the algebra $C(A, \sigma) \otimes_F A$ is similar to the conjugate of $C(A, \sigma)$ over L (section 0). But A is split over E and by the first part of the proof the group $\mathbf{PSim}_+(A_E, \sigma_E)$ (and hence $\mathbf{PSim}_+(A, \sigma)$) is not R-trivial. ■

The following Corollary is a particular case of the Theorem 2 when algebra A is split.

Corollary. — Let q be a non-degenerate quadratic form of even dimension over a field F. *If the discriminant of q is not trivial and* ind $C_0(q) \geqslant 4$, *then the group* $\mathbf{PGO}_+(q)$ *is not* R-*trivial and hence is not stably rational.* ■

3. Absolutely simple adjoint groups of rank 3

In this section we consider absolutely simple adjoint classical groups of rank 3. Groups of the types B_3 and C_3 are R-trivial (section 2), hence we will consider the type $D_3 = A_3$.

An arbitrary absolutely simple adjoint group of type D_3 is isomorphic to $\mathbf{PSim}_+(A, \sigma)$ for some central simple algebra A over F of degree 6 with an orthogonal involution σ. As in section 2 denote by C the Clifford algebra of (A, σ). It is a central Azumaya algebra of degree 4 with involution τ of the second kind over the discriminant etale quadratic extension L/F. The natural group homomorphism

$$\mathbf{PSim}_+(A, \sigma) \to \mathbf{PSim}(C, \tau)$$

is an isomorphism of simple adjoint groups of type $D_3 = A_3$ ([23, Prop. 5.5]).

Denote disc(σ) by $dF^{\times 2}$, $d \in F^\times$; then $L = F(\sqrt{d})$. If the discriminant is trivial, $dF^{\times 2} = F^{\times 2}$, then $L = F \times F$ and C is the product $C^+ \times C^-$ of two central simple algebras of degree 4. In this case $\mathbf{PSim}_+(A, \sigma) \simeq \mathbf{PGL}_1(C^+)$ is a rational group (Proposition 6).

We will assume that the discriminant $dF^{\times 2}$ is not trivial, i.e. L is a field. By the Corollary of Lemma 5

$$\mathbf{PSim}_+(A, \sigma)\,(F)/R = G_+(A, \sigma)/F^{\times 2}.N_{L/F}\,\mathrm{Nrd}\, C.$$

Consider the following four cases:

1. The algebra C is split.
2. Ind C $= 2$ and A is split.
3. Ind C $= 2$ and A is not split.
4. Ind C $= 4$.

Case 1: *The algebra* C *is split.*

By the formula above $\mathbf{PSim}_+(A, \sigma)(F)/R = G_+(A, \sigma)/N_{L/F}(L^\times)$. On the other hand, it is known that $G_+(A, \sigma) \subset N_{L/F}(L^\times)$ ([26, Theorem A]), hence $G_+(A, \sigma) = N_{L/F}(L^\times)$ and the group $\mathbf{PSim}_+(A, \sigma)$ is R-trivial in the case 1. Applying Corollary 2 of Proposition 3 to the torus $G = R_{L/F}(\mathbf{G}_{m, L})$ and the norm homomorphism $N_{L/F} = \alpha : G \to \mathbf{G}_{m, F}$ one sees immediately that the variety of the group $\mathbf{PSim}_+(A, \sigma)$ is stably rational in this case.

Case 2: Ind C $= 2$ *and* A *is split.*

The involution σ is adjoint to some quadratic form q of dimension 6 and discriminant $dF^{\times 2}$.

We claim that there is a quadratic form q' of dimension 4 and the same discriminant $dF^{\times 2}$ such that $G(q) = G(q')$ and $C_0(q) \sim C_0(q')$. Indeed, if q is isotropic then $q \simeq q' \perp \mathbf{H}$ and q' clearly satisfies the properties we need.

Assume that q is anisotropic. Since by assumption, the degree 4 algebra $C \sim C(q_L)$ is not a skewfield, it follows that the Albert form q_L of C is isotropic ([1]). Hence, $q \simeq q'' \perp \langle x, -xd \rangle$ for some $x \in F^\times$ and some 4-dimensional quadratic form q'' of trivial discriminant ([18, Lemma VII.3.1]). Replacing q by some multiple we may assume that $q'' = \langle\langle b, c \rangle\rangle$ for $b, c \in F^\times$. Consider the quadratic form $q' = \langle 1, -b, -c, bcd \rangle$ of discriminant $dF^{\times 2}$. By the theorem of Dieudonné [10, Th. 2], a similarity factor of a quadratic form of even dimension is a norm of the discriminant extension, so that $G(q), G(q') \subset N_{L/F}(L^\times)$. Since in the Witt ring $q - q''$ and $q' - q''$ is a multiple of $\langle\langle d \rangle\rangle$ and $G(\langle\langle d \rangle\rangle) = N_{L/F}(L^\times)$, it follows that $G(q) = N_{L/F}(L^\times) \cap G(q'') = G(q')$. On the other hand, $C = C_0(q) \sim (b, c)_L \simeq C_0(q')$.

By Proposition 5,

$$G_+(A, \sigma) = G(q) = G(q') = F^{\times 2}.N_{L/F}\,\mathrm{Nrd}(C_0(q')) = F^{\times 2}.N_{L/F}\,\mathrm{Nrd}(C_0(q)) = F^{\times 2}.N_{L/F}\,\mathrm{Nrd}\,C,$$

hence by the Corollary of Lemma 5 the group $\mathbf{PSim}_+(A, \sigma)$ is R-trivial in case 2.

Proposition 8. — *In case 2 the group* $\mathbf{PSim}_+(A, \sigma)$ *is stably rational.*

Proof. — Denote by D the quaternion algebra $(b, c)_F$, so that

$$G_+(A, \sigma) = G(q) = N_{L/F}(L^\times) \cap \mathrm{Nrd}(D).$$

Let G be a subgroup in $R_{L/F}(\mathbf{G}_{m,L}) \times \mathbf{GL}_1(D)$ consisting of pairs (x, y) such that $N_{L/F}(x) = \mathrm{Nrd}(y)$. The kernel of the homomorphism

$$G \to \mathbf{G}_{m,F}, \quad (x, y) \mapsto N_{L/F}(x) = \mathrm{Nrd}(y)$$

equals $R^1_{L/F}(\mathbf{G}_{m,L}) \times \mathbf{SL}_1(D)$ and hence is a rational variety. The variety of G is defined by

$$u^2 - dv^2 = x^2 - by^2 - cz^2 + bct^2 \neq 0$$

which is an open subset in a smooth affine quadric with an F-rational point, hence is an F-rational variety. By Corollary 2 of Proposition 3, $\mathbf{PSim}_+(A, \sigma)$ is stably rational. ∎

Case 3: Ind C = 2 *and* A *is not split.*

Since the index of A is a 2-power, in this case $A = \mathrm{End}_Q(V)$ where Q is a quaternion division algebra over F and V is the right Q-vector space of dimension 3.

By the results of section 0, the algebra A_L is similar to $C \otimes_L C$ which splits by assumption. Hence, A is split over L, therefore $Q = (d, b)_F$ for some $b \in F^\times$ is a quaternion division algebra over F and V is the right Q-vector space of dimension 3. Denote the generators of Q by i and $j : i^2 = d$, $j^2 = b$, $ij = -ji$.

The algebra A splits over L, hence the involution σ_L is adjoint to some quadratic form q over L of dimension 6 and trivial discriminant. Since $C(q)$ is similar to C over L, it follows that q is an Albert form for C. By assumption, C is not a division algebra, hence q is an isotropic form ([1]), therefore, the involution σ_L is isotropic.

Lemma 9. — Let (V, h) *be a* (-1)*-hermitian form over* Q *with respect to the canonical involution. Then* h_L *is isotropic over* L *if and only if* h *over* F *represents* xi *for some* $x \in F^\times$.

Proof. — Assume that h_L is isotropic. If h is isotropic over F, then it represents xi for any $x \in F^\times$. Let h be anisotropic over F. By assumption, there exist $u, v \in V$ such that $u + v\sqrt{d} \neq 0$ and $h(u + v\sqrt{d}, u + v\sqrt{d}) = 0$ or, equivalently, $h(u, u) + dh(v, v) = 0$ and $h(u, v) \in F$.

Assume first that $u + vi = 0$. Then $v \neq 0$, $u = -vi$ and

$$0 = h(u, u) + dh(v, v) = -ih(v, v)\, i + dh(v, v).$$

It follows that $h(v, v)$ commutes with i in Q, therefore, $0 \neq h(v, v) \in F(i) = F + F.i$, and $h(v, v) \in F.i$ since $\overline{h(v, v)} = -h(v, v)$.

If $u + vi \neq 0$ then

$$0 \neq h(u + vi, u + vi) = -dh(v, v) - ih(v, v)\, i + h(u, v)\, i - ih(v, u) = xi$$

for some $x \in F^\times$ since $h(u, v) = -h(v, u) \in F$ and $dq + iqi \in F.i$ for any pure quaternion q.

Conversely, if $h(u, u) = xi$ for some $u \in V$ and $x \in F^\times$, then one easily checks that $h(ui + u\sqrt{d}, ui + u\sqrt{d}) = 0$. ∎

The canonical involution on Q is symplectic, so that the orthogonal involution σ on A is adjoint to some (-1)-hermitian form h of dimension 3 on V over Q with respect to the canonical involution on Q. Since the involution σ_L is isotropic, it follows that h is isotropic over L. By Lemma 9, $h = h' \perp \langle xi \rangle$ for some $x \in F^\times$ and some (-1)-hermitian form h' on 2-dimensional subspace $V' \subset V$. It follows from the equality $\operatorname{disc}(\langle xi \rangle) = dF^{\times 2}$ that $\operatorname{disc}(h') \in F^{\times 2}$. Note also that if $c = y^2 - dz^2 \in N_{L/F}(L^\times)$ for $y, z \in F$ then $\overline{(y + zi)}.xi.(y + zi) = cxi$, hence $c\langle xi \rangle \simeq \langle xi \rangle$, therefore, $N_{L/F}(L^\times) \subset G(\langle xi \rangle)$.

Consider a central simple algebra $A' = \operatorname{End}_Q V'$ with the orthogonal involution σ' adjoint to a (-1)-hermitian form h' of trivial discriminant. Then $A' = Q_1 \otimes_F Q_2$, $\sigma = \sigma_1 \otimes \sigma_2$ where Q_i are quaternion algebras and σ_i are canonical involutions ([16]). The Clifford algebra $C(A', \sigma')$ is isomorphic to $Q_1 \times Q_2$ ([33, Prop. 4.5]). Note that $Q_1 \otimes_F Q_2 \simeq A' \sim Q$.

Let q' be a 4-dimensional quadratic form associated to the involution σ' over L. Since the form $\langle xi \rangle$ is hyperbolic over L by Lemma 9, we have: $q \simeq q' \perp \mathbf{H}$. Hence,

$$C = C(A, \sigma) \sim C(q) \sim C(q') \sim C^{\pm}(A'_L, \sigma'_L) \simeq (Q_i)_L \quad \text{for } i = 1, 2.$$

Lemma 10. — We have $F^{\times 2}.N_{L/F} \operatorname{Nrd} C = N_{L/F}(L^\times) \cap \operatorname{Nrd} Q_i$ *for* $i = 1, 2$.

Proof. — Assume that $c \in N_{L/F} \operatorname{Nrd} C$. Then there exists a finite extension E/L such that E splits C and c is a norm of E/F. The algebra C is similar to $(Q_i)_L$, hence E splits Q_i and $c \in \operatorname{Nrd} Q_i$.

Conversely, assume that $c \in N_{L/F}(L^\times) \cap \operatorname{Nrd} Q_i$. Then c is the norm of some quadratic extension E over F which splits Q_i. It is known that

$$N_{L/F}(L^\times) \cap N_{E/F}(E^\times) = F^{\times 2}.N_{P/F}(P^\times)$$

where $P = L \otimes_F E$ ([11, 2.13]). Hence,

$$c \in F^{\times 2}.N_{P/F}(P^\times) \subset F^{\times 2}.N_{L/F} \operatorname{Nrd} C,$$

since P splits C. ■

Proposition 9. — The following equality holds for $i = 1, 2$:

$$\mathbf{PSim}_+(A, \sigma)(F)/R = (N_{L/F}(L^\times) \cap \operatorname{Nrd} Q_1.\operatorname{Nrd} Q_2)/(N_{L/F}(L^\times) \cap \operatorname{Nrd} Q_i).$$

Proof. — We prove first that

$$G_+(A, \sigma) = N_{L/F}(L^\times) \cap G_+(A', \sigma').$$

If $c \in N_{L/F}(L^\times) \cap G_+(A', \sigma')$, then $c.h' \simeq h'$ and $c.\langle xi \rangle \simeq \langle xi \rangle$, hence $c.h \simeq h$ and $c \in G(A, \sigma)$. Since $c \in N_{L/F}(L^\times)$ and A is not split, it follows that $c \in G_+(A, \sigma)$ by [26, Th. A]. Coversely, if $c \in G_+(A, \sigma)$ then $c.h \simeq h$ and $c \in N_{L/F}(L^\times)$ ([26, Th. A]), hence $c.\langle xi \rangle \simeq \langle xi \rangle$ and $c.h' \simeq h'$, $c \in G(A', \sigma') = G_+(A', \sigma')$ ([26, Cor. 1.16]).

It follows from the Corollary of Proposition 5 that

$$G_+(A', \sigma') = \mathrm{Nrd}\, Q_1 . \mathrm{Nrd}\, Q_2,$$

hence $$G_+(A, \sigma) = N_{L/F}(L^\times) \cap \mathrm{Nrd}\, Q_1 . \mathrm{Nrd}\, Q_2.$$

The statement follows now from the Corollary of Lemma 5 and Lemma 10. ■

Lemma 11. — There exists a field extension E/F, *linearly disjoint from* L/F, *such that* $N_{EL/E}(EL^\times) \not\subset \mathrm{Nrd}(Q_1)_E$ *and* $\mathrm{Nrd}(Q_2)_E \not\subset \mathrm{Nrd}(Q_1)_E$.

Proof. — The algebra $(Q_1)_L$ is similar to C and hence is not split. Since $A \sim Q_1 \otimes_F Q_2$ is not split, it follows that $Q_1 \not\simeq Q_2$.

Let

$$Q_2 = (a_2, b_2)_F, \quad E = F(t_1, t_2, t_3, t_4), \quad f = t_1^2 - dt_2^2 \in N_{EL/E}(E^\times),$$

$$g = t_1^2 - a_2 t_2^2 - b_2 t_3^2 + a_2 b_2 t_3^2 \in \mathrm{Nrd}(Q_2)_E.$$

Assume that $f \in \mathrm{Nrd}(Q_1)_E$. By the Subform Theorem ([18, Th. 2.8]), the form $\langle 1, -d \rangle$ is the subform in the reduced norm form of Q_1, hence Q_1 is split by L, a contradiction. If $g \in \mathrm{Nrd}(Q_1)_E$ then again by the Subform Theorem, the reduced norm forms of Q_1 and Q_2 are isomorphic, hence $Q_1 \simeq Q_2$, a contradiction. ■

Denote by $H^n(F)$ the Galois cohomology group $H^n(\mathrm{Gal}(F_{sep}/F), \mathbf{Z}/2\mathbf{Z})$. For any $x \in F^\times$ denote by (x) the class in $H^1(F)$ corresponding to $xF^{\times 2}$ with respect to the canonical isomorphism $F^\times/F^{\times 2} \simeq H^1(F)$.

Lemma 12 ([3, Satz 5.6]). — *Let* $Q_1 = (a_1, b_1)_F$ *and* Z *be a projective quadric given by the form* $\langle 1, -a_1, -b_1, a_1 b_1, -c \rangle$, $c \in F^\times$. *Then the kernel of the canonical homomorphism* $f: H^3(F) \to H^3(F(Z))$ *is generated by* $(a_1) \cup (b_1) \cup (c)$. ■

Corollary. — The intersection $F^\times \cap \mathrm{Nrd}(Q_1 \otimes_F F(Z))$ *is generated by* $\mathrm{Nrd}\, Q_1$ *and* c.

Proof. — If $e \in F^\times \cap \mathrm{Nrd}(Q_1 \otimes_F F(Z))$, then $(a_1) \cup (b_1) \cup (e) \in \ker(f)$, i.e.

$$(a_1) \cup (b_1) \cup (e) = (a_1) \cup (b_1) \cup (c^k)$$

by Lemma 12, hence $e \in c^k.\mathrm{Nrd}\, Q_1$ by [25, Th. 12.1]. ■

Proposition 10. — In case 3 *the group* $\mathbf{PSim}_+(A, \sigma)$ *is not* R-*trivial.*

Proof. — It follows from Lemma 11 that there is a field extension E/F and elements $x \in N_{EL/E}(EL^\times)$ and $y \in \mathrm{Nrd}(Q_2 \otimes_F E)$ which do not belong to $\mathrm{Nrd}(Q_1 \otimes_F E)$. Replacing F by E we may assume that $x \in N_{L/F}(L^\times)$, $y \in \mathrm{Nrd}\, Q_2$ but $x, y \notin \mathrm{Nrd}\, Q_1$.

Let Z be the quadric given by the quadratic form $\langle 1, -a_1, -b_1, a_1 b_1, -xy \rangle$. Since x and y do not belong to $\mathrm{Nrd}\, Q_1$, it follows from the Corollary of Lemma 12 that $y \notin \mathrm{Nrd}(Q_1 \otimes_F F(Z))$. Therefore, replacing F by F(Z) we may assume that $xy \in \mathrm{Nrd}\, Q_1$, but $y \notin \mathrm{Nrd}\, Q_1$ and hence $x \notin \mathrm{Nrd}\, Q_1$. On the other hand,

$$x = xy.y^{-1} \in N_{L/F}(L^\times) \cap \mathrm{Nrd}\, Q_1.\mathrm{Nrd}\, Q_2,$$

i.e. by Proposition 9, x represents a non-trivial element in $\mathbf{PSim}_+(A, \sigma)$ (F)/R. ∎

Case 4: Ind C = 4.

It follows from Theorem 2 that the group $\mathbf{PSim}_+(A, \sigma)$ is not R-trivial in this case. We have proved the following

Theorem 3. — Let A *be a central simple algebra of degree* 6 *over a field* F *with orthogonal involution* σ, *and* $C = C(A, \sigma)$ *be the Clifford algebra of* (A, σ). *Then*

I. *If* $\mathrm{disc}(\sigma)$ *is trivial, then the group* $\mathbf{PSim}_+(A, \sigma)$ *is rational and hence* R-*trivial.*

II. *If* $\mathrm{disc}(\sigma)$ *is not trivial, then*

(1) *if* C *splits, or if* ind C = 2 *and* A *splits, then the group* $\mathbf{PSim}_+(A, \sigma)$ *is stably rational and hence* R-*trivial*;

(2) *if* ind C = 4, *or if* ind C = 2 *and* A *is not split, then the group* $\mathbf{PSim}_+(A, \sigma)$ *is not* R-*trivial and hence is not stably rational.* ∎

Corollary. — Let q be a non-degenerate quadratic form of dimension 6 *over* F. *Then* $\mathbf{PGO}_+(q)$ *is not stably rational if and only if the discriminant of q is not trivial and* $C_0(q)$ *is a division algebra.* ∎

4. Examples

In this section we give examples of not stably rational adjoint simple groups of type D_n, $n \geqslant 3$ over " small " fields.

Let $Q = (a, b)_F$ be a division quaternion algebra over a field F, $A = M_n(Q)$. Denote by Q^+ the subspace of dimension 3 in Q of pure quaternions (elements of trace zero). For a (-1)-hermitian form $h = \langle x_1, x_2, \ldots, x_n \rangle$ with $x_1, x_2, \ldots, x_n \in Q^+$ over Q with respect to the canonical involution on Q we consider the adjoint orthogonal involution σ_h on A.

Proposition 11. — Let n be an odd integer, $n \geqslant 3$ *and* $d \in -\mathrm{Nrd}(Q)$. *Then*

(1) *There exists a* (-1)-*hermitian form as above such that* $\mathrm{disc}(\sigma_h) = d.F^{\times 2}$.

(2) *If* $d \notin D(\langle a, b, -ab \rangle) \cup F^{\times 2}$ *and* $\mathrm{disc}(\sigma_h) = d.F^{\times 2}$, *then* $\mathbf{PSim}_+(A, \sigma_h)$ *is a not stably rational adjoint simple group of type* D_n.

Proof. — 1. It is easy to see that any element in Q is a product of two and hence n pure quaternions. Therefore, if $d = -\mathrm{Nrd}(x)$ for some $x \in Q^\times$, there exist $x_1, x_2, \ldots, x_n \in Q^+$ such that x is the product of all x_i. Taking $h = \langle x_1, x_2, \ldots, x_n \rangle$, we have $\mathrm{disc}(\sigma_h) = x_1^2 x_2^2 \ldots x_n^2 F^{\times 2} = -\mathrm{Nrd}(x) F^{\times 2} = dF^{\times 2}$ (see example in section 0).

2. By assumption, the quadratic discriminant extension $L = F(\sqrt{d})$ does not split Q and hence A. The tensor square of the Clifford algebra C of (A, σ_h) over L is similar to A_L (section 0) and therefore is not split. Thus, the exponent (and hence the index) of C is at least 4 and by Theorem 2 the group $\mathbf{PSim}_+(A, \sigma_h)$ is not stably rational. ∎

The situation described in the Proposition 11 can be realized over an arbitrary number field F. Consider any quaternion algebra Q which is ramified at some finite place v. Then any totally negative $d \in F^\times - F^{\times 2}$ being a square in the completion F_v satisfies the condition of the Proposition 11.

Example. — Consider the field F of rational numbers, $Q = (-1, -1)_F$ with the standard base $1, i, j, k$ ramified at $v = 2$; $d = -28$; $h = \langle 3i + 2j + k, 2i, i \rangle \perp \frac{n-3}{2} \mathbf{H}$. Then the group $\mathbf{PSim}_+(M_n(Q), \sigma_h)$ is an adjoint simple not stably rational group of the type D_n over F.

Now consider the case of an even number $n \geqslant 4$. Since the tensor square of the Clifford algebra C of a central simple algebra of degree $2n$ with an orthogonal involution splits, over a number field we always have ind $C \leqslant 2$ and we cannot apply Theorem 2. Instead of a number field, we consider the field of rational functions $F = K(t)$ over some field K. Assume that we are given $a, b, c, d \in K^\times$ such that the quaternion algebra $(a, b)_K$ does not split over the biquadratic extension $K(\sqrt{c}, \sqrt{d})$ (such examples clearly exist over an arbitrary number field K). Hence by [34, Prop. 2.4] $C = (a, b)_F \otimes_F (c, t)_F \otimes_F F(\sqrt{d})$ is a division algebra of degree 4 over $F(\sqrt{d})$. It is clear that C is similar to the even Clifford algebra of the quadratic form $q = \langle ad, b, -ab, -c, -t, ct \rangle \perp (n-3)\,\mathbf{H}$ over F. Hence, by Corollary of Theorem 2, $\mathbf{PGO}_+(q)$ is an adjoint simple not stably rational group of type D_n over $F = K(t)$.

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Department of Mathematics and Mechanics,
St. Petersburg State University,
St. Petersburg, 198904 Russia
E-mail address: merkurev@math.lgu.spb.su

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